

MODIFICATION OF THE SIMPSON MODULI SPACE $M_{3m+1}(\mathbb{P}_2)$ BY VECTOR BUNDLES (I)

OLEKSANDR IENA AND GÜNTHER TRAUTMANN

ABSTRACT. We consider the moduli space of stable vector bundles on curves embedded in \mathbb{P}_2 with Hilbert polynomial $3m + 1$ and construct a compactification of this space by vector bundles. The result \widetilde{M} is a blow up of the Simpson moduli space $M_{3m+1}(\mathbb{P}_2)$.

CONTENTS

1. Introduction	1
2. Moduli space $M := M_{3m+1}(\mathbb{P}_2)$	3
3. Definition of R -bundles and their properties	5
4. R -bundles as 1-dimensional degenerations of $(3m + 1)$ -sheaves.	8
5. Classification result (main result)	13
6. Examples	15
7. Universal families of R -bundles	19
References	22

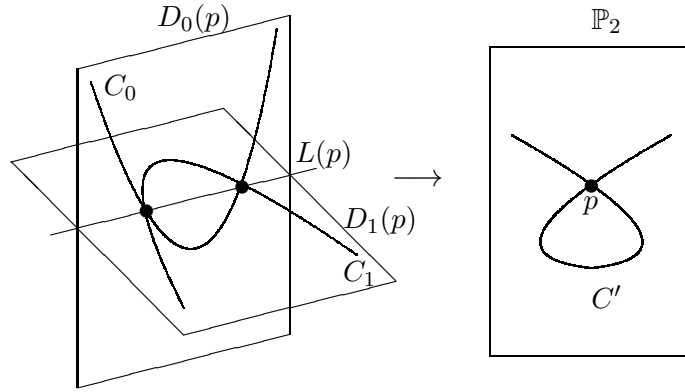
1. INTRODUCTION

1.1. **Motivation.** Simpson showed in [12] that for an arbitrary smooth projective variety X and for an arbitrary numerical polynomial $P \in \mathbb{Q}[m]$ there is a coarse moduli space $M := M_P(X)$ of semi-stable sheaves on X with Hilbert polynomial P , which turns out to be a projective variety.

In many cases M contains an open dense subset M_B whose points consists of sheaves which are locally free on their support. So, one could consider M as a compactification of M_B . We call sheaves in the boundary $M \setminus M_B$ *singular*. It is an interesting question whether and how one could replace the boundary of singular sheaves by one which consists entirely of vector bundles with varying and possibly reducible supports.

In this paper we answer this question for the moduli space $M = M_{3m+1}(\mathbb{P}_2)$ of stable sheaves supported on cubic curves in the projective plane having Hilbert polynomial $P(m) = 3m + 1$. This is the first non-trivial case of 1-dimensional sheaves on surfaces. It turns out that the blow up \widetilde{M} of M along the locus of singular sheaves is a compactification in the above sense. Even so this is only a first example it leads to several interesting constructions which might be helpful in more general situations.

1.2. **Summary of the paper.** The moduli space $M = M_{3m+1}(\mathbb{P}_2)$ is completely understood see [6], [3], [2]. The sheaves in M will be called $(3m + 1)$ -*sheaves*. Their supports are defined by the their Fitting ideals. M is smooth of dimension 10 and isomorphic to the universal cubic curve. The subvariety M' of singular sheaves is a smooth subvariety in M of dimension 8. To indicate this, we will denote it by M_8 .

FIGURE 1. Surface $D(p)$ and support of R -bundles.

For a singular $(3m + 1)$ -sheaf \mathcal{F} there is only one point $p \in C' = \text{Supp } \mathcal{F}$, where \mathcal{F} is not $\mathcal{O}_{C'}$ -free. This point is a singular point of the curve C' .

For each point p of the projective plane, we introduce a reducible surface $D(p)$ consisting of two irreducible components $D_0(p)$ and $D_1(p)$, $D_0(p)$ being the blow up of the projective plane at p and $D_1(p)$ being another projective plane, such that these components intersect along the line $L(p)$ which is the exceptional divisor of $D_0(p)$. Each surface $D(p)$ can be defined as the subvariety in $\mathbb{P}_2 \times \mathbb{P}_2$ with equations $u_0x_1, u_0x_2, u_1x_2 - u_2x_1$ where the x_i resp. u_i are the homogeneous coordinates of the first resp. second \mathbb{P}_2 , such that the first projection contracts $D_1(p)$ to p and describes $D_0(p)$ as the blow up. We let $\mathcal{O}_{D(p)}(a, b)$ denote the invertible sheaf induced by $\mathcal{O}_{\mathbb{P}_2}(a) \boxtimes \mathcal{O}_{\mathbb{P}_2}(b)$.

For every singular $(3m + 1)$ -sheaf we introduce a family of coherent 1-dimensional sheaves on $D(p)$, locally free on their (Fitting-)support. We call the objects of this family R -bundles.

Definition 1.1. *An R -bundle associated to a singular $(3m + 1)$ -sheaf \mathcal{F} with singular point p is a coherent 1-dimensional sheaf \mathcal{E} on $D(p)$ subject to the following conditions.*

- \mathcal{E} is locally free on its support $C = \text{Supp } \mathcal{E}$.
- There is an exact sequence

$$(1) \quad 0 \rightarrow 2\mathcal{O}_{D_1(p)}(-L) \xrightarrow{\varrho} \sigma_p^*(\mathcal{F}) \xrightarrow{\theta} \mathcal{E} \rightarrow 0.$$

- $\mathcal{E}|_{D_0(p)}$ has Hilbert polynomial $4m + 1$ with respect to the sheaf $\mathcal{O}_{D_0(p)}(1, 1)$.

R -bundles turn out to be flat limits of non-singular $(3m + 1)$ -sheaves, hence they can be seen as reasonable replacements for singular $(3m + 1)$ -sheaves. R -bundles are supported on reducible curves of the type $C = C_0 \cup C_1$, where $C_i = C \cap D_i(p)$. The curve C_0 coincides in most of the cases with the proper transform of $C' = \text{Supp } \mathcal{F}$ under the blow up $D_0(p) \rightarrow \mathbb{P}_2$, in general it is a subvariety of the total transform of C' . The curve C_1 is a conic in the projective plane $D_1(p)$ bearing the degree of an R -bundle.

For R -bundles we introduce the following equivalence relation, which is an “embedded” version of Definition 4.1, (ii) from [11] (see also [7] and [8]).

Definition 1.2. *Let \mathcal{E}_1 and \mathcal{E}_2 be two R -bundles on $D(p)$ associated to the same singular sheaf. We call them equivalent if there exists an automorphism ϕ of $D(p)$ that acts identically on $D_0(p)$ and such that $\phi^*(\mathcal{E}_1) \cong \mathcal{E}_2$.*

The main result of this paper is the following theorem.

Theorem 1.3. *The set of equivalence classes of R -bundles on $D(p)$ associated to the same singular $(3m + 1)$ -sheaf \mathcal{F} is in a natural bijection with the projectivised normal space $\mathbb{P}N_{[\mathcal{F}]}$ of the normal space $N_{[\mathcal{F}]} = T_{[\mathcal{F}]}(M)/T_{[\mathcal{F}]}(M_8)$ to M_8 at the point $[\mathcal{F}] \in M_8$.*

This justifies the following interpretation.

Corollary 1.4. *The blow up \widetilde{M} of M along M_8 is the space of all the isomorphism classes of the non-singular $(3m + 1)$ -sheaves together with all the equivalence classes of all R -bundles, which are the points of the exceptional divisor.*

In section 7 we construct a “universal” flat family of non-singular $(3m + 1)$ -sheaves and R -bundles over \widetilde{M} whose members are representatives of the equivalence classes of those sheaves. One can expect that under an appropriate notion of families of non-singular $(3m + 1)$ -sheaves together with R -bundles the blow up \widetilde{M} represents the corresponding moduli functor.

1.3. Structure of the paper. In Section 2 we collect the essential facts about the moduli space $M_{3m+1}(\mathbb{P}_2)$. Details and technicalities necessary to define R -bundles are discussed in Section 3. In Section 4 we discuss the most important properties of R -bundles. Using them we prove then the main result in Section 5. Examples of R -bundles are considered in Section 6. In Section 7 we construct parameter spaces for all non-singular $(3m + 1)$ -sheaves (up to isomorphism) and all R -bundles (up to equivalence).

1.4. Some notations and conventions. In this paper \mathbb{k} is an algebraically closed field of characteristic zero. We work in the category of separated schemes of finite type over \mathbb{k} and call them varieties, using only their closed points. Note that we do not restrict ourselves to reduced or irreducible varieties. Dealing with homomorphism between direct sums of line bundles and identifying them with matrices, we consider the matrices acting on elements from the right, i. e, the composition $X \xrightarrow{A} Y \xrightarrow{B} Z$ is given by the matrix $A \cdot B$.

2. MODULI SPACE $M := M_{3m+1}(\mathbb{P}_2)$

Let us recall here some of the results from [3]. We consider stable sheaves on \mathbb{P}_2 with Hilbert polynomial $3m + 1$ and call them simply $(3m + 1)$ -sheaves. Every $(3m + 1)$ -sheaf \mathcal{F} defines a non-trivial extension

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{F} \rightarrow \mathbb{k}_p \rightarrow 0,$$

where C is a cubic curve supporting \mathcal{F} and \mathbb{k}_p is the skyscraper sheaf at $p \in C$ of length 1, whereby $h^0(\mathcal{F}) = 1$.

There exists a fine moduli space $M = M_{3m+1}(\mathbb{P}_2)$ of $(3m + 1)$ -sheaves. M is projective, nonsingular of dimension 10 and is isomorphic to the universal cubic

$$\{(\langle f \rangle, \langle x \rangle) \in \mathbb{P}_9 \times \mathbb{P}_2 \mid f(x) = 0\},$$

where \mathbb{P}_9 is identified with the space of cubic curves in \mathbb{P}_2 . The map underlying this isomorphism is given by $[\mathcal{F}] \mapsto (C, p)$.

The $(3m + 1)$ -sheaves on \mathbb{P}_2 are exactly the sheaves given by locally free resolutions

$$0 \rightarrow 2\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{A} \mathcal{O}_{\mathbb{P}_2}(-1) \oplus \mathcal{O}_{\mathbb{P}_2} \rightarrow \mathcal{F} \rightarrow 0,$$

where

$$(2) \quad A = \begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix}$$

with linear independent linear forms $z_1, z_2 \in \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1))$ and non-zero determinant. The space of all such matrices is a parameter space of M and is denoted by X . X is isomorphic to an open subset in \mathbb{k}^{18} and is acted on by the group

$$G = \mathrm{GL}_2(\mathbb{k}) \times H,$$

where H is the group of 2×2 matrices

$$\begin{pmatrix} \lambda & z \\ 0 & \mu \end{pmatrix}, \quad \lambda, \mu \in \mathbb{k}, \quad \lambda\mu \neq 0, \quad z \in \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1)).$$

The action is defined by the rule $(g, h) \cdot A = gAh^{-1}$. M is a geometric quotient of X by G , the quotient morphism $X \xrightarrow{\nu} M$ is

$$A = \begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix} \mapsto \langle \det A \rangle \times p(A),$$

where $p(A) := \langle z_1 \wedge z_2 \rangle$ denotes the common zero point of z_1 and z_2 in \mathbb{P}_2 .

A $(3m + 1)$ -sheaf is called singular if it is not locally free on its support. A point $\langle f \rangle \times p \in M$ represents an isomorphism class of a singular sheaf if and only if p is a singular point of the curve $\{f = 0\} \subset \mathbb{P}_2$. The subvariety of all singular sheaves in M is denoted by M_8 . It is easy to verify that M_8 is smooth of codimension 2 in M .

The corresponding subvariety in X is denoted by X_8 . A matrix A as in (2) belongs to X_8 if and only if $q_1(p(A)) = q_2(p(A)) = 0$. These two conditions give two global equations of X_8 in X and one concludes that X_8 is a global complete intersection in X , smooth of codimension 2.

Let $x_0, x_1, x_2 \in \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1))$ be fixed coordinates of \mathbb{P}_2 . Then a matrix A from (2) with $z_1 = x_1, z_2 = x_2$ belongs to X_8 if and only if

$$(3) \quad A = \begin{pmatrix} x_1 & x_1y_1 + x_2y_2 \\ x_2 & x_1z_1 + x_2z_2 \end{pmatrix}$$

for some linear forms $y_i, z_i \in \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1)), i = 1, 2$.

Let $\widetilde{M} \rightarrow M$ be the blow up of M along M_8 . Since M_8 is smooth of codimension 2 in M , the exceptional divisor E_M of the blow up $\widetilde{M} \rightarrow M$ is isomorphic to the projective normal bundle $\mathbb{P}N_{M_8/M}$. Let $\widetilde{X} \xrightarrow{\alpha} X$ be the blowing up of X along X_8 . Since X_8 is defined by two global equations, \widetilde{X} may be considered as a subvariety in $X \times \mathbb{P}_1$ such that the exceptional divisor E_X of $\widetilde{X} \xrightarrow{\alpha} X$ may be identified with $X_8 \times \mathbb{P}_1$.

Note that X_8 is invariant under the action of G . Therefore, since the blowing up $\alpha : \widetilde{X} \rightarrow X$ is an isomorphism over $X \setminus X_8$, we obtain an action of G on $\widetilde{X} \setminus E_X$. This action can be uniquely extended to an action of G on \widetilde{X} . An element $(g, h) \in G$ acts by the rule

$$(g, h) \cdot (A, \langle t_3, t_4 \rangle) = (gAh^{-1}, \langle (t_3, t_4)g^T \rangle).$$

We obtain the following commutative diagram

$$\begin{array}{ccc} G \times \widetilde{X} & \longrightarrow & \widetilde{X} \\ \mathrm{id} \times \alpha \downarrow & & \downarrow \alpha \\ G \times X & \longrightarrow & X. \end{array}$$

Note that for an arbitrary point $(A, \langle t_3, t_4 \rangle) \in \tilde{X}$ its stabilizer is the subgroup

$$St = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \times \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \mid \lambda \in \mathbb{k}^* \right\}.$$

Therefore, we can consider the corresponding free action of the group $\mathbb{P}G = G/St$ on \tilde{X} .

Note that since $\nu^{-1}(M_8) = X_8$, we obtain a unique lifting $\tilde{\nu}$ of ν , i. e., the commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\exists! \tilde{\nu}} & \tilde{M} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\nu} & M. \end{array}$$

Then $\tilde{\nu} : \tilde{X} \rightarrow \tilde{M}$ is G -invariant and the set of the orbits coincides with the set of the fibres $\tilde{\nu}^{-1}(\xi)$, $\xi \in \tilde{M}$. In a neighbourhood of every point of \tilde{M} there is a local section of $\tilde{\nu}$. Using this and Zariski main theorem one shows that \tilde{X} is a principal $\mathbb{P}G$ -bundle over \tilde{M} . Hence \tilde{M} is a geometrical quotient.

3. DEFINITION OF R -BUNDLES AND THEIR PROPERTIES

Surfaces $D(p)$. Let p be a point in \mathbb{P}_2 . Let $\sigma_p : \text{Bl}_{0 \times p}(\mathbb{k} \times \mathbb{P}_2) \rightarrow \mathbb{k} \times \mathbb{P}_2$ be the blow up of $\mathbb{k} \times \mathbb{P}_2$ at $0 \times p$. Consider the composition of σ_p with the canonical projection onto \mathbb{k}

$$\text{Bl}_{0 \times p}(\mathbb{k} \times \mathbb{P}_2) \xrightarrow{\sigma_p} \mathbb{k} \times \mathbb{P}_2 \xrightarrow{pr_1} \mathbb{k}.$$

Denote by $D(p)$ the fibre over 0. It is a reducible projective surface consisting of two irreducible components $D_0(p)$ and $D_1(p)$. The first is isomorphic to the blow up $\text{Bl}_p \mathbb{P}_2$ of \mathbb{P}_2 at p , the second is a projective plane \mathbb{P}_2 . Their intersection $L(p) = D_0(p) \cap D_1(p)$ is the exceptional divisor of $D_0(p)$ and is a projective line in $D_1(p)$ (see Figure 1). The restriction $\sigma_p|_{D(p)}$ contracts $D_1(p)$ to p and is the blow up $D_0(p) \rightarrow \mathbb{P}_2$ at p .

Invertible sheaves on $D(p)$ and their cohomology. Note that all surfaces $D(p)$ are isomorphic to each other. Each surface $D(p)$ comes as a closed subvariety of $\mathbb{P}_2 \times \mathbb{P}_2$ such that σ_p is the restriction of the first projection to \mathbb{P}_2 .

$$\begin{array}{ccccc} & & \sigma_p & & \\ & & \curvearrowright & & \\ & & & & \\ D_0(p) & \longrightarrow & D(p) & \longleftarrow & D_1(p) \\ \downarrow \sigma_p & & \swarrow \sigma_p & & \downarrow \cong \\ \{p\} & \hookrightarrow & \mathbb{P}_2 & \xleftarrow{pr_1} & \mathbb{P}_2 \times \mathbb{P}_2 & \xrightarrow{pr_2} & \mathbb{P}_2 \end{array}$$

As a subvariety in $\mathbb{P}_2 \times \mathbb{P}_2$ every surface $D(p)$ has two different twisting sheaves on $D(p)$, namely $\mathcal{O}_{D(p)}(1, 0) = \mathcal{O}_{\mathbb{P}_2 \times \mathbb{P}_2}(1, 0)|_{D(p)}$, and $\mathcal{O}_{D(p)}(0, 1) = \mathcal{O}_{\mathbb{P}_2 \times \mathbb{P}_2}(0, 1)|_{D(p)}$. We can also define the following two divisors H and F on $D(p)$ by $\mathcal{O}_{D(p)}(H) = \mathcal{O}_{D(p)}(1, 0)$ and $\mathcal{O}_{D(p)}(F) = \mathcal{O}_{D(p)}(0, 1)$. In other words H is defined by the pull-back of a line $h \subset \mathbb{P}_2$ in the first \mathbb{P}_2 and F is defined by the pull-back of

a line $f \subset \mathbb{P}_2$ in the second \mathbb{P}_2 .

Let u_0, u_1, u_2 be the coordinates of the second \mathbb{P}_2 and let us choose the coordinates x_0, x_1, x_2 of the first \mathbb{P}_2 such that $p = \langle 1, 0, 0 \rangle$. Then the surface $D(p)$ is given by the equations

$$x_1 u_2 - x_2 u_1 = 0, \quad x_1 u_0 = 0, \quad x_2 u_0 = 0$$

with $D_0(p) = \{u_0 = 0\}$, $D_1(p) = \{x_1 = x_2 = 0\}$. The canonical lifting homomorphisms

$$\Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(h)) \rightarrow \Gamma(\mathbb{P}_2, \mathcal{O}_{D(p)}(H)), \quad \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(f)) \rightarrow \Gamma(\mathbb{P}_2, \mathcal{O}_{D(p)}(F))$$

are isomorphisms. By abuse of notation we denote the images of the homogeneous coordinates also by $x_0, x_1, x_2 \in \Gamma(\mathbb{P}_2, \mathcal{O}_{D(p)}(H))$ respectively $u_0, u_1, u_2 \in \Gamma(\mathbb{P}_2, \mathcal{O}_{D(p)}(F))$, each

forming a basis. Then we still have the equations $x_1u_2 - x_2u_1 = 0$, $x_1u_0 = 0$, $x_2u_0 = 0$ on $D(p)$ with $D_0(p)$, $D_1(p)$ as above.

One can show (cf. [5]) that the line bundles $\mathcal{O}_{D(p)}(H)$ and $\mathcal{O}_{D(p)}(F)$ are free generators of the Picard group of $D(p)$. More precisely, the map

$$\mathbb{Z} \oplus \mathbb{Z} \rightarrow \text{Pic}(D(p)), \quad (a, b) \mapsto [\mathcal{O}_{D(p)}(aH + bF)].$$

is a group isomorphism.

We will denote the restrictions of the divisors H and F to $D_i(p)$, $i = 0, 1$, by H_i and F_i respectively. Note that $H_1 \sim 0$, so $H = H_0$. If it does not cause any misunderstandings, we will often write just H and F for the restrictions H_i and F_i . The intersection line $L(p)$ as a divisor in $D_0(p)$ respectively in $D_1(p)$ will be denoted by $L_0(p)$ respectively $L_1(p)$. There are equivalences of divisors $L_1(p) \sim F_1$ and $L_0(p) \sim H_0 - F_0$. The intersection numbers are given by

$$\begin{aligned} L_0(p)^2 &= -1, & L_1(p)^2 &= 1, & H_0 \cdot F_0 &= 1, & H_0 \cdot L_0(p) &= 0, \\ F_0 \cdot L_0(p) &= 1, & F_1 \cdot L_1(p) &= 1, & F_0^2 &= 0, & F_1^2 &= 1. \end{aligned}$$

Proposition 3.1. (i) *The Euler characteristic of the invertible sheaf $\mathcal{O}_{D(p)}(aH + bF)$ is given by the formula $\chi(\mathcal{O}_{D(p)}(aH + bF)) = \frac{1}{2}(a + b)^2 + \frac{3}{2}(a + b) + 1$.*

(ii) *The Hilbert polynomial of $\mathcal{O}_{D(p)}(aH + bF)$ with respect to the invertible sheaf $\mathcal{L} = \mathcal{O}_{D(p)}(H + F)$ equals*

$$(4) \quad 2m^2 + [2(a + b) + 3] \cdot m + \frac{1}{2}(a + b)^2 + \frac{3}{2}(a + b) + 1.$$

(iii) *Higher cohomology groups vanish for the following sheaves:*

$$\begin{aligned} &\mathcal{O}_{D(p)}(-2F), \quad \mathcal{O}_{D(p)}(-H), \quad \mathcal{O}_{D(p)}(-F), \quad \mathcal{O}_{D(p)}, \quad \mathcal{O}_{D(p)}(H - F), \quad \mathcal{O}_{D(p)}(-H + F), \\ &\mathcal{O}_{D(p)}(H), \quad \mathcal{O}_{D(p)}(F), \quad \mathcal{O}_{D(p)}(-H - F), \quad \mathcal{O}_{D(p)}(H + F), \quad \mathcal{O}_{D(p)}(H + 2F), \end{aligned}$$

hence their h^0 can be computed using the formula above.

Some key tools for the proof. All the statements of Proposition 3.1 can be directly verified using the following observations.

For a locally free sheaf \mathcal{G} on $D(p)$ there is the “gluing” exact sequence

$$0 \rightarrow \mathcal{G} \rightarrow \mathcal{G}|_{D_0(p)} \oplus \mathcal{G}|_{D_1(p)} \rightarrow \mathcal{G}|_{L(p)} \rightarrow 0$$

and in particular the exact sequences

$$(5) \quad 0 \rightarrow \mathcal{O}_{D(p)}(aH + bF) \rightarrow \mathcal{O}_{D_0(p)}(aH_0 + bF_0) \oplus \mathcal{O}_{D_1(p)}(bF_1) \rightarrow \mathcal{O}_L(b) \rightarrow 0, \quad a, b \in \mathbb{Z}.$$

The cohomology of the sheaves $\mathcal{O}_{D(p)}(aH + bF)$ can be deduced from (5) and the exact sequences

$$0 \rightarrow \mathcal{O}_{\mathbb{P}_2 \times \mathbb{P}_1}(a - 1, b - 1) \rightarrow \mathcal{O}_{\mathbb{P}_2 \times \mathbb{P}_1}(a, b) \rightarrow \mathcal{O}_{D_0(p)}(aH_0 + bF_0) \rightarrow 0$$

describing the embedding $D_0(p) \subset \mathbb{P}_2 \times \mathbb{P}_1$ with equation $u_1x_2 - u_2x_1 = 0$ using the Künneth formulas (see [10]) for $\mathbb{P}_2 \times \mathbb{P}_1$. \square

Some canonical homomorphisms. In the following we need some canonical homomorphisms related to the reducible surface $D(p)$:

- There is a canonical section $\mathcal{O}_{D(p)} \xrightarrow{s} \mathcal{O}_{D(p)}(H-F)$ induced by the canonical section of the canonical divisor $L_0 \sim H - F_0$ via the gluing sequence (5), vanishing along $D_1(p)$.
- For any $a \in \mathbb{Z}$ there is the homomorphism $\mathcal{O}_{D_1(p)} \xrightarrow{u_0} \mathcal{O}_{D(p)}(aH + F)$ induced by the diagram

$$\begin{array}{c}
 \mathcal{O}_{D_1(p)} \\
 \swarrow \text{---} u_0 \text{---} \quad \downarrow (0, u_0) \\
 0 \rightarrow \mathcal{O}_{D(p)}(aH + F) \longrightarrow \mathcal{O}_{D_0(p)}(aH_0 + F_0) \oplus \mathcal{O}_{D_1(p)}(F_1) \rightarrow \mathcal{O}_L(1) \longrightarrow 0
 \end{array}$$

because u_0 is the equation of L in $D_1(p)$.

- Using the gluing sequence (5) one obtains the exact sequence

$$0 \rightarrow \mathcal{O}_{D_1(p)}(-F_1) \xrightarrow{u_0} \mathcal{O}_{D(p)}(aH) \xrightarrow{s} \mathcal{O}_{D(p)}((a+1)H - F) \xrightarrow{r_1} \mathcal{O}_{D_1(p)}(-F_1) \rightarrow 0,$$

where r_1 denotes the restriction homomorphism to $D_1(p)$.

- The sections x_1, x_2 of $\mathcal{O}_{D(p)}(H)$ factorize as $x_\nu = u_\nu \circ s$:

$$(6) \quad \begin{array}{ccc} \mathcal{O}_{D(p)}(-H) & \xrightarrow{x_\nu} & \mathcal{O}_{D(p)} \\ & \searrow s & \nearrow u_\nu \\ & \mathcal{O}_{D(p)}(-F) & \end{array}$$

R -bundles and their properties. We define R -bundles as in Definition 1.1. In addition to the three items in the definition, R -bundles have four other properties, which will be derived at the end of Section 4.

Proposition 3.2. *Let \mathcal{E} be an R -bundle and let $C_i = D_i(p) \cap C$, $i = 0, 1$, be the components of its support. Then*

- 1) \mathcal{E} has a locally free resolution

$$(7) \quad 0 \rightarrow 2\mathcal{O}_{D(p)}(-H - F) \rightarrow \mathcal{O}_{D(p)}(-H) \oplus \mathcal{O}_{D(p)} \rightarrow \mathcal{E} \rightarrow 0.$$

- 2) The restriction of (7) to $D_1 = D_1(p)$ induces a resolution

$$0 \rightarrow 2\mathcal{O}_{D_1(p)}(-L) \rightarrow 2\mathcal{O}_{D_1(p)} \rightarrow \mathcal{E}|_{D_1(p)} \rightarrow 0$$

such that $\mathcal{E}|_{D_1(p)}$ is a semistable $(2m+2)$ -sheaf on $D_1(p) \cong \mathbb{P}_2$ with $\text{Supp}(\mathcal{E}|_{D_1(p)})$ a conic.

- 3) $\mathcal{E}|_{D_0(p)}$ is isomorphic to the structure sheaf \mathcal{O}_{C_0} with the resolution

$$0 \rightarrow \mathcal{O}_{D_0(p)}(-2F_0 - H_0) \rightarrow \mathcal{O}_{D_0(p)} \rightarrow \mathcal{O}_{C_0} \rightarrow 0.$$

- 4) $h^0\mathcal{E} = 1$, and the non-zero section gives rise to a non-trivial extension sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{E} \rightarrow \mathbb{k}_q \rightarrow 0,$$

where $q \in C_1 \setminus L$ is uniquely determined by \mathcal{E} .

Remark 3.3. Let \mathcal{E} be an R -bundle. Let C' be the cubic curve $C' = \text{Supp } \mathcal{F}$. Note that the curve C_0 is a subvariety of the total transform of C' under the blow up $D_0(p) \rightarrow \mathbb{P}_2$. In most of the cases it is the proper transform of C' . Therefore, the restriction map $\sigma_p : C_0 \rightarrow C'$ may be considered as a ‘‘partial normalization’’ of C' .

Remark 3.4. Note that the properties 1) and 4) of Proposition 3.2 are analogous to those of the $(3m+1)$ -sheaves. The degree 1 sheaf \mathcal{F} on C' is replaced by \mathcal{E} whose degree has been shifted to the additional curve C_1 with \mathcal{E}_{D_1} a $(2m+2)$ -sheaf and a vector bundle of degree 1 on C_1 .

Remark 3.5. 1) Note that the restriction of resolution (7) to the component $D_1(p) = \mathbb{P}_2$ is a Beilinson resolution of $\mathcal{E}|_{D_1(p)}$ on \mathbb{P}_2 .

2) One can also show that the restriction of (7) to $D_0(p)$ is the resolution of Beilinson type, see [1], Theorem 8.

Remark 3.6. Note that $\sigma_{p_*}\sigma_p^*(\mathcal{F}) \cong \mathcal{F}$ for every $(3m+1)$ -sheaf \mathcal{F} . Applying σ_{p_*} to sequence (1) and using that $R^0\sigma_{p_*}\mathcal{O}_{D_1(p)}(-L) = R^1\sigma_{p_*}\mathcal{O}_{D_1(p)}(-L) = 0$ we obtain the isomorphism $\sigma_{p_*}\mathcal{E} \cong \mathcal{F}$.

The proof of Proposition 3.2 will be a consequence of the description of R -bundles as flat limits of non-singular $(3m+1)$ -sheaves in the following section.

4. R -BUNDLES AS 1-DIMENSIONAL DEGENERATIONS OF $(3m+1)$ -SHEAVES.

Let A be a matrix in X_8 and let B be a matrix representing a morphism $2\mathcal{O}_{\mathbb{P}_2}(-2) \rightarrow \mathcal{O}_{\mathbb{P}_2}(-1) \oplus \mathcal{O}_{\mathbb{P}_2}$. Recall (cf. Section 2) that A and B can be considered as elements in \mathbb{k}^{18} . Consider the morphism

$$l_B : \mathbb{k} \rightarrow \mathbb{k}^{18}, \quad t \mapsto A + tB.$$

Let $T = l_B^{-1}(X)$. This way we obtain the morphism

$$(8) \quad l_B := l_B|_T : T \rightarrow X.$$

By the property of the space M we obtain a $(3m+1)$ -family \mathcal{F} over T with the resolution

$$(9) \quad 0 \rightarrow 2\mathcal{O}_{T \times \mathbb{P}_2}(-2H) \xrightarrow{A+tB} \mathcal{O}_{T \times \mathbb{P}_2}(-H) \oplus \mathcal{O}_{T \times \mathbb{P}_2} \rightarrow \mathcal{F} \rightarrow 0.$$

Here H is represented by the pull-back of a line $h \subset \mathbb{P}_2$. We choose h such that the point p does not lie on h . By shrinking T we can also assume that $A + tB \in X \setminus X_8$ for all $t \in T$, $t \neq 0$. In other words, the restrictions \mathcal{F}_t of the sheaf \mathcal{F} to the fibres $t \times \mathbb{P}_2 \cong \mathbb{P}_2$ are non-singular $(3m+1)$ -sheaves for all $t \in T$, $t \neq 0$. So the singular $(3m+1)$ -sheaf \mathcal{F}_0 is a flat 1-parameter degeneration of non-singular $(3m+1)$ -sheaves.

Let $p = p(A)$ and consider the blow up $Z := \text{Bl}_{0 \times p}(T \times \mathbb{P}_2) \xrightarrow{\sigma} T \times \mathbb{P}_2$. As above we denote the exceptional divisor of σ by $D_1 = D_1(p)$. By abuse of notation the lifting of the divisor $H \subset T \times \mathbb{P}_2$ is again denoted by H .

Remark 4.1. Letting x_i denote the homogeneous coordinates of \mathbb{P}_2 , such that the point p has the equations tx_0, x_1, x_2 , Z is embedded in $T \times \mathbb{P}_2 \times \mathbb{P}_2$ with equations

$$tx_0u_1 - x_1u_0, \quad tx_0u_2 - x_2u_0, \quad x_1u_2 - x_2u_1,$$

where the u_i are the coordinates of the second \mathbb{P}_2 . It follows that the fibre Z_0 for $t = 0$ equals $D(p)$, see section 3.

Note that the morphism $Z \xrightarrow{\sigma} T \times \mathbb{P}_2 \xrightarrow{pr_1} T$ is flat. Indeed, since both Z and T are regular, $\dim Z = 3$, $\dim T = 1$, and $\dim Z_t = 2 = \dim Z - \dim T$ for all $t \in T$, this follows from [4], 6.1.5. Applying σ^* to sequence (9) we obtain the sequence

$$0 \rightarrow 2\mathcal{O}_Z(-2H) \xrightarrow{\sigma^*(A+tB)} \mathcal{O}_Z(-H) \oplus \mathcal{O}_Z \rightarrow \sigma^*(\mathcal{F}) \rightarrow 0,$$

which remains exact because the sheaf $\mathcal{O}_Z(-2H)$ is locally free and, therefore, has no torsion. There is a canonical section $s \in \Gamma(Z, \mathcal{O}_Z(D_1))$, which gives us the exact sequence

$$0 \rightarrow \mathcal{O}_Z(-D_1) \xrightarrow{s} \mathcal{O}_Z \rightarrow \mathcal{O}_{D_1} \rightarrow 0.$$

Tensoring with $\mathcal{O}_Z(D_1 - 2H)$ one gets the exact sequence

$$0 \rightarrow 2\mathcal{O}_Z(-2H) \xrightarrow{\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}} 2\mathcal{O}_Z(-2H + D_1) \rightarrow 2\mathcal{O}_{D_1}(-L) \rightarrow 0.$$

We use here that H and D_1 do not meet (our choice of H) and that $\mathcal{O}_{D_1} \otimes \mathcal{O}_Z(D_1) \cong \mathcal{O}_{D_1}(-L)$ (properties of blow ups).

Note that $A + tB$ vanishes at $0 \times p$. Therefore, the morphism $\sigma^*(A + tB)$ vanishes on D_1 and hence factorizes uniquely through s , i. e., there exists

$$2\mathcal{O}_Z(-2H + D_1) \xrightarrow{\phi(A,B)} \mathcal{O}_Z(-H) \oplus \mathcal{O}_Z$$

such that the diagram

$$\begin{array}{ccc} 2\mathcal{O}_Z(-2H) & \xrightarrow{\sigma^*(A+tB)} & \mathcal{O}_Z(-H) \oplus \mathcal{O}_Z \\ & \searrow \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} & \nearrow \phi(A,B) \\ & & 2\mathcal{O}_Z(-2H + D_1) \end{array}$$

commutes.

Note that $\phi(A, B)$ is injective since $2\mathcal{O}_Z(-2H + D_1)$ is torsion free and since $\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$ is an isomorphism outside of the exceptional divisor D_1 .

Note also that the exceptional divisor D_1 is equivalent to the difference $H - F$, where F is the pull-back of a line in the second \mathbb{P}_2 along the standard embedding $Z \subset T \times \mathbb{P}_2 \times \mathbb{P}_2$. Hence we obtain the following commutative diagram with exact rows and columns.

$$(10) \quad \begin{array}{ccccccc} & & & & & 0 & \\ & & & & & \downarrow & \\ & & & & & 2\mathcal{O}_{D_1}(-L) & \\ & & & & & \downarrow & \\ & & & & & \sigma^* \mathcal{F} & \longrightarrow 0 \\ 0 & \longrightarrow & 2\mathcal{O}_Z(-2H) & \xrightarrow{\sigma^*(A+tB)} & \mathcal{O}_Z(-H) \oplus \mathcal{O}_Z & \longrightarrow & \sigma^* \mathcal{F} \longrightarrow 0 \\ & & \downarrow \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} & & \parallel & & \downarrow \\ 0 & \longrightarrow & 2\mathcal{O}_Z(-H - F) & \xrightarrow{\phi(A,B)} & \mathcal{O}_Z(-H) \oplus \mathcal{O}_Z & \longrightarrow & \tilde{\mathcal{F}} \longrightarrow 0 \\ & & \downarrow & & & & \downarrow \\ & & 2\mathcal{O}_{D_1}(-L) & & & & 0 \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

where $\tilde{\mathcal{F}}$ is defined as cokernel.

Proposition 4.2. *The sheaf $\tilde{\mathcal{F}}$ is locally free on its support if and only if B is a normal vector to X_8 at A , i. e., if and only if $B \in T_A X \setminus T_A X_8$.*

Sketch of the proof. The sheaf $\tilde{\mathcal{F}}$ is not locally free at some point if and only if the morphism $\phi(A, B)$ vanishes at this point. Since the only zero point of the morphism $A + tB$ is $(0, p)$ and since the preimage of $(0, p)$ is the exceptional divisor D_1 , we conclude that $\phi(A, B)$ may only vanish at points lying in D_1 .

To simplify the considerations one can assume without loss of generality that $p = \langle 1, 0, 0 \rangle$ and that A is of the form (3). Let

$$(11) \quad A = \begin{pmatrix} x_1 & A_{01}x_0x_1 + \cdots + A_{22}x_2^2 \\ x_2 & B_{01}x_0x_1 + \cdots + B_{22}x_2^2 \end{pmatrix}, \quad B = \begin{pmatrix} \xi_0x_0 + \xi_1x_1 + \xi_2x_2 & \xi_{00}x_0^2 + \cdots + \xi_{22}x_2^2 \\ \eta_0x_0 + \eta_1x_1 + \eta_2x_2 & \eta_{00}x_0^2 + \cdots + \eta_{22}x_2^2 \end{pmatrix},$$

then straightforward calculations show that vanishing of $\phi(A, B)$ at a point $q \in D_1(p)$ is equivalent to the system

$$(12) \quad \begin{cases} \xi_{00} = A_{01}\xi_0 + A_{02}\eta_0, \\ \eta_{00} = B_{01}\xi_0 + B_{02}\eta_0. \end{cases}$$

One easily checks that (12) are tangent equations at A . This proves the required statement. \square

The sheaf $\tilde{\mathcal{F}}$ is flat over T , because its resolution remains exact after restriction to fibres. So we obtain for every normal vector B a sheaf

$$(13) \quad \mathcal{E} = \mathcal{E}(A, B) := \tilde{\mathcal{F}}|_{D(p)}$$

on $D(p)$. This sheaf is locally free on its support and is a flat 1-dimensional degeneration of non-singular $(3m + 1)$ -sheaves. Using flatness of $\tilde{\mathcal{F}}$ and restricting diagram (10) to $D(p)$ one obtains an exact sequence

$$0 \rightarrow 2\mathcal{O}_{D_1(p)}(-L) \rightarrow \sigma_p^*(\mathcal{F}_0) \rightarrow \mathcal{E} \rightarrow 0$$

and the locally free resolution of \mathcal{E} on $D(p)$,

$$(14) \quad 0 \rightarrow 2\mathcal{O}_{D(p)}(-H - F) \xrightarrow{\Phi(A, B)} \mathcal{O}_{D(p)}(-H) \oplus \mathcal{O}_{D(p)} \rightarrow \mathcal{E} \rightarrow 0, \quad \Phi(A, B) := \phi(A, B)|_{D(p)}.$$

Straightforward calculations show now that the sheaf \mathcal{E} satisfies Definition 1.1.

We have obtained a construction that produces R -bundles. The following proposition shows that every R -bundle can be obtained by this construction for some A and B and yields at the same time a proof of Proposition 3.2.

Proposition 4.3. *Each R -bundle \mathcal{E} is part of an exact diagram*

$$(15) \quad \begin{array}{ccccccc} & & & & & & 0 \\ & & & & & & \downarrow \\ & & & & & & 2\mathcal{O}_{D_1}(-L) \\ & & & & & & \downarrow \varrho \\ & & & & & & \mathcal{O}_{D(p)}(-H) \oplus \mathcal{O}_{D(p)} \\ & & & & & & \downarrow \theta \\ 0 \rightarrow & 2\mathcal{O}_{D_1}(-L) & \xrightarrow{\begin{pmatrix} u_0 & 0 \\ 0 & u_0 \end{pmatrix}} & 2\mathcal{O}_{D(p)}(-2H) & \xrightarrow{\sigma^*A} & \mathcal{O}_{D(p)}(-H) \oplus \mathcal{O}_{D(p)} & \xrightarrow{\pi} & \sigma^*\mathcal{F} \rightarrow 0 \\ & \searrow & & \downarrow \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} & & \parallel & & \downarrow \theta \\ & & & 0 \rightarrow & 2\mathcal{O}_{D(p)}(-H - F) & \xrightarrow{\Phi} & \mathcal{O}_{D(p)}(-H) \oplus \mathcal{O}_{D(p)} & \xrightarrow{\pi'} & \mathcal{E} \rightarrow 0 \\ & & & & \downarrow & & \downarrow & & \downarrow \\ & & & & 2\mathcal{O}_{D_1}(-L) & & & & 0 \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

with $\Phi = \Phi(A, B)$ for some $A \in X_8$ and $B \in T_A X \setminus T_A X_8$.

If $p = \langle 1, 0, 0 \rangle$ and A is as in (3), then

$$\Phi = \begin{pmatrix} u_1 & u_1y_1 + u_2y_2 \\ u_2 & u_1z_1 + u_2z_2 \end{pmatrix} + \begin{pmatrix} \xi_0 & \xi_{00}x_0 \\ \eta_0 & \eta_{00}x_0 \end{pmatrix} u_0.$$

Proof. We divide the proof into the following steps.

1) Let \mathcal{E} be an R -bundle as in Definition 1.1. For the proof we may assume that \mathcal{F} is the cokernel of an A as in (3). Let $\sigma = \sigma_p$, $D = D(p)$, $D_i = D_i(p)$. Then $\sigma^*(\mathcal{F})$ has the resolution

$$0 \rightarrow 2\mathcal{O}_{D_1(-L)} \xrightarrow{\begin{pmatrix} u_0 & 0 \\ 0 & u_0 \end{pmatrix}} 2\mathcal{O}_D(-2H) \xrightarrow{\sigma^*A} \mathcal{O}_D(-H) \oplus \mathcal{O}_D \xrightarrow{\pi} \sigma^*\mathcal{F} \rightarrow 0.$$

2) The homomorphism ϱ can be uniquely lifted to a morphism of resolutions

$$(16) \quad \begin{array}{ccccccc} 0 \rightarrow 2\mathcal{O}_{D_1(-L)} & \xrightarrow{\begin{pmatrix} u_0 & 0 \\ 0 & u_0 \end{pmatrix}} & 2\mathcal{O}_D(-2H) & \xrightarrow{\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}} & 2\mathcal{O}_D(-H-F) & \rightarrow & 2\mathcal{O}_{D_1(-L)} \rightarrow 0 \\ & \downarrow B & \downarrow B & & \tilde{A} \downarrow & & \downarrow \varrho \\ 0 \rightarrow 2\mathcal{O}_{D_1(-L)} & \xrightarrow{\begin{pmatrix} u_0 & 0 \\ 0 & u_0 \end{pmatrix}} & 2\mathcal{O}_D(-2H) & \xrightarrow{\sigma^*A} & \mathcal{O}_D(-H) \oplus \mathcal{O}_D & \xrightarrow{\pi} & \sigma^*\mathcal{F} \rightarrow 0 \end{array}$$

because of the vanishing of the relevant Ext-groups, following from Proposition 3.1.

Claim. B is an isomorphism.

Proof of the Claim. The restriction of (16) to D_0 becomes the exact diagram

$$\begin{array}{ccccccc} 0 \rightarrow 2\mathcal{O}_{D_0}(-2H) & \xrightarrow{\begin{pmatrix} s_0 & 0 \\ 0 & s_0 \end{pmatrix}} & 2\mathcal{O}_{D_0}(-H-F_0) & \rightarrow & 2\mathcal{O}_L(-1) & \rightarrow & 0 \\ & \downarrow B & \tilde{A}_{D_0} \downarrow & & \downarrow \varrho_{D_0} & & \\ 0 \rightarrow 2\mathcal{O}_{D_0}(-2H) & \xrightarrow{\sigma_0^*A} & \mathcal{O}_{D_0}(-H) \oplus \mathcal{O}_{D_0} & \xrightarrow{\pi} & \sigma_0^*\mathcal{F} & \rightarrow & 0, \end{array}$$

where $\sigma_0 : D_0 \rightarrow \mathbb{P}_2$ is the blow up map and s_0 is the canonical section of $\mathcal{O}_{D_0}(L)$. Using the bottom row of this diagram one concludes that the Hilbert polynomial of $\sigma_0^*(\mathcal{F})$ with respect to the sheaf $\mathcal{O}_{D_0}(1, 1)$ is $6m + 1$. The restriction of (1) to D_0 becomes the exact sequence

$$2\mathcal{O}_L(-1) \xrightarrow{\varrho_{D_0}} \sigma_0^*(\mathcal{F}) \xrightarrow{\theta_{D_0}} \mathcal{E}|_{D_0} \rightarrow 0.$$

Therefore, we conclude that the Hilbert polynomial of the kernel of ϱ_{D_0} is zero, hence ϱ_{D_0} is injective.

By the shape of the original matrix A we conclude that $\tilde{A}_{D_0} = B \cdot \begin{pmatrix} u_1 & u_1y_1 + u_2y_2 \\ u_2 & u_1z_1 + u_2z_2 \end{pmatrix}$, where B is identified with its corresponding 2×2 matrix. If $B = 0$, then $\tilde{A}_{D_0} = 0$ and $\varrho_{D_0} = 0$, which contradicts the injectivity of ϱ_{D_0} .

Assume that the rank of B is 1 and that

$$B = \begin{pmatrix} \lambda\alpha & \lambda\beta \\ \mu\alpha & \mu\beta \end{pmatrix}, \quad (\alpha, \beta) \neq (0, 0), (\lambda, \mu) \neq (0, 0).$$

In this case the kernel of B is isomorphic to $\mathcal{O}_D(-2H)$ and is generated by the matrix $(\mu \ -\lambda)$. Then $\tilde{A}_{D_0} = \begin{pmatrix} \lambda u & \lambda q \\ \mu u & \mu q \end{pmatrix}$ for $u = \alpha u_1 + \beta u_2$ and $q = \alpha(u_1y_1 + u_2y_2) + \beta(u_1z_1 + u_2z_2)$. Then the kernel of \tilde{A}_{D_0} is generated by $(\mu \ -\lambda)$ and is isomorphic to $\mathcal{O}_D(-H-F)$. As ϱ_{D_0} is injective, we conclude that the kernels of B and \tilde{A}_{D_0} must be isomorphic, which is impossible since $H \not\sim F$. \square

Since B is an isomorphism, using the action of $\mathrm{GL}_2(\mathbb{k})$ on the upper row of diagram (16), we may assume that $B = \mathrm{id}$. Then \tilde{A} can be written as

$$\tilde{A} = \begin{pmatrix} u_1 & u_1y_1 + u_2y_2 \\ u_2 & u_1z_1 + u_2z_2 \end{pmatrix} + \begin{pmatrix} \xi_0 & \xi_{00}x_0 \\ \eta_0 & \eta_{00}x_0 \end{pmatrix} u_0$$

using decomposition (6).

3) By the resolution of $\sigma^*\mathcal{F}$ we obtain the following exact diagram

$$(17) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & 2\mathcal{O}_{D_1}(-L) & & 2\mathcal{O}_{D_1}(-L) & & \\ & \nearrow & \downarrow \begin{pmatrix} u_0 & 0 \\ 0 & u_0 \end{pmatrix} & & \downarrow \varrho & & \\ 0 \rightarrow 2\mathcal{O}_{D_1}(-L) & \xrightarrow{\begin{pmatrix} u_0 & 0 \\ 0 & u_0 \end{pmatrix}} & 2\mathcal{O}_D(-2H) & \xrightarrow{\sigma^*A} & \mathcal{O}_D(-H) \oplus \mathcal{O}_D & \xrightarrow{\pi} & \sigma^*\mathcal{F} \rightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow \theta \\ 0 & \longrightarrow & \mathcal{K} & \xrightarrow{\Phi} & \mathcal{O}_D(-H) \oplus \mathcal{O}_D & \xrightarrow{\pi'} & \mathcal{E} \rightarrow 0 \\ & & \downarrow & & & & \downarrow \\ & & 2\mathcal{O}_{D_1}(-L) & & & & 0 \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

with \mathcal{K} the kernel of $\theta \circ \pi$.

By the construction of \tilde{A} , $\theta \circ \pi \circ \tilde{A} = 0$, and hence \tilde{A} factorizes uniquely through \mathcal{K} , with the commutative diagram

$$\begin{array}{ccc} & 2\mathcal{O}_D(-H-F) & \\ & \downarrow \alpha & \nearrow \tilde{A} \\ \exists! & \mathcal{K} & \xrightarrow{\Phi} \mathcal{O}_D(-H) \oplus \mathcal{O}_D. \end{array}$$

Because Φ and \tilde{A} are injective, α is injective as well. On the other hand, \mathcal{K} and $2\mathcal{O}_D(-H-F)$ have the same Hilbert polynomial with respect to $\mathcal{O}_D(H+F)$, namely $2(2m^2 - m)$. Hence α is an isomorphism. Replacing \mathcal{K} by $2\mathcal{O}_D(-H-F)$ yields the diagram of the proposition. Note that $\tilde{A} = \Phi(A, B)$ for $B = \begin{pmatrix} \xi_{0x_0} & \xi_{00}x_0^2 \\ \eta_{0x_0} & \eta_{00}x_0^2 \end{pmatrix}$. This completes the proof of Proposition 4.3. \square

Proof of Proposition 3.2. 1) Follows directly from Proposition 4.3.

2) Restricting the resolution of \mathcal{E} to D_1 one obtains the required statement.

3) Restricting to D_0 we see that $\mathcal{E}|_{D_0}$ is the cokernel of $\begin{pmatrix} u_1 & u_1y_1 + u_2y_2 \\ u_2 & u_1z_1 + u_2z_2 \end{pmatrix}$.

Because $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is surjective, one obtains that $\mathcal{E}|_{D_0}$ is the structure sheaf of the curve given by the determinant of this matrix.

4) Since Φ in (15) is of the form $\begin{pmatrix} l_1 & q_1 \\ l_2 & q_2 \end{pmatrix}$ such that $l_1, l_2 \in \Gamma(D, \mathcal{O}_D(F))$ are linearly independent with a single common zero $q \in D_1 \setminus L$, there is the exact sequence

$$0 \rightarrow \mathcal{O}_D(-2F-H) \xrightarrow{\begin{pmatrix} l_2 & -l_1 \end{pmatrix}} 2\mathcal{O}_D(-F-H) \xrightarrow{\begin{pmatrix} l_1 \\ l_2 \end{pmatrix}} \mathcal{O}_D(-H) \rightarrow \mathbb{k}_q \rightarrow 0.$$

Splitting this sequence into short exact sequences one obtains the exact diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & \mathcal{O}_D(-2F - H) & \xrightarrow{-(l_1 q_2 - l_2 q_1)} & \mathcal{O}_D & \longrightarrow & \mathcal{O}_C \rightarrow 0 \\
 & & \downarrow (l_2 \ -l_1) & & \downarrow (0 \ 1) & & \\
 0 & \rightarrow & 2\mathcal{O}_D(-H - F) & \xrightarrow{\begin{pmatrix} l_1 & q_1 \\ l_2 & q_2 \end{pmatrix}} & \mathcal{O}_D(-H) \oplus \mathcal{O}_D & \longrightarrow & \mathcal{E} \rightarrow 0 \\
 & & \downarrow & \searrow & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \\
 0 & \longrightarrow & \mathcal{A} & \xrightarrow{\begin{pmatrix} l_1 \\ l_2 \end{pmatrix}} & \mathcal{O}_D(-H) & \longrightarrow & \mathbb{k}_q \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

and then, by the snake lemma, the required extension

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{E} \rightarrow \mathbb{k}_q \rightarrow 0.$$

If this extension is trivial, then $\sigma_* \mathbb{k}_q \cong \mathbb{k}_p$ must be a direct summand of the $(3m+1)$ -sheaf $\sigma_* \mathcal{E} \cong \mathcal{F}$ (cf. Remark 3.6). This contradicts the stability of \mathcal{F} .

Remark 4.4. The point q from Proposition 3.2, 4) arises from a flat degeneration as above as follows. The points $p(t)$ of the fibres \mathcal{F}_t of the flat family \mathcal{F} form a section S of $T \times \mathbb{P}_2$ over T passing through $(0, \langle 1, 0, 0 \rangle)$. After blowing up its proper transform meets D in the point q .

Remark 4.5. Let \mathcal{F} be a $(3m+1)$ -sheaf as in the proof of Proposition 4.3. Then already \mathcal{F} induces an exact commutative diagram of type (15). However in this case $\Phi = \begin{pmatrix} u_1 & u_1 y_1 + u_2 y_2 \\ u_2 & u_1 z_1 + u_2 z_2 \end{pmatrix}$ and its cokernel is a one-dimensional sheaf $\tilde{\mathcal{F}}$ such that $\tilde{\mathcal{F}}|_{D_1}$ is a singular $(2m+2)$ -sheaf on $D_1 \cong \mathbb{P}_2$ whereas $\tilde{\mathcal{F}}|_{D_0}$ is the structure sheaf of the supporting curve C_0 as before.

Remark 4.6. Note that because of the vanishing of the relevant *Ext*-groups every morphism between sheaves on $D(p)$ with resolutions of the type (7) can be uniquely lifted to a morphism of the corresponding resolutions. In particular this holds for R -bundles.

5. CLASSIFICATION RESULT (MAIN RESULT)

In this section we are going to prove theorem 1.3. First of all note that the relation “to be equivalent” defined in Definition 1.2 is in fact an equivalence relation on the set of R -bundles associated to a given singular $(3m+1)$ -sheaf.

Proof of Theorem 1.3. Since the morphism $X \rightarrow M$ induces an isomorphism $N_{[\mathcal{F}]} \cong N_A$, where $N_A = N_A(X_8) = T_A(X)/T_A(X_8)$ is the normal space to X_8 at the point A , it is enough to show that every two R -bundles $\mathcal{E}_1 = \mathcal{E}(A, B_1)$ and $\mathcal{E}_2 = \mathcal{E}(A, B_2)$ on $D(p)$ as in (13) are equivalent if and only if B_1 and B_2 represent the same point in $\mathbb{P}N_A$.

Without loss of generality one can assume that $p = \langle 1, 0, 0 \rangle$ and that A is of the form (3). We can write A as in (11). Adding a suitable multiple of the first column of A to the second column (this gives us an affine automorphism of X) we can assume without loss of generality that the coefficients A_{01} , A_{11} , and A_{12} are zero.

Let $\mathcal{E}_1 = \mathcal{E}(A, B_1)$ and $\mathcal{E}_2 = \mathcal{E}(A, B_2)$ be two equivalent R -bundles, then the sheaves \mathcal{E}_1 and \mathcal{E}_2 possess locally free resolutions of type (14), they are cokernels of $\Phi_1 = \Phi(A, B_1)$ and $\Phi_2 = \Phi(A, B_2)$ respectively.

Equivalence of \mathcal{E}_1 and \mathcal{E}_2 means that there exists an isomorphism $\phi : D(p) \rightarrow D(p)$ identical on $D_0(p)$ such that there is an isomorphism $\mathcal{E}_2 \xrightarrow{\xi} \phi^*(\mathcal{E}_1)$. By Remark 4.6 ξ can be uniquely lifted to a morphism of resolutions

$$(18) \quad \begin{array}{ccccccc} 0 & \longrightarrow & 2\mathcal{O}_{D(p)}(-H-F) & \xrightarrow{\Phi_2} & \mathcal{O}_{D(p)}(-H) \oplus \mathcal{O}_{D(p)} & \longrightarrow & \mathcal{E}_2 \longrightarrow 0 \\ & & \downarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} & & \downarrow \begin{pmatrix} \bar{a} & \bar{b} \\ 0 & \bar{d} \end{pmatrix} & & \downarrow \xi \\ 0 & \longrightarrow & 2\mathcal{O}_{D(p)}(-H-F) & \xrightarrow{\phi^*(\Phi_1)} & \mathcal{O}_{D(p)}(-H) \oplus \mathcal{O}_{D(p)} & \longrightarrow & \phi^*(\mathcal{E}_1) \longrightarrow 0. \end{array}$$

Note that from the uniqueness of the lifting it follows that isomorphisms between R -bundles lift to automorphisms of $\mathcal{O}_{D(p)}(-H) \oplus \mathcal{O}_{D(p)}$ and thus the induced endomorphisms of $2\mathcal{O}_{D(p)}(-H-F)$ are also automorphisms in this case. Therefore, both matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} \bar{a} & \bar{b} \\ 0 & \bar{d} \end{pmatrix}$ are invertible.

Straightforward verifications (cf. [5]) show that for some $\mu \in \mathbb{k}^*$ the matrix $B_2 - \mu B_1$ satisfies the tangent equations (12), i. e., $B_2 - \mu B_1 \in T_A(X_8)$. So B_1 and B_2 represent the same element in $\mathbb{P}N_A$.

Let now B_1 and B_2 be two equivalent normal vectors at $A \in X_8$. Let $\Phi_1 = \Phi(A, B_1)$ and $\Phi_2 = \Phi(A, B_2)$ be the matrices defining as in (14) the sheaves \mathcal{E}_1 and \mathcal{E}_2 respectively. Since B_1 and B_2 define the same point in $\mathbb{P}N_A$, it follows that

$$B_2 - \alpha \cdot B_1 \in T_A(X_8)$$

for some $\alpha \in \mathbb{k}^*$.

Let

$$B_1 = \begin{pmatrix} \xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2 & \xi_{00} x_0^2 + \cdots + \xi_{22} x_2^2 \\ \eta_0 x_0 + \eta_1 x_1 + \eta_2 x_2 & \eta_{00} x_0^2 + \cdots + \eta_{22} x_2^2 \end{pmatrix}$$

and

$$B_2 = \begin{pmatrix} \mu_0 x_0 + \mu_1 x_1 + \mu_2 x_2 & \mu_{00} x_0^2 + \cdots + \mu_{22} x_2^2 \\ \nu_0 x_0 + \nu_1 x_1 + \nu_2 x_2 & \nu_{00} x_0^2 + \cdots + \nu_{22} x_2^2 \end{pmatrix}.$$

Take

$$\beta = \mu_0 - \xi_0 \alpha, \quad \gamma = \nu_0 - \eta_0 \alpha,$$

and let

$$(19) \quad \phi_1 = \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \mathbb{P}_2 \rightarrow \mathbb{P}_2, \quad \langle u_0, u_1, u_2 \rangle \mapsto \langle (u_0, u_1, u_2) \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rangle.$$

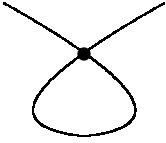
Note that the automorphisms of the form (19) are exactly the automorphisms of $D_1 \cong \mathbb{P}_2$ acting identically on L . Consider now such an automorphism $\phi : D(p) \rightarrow D(p)$ with $\phi|_{D_1} = \phi_1$ and $\phi|_{D_0} = \text{id}_{D_0}$. Using the tangent equations (12) one checks that $\phi^*(\Phi_1) = \Phi_2$. Therefore, there is an isomorphism $\phi^*(\mathcal{E}_1) \cong \mathcal{E}_2$, which means that the sheaves \mathcal{E}_1 and \mathcal{E}_2 are equivalent. This completes the proof. \square

Remark 5.1. When one considers isomorphism classes of the R -bundles one finds that they depend on the point q in Proposition 3.2. However, the isomorphism classes of the restrictions of the R -bundles to the plane $D_1(p)$ don't depend on that point, as well as the equivalence classes of the R -bundles.

6. EXAMPLES

Let us fix some $A \in X_8$ and let us consider the R -bundles $\mathcal{E}(A, B)$. The curve C_0 is uniquely defined by the matrix A . The curve C_1 depends on B . Let us fix C_1 . Then by Proposition 3.2, 4), points of $C_1 \setminus L$ parameterize the isomorphism classes of R -bundles with the support $C_0 \cup C_1$. To be more precise: there is a one-to-one correspondence between the isomorphism classes of R -bundles supported on $C_0 \cup C_1$ and non-singular points of $C_1 \setminus L$.

It may however happen that there are two different points of the curve C_1 that define the same equivalence class of R -bundles. This is the case when the curve C_1 has a non-trivial stabilizer under the action of the automorphisms of D_1 identical on L (automorphisms from (19)). There is a natural action of this stabilizer on $C_1 \setminus L$ (and also on its non-singular locus) and the orbits of this action are clearly in one to one correspondence with the equivalence classes of R -bundles $\mathcal{E}(A, B)$ with support $C_0 \cup C_1$.



$$x_1^2 x_0 - x_2(x_2 + x_0) = 0$$

Generic case: $C_0 \cap L$ consists of two points. Let us fix the matrix $A = \begin{pmatrix} x_1 & x_2(x_0 + x_2) \\ x_2 & x_1 x_0 \end{pmatrix}$. Let C be a curve in \mathbb{P}_2 given by the determinant of this matrix. This is an irreducible cubic curve with an ordinary double point singularity.

Then for all directions B the curve C_0 is given by the determinant of the matrix $\begin{pmatrix} u_1 & u_2(x_0 + x_2) \\ u_2 & u_1 x_0 \end{pmatrix}$. The intersection of C_0 with the line L is given by the equation $u_1^2 - u_2^2 = 0$ and consists of two points, say q_1 and q_2 . For a direction

$$B = \begin{pmatrix} \xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2 & \xi_{00} x_0^2 + \dots + \xi_{22} x_2^2 \\ \eta_0 x_0 + \eta_1 x_1 + \eta_2 x_2 & \eta_{00} x_0^2 + \dots + \eta_{22} x_2^2 \end{pmatrix}$$

the restriction of $\mathcal{E}(A, B)$ to D_1 is given as the cokernel of the matrix

$$\begin{pmatrix} u_1 + \xi_0 u_0 & u_2 + \xi_{00} u_0 \\ u_2 + \eta_0 u_0 & u_1 + \eta_{00} u_0 \end{pmatrix}.$$

Its support is then the conic in D_1 through the points q_1 and q_2 given by the determinant of this matrix.

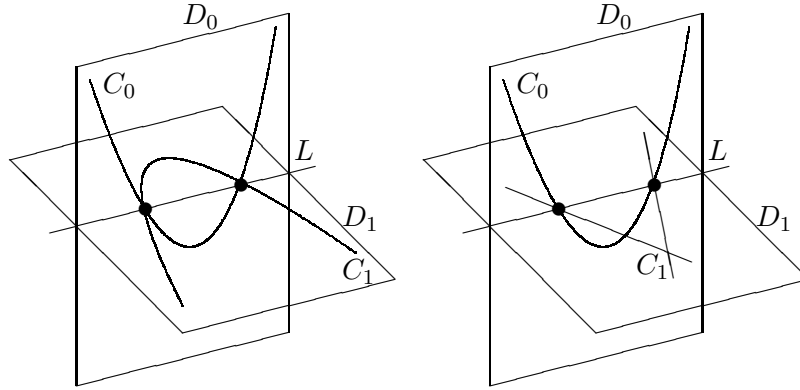
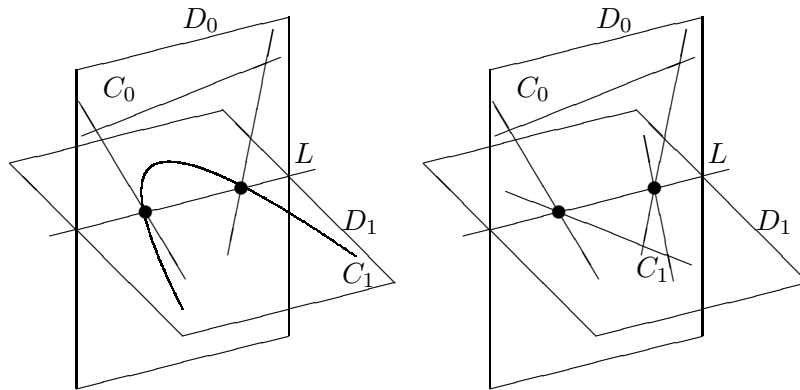
Remark 6.1 (comparison with [11]). One sees that C_0 is a normalization of C (C_0 is a proper transform of C).

If the conic C_1 is smooth, then it is isomorphic to \mathbb{P}_1 and thus the support of an R -bundle in this situation is a curve of type X_1 (see [11], pp. 212–213).

If C_1 is singular, then it is just a union of two lines and thus the whole support $C_0 \cup C_1$ is a curve of type X_2 .

Let us fix C_1 .

Smooth curve C_1 . As already noticed, the points of $C_1 \setminus L$ are in one-to-one correspondence with the isomorphism classes of the R -bundles supported on $C_0 \cup C_1$. One sees in this case that the stabilizer group of C_1 consists of two elements. The non-trivial one is the central symmetry with respect to the intersection point of the tangent lines to C_1 at q_1 and q_2 respectively (axes of C_1).

FIGURE 2. Support of R -bundles in the generic case.FIGURE 3. Support of R -bundles for three lines with simple intersections.

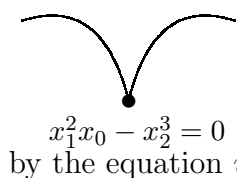
The orbits of the action of the stabilizer group on $C_1 \setminus L$ consist clearly of two points, the orbit space may be identified with the curve $C_1 \setminus L$ itself, the quotient map being induced by the $2 : 1$ self-covering $\mathbb{k}^* \rightarrow \mathbb{k}^*$, $t \mapsto t^2$, via an isomorphism $C_1 \setminus L \cong \mathbb{k}^*$.

Singular curve C_1 . In this case C_1 is a union of two lines. The stabilizer group of C_1 is a group isomorphic to k^* and there are only two orbits in the non-singular locus of $C_1 \setminus L$ each being one of the components of C_1 without their intersection point and without the points on L . Each orbit is isomorphic to \mathbb{k}^* .

Another cases with two points in the intersection $C_0 \cap L$ are the following.

Three lines with simple intersections. As a typical example one can choose the matrix $A = \begin{pmatrix} x_1 & 0 \\ x_2 & x_2x_0 \end{pmatrix}$. We present the pictures for different types of C_1 in Figure 3.

Transversal intersection of a line with a smooth conic. A typical example comes from the matrix $A = \begin{pmatrix} x_1 & x_0x_1 \\ x_2 & x_1^2 \end{pmatrix}$. The corresponding pictures can be seen on Figure 4



by the equation $u_1^2 = 0$ and consists of a single point. Let us fix some C_1 .

$C_0 \cap L$ **consists of a single point.** Let us fix the matrix $A = \begin{pmatrix} x_1 & x_2^2 \\ x_2 & x_1x_0 \end{pmatrix}$. The curve C is in this case a cuspidal cubic curve. The intersection of C_0 with the line L is given

by the equation $u_1^2 = 0$ and consists of a single point. Let us fix some C_1 .

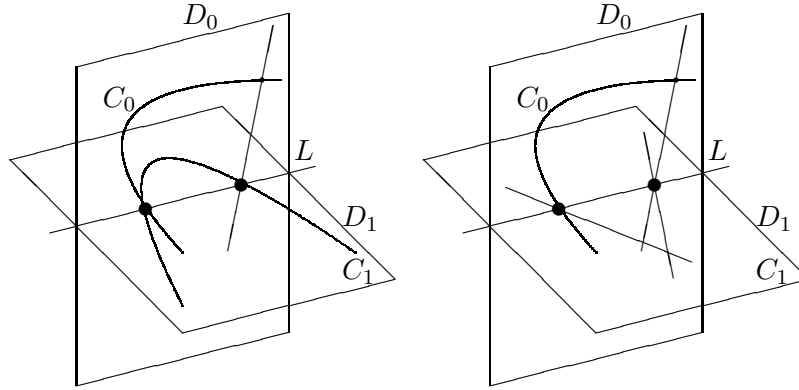


FIGURE 4. Support of R -bundles for a simple intersection of a line and a smooth conic.

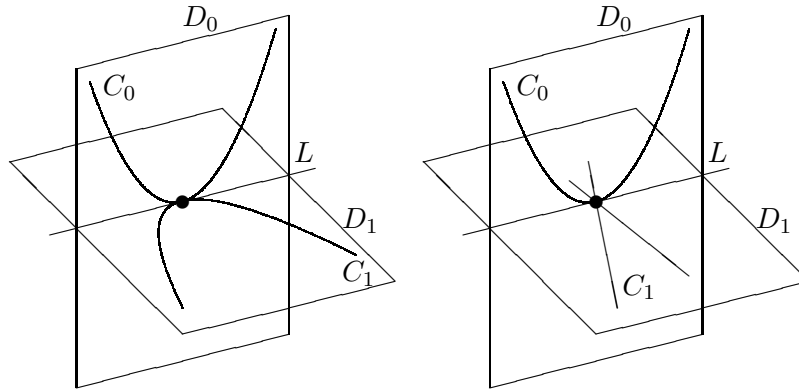


FIGURE 5.

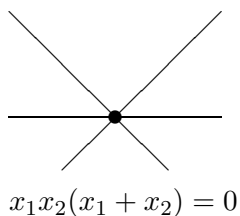
Smooth curve C_1 . One sees that the stabilizer of C_1 is trivial in this case. Hence the points of $C_1 \setminus L$ are in one-to-one correspondence with the isomorphism and simultaneously with the equivalence classes of R -bundles supported on $C_0 \cup C_1$.

Singular curve C_1 . In this case the stabilizer of C_1 is a 1-dimensional group acting transitively on $C_1 \setminus L$.

There are also the following similar cases.

Tangent intersection of a line with a smooth conic. An example for this case is provided by the matrix $A = \begin{pmatrix} x_1 & x_0x_2 \\ x_2 & x_1x_2 \end{pmatrix}$. The pictures are given in Figure 6.

Point on a double line. We can consider the matrix $A = \begin{pmatrix} x_1 & 0 \\ x_2 & x_0x_1 \end{pmatrix}$. The pictures are given in Figure 7.



$C_0 \cap L$ is the whole line L . We start here from the matrix $A = \begin{pmatrix} x_1 & 0 \\ x_2 & x_2(x_1 + x_2) \end{pmatrix}$, $A = \begin{pmatrix} x_1 & 0 \\ x_2 & x_2^2 \end{pmatrix}$ or $A = \begin{pmatrix} x_1 & 0 \\ x_2 & x_1^2 \end{pmatrix}$. Then the curve C consists of three different lines that all intersect in the same point, of a line and a double line, or of a triple line respectively.

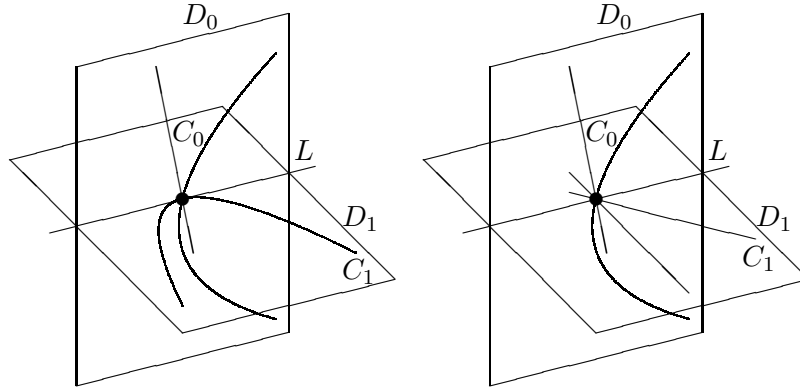


FIGURE 6.

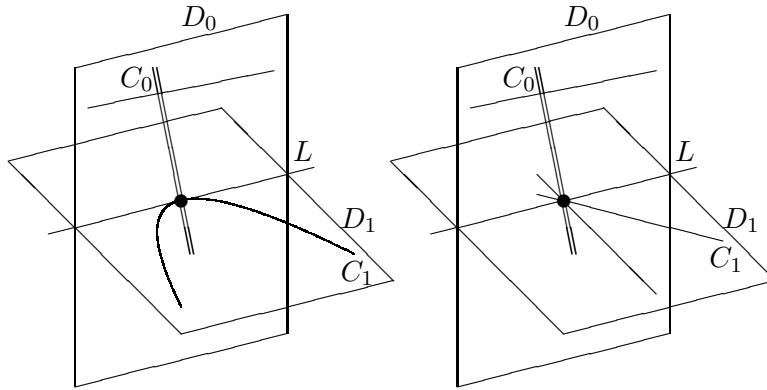


FIGURE 7.

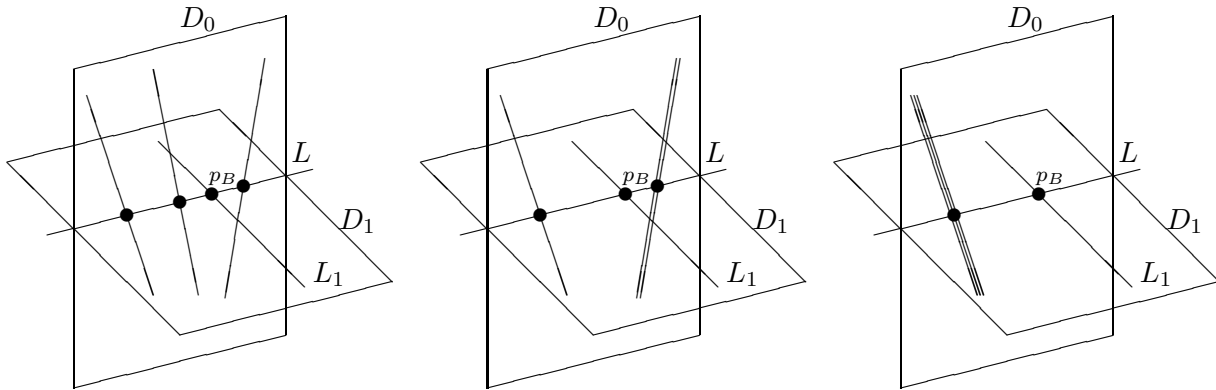


FIGURE 8.

In this case C_1 is the union the line L with another line L_1 , which intersects L at a point p_B that depends on the direction B . It is clear that the stabilizer of C_1 acts transitively on L_1 , hence there is only one equivalence class of R -bundles with fixed C_1 in this case. Two directions B and B' with different intersection points p_B and $p_{B'}$ define non-equivalent sheaves because all the allowed automorphism are identities on L . See Figure 8 for the corresponding pictures.

7. UNIVERSAL FAMILIES OF R -BUNDLES

We are going to construct flat families parameterizing all R -bundles as well as all non-singular $(3m+1)$ -sheaves.

Family over \tilde{X} , construction of the space. Let $\tilde{X} \xrightarrow{\alpha} X$ be the blow up of X along X_8 and let $H = X \times h$ for a line $h \subset \mathbb{P}_2$. Then there is the universal $(3m+1)$ -sheaf \mathcal{U} on $X \times \mathbb{P}_2$ given by the locally free resolution (cf. [3], 6.1)

$$(20) \quad 0 \rightarrow 2\mathcal{O}_{X \times \mathbb{P}_2}(-2H) \xrightarrow{A_X} \mathcal{O}_{X \times \mathbb{P}_2}(-H) \oplus \mathcal{O}_{X \times \mathbb{P}_2} \rightarrow \mathcal{U} \rightarrow 0,$$

where A_X is the universal matrix: $A_X|_{\{A\} \times \mathbb{P}_2} = A$ for all $A \in X$.

By pulling back we obtain the family $\tilde{\mathcal{U}} := (\alpha \times \text{id})^* \mathcal{U}$ of $(3m+1)$ -sheaves over \tilde{X} . Let $S_8 = \text{Sing} \mathcal{U}$ be the closed subvariety of $X \times \mathbb{P}_2$ where \mathcal{U} is not locally free, i. e., the zero locus of A_X . Since X is the parameter space of the universal cubic, the subvariety S_8 is a section of $X \times \mathbb{P}_2$ over X_8 and so isomorphic to X_8 . In particular S_8 is smooth of codimension 3. Then $\tilde{S}_8 := (\alpha \times \text{id})^{-1}(S_8)$ is the set of points in $\tilde{X} \times \mathbb{P}_2$ where the sheaf $\tilde{\mathcal{U}}$ is not locally free. \tilde{S}_8 is isomorphic to the exceptional divisor $\tilde{X}_8 = \alpha^{-1}(X_8)$ of the blow up $\tilde{X} \xrightarrow{\alpha} X$, in particular \tilde{S}_8 is smooth.

Let $\tau : Y \rightarrow \tilde{X} \times \mathbb{P}_2$ be the blowing up of $\tilde{X} \times \mathbb{P}_2$ along \tilde{S}_8 and let \tilde{D}_1 denote the exceptional divisor of τ . Let \tilde{D}_0 be the proper transform of $\tilde{X}_8 \times \mathbb{P}_2$. It is isomorphic to the blow up of $\tilde{X}_8 \times \mathbb{P}_2$ along \tilde{S}_8 . Then $\tilde{D} = \tilde{D}_0 \cup \tilde{D}_1$ is the preimage of $\tilde{X}_8 \times \mathbb{P}_2$.

$$\begin{array}{ccc} \tilde{D}_1 & \longrightarrow & Y \\ \downarrow & & \downarrow \tau \\ \tilde{S}_8 & \longrightarrow & \tilde{X} \times \mathbb{P}_2 \end{array} \qquad \begin{array}{ccccc} & & \tilde{X} \times \mathbb{P}_2 & \xrightarrow{\alpha \times \text{id}} & X \times \mathbb{P}_2 \\ \tilde{S}_8 & \xrightarrow{\quad} & S_8 & \xrightarrow{\quad} & \downarrow \text{pr}_1 \\ \downarrow \cong & & \downarrow \text{pr}_1 & & \downarrow \text{pr}_1 \\ \tilde{X}_8 & \xrightarrow{\quad} & \tilde{X} & \xrightarrow{\alpha} & X \\ & & \downarrow \cong & & \downarrow \text{pr}_1 \\ & & X_8 & & \end{array}$$

A fibre Y_x of the morphism $Y \rightarrow \tilde{X} \times \mathbb{P}_2 \rightarrow \tilde{X}$ for a point $x \in \tilde{X}_8$ is a surface $D(p)$ with $p \in \tilde{S}_8$ corresponding to x . Moreover, the composed morphism $Y \rightarrow \tilde{X}$ is flat. Hence the situation is analogous to that of the blow up Z in section 4. There is also an embedding of Y analogous to that of Z :

Remark 7.1. The ideal sheaf \mathcal{I} of \tilde{S}_8 is the quotient of a decomposable rank-3 vector bundle as follows. Let $x_1, x_2 \in \Gamma \mathcal{O}_{\tilde{X} \times \mathbb{P}_2}(H)$ be the independent linear entries of the lifted matrix A . Then \mathcal{I} is generated by x_1, x_2 and the lifted equation of \tilde{X}_8 , giving rise to a surjection

$$\mathbb{E} := \mathcal{O}_{\tilde{X}}(-\tilde{X}_8) \boxtimes \mathcal{O}_{\mathbb{P}_2} \oplus 2\mathcal{O}_{\tilde{X} \times \mathbb{P}_2}(-H) \rightarrow \mathcal{I}.$$

It follows that $Y = \mathbb{P}(\mathcal{I})$ is embedded in the \mathbb{P}_2 -bundle $\mathbb{P}(\mathbb{E})$ with local equations similar to those of Z .

Family over \tilde{X} , construction of the sheaf. Pulling back sequence (20) along the morphism

$$Y \xrightarrow{\tau} \tilde{X} \times \mathbb{P}_2 \xrightarrow{\alpha \times \text{id}} X \times \mathbb{P}_2,$$

we obtain the sequence

$$0 \rightarrow 2\mathcal{O}_Y(-2H) \rightarrow \mathcal{O}_Y(-H) \oplus \mathcal{O}_Y \rightarrow \tau^*(\tilde{\mathcal{U}}) \rightarrow 0,$$

where H also denotes the pull-back of $\tilde{X} \times h$. This remains exact because the sheaf $\mathcal{O}_Y(-2H)$ is locally free and, therefore, has no torsion. Similar to diagram (10) one obtains the commutative diagram with exact rows and columns

$$(21) \quad \begin{array}{ccccccc} & & & & & & 0 \\ & & & & & & \downarrow \\ & & & & & & \mathcal{C} \\ & & & & & & \downarrow \\ & & & & & & 0 \\ 0 & \longrightarrow & 2\mathcal{O}_Y(-2H) & \longrightarrow & \mathcal{O}_Y(-H) \oplus \mathcal{O}_Y & \longrightarrow & \tau^*\tilde{\mathcal{U}} \rightarrow 0, \\ & & \downarrow & & \parallel & & \downarrow \\ 0 & \longrightarrow & 2\mathcal{O}_Y(-2H + \tilde{D}_1) & \longrightarrow & \mathcal{O}_Y(-H) \oplus \mathcal{O}_Y & \longrightarrow & \tilde{\mathcal{U}} \rightarrow 0 \\ & & \downarrow & & & & \downarrow \\ & & \mathcal{C} & & & & 0 \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

by multiplying with the canonical section of $\mathcal{O}_Y(\tilde{D}_1)$, with $\mathcal{C} = \mathcal{O}_{\tilde{D}_1} \otimes \mathcal{O}_Y(-2H + \tilde{D}_1)$, where $\tilde{\mathcal{U}}$ is the quotient.

Proposition 7.2. 1) *The sheaf $\tilde{\mathcal{U}}$ is flat over \tilde{X} and the fibres of $\tilde{\mathcal{U}}$ are either non-singular $(3m+1)$ -sheaves or R -bundles on some $D(p)$.*

2) *Any non-singular $(3m+1)$ -sheaf and any R -bundle on some $D(p)$ is equivalent to a fibre of $\tilde{\mathcal{U}}$.*

Proof. Consider l_B as in (8), an embedding of an open set of \mathbb{k} in X along a normal direction $B \in \mathbb{k}^{18}$ such that $0 \in T$ is the only point in T with the image in X_8 . Then from the universal property of blow-ups it follows that l_B uniquely factorizes through $\tilde{X} \xrightarrow{\alpha} X$, i. e., there exists the commutative diagram

$$\begin{array}{ccc} & & \tilde{X} \\ & \exists! \tilde{l}_B \nearrow & \downarrow \alpha \\ U & \xrightarrow{l_B} & X. \end{array}$$

Then using again the universal property of blow-ups we obtain the commutative diagram with cartesian squares

$$(22) \quad \begin{array}{ccccc} & & Z & \xrightarrow{\exists! L_B} & Y \\ & & \downarrow \sigma & & \downarrow \tau \\ T \times \mathbb{P}_2 & \xrightarrow{\tilde{l}_B \times \text{id}} & \tilde{X} \times \mathbb{P}_2 & \xrightarrow{\alpha \times \text{id}} & X \times \mathbb{P}_2 \\ & & \downarrow & & \downarrow \\ & & T & \xrightarrow{\tilde{l}_B} & \tilde{X} & \xrightarrow{\alpha} & X. \end{array}$$

Comparing diagram (22) with the construction of the blow up Z in Section 3, one finds that the restriction of the exceptional divisor and the proper transform of $\tilde{X}_8 \times \mathbb{P}_2$ to the image of T in \tilde{X} is the one of Z . It follows that the restriction of diagram (21) to this image is isomorphic to diagram (10). It follows that the sheaf $\tilde{\mathcal{U}}$ is flat over \tilde{X} and locally free on its support.

In particular, the construction of R -bundles (diagram (10)) is obtained by pulling back diagram (21) from Y along L_B as in (22). In particular this means that the fibres of the sheaf $\tilde{\mathcal{U}}$ are either non-singular $(3m+1)$ -sheaves or R -bundles on some $D(p)$. By the above construction of l_B every non-singular $(3m+1)$ -sheaf (up to isomorphy) as well as every R -bundle (up to equivalence) appear as fibres of $\tilde{\mathcal{U}}$. \square

Remark 7.3. Because $\tilde{\mathcal{U}}$ is flat over \tilde{X} it is easy to show that the sheaf \mathcal{C} is the “relative torsion” of $\tau^*\tilde{\mathcal{U}}$ over \tilde{S}_8 . The same holds for diagram (10).

The universal family over \tilde{M} . Let $Y_M = \text{Bl}_{\tilde{\nu}(\tilde{S}_8)}(\tilde{M} \times \mathbb{P}_2)$ and let $Y_M \xrightarrow{\tau_M} \tilde{M} \times \mathbb{P}_2$ be the corresponding morphism. Let \tilde{D}_M be the exceptional divisor. Then by the universal property of blow ups there exists a unique morphism $Y \xrightarrow{\xi} Y_M$ such that the diagram

$$(23) \quad \begin{array}{ccc} Y & \xrightarrow{\xi} & Y_M \\ \downarrow \tau & & \downarrow \tau_M \\ \tilde{X} \times \mathbb{P}_2 & \xrightarrow{\tilde{\nu} \times \text{id}} & \tilde{M} \times \mathbb{P}_2 \end{array}$$

commutes.

Proposition 7.4. *There exists a flat sheaf $\tilde{\mathcal{V}}$ on Y_M such that $\tilde{\mathcal{U}}$ is a pull back of $\tilde{\mathcal{V}}$ along ξ up to a twist by a pull back of a line bundle on \tilde{X} .*

Proof. Let \mathcal{V} be the universal $(3m+1)$ -family on $M \times \mathbb{P}_2$. Let $p_1 : M \times \mathbb{P}_2 \rightarrow M$ and $p_2 : M \times \mathbb{P}_2 \rightarrow \mathbb{P}_2$ be the canonical projections. Then there exists a relative Beilinson resolution (cf. [9])

$$(24) \quad 0 \rightarrow p_1^* \mathcal{A}_2 \otimes p_2^* \mathcal{O}_{\mathbb{P}_2}(-2) \rightarrow (p_1^* \mathcal{A}_1 \otimes p_2^* \mathcal{O}_{\mathbb{P}_2}(-1)) \oplus (p_1^* \mathcal{A}_0 \otimes p_2^* \mathcal{O}_{\mathbb{P}_2}) \rightarrow \mathcal{V} \rightarrow 0,$$

where $\mathcal{A}_2 = R^1 p_{1*}(\mathcal{V} \otimes p_2^* \Omega_{\mathbb{P}_2}^2(2))$, $\mathcal{A}_1 = R^1 p_{1*}(\mathcal{V} \otimes p_2^* \Omega_{\mathbb{P}_2}^1(1))$, and $\mathcal{A}_0 = R^0 p_{1*}(\mathcal{V})$ are locally free sheaves on S of rank 2, 1, and 1 respectively.

Consider the blow up $\alpha_M : \tilde{M} \rightarrow M$. Let $\tilde{\mathcal{V}}$ be the pull-back of \mathcal{V} to $\tilde{M} \times \mathbb{P}_2$ along $\alpha_M \times \text{id}$. Pulling back once more along τ_M one obtains the exact sequence

$$0 \rightarrow \tau_M^*(\mathcal{A}_2 \boxtimes \mathcal{O}_{\mathbb{P}_2}(-2)) \rightarrow \tau_M^*(\mathcal{A}_1 \boxtimes \mathcal{O}_{\mathbb{P}_2}(-1) \oplus \mathcal{A}_0 \boxtimes \mathcal{O}_{\mathbb{P}_2}) \rightarrow \tau_M^*(\tilde{\mathcal{V}}) \rightarrow 0.$$

Using the canonical section of $\mathcal{O}_{Y_M}(\tilde{D}_M)$, similarly to the construction of (21) we obtain the commutative diagram with exact rows

$$(25) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \tau_M^*(\mathcal{A}_2 \boxtimes \mathcal{O}_{\mathbb{P}_2}(-2)) & \longrightarrow & \tau_M^*(\mathcal{A}_1 \boxtimes \mathcal{O}_{\mathbb{P}_2}(-1) \oplus \mathcal{A}_0 \boxtimes \mathcal{O}_{\mathbb{P}_2}) & \longrightarrow & \tau_M^* \tilde{\mathcal{V}} \rightarrow 0, \\ & & \downarrow & & \parallel & & \downarrow \\ 0 & \rightarrow & \tau_M^*(\mathcal{A}_2 \boxtimes \mathcal{O}_{\mathbb{P}_2}(-2)) \otimes \mathcal{O}_{Y_M}(\tilde{D}_M) & \rightarrow & \tau_M^*(\mathcal{A}_1 \boxtimes \mathcal{O}_{\mathbb{P}_2}(-1) \oplus \mathcal{A}_0 \boxtimes \mathcal{O}_{\mathbb{P}_2}) & \rightarrow & \tilde{\mathcal{V}} \rightarrow 0 \end{array}$$

Note that since the morphism $X \xrightarrow{\nu} M$ is flat, the pull-back of (24) along $\nu \times \text{id}$ is the relative Beilinson resolution of \mathcal{U} up to a twist by $p_1^* \mathcal{L}$ for some line bundle \mathcal{L} on X , i. e., resolution (20) tensorized by $p_1^* \mathcal{L}$. Therefore, the commutativity of (23) implies that the pull-back of (25) along ξ coincides with (21) twisted by the pull back of the line bundle $\nu^* \mathcal{L}$ on \tilde{X} . This completes the proof of Proposition 7.4. \square

Corollary 7.5. *The family $\tilde{\mathcal{V}}$ is universal in the following sense. Each equivalence class of R -bundles has a unique representative as the fibre of $\tilde{\mathcal{V}}$ over its corresponding point in \tilde{M}_8*

and each isomorphism class of non-singular $(3m + 1)$ -sheaves has a unique representative as the fibre of $\widetilde{\mathcal{V}}$ over its corresponding point in $\widetilde{M} \setminus \widetilde{M}_8$. Moreover, the sheaves on Z as in Section 4 (one-dimensional deformations of R -bundles) are obtained as pull-backs of $\widetilde{\mathcal{V}}$ via unique morphisms $T \rightarrow \widetilde{M}$.

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SISSA, VIA BONOMEA, 265, 34136 TRIESTE, ITALY

E-mail address: iena@sissa.it

FB MATHEMATIK, TECHNISCHE UNIVERSITÄT KAISERSLAUTERN, POSTFACH 3049, D-67653 KAISERSLAUTERN, GERMANY

E-mail address: trm@mathematik.uni-kl.de