The Kelmans-Seymour conjecture for apex graphs

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Abstract. We provide a short proof that a 5-connected nonplanar apex graph contains a subdivided K_5 or a K_4^- (= K_4 with a single edge removed) as a subgraph. Together with a recent result of Ma and Yu that *every nonplanar 5-connected graph containing K_4^- as a subgraph has a subdivided K_5*; this settles the Kelmans-Seymour conjecture for apex graphs.

Keywords. Subdivided K_5, Apex graphs.

Preamble. Whenever possible notation and terminology are that of [1]. Throughout, a graph is always simple, undirected, and finite. G always denotes a graph. A subdivided G is denoted TG. K_4^- denotes K_4 with a single edge removed. We write $\delta(G)$ and $d_G(v)$ to denote the minimum degree of G and the degree of a vertex $v \in V(G)$, respectively. The *k-wheel* graph consists of a k-circuit C and an additional vertex, called the *hub*, adjacent to every vertex of C through edges called the *spokes*. C is called the *rim* of the wheel.

§1 Introduction. A refinement of Kuratowski's theorem postulated by the Kelmans-Seymour conjecture (1975) is that: *the 5-connected nonplanar graphs contain a TK_5*. As this conjecture is open for many years now, it does not stand to reason that certain special cases of this conjecture be considered. If to pick a special case, then we contend that the apex graphs are a natural choice; where a graph is *apex* if it has a vertex, referred to as an *apex vertex*, removal of which results in a planar graph. In this paper, we prove in a short manner that:

1.1. *A 5-connected nonplanar apex graph contains a TK_5 or a K_4^- as a subgraph.*

Recently, Ma and Yu [2, 3] proved that:

1.2. ***(Ma-Yu [2, 3])***
A 5-connected nonplanar graph containing K_4^- as a subgraph, contains a TK_5.

By **1.1** and **1.2**, it follows that the Kelmans-Seymour conjecture holds for apex graphs.

1.3. *A 5-connected nonplanar apex graph contains a TK_5.*

Our proof of **1.1**. By Euler's formula, a 2-connected planar graph with minimum degree 5 contains K_4^- as a subgraph [4, Lemma 2]. Consequently, a 5-connected nonplanar apex graph G satisfying $K_4^- \not\subseteq G$ has $\delta(G - v) = 4$, where v is an apex vertex of G. Thus, a 5-connected

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nonplanar apex graph contains K_4^- as a subgraph or has an apex vertex that is part of a 5-(vertex)-disconnector of G. Thus, to prove **1.1**, suffices that we prove the following.

1.4. *A* 5*-connected nonplanar apex graph* G *with an apex vertex contained in a* 5*-(vertex)-disconnector of* G *satisfies* $TK_5 \subseteq G$ *or* $K_4^- \subseteq G$.

Adjourning technical details until later sections, we outline here the sole manner in which we construct a TK_5 in our proof of **1.4** assuming $K_4^- \not\subseteq G$ and v is an apex vertex of G satisfying the premise of **1.4**.

(S.1) We fix an embedding of $G - v$ and identify it with its embedding. We then pick a "suitable" 5-(vertex)-disconnecter D containing v such that $G = G_1 \cup G_2$ and $G[D] = G_1 \cap G_2$.

(S.2) In one of the sides of this disconnector, say G_1, we find a 4-valent vertex u such that together with u the vertices cofacial with u in $G - v$ induce a subdivided $d(u)$-wheel $S \subseteq G_1$ whose spokes are preserved and coincide with the edges incident with u.

(S.3) In $G_1 - v$, we construct 3 pairwise vertex-disjoint paths (i.e., a 3-linkage) linking $D - v$ and the rim of S (not meeting u) so that these paths meet the rim of S only at $N_G(u)$.

(S.4) We choose an arbitrary vertex in $G_2 - D$ and connect it to D through a 5-fan contained in G_2.

(S.5) $uv \in E(G)$ as u is 4-valent.

(S.6) $TK_5 \subseteq$ the union of S, the 3-linkage, the 5-fan, and uv.

Essentially, the remainder of this paper consists of our preparation for this single construction. The accurate form of this construction can be found in §5. We use the discharging method for finding the wheel S in (S.2).

§2 Preliminaries.

SUBGRAPHS. Let H be a subgraph of G, denoted $H \subseteq G$. The boundary of H, denoted by $\text{bnd}_G H$ (or simply $\text{bnd}H$), is the set of vertices of H incident with $E(G) \setminus E(H)$. By $int_G H$ (or simply $intH$) we denote the subgraph induced by $V(H) \setminus bndH$. If $v \in V(G)$, then $N_H(v)$ denotes $N_G(v) \cap V(H)$.

PATHS AND CIRCUITS. For $X, Y \subseteq V(G)$, an (X,Y)-path is a simple path with one end in X and the other in Y internally-disjoint of $X \cup Y$. If $X = \{x\}$, we write (x, Y)-path. If $|X| = |Y| = k \geq 1$, then a set of k pairwise vertex-disjoint (X,Y)-paths is called an (X,Y)-*k-linkage*. Throughout this paper, a linkage is always of size 4.

If $x \in V(G)$ and $Y \subseteq V(G) \setminus \{x\}$, then by (x,Y)-*k-fan* we mean a set of $k \geq 1$ (x,Y)-paths with only x as a common vertex.

The *interior* of an xy-path P is the set $V(P) \setminus \{x,y\}$ and is denoted $intP$. For $u, v \in V(P)$, we write $[uPv]$ to denote the uv-subpath of P. We write (uPv) to denote $int[uPv]$, and in a similar manner the semi-open segments $[uPv)$ and $(uPv]$.

If C is a circuit of a plane graph G and $A = \{a_1, a_2, a_3, a_4\} \subseteq V(C)$ appear in this clockwise order along C, then $[a_i C a_{i+1}]$, $1 \leq i \leq 4$, denotes the segment of C whose ends are a_i

and a_{i+1} and such that its interior, denoted $(a_i C a_{i+1})$, does not meet A (clearly, $a_5 = a_1$). Semi-open segments $[a_i C_u a_{i+1})$ are defined accordingly. Two members of A are called *consecutive* if these are consecutive in the clockwise ordering of A along C.

BRIDGES. Let $H \subseteq G$. By *H-bridge* we mean either an edge $uv \notin E(H)$ and $u, v \in V(H)$ or a connected component of $G - H$. In the latter case, the H-bridge is called *nontrivial.* The vertices of H adjacent to an H-bridge B are called the *attachment vertices* of B. A uv-path internally-disjoint of H with $u, v \in V(H)$, is called an *H-ear.*

HAMMOCKS. A *k-hammock* of G is a connected subgraph H satisfying $|bndH| = k \geq 1$. A hammock H coinciding with its boundary is called *trivial*, *degenerate* if $|V(H)| = |bndH| + 1$, and *fat* if $|V(H)| \geq |bndH| + 2$. We call a 4-hammock *minimal* if all its proper 4-hammocks, if any, are trivial or degenerate.

2.1. *A minimal fat* 4*-hammock H of a* 4*-connected graph G, $K_4^- \not\subseteq G$, satisfies $\kappa(H) \geq 2$.*

Proof. Assume, to the contrary, that $H = H_1 \cup H_2$ such that $\{x\} = V(H_1) \cap V(H_2)$ and $V(H_i) \setminus \{x\} \neq \emptyset$, for $i = 1, 2$. Clearly, $bndH_i = \{x\} \cup X_i$, where $X_i \subset bndH$, for $i = 1, 2$.

Consequently, if H_i is a 4-hammock of G, then H_{3-i} consists of a single edge; implying that H_i is degenerate, by fatness of H. As $d_G(x) \geq 4$, $K_4^- \subseteq G$.

Next, if each of H_i, $i = 1, 2$, is a k-hammock of G with $k \leq 3$, then both are trivial, by 4-connectivity of G. This in turn implies that H is degenerate satisfying $\{x\} = V(H) \setminus bndH$; contradiction to the fatness of H.■

SUBDIVIDED WHEELS. For $u \in V(G)$, we write S_u to denote a subdivided $d(u)$-wheel with hub u, the spokes preserved and coinciding with $\{uv : v \in N(u)\}$. Its rim, denoted C_u, is an induced circuit of G separating u from the rest of G.

If G is a 4-connected plane graph, then such an S_u exists for every $u \in V(G) \setminus V(X_G)$, where X_G is the infinite face of G. Indeed, the set of vertices cofacial with u form C_u. Consequently, if G is a plane graph and $u \in V(G) \setminus V(X_G)$ we refer to S_u as the *facial wheel of u*. Such a subdivided wheel is called *short* if:

(SH.1) $d(u) = 4$ and u is the common vertex of two edge disjoint triangles, say T and T'; and

(SH.2) the two segments of $C_u - (E(C_u) \cap E(T)) - (E(C_u) \cap E(T))$, say Q and Q', satisfy:

(SH.2.a) $2 \leq |V(Q)|, |V(Q')| \leq 4$; and

(SH.2.b) if one segment is of order 4, then the other is of order ≤ 3.

A short wheel is called *imbalanced* if one of its segments is of order 4. An imbalanced wheel $S_u \subseteq H$, where H is a 4-hammock of a 4-connected graph, is called *proper with respect to H* if the interior of its segment of order 4 does not meet $bndH$. If H is understood, then we write *proper.*

FACES OF PLANE GRAPHS. Let G be a 2-connected plane graph. By $F(G)$ we denote the set of faces of a plane graph G. A face f of length k is called a *k-face* and its length is denoted $|f|$. We write $(\geq k)$*-face* and $(\leq k)$*-face* to denote a face of length $\geq k$ and $\leq k$, respectively. A 4-valent vertex is called an (f_1, f_2, f_3, f_4)*-vertex*, if the faces incident with v are of length f_i, $1 \leq i \leq 4$, and these are met in a clockwise order around v.

§3 Linkages and wheels. Throughout this section, G is a 4-connected plane graph, and

$$u \in V(G) \setminus V(X_G),\ S_u \subseteq H \text{ is the facial wheel of } u, \text{ where } H \text{ is a 4-hammock of } G. \quad (3.1)$$

By a C_u-*linkage* we mean a $(bndH, C_u)$-linkage in H; such clearly does not meet u, by planarity. By $end\mathcal{P}$ we refer to the end vertices on C_u of members of a C_u-*linkage* $\mathcal{P}$. For such a $\mathcal{P}$, put $\alpha(\mathcal{P}) = |end\mathcal{P} \cap V(C_u) \cap N(u)|$. Also, if $end\mathcal{P} = \{a_1, a_2, a_3, a_4\}$, then we always assume these appear in this clockwise order along C_u and denote by P_i the member of $\mathcal{P}$ meeting a_i.

By planarity and since $N(u) \subseteq V(C_u)$, every $S_u \cup \mathcal{P}$-bridge does not meet or attach to u. Let $P \in \mathcal{P}$ and let P' be a member of $\mathcal{P}$ or a segment of C_u. By P-*ear* we mean an $S_u \cup \mathcal{P}$-ear with both its ends in P. By (P, P')-*ear* we mean an $S_u \cup \mathcal{P}$-ear with one end in P and the other in P'.

If for any $b \in (a_i C_u a_{i+1})$ there exists a C_u-linkage $\mathcal{P}'$ satisfying $end\mathcal{P}' = (end\mathcal{P} \setminus \{a_i\}) \cup \{b\}$ or $end\mathcal{P}' = (end\mathcal{P} \setminus \{a_{i+1}\}) \cup \{b\}$, then we call $\mathcal{P}$ *slippery with respect to* $[a_i C_u a_{i+1}]$, where $1 \leq i \leq 4$, and $a_5 = a_1$. We say that $\mathcal{P}$ is *slippery* if it is slippery with respect to each segment $[a_i C_u a_{i+1}]$ satisfying $a_i C_u a_{i+1} \neq \emptyset$.

3.2. *A C_u-linkage is slippery.*

Proof. Let $\mathcal{P}$ denote such a linkage, and let $w \in (a_i C_u a_{i+1})$ such that $1 \leq i \leq 4$. Planarity and C_u being induced assert that there is an $S_u \cup \mathcal{P}$-bridge B with w as an attachment. Such a bridge attaches to at least one of $P_i - a_i$ or $P_{i+1} - a_{i+1}$. This is clearly true if B is trivial, as C_u is induced. If nontrivial, then having all attachments of B in $[a_i C_u a_{i+1}]$ implies that the 3-set consisting of u and the two extremal attachments of B on $[a_i C_u a_{i+1}]$ is a 3-disconnector of G, by planarity. ■

It follows now from **3.2** that:

3.3. *A C_u-linkage satisfying $\alpha \geq 1$ exists.*

Our main tool for proving subsequent claims is the following.

3.4. *Suppose that:*

*(**3.4**.a) H is a minimal fat 4-hammock; and*

*(**3.4**.b) $\mathcal{P}$ is a C_u-linkage with $end\mathcal{P} = \{a_1, a_2, a_3, a_4\}$ satisfying:*

*(**3.4**.b.1) $\alpha(\mathcal{P}) = k > 0$, k an integer; and*

*(**3.4**.b.2) $a_1, a_3 \notin N(u)$; and*

*(**3.4**.b.3) $a_2 \in N(u)$; and*

*(**3.4**.b.4) $|N(u) \cap [a_1 C_u a_2]| \geq 2$.*

Then, $K_4^- \subseteq G$ or there exists a C_u-linkage satisfying $\alpha \geq k+1$.

Proof. Assume towards contradiction that

$$\text{a } C_u\text{-linkage with } \alpha \geq k+1 \text{ does not exist.} \quad (3.5)$$

Let P be the $a_1 a_3$-segment of C_u not containing a_4. By (3.5) and planarity, for any $\mathcal{P}$ satisfying (**3.4**.b), every member of $(N(u) \setminus \{a_2\}) \cap P$ is an attachment of an $S_u \cup \mathcal{P}$-bridge attaching to C_u and P_2 only. Such bridges exist by (**3.4**.b.4), (3.5), and since C_u is induced. Consequently, $a_2 \notin bndH$.

Choose a $\mathcal{P}$ satisfying (**3.4**.b) such that

$$\text{no } P_2\text{-ears are embedded in the region of the plane interior to } [a_2 C_u a_3] \cup P_2 \cup P_3. \quad (3.6)$$

By (**3.4**.b.4), let $z \in N(u) \cap (a_1 C_u a_2)$ such that $[a_1 P z]$ is minimal. Let B be an $S_u \cup \mathcal{P}$-bridge attached to z; such is embedded in the region of the plane interior to $[a_1 C_u a_2] \cup P_1 \cup P_2$. By (3.5),

$$B \text{ has no attachment on } P_1. \quad (3.7)$$

Connectivity and existence of z then imply that there are vertices $x \in [a_1 C_u a_2]$ (possibly $x = z$) and $y \in V(P_2)$ attachments of B such that $[a_1 P x]$ and $[y P_2 v]$ are minimal, where $v \in V(P_2) \cap bndH$.

By (3.5),

$$\text{there are no } (P_2, P_3 - a_3)\text{-ears with an end in } [a_2 P_2 y). \quad (3.8)$$

Indeed, if such an ear exists, then P_2 can be rerouted through y and B to meet z, and P_3 can be rerouted through the ear and $[a_2 P_2 y)$ to meet a_2; contradicting (3.5).

Let $\ell \in [a_2 C_u a_3]$ be defined as follows. If there exist an $((a_2 P_2 y), [a_2 C_u a_3])$-ear, then ℓ is an end of such an ear such that $[\ell P a_3]$ is minimal. Otherwise, $\ell = a_2$.

By planarity, (3.6), (3.7), and (3.8), $\{u, x, y, \ell\}$ form the boundary of a 4-hammock of H; such is trivial or degenerate, by minimality of H. In either case, x coincides with z and B consists of the single edge xy (otherwise, there is a k-disconnector, $k \leq 3$, separating B from the rest of G) implying that $\{x, u, a_2, y\}$ induce a K_4^-. ■

We infer the following from **3.4**.

3.9. *Suppose H is a minimal fat 4-hammock. Then, $K_4^- \subseteq G$ or there is a C_u-linkage $\mathcal{P}$ satisfying:*

*(**3.9**.a) $\alpha(\mathcal{P}) \geq 2$; and*

*(**3.9**.b) if $\alpha \leq 2$ for every C_u-linkage, then every C_u-linkage with $\alpha = 2$ meets $N(u)$ at consecutive members of $end\mathcal{P}$.*

Proof. A C_u-linkage satisfying $\alpha \geq 1$ exists, by **3.3**. To show that such a linkage with $\alpha \geq 2$ exists, assume, towards contradiction, that every C_u-linkage has $\alpha \leq 1$. Let $\mathcal{P}$ be a C_u-linkage with $\alpha(\mathcal{P}) = 1$ and $end\mathcal{P} = \{a_1, a_2, a_3, a_4\}$; choose such notation so that $a_2 \in N(u)$. As, by assumption, a linkage with $\alpha \geq 2$ does not exist, each vertex in $N(u) \setminus \{a_2\}$ is an attachment vertex of an $S_u \cup \mathcal{P}$-bridge that has attachments on C_u and P_2 only. Since $d(u) \geq 4$ and C_u is induced, such bridges exist and thus $a_2 \notin bndH$. By rerouting P_2 through such bridges we may choose such a $\mathcal{P}$ such that $N(u) \subseteq (a_1 C_u a_2]$. Thus, by **3.4** the claim follows.

Suppose next, that $\alpha \leq 2$ for every C_u-linkage, and suppose $\mathcal{P}$ is such a linkage with $\alpha(\mathcal{P}) = 2$ so that $N(u)$ is met by nonconsecutive members of $end\mathcal{P}$, say, a_2, a_4. As, by assumption, there is no linkage with $\alpha > 2$, each vertex in $N(u) \setminus \{a_2, a_4\}$ is an attachment vertex of an $S_u \cup \mathcal{P}$-bridge that has attachments on C_u and P_2 only, or on C_u and P_4 only (both options do not occur together). Since $d(u) \geq 4$ and C_u is induced, such bridges exist; hence $|bndH \cap \{a_2, a_4\}| \leq 1$. By rerouting P_2 and/or P_4 through such bridges, we may choose $\mathcal{P}$ so that $|N(u) \cap (a_1 C_u a_2]| \geq 2$ or $|N(u) \cap [a_4 C_u a_1)| \geq 2$. The claim then follows by **3.4**.■

We conclude this section with the following.

3.10. *Let H be minimal and fat and suppose S_u is short such that if it is imbalanced then it is proper. Then, a C_u-linkage satisfying $\alpha \geq 3$ exists.*

Proof. Assume, to the contrary, that

$$\text{a } C_u\text{-linkage satisfying } \alpha \geq 3 \text{ does not exist.} \tag{3.11}$$

By **3.9**, a linkage with $\alpha = 2$ exists; moreover, any C_u-linkage satisfying $\alpha = 2$ meets $N(u)$ at consecutive ends. Suppose $\mathcal{P}$ is such a linkage where $end\mathcal{P} = \{a_1, a_2, a_3, a_4\}$, and choose the notation so that the members of $end\mathcal{P}$ meeting $N(u)$ are a_2 and a_3.

Since C_u is induced,

$$(a_4 C_u a_1) \cap N(u) = \emptyset. \tag{3.12}$$

Indeed, otherwise, a bridge attached to a member of $(a_4 C_u a_1) \cap N(u)$ has an attachment on at least one of P_1 or P_4, by planarity and 4-connectivity (see argument of **3.2**); contradicting (3.11).

Let T, T' be as in (SH.1). S_u being short and (3.12) imply that either

$$|V(T) \cap \{a_2, a_3\}| = |V(T') \cap \{a_2, a_3\}| = 1, \tag{3.13}$$

or

$$|V(T'') \cap \{a_2, a_3\}| = 2, T'' \in \{T, T'\}. \tag{3.14}$$

In either case, (3.12) implies that $\{a_1, a_4\} \subseteq intQ''$, $Q'' \in \{Q, Q'\}$, where Q, Q' are as in (SH.2). Consequently, S_u is imbalanced and consequently proper, by assumption. That is, $bndH \cap \{a_1, a_4\} = \emptyset$.

An $(S_u - a_1, bndH)$-linkage $\mathcal{P}'$ exists in $H - a_1$; otherwise a_1 and a k-disconnector, $k \leq 3$, separating $bndH$ and $S_u - a_1$ in $H - a_1$ form a proper 4-hammock of H that is neither trivial nor degenerate; contradicting the minimality of H. As, by assumption, $d(u) = 4$, $\mathcal{P}'$ does not meet u and is a C_u-linkage in H.

Since S_u is short, $\alpha(\mathcal{P}') = |end\mathcal{P}' \cap N(u)| \geq 2$. We may assume equality holds or the claim follows. If $N(u)$ is met by consecutive members of $\mathcal{P}'$, then these are not contained in a single triangle T or T', as this would contradict (3.12) (which applies to any C_u-linkage with $\alpha = 2$ meeting $N(u)$ at consecutive members). On the other hand, if $N(u)$ is met by nonconsecutive members of $\mathcal{P}'$ (so that (3.13) is satisfied by $\mathcal{P}'$), then a C_u-linkage satisfying the premise of **3.4** exists (see argument of **3.9**) and the claim follows by **3.4**.■

§4 Short wheels in minimal fat hammocks. The purpose of this section is to prove **4.1**. Let H be a minimal fat 4-hammock of a 4-connected plane graph G; such is 2-connected, by **2.1**. Consequently, every member of $F(H)$ is a circuit of H, each edge of H is contained in precisely 2 faces (we use this in the proof of (4.2) below), and each $v \in V(H)$ is incident with $d_H(v)$ distinct faces. A vertex $v \in V(H)$ is called *good* if $d_H(v) \geq 5$ or $v \in bndH$.

4.1. *Let H be a minimal fat 4-hammock of a 4-connected plane graph G satisfying:*

*(**4.1**.a) $K_4^- \nsubseteq G$; and*

*(**4.1**.b) every $P_3 \cong P \subset H$ contains a good vertex; and*

*(**4.1**.c) every $K_3 \cong K \subset H$ contains ≥ 2 good vertices.*
Then, H contains a short facial wheel S_u for some $u \in V(H) \setminus V(X_H)$ such that if S_u is imbalanced, then it is proper.

We shall use the well-known "discharging method" in order to prove **4.1**. Such a method involves four main steps: (i) distributing initial charges to elements of the graph, (ii) calculating the total charge distributed using Euler's formula, (iii) redistributing charges according to a set of discharging rules, and finally (iv) estimating the resultant charge of each element. In our case, we shall employ the following charging-discharging schemes.

CHARGING SCHEME. For $x \in V(H) \cup F(H)$, define the *charge* $ch(x)$ as follows:
(CH.1) $ch(v) = 6 - d_H(v)$, for any $v \in V(H)$.
(CH.2) $ch(f) = 6 - 2|f|$, for any $f \in F(H) \setminus \{X_H\}$.
(CH.3) $ch(X_H) = -5\frac{2}{3} - 2|X_H|$.

Next, we show that

$$\sum_{x \in V(H) \cup F(H)} ch(x) = \frac{1}{3}. \tag{4.2}$$

Proof.

$$\begin{aligned}
\sum_{x \in V(H) \cup F(H)} ch(x) &= -5\frac{2}{3} - 2|X_H| + \sum_{f \in F(H) \setminus X_H} (6 - 2|f|) + \sum_{v \in V(H)} (6 - d(v)) \\
&= -5\frac{2}{3} - 2|X_H| + 6(|f(H)| - 1) + \sum_{f \in F(H) \setminus X_H} (-2|f|) + \sum_{v \in V(H)} (6 - d(v)) \\
&= -5\frac{2}{3} + 6(|f(H)| - 1) - 2(2|E|) + 6|V(H)| - 2|E(H)| \\
&= 6(F(H) - E(H) + V(H)) - 11\frac{2}{3} = \frac{1}{3}
\end{aligned}$$

■

DISCHARGING SCHEME. In what follows, by *send* we mean "discharge" or "pass charge".

(DIS.1) Let $v \in V(X_H)$, such that $2 \leq d_H(v) \leq 4$.

(DIS.1.a) If $d_H(v) = 2$, then let g the face incident with v other than X_H. If $|g| = 3$, then v sends 4 to X_H. Otherwise v sends $3\frac{2}{3}$ to X_H and $\frac{1}{3}$ to g.

(DIS.1.b) If $d_H(v) = 3$, then v sends $2\frac{2}{3}$ to X_H and $\frac{1}{3}$ to every incident (≥ 4)-face.

(DIS.1.c) If $d_H(v) = 4$, then v sends $1\frac{2}{3}$ to X_H and $\frac{1}{3}$ to every incident (≥ 4)-face.

(DIS.2) If $v \in V(H)$ is at least 5-valent, then, v sends $\frac{1}{3}$ to every incident (≥ 4)-face.

(DIS.3) If $v \in V(H) \setminus V(X_H)$ is 4-valent, then:

(DIS.3.a) v sends $\frac{2}{3}$ to every incident 4-face.

(DIS.3.b) v sends 1 to every incident 5-face, unless v is a $(3,4,3,5)$-vertex, and then v sends $1\frac{1}{3}$ to its single incident 5-face.

(DIS.3.c) v sends $1\frac{1}{3}$ to every (≥ 6)-face.

Proof of 4.1. Assume, to the contrary, that the claim is false and apply (CH.1-3) and (DIS.1-3) to members of $V(H) \cup F(H)$. Let $ch^*(x)$ denote the charge of a member of $V(H) \cup F(H)$ after applying (DIS.1-3). We obtain a contradiction to (4.2) by showing that $ch^*(x) \leq 0$ for every $x \in V(H) \cup F(H)$. This is clearly implied by the following claims proved below.

(**4.1**.A) $ch^*(v) \leq 0$, *for each* $v \in V(H)$.
(**4.1**.B) $ch^*(f) \leq 0$, *for each* $f \in F(H) \setminus \{X_H\}$.
(**4.1**.C) $ch^*(X_H) \leq 0$.

Observe that according to (DIS.1-3), faces do not send charge and vertices do not receive charge.

Proof of (**4.1**.A). It is sufficient to consider vertices v satisfying $2 \leq d_H(v) \leq 4$. Indeed, if $d_H(v) \geq 6$, then $ch(v) = ch^*(v) \leq 0$ by (CH.1); and, if $d_H(v) = 5$, then v is incident with at least three (≥ 4)-faces, as $K_4^- \nsubseteq G$, implying that $ch^*(v) \leq 0$ by (DIS.2).

By (DIS.1.a-c), $ch^*(v) \leq 0$ for every $v \in V(X_H)$ with $2 \leq d_H(v) \leq 4$. This is clear if v is 2-valent; and true in case v is at least 3-valent as such a vertex is incident with at least one (≥ 4)-face distinct of X_H, since $K_4^- \nsubseteq G$.

It remains to consider $v \notin V(X_H)$ satisfying $2 \leq d_H(v) \leq 4$; such is clearly 4-valent, as $\kappa(G) \geq 4$. We may assume v is not incident with at least three (≥ 4)-faces, for otherwise $ch^*(v) \leq 0$ since by (DIS.3.a-c), v sends at least $\frac{2}{3}$ to each (≥ 4)-face. Consequently, since $K_4^- \nsubseteq G$, v is incident with precisely two 3-faces that are edge disjoint. Next, at least one of the remaining faces incident with v, say f, is a 4-face for otherwise $ch^*(v) \leq 0$ by (DIS.3.b-c). The remaining face incident with v, say g, is a (≥ 5)-face for otherwise H contains a short facial wheel; contradictory to our assumption. By (DIS.3.a), v sends $2/3$ to f. Hence, $ch^*(v) \leq 0$ by (DIS.3.b) if $|g| = 5$, and by (DIS.3.c) if $|g| \geq 6$.□

Proof of (**4.1**.B). If $|f| = 3$, then, $ch(f) = ch^*(f) = 0$ for any 3-face f, by (CH.2). It remains to consider (≥ 4)-faces. If f is such a face, then put $A_f = \{v \in V(f) \setminus bndH : d_H(v) = 4\}$ and note that (**4.1**.b) implies:

$$|A_f| \leq |f| - 2. \tag{4.3}$$

Clearly,

$$ch^*(f) = ch(f) + c(A_f) + c(V(f) \setminus A_f), \tag{4.4}$$

where $c(X)$, $X \subseteq V(f)$, is the total charge sent to f from members of X.

We may assume that f is a 5-face. Indeed, if $|f| = 4$, then $c(A_f) \leq \frac{2}{3}|A_f|$, by (DIS.1.c) and (DIS.3.a), $c(V(f) \setminus A_f) \leq \frac{1}{3}(|f| - |A_f|)$, by (DIS.1.a) and (DIS.2), and $|A_f| \leq 2$, by (4.3). Thus, $ch^*(f) \leq 0$, by (4.4). Next, if $|f| \geq 6$, then $c(A_f) \leq 1\frac{1}{3}|A_f|$, by (DIS.1.c) and (DIS.3.c), $c(V(f) \setminus A_f) \leq \frac{1}{3}(|f| - |A_f|)$, by (DIS.1.a) and (DIS.2), and $|A_f| \leq 4$, by (4.3). Hence, $ch^*(f) \leq 0$, by (4.4).

Assume then that $|f| = 5$ so that $|A_f| \leq 3$, by (4.3). We may assume that f is incident with a $(3,4,3,5)$-vertex not in $V(X_H)$; otherwise, $c(A_f) \leq 1 \times |A_f|$, by (DIS.1.c) and (DIS.3.b), $c(V(f) \setminus A_f) = \frac{1}{3}(|f| - |A_f|)$, by (DIS.1-2). By (4.4) (and as $|A_f| \leq 3$), $ch^*(f) \leq 0$.

Let then $v \in V(f) \setminus V(X_{_H})$ be a $(3,4,3,5)$-vertex. The members of $V(f)$ adjacent to v, say v', v'', are good by (**4.1**.c); and $|(V(f) \setminus \{v, v', v''\}) \cap bndH| \geq 1$ or $S_{_u}$ is proper contradicting the assumption that such wheels do not exist in H. Let $v''' \in (V(f) \setminus \{v, v', v''\}) \cap bndH$. v sends $1\frac{1}{3}$ to f, By (DIS.3.c). Each of $\{v', v'', v'''\}$ sends $\frac{1}{3}$ to f, by (DIS.1-2) and since $f \neq X_{_H}$. The remaining vertex $V(f) \setminus \{v, v', v'', v'''\}$ sends at most $1\frac{1}{3}$ to f, by (DIS.1-3) and since $f \neq X_H$. Consequently, $ch^*(f) = ch(f) + 2 \times 1\frac{1}{3} + 2 \times \frac{1}{3} \leq 0$ (as $ch(f) = -4$).□

Proof of (**4.1**.C). For $i = 2, \ldots, 5$, let $A_i = \{v \in V(X_H) : d_H(v) = i\}$; $B = \{v \in V(X_H) : d_H(v) \geq 5\}$; $A_2' = \{v \in A_2 : v$ is incident with a 3-face$\}$; and put $A_2'' = A_2 \setminus A_2'$. Clearly, $A_i \subseteq bndH$ for $i < 4$. Hence, since H is a 4-hammock of G and $\kappa(G) \geq 4$,

$$|A_2| + |A_3| \leq 4. \tag{4.5}$$

By definition,

$$|A_2| + |A_3| + |A_4| \leq |X_H|. \tag{4.6}$$

By (CH.3) and (DIS.1-2),

$$ch^*(X_H) = -5\tfrac{2}{3} - 2|X_H| + 4|A_2'| + 3\tfrac{2}{3}|A_2''| + 2\tfrac{2}{3}|A_3| + 1\tfrac{2}{3}|A_4| + \tfrac{1}{3}|A_5| \tag{4.7}$$

By (4.7), (4.5), and (4.6), it can be easily verified that $ch^*(X_{_H}) \leq 0$ in the following cases: (i) $|X_H| \geq 11$; (ii) $7 \leq |X_H| \leq 10$ and $|A_2| \neq 4$; and (iii) $4 \leq |X_H| \leq 6$ and $|A_2| \leq 2$.

It remains to show that $ch^*(X_H) \leq 0$ in the cases: (I) $7 \leq |X_H| \leq 10$ and $|A_2| = 4$ and (II) $4 \leq |X_H| \leq 6$, and $|A_2| \geq 3$. In the latter case, $V(X_H) \setminus A_2$ is a k-disconnector, $k \leq 3$, of G; this is so since $V(H) \setminus V(X_H) \neq \emptyset$ by the fatness of H and each vertex in $intH$ being at least 4-valent.

Suppose then that (I) occurs. Then, $|B| = 0$ can be assumed; indeed, if $|B| \geq 1$, then $|A_2| + |A_3| + |A_4| \leq |X_H| - 1$ implying that $ch^*(f) \leq 0$, by (4.7) and (4.5). We may also assume that $|A_2'| \geq 1$; otherwise $|A_2''| = |A_2| = 4$, and $ch^*(f) \leq 0$, by (4.7) and (4.5). Let then $x \in A_2' \subseteq bndH$. $\{x\} \cup N_H(x)$ induce a 3-face implying that at least one member of $N_H(x)$ is a good verex, by (**4.1**.c), and consequently in $bndH$ as $A_5 \subseteq B = \emptyset$ (see above). As $|X_H| \geq 7$ and thus $|V(H) \setminus \{x\}| \geq 6$, it follows that $(bndH \setminus \{x\}) \cup N_H(x)$ is either a 3-disconnector of G or a 4-hammock of H with its interior containing at least 2 vertices; contradicting $\kappa(G) \geq 4$ and H being minimal, respectively.□
■

§5 Proof of 1.4. Suppose $K_{_4}^- \not\subseteq G$ and let v be an apex vertex of G contained in some 5-disconnector of G. Fix an embedding of G and identify G with its embedding.

By [4, Lemma 2](see Introduction), $\delta(G - v) = 4$; implying that we may assume that $G - v$ has a minimal fat 4-hammock H. To see this, let $u \in V(G - v)$ be 4-valent. $N_{G-v}(u)$ is the boundary of two 4-hammocks of $G - v$. If each of these two hammocks is degenerate, then G is a 7-vertex graph which contains a $TK_{_5}$. Thus, we may assume that at least one of these hammocks is fat; implying that minimal fat 4-hammocks exist in $G - v$.

H satisfies (**4.1**.b-c) or $K_{_4}^- \subseteq G$; hence, by **4.1**, there is a short facial wheel $S_{_u} \subseteq H$ with some 4-valent vertex $u \notin V(X_{_H})$ as a hub; and such that $S_{_u}$ is proper if it is imbalanced. Let

$\mathcal{P}$ be a C_u-linkage in H satisfying $\alpha(\mathcal{P}) \geq 3$, by **3.10**. The set $\{v\} \cup bndH$ forms the boundary of a 5-hammock H' of G satisfying $S_u \subseteq H'$; let $w \notin V(H')$ and let F be a $(w, bndH')$-5-fan in G, such clearly does not meet $intH'$. Observing that $uv \in E(G)$, as u is 4-valent in $G - v$, it follows that $TK_5 \subseteq S_u \cup \mathcal{P} \cup F \cup \{uv\} \subseteq G$. ■

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