STATISTICALLY QUASI-CAUCHY SEQUENCES

HÜSEYİN ÇAKALLI
MALTEPE UNIVERSITY,TR 34857, MALTEPE, ISTANBUL, TURKEY
PHONE:(+90216)6261050 EXT:1206, FAX:(+90216)6261113

Abstract. A subset E of a metric space (X,d) is totally bounded if and only if any sequence of points in E has a Cauchy subsequence. We call a sequence (x_n) statistically quasi-Cauchy if $st-\lim_{n\to\infty} d(x_{n+1},x_n)=0$, and lacunary statistically quasi-Cauchy if $S_\theta-\lim_{n\to\infty} d(x_{n+1},x_n)=0$. We prove that a subset E of a metric space is totally bounded if and only if any sequence of points in E has a subsequence which is any type of the following, statistically quasi-Cauchy, lacunary statistically quasi-Cauchy, quasi-Cauchy, and slowly oscillating. It turns out that a function defined on a subset E of a metric space is uniformly continuous if and only if it preserves either quasi-Cauchy sequences or slowly oscillating sequences of points in E.

1. Introduction

The concept of a metric, and any concept in a metric space play a very important role not in functional analysis, and topology but also in other branches of sciences involving mathematics especially in computer sciences, information theory, biological sciences, and dynamical systems.

A subset E of a metric space (X,d) is totally bounded if it has a finite ε-net for each $\varepsilon>0$ where a subset A of E is called to be an ε-net in E if $E=\bigsqcup_{a\in A}[E\bigcap B(a,\varepsilon)]$. This is equivalent to the statement that any sequence of points in E has a Cauchy subsequence. This suggests us to ask what happens if we replace the term "Cauchy" with another term "quasi-Cauchy". In fact we could interchangeably put any of the terms "quasi-Cauchy", "statistically quasi-Cauchy",

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"lacunary statistically quasi-Cauchy", and "slowly oscillating" instead of the term "Cauchy".

The purpose of this paper is to investigate characterizations of totally boundedness of a subset of a metric space X, and characterizations of uniform continuity a function defined on a subset of X via sequences mentioned above.

2. Priliminaries

Throughout this paper, $\mathbf{N}$, R, and X will denote the set of positive integers, the set of real numbers, and a metric space with a metric d, respectively. We will use boldface letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, ... for sequences $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n)$, $\mathbf{z} = (z_n)$, ... of points in X.

Recall that a subset E of a metric space (X, d) is called bounded if

$$\delta(E) = sup\{d(a, b) : a, b \in E\} \leq M$$

where M is a positive real constant number. A subset A of a metric space X is said to be an ε-net in X if

$$X = \bigsqcup_{a \in A} B(a, \varepsilon).$$

The metric space (X, d) is called totally bounded if it has a finite ε-net in X for each $\varepsilon > 0$. A subspace (E, d_E) of (X, d) is said to be totally bounded if it is totally bounded as a metric space in its own right. A subset E of a metric space (X, d) is said to be totally bounded if it is totally bounded as a metric subspace. The definition of totally bounded sets in arbitrary metric spaces is consistent with that of bounded sets in R, R^2, R^3, and R^n, when equipped with the usual metric, but need not be consistent when equipped with an arbitrary metric. Moreover, note that the set $E = \{e_1, e_2, ..., e_n, ...\} \subset \ell^2$ although bounded with $\delta(E) = \sqrt{2}$, is still quite big in the sense that there are infinite many points, in fact, in E none of which is close to any of the others, in other words it is not a totally bounded set. Such sets in arbitrary metric space fail to give certain properties, namely totally boundedness, which are expected in analogy with boundededness in R, the set of real numbers with the usual topology generated by the absolute value metric.

The notion of statistical convergence was introduced by Fast [1], has been investigated by Fridy in [2], and has been extended to metrizable topological Hausdorff groups in [3]. A sequence (x_k) of points in X is called to be statistically convergent to an element ℓ of X if for each positive real number ε

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : d(x_k, \ell) \geq \varepsilon\}| = 0,$$

and this is denoted by $st - \lim_{n\to\infty} x_n = \ell$.

The notion of lacunary statistical convergence was introduced, and studied by Fridy and Orhan in [4], and has been extended to metrizable topological Hausdorff groups in [5]. A sequence (x_k) of points in X is called lacunary statistically convergent to an element ℓ of X if

$$\lim_{r\to\infty} \frac{1}{h_r}|\{k \in I_r : d(x_k, \ell) \geq \varepsilon\}| = 0,$$

for every positive real number ε where $I_r = (k_{r-1}, k_r]$ and $k_0 = 0$, $h_r : k_r - k_{r-1} \to \infty$ as $r \to \infty$ and $\theta = (k_r)$ is an increasing sequence of positive integers. Throughout this paper, we assume that $\liminf_r \frac{k_r}{k_{r-1}} > 1$.

A sequence (x_n) of points in X is called quasi-Cauchy if $\lim_{n\to\infty} \Delta x_n = 0$ where $\Delta x_n = d(x_{n+1}, x_n)$ (see [6]), and a sequence (x_n) of points in X is called slowly oscillating if for any given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and $N = N(\varepsilon)$ such that $d(x_m, x_n) < \varepsilon$ if $n \geq N(\varepsilon)$ and $n \leq m \leq (1+\delta)n$ (see [7]).

The term "quasi-Cauchy" was used by Burton and Coleman while those sequences were studied in both [8], and [9] in which those sequences were called as forward convergent to zero. Trivially, Cauchy sequences are slowly oscillating. It is easy to see that any slowly oscillating sequence is quasi-Cauchy, and therefore any Cauchy sequence is quasi-Cauchy. The converses are not always true. There are quasi-Cauchy sequences which are not Cauchy. There are quasi-Cauchy sequences which are not slowly oscillating. Furthermore any quasi-slowly oscillating sequence ([10] is δ-quasi-Cauchy ([11]). Any subsequence of a Cauchy sequence is Cauchy. The analogous property fails for quasi-Cauchy sequences, and slowly oscillating sequences as well.

3. SEQUENTIAL DEFINITIONS OF TOTALLY BOUNDEDNESS

We call a sequence (x_n) of points in X statistically quasi-Cauchy if $st-\lim_{n\to\infty} \Delta x_n = 0$, and lacunary statistically quasi-Cauchy if $S_\Theta - \lim_{n\to\infty} \Delta x_n = 0$.

In this section, we give a further investigation of quasi-Cauchy sequences, and slowly oscillating sequences; and obtain some characterizations of totally boundedness of a subset of X by using the concepts of a quasi-Cauchy sequence, a slowly oscillating sequence, a statistically quasi-Cauchy sequence, and a lacunary statistically quasi-Cauchy sequence of points in X.

A subset E of X is called slowly oscillating compact if whenever $\mathbf{x} = (x_n)$ is a sequence of points in E there is a slowly oscillating subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of $\mathbf{x}$ (see also [12]); and a subset E of X is called ward compact if whenever $\mathbf{x} = (x_n)$ is a sequence of points in E there is a quasi-Cauchy subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of $\mathbf{x}$ (see [13], and [14]).

Now we give the following two lemmas which will have quite importance in our proofs.

Lemma 1. ([3]) Any statistically convergent sequence of points in X with a statistical limit ℓ has a convergent subsequence with the same limit ℓ in the ordinary sense.

Lemma 2. ([5]) Any lacunary statistically convergent sequence of points in X with a lacunary statistical limit ℓ has a convergent subsequence with the same limit ℓ in the ordinary sense.

Now we give one of our main results, which enables us to see certain characterizations of totally boundedness of a subset of X.

Theorem 3. Let E be a subset of X. The following statements are equivalent.
(a) E is totally bounded.
(b) E is ward compact.
(c) E is slowly oscillating compact.
(d) Any sequence of points in E has a statistically quasi-Cauchy subsequence.
(e) Any sequence of points in E has a lacunary statistically quasi-Cauchy subsequence.

Proof. Let us first prove that (a) implies (b). Take any sequence (x_n) of points in E. Since E can be covered by a finite number of subsets of X of diameter less than 1, one of these sets , which we denote by A_1, must contain x_n for infinitely many values of n. We may choose a positive integer n_1 such that $x_{n_1} \in A_1$. Then A_1 is totally bounded and hence it can be covered by a finite number of subsets of A_1 of diameter less than $\frac{1}{2}$. One of these subsets of A_1, which we denote by A_2, contains x_n for infinitely many n. choose a positive integer n_2 such that $n_2 > n_1$ and $x_{n_2} \in A_2$. Since $A_2 \subset A_1$, it follows that $x_{n_2} \in A_1$ as well. Continuing in this way, we obtain, for any positive integer k, a subset A_k of A_{k-1} with *diameter* less than $\frac{1}{k}$ and a term $x_{n_k} \in A_k$ of the sequence (x_n) , where $n_k > n_{k-1}$. Since all x_{n_k}, $x_{n_{k+1}}$, $x_{n_{k+2}}$,...,$x_{n_{k+j}}$,... lie in A_k and the diameter of A_k is less than $\frac{1}{k}$, it follows that (x_{n_k}) is a quasi-Cauchy subsequence of the sequence (x_n). To prove that (b) implies (a), suppose that E is not totally bounded. Then there exists an $\varepsilon > 0$ such that there does not exist a finite ε-net. Take any $x_1 \in E$. By the assumption that E is not totally bounded, the open ball $B_E(x_1, \varepsilon)$ is not equal to E, i.e. $B_E(x_1, \varepsilon) \neq E$, so there exists an $x_2 \in E$ such that $d_E(x_1, x_2) \geq \varepsilon$, i.e. $x_2 \notin B_E(x_1, \varepsilon)$, and $x_2 \in E$. Then $B_E(x_1, \varepsilon) \cup B_E(x_2, \varepsilon) \neq E$ otherwise $\{x_1, x_2\}$ would be a finite ε-net in E. Let $x_3 \notin B_E(x_1, \varepsilon) \cup B_E(x_2, \varepsilon)$ i.e. $d_E(x_1, x_2) \geq \varepsilon$, $d_E(x_1, x_3) \geq \varepsilon$, and $d_E(x_2, x_3) \geq \varepsilon$. Continuing the process in this manner, one can obtain a sequence (x_n) of points in E such that

$$x_n \notin B_E(x_1, \varepsilon) \cup B_E(x_2, \varepsilon) \cup ... \cup B_E(x_{n-1}, \varepsilon),\ (n = 2, 3, ...)$$

$$i.e.\ d_E(x_i, x_n) \geq \varepsilon\ (i = 1, 2, ..., n-1)\ and\ (n = 1, 2, ...),\ n \neq i.$$

The sequence (x_n) constructed in this manner has no quasi-Cauchy subsequence. This contradiction completes the proof that (b) implies (a). If E is ward compact, then any sequence (x_n) of points in E has a quasi-Cauchy subsequence, which is also statistically quasi-Cauchy. Thus (b) implies (d). Let any sequence of points in E have a statistically quasi-Cauchy subsequence. Take any sequence (x_n) of points in E. Thus (x_n) has a statistically quasi-Cauchy subsequence (x_{k_n}). By Lemma 1, there exists a subsequence (z_j) of the sequence (Δx_{k_n}) such that $\lim_{j\to\infty} z_j = \lim_{n\to\infty} \Delta x_{k_{n_j}} = 0$. This means that the subsequence $(x_{k_{n_j}})$ is quasi-Cauchy. So we get that (d) implies (b). Since any convergent sequence is lacunary statistically convergent with the same limit, it follows that (b) implies (e). Now suppose that any sequence of points in E has a statistically quasi-Cauchy subsequence. If (x_n) is any any sequence of points in E, then it has a lacunary statistically quasi-Cauchy subsequence, x_{k_n}), say. Thus $S_\theta - \lim_{n\to\infty} \Delta x_{k_n} = 0$. By Lemma 2, there exists a subsequence (z_m) of the sequence (Δx_{k_n}) such that $\lim_{n\to\infty} z_m = 0$. This means that the subsequence we have obtained, $(x_{k_{n_m}})$, is quasi-Cauchy. So we get that (e) implies (b). As Cauchy sequences are slowly oscillating, we see that (a) implies (c). Finally, let E be slowly oscillating compact. Take any sequence $\mathbf{x} = (x_n)$ of points in E. Then $\mathbf{x}$ has a slowly oscillating subsequence, say $\mathbf{z}$. Since any slowly oscillating sequence is quasi-Cauchy it follows that E is ward compact. This completes the proof of the theorem. □

We see that for any regular subsequential method G defined on X, if a subset E of X is G-sequentially compact, then any one of the conditions of Theorem 1 is satisfied (see [15] for the definition of G-sequentially compactness, see also [16]). But the converse is not always true.

4. SEQUENTIAL DEFINITIONS OF UNIFORM CONTINUITY

Slowly oscillating and ward continuity concepts were introduced by Cakalli in [17] for real functions, and further investigation of slowly oscillating continuity was done by Canak, and Dik in [12], and further invesigation of ward continuity was done by Burton and Coleman in [6]. Now we modify Lemma 1 in [6] to the metric space setting.

Lemma 4. If (ξ_n, η_n) is a sequence of ordered pairs of points in a subset E of X such that $\lim_{n\to\infty} d(\xi_n, \eta_n) = 0$, then there exists a quasi-Cauchy sequence (x_n) with the property that for any positive integer i there exists a positive integer j such that $(\xi_i, \eta_i) = (x_{j-1}, x_j)$.

Proof. For each positive integer k, fix $z_0^k, z_1^k, ..., z_{n_k}^k$ in E with $z_0^k = \eta_k$, $z_{n_k}^k = \xi_{k+1}$, and $d(z_i^k, z_{i-1}^k) < \frac{1}{k}$ for $1 \le i \le n_k$. Now write

$$(\xi_1, \eta_1, z_1^1, ..., z_{n_1-1}^1, \xi_2, \eta_2, z_1^2, ..., z_{n_2-1}^2, \xi_3, \eta_3, ..., \xi_k, \eta_k, z_1^k, ..., z_{n_{k-1}}^k, \xi_{k+1}, \eta_{k+1}, ...)$$

Then denoting this sequence by (x_n) we obtain that for any positive integer i there exists a positive integer j such that $(\xi_i, \eta_i) = (x_{j-1}, x_j)$. This completes the proof of the lemma.

In [12] it was proved that any slowly oscillating continuous function defined on a slowly oscillating compact subset A of R is uniformly continuous. Since any slowly oscillating compact subset of R is bounded it follows that any slowly oscillating continuous function defined on any bounded subset of R is uniformly continuous. Recently in [13], (see also [14], and [18]) it was proved that a slowly oscillating continuous function is uniformly continuous on any subset of R. We see below that is also the case that any slowly oscillating continuous function on a subset of a metric space is uniformly continuous. In [8] it was proved that any ward continuous function defined on a ward compact subset A of R is uniformly continuous. Since any ward compact subset of R is bounded it follows that any ward continuous function defined on any bounded subset of R is uniformly continuous. Recently in [7] (see also [13], and [14]) it was proved that a ward continuous function defined on any subset of R is uniformly continuous. It turns out that it is also the case that any ward continuous function on a subset of a metric space is uniformly continuous. □

Theorem 5. Let E be a subset of X, and f be a function defined on E. Then the following statements are equivalent.
(UC) f is uniformly continuous on E.
(WC) f is ward continuous on E.
(SOC) f is slowly oscillating continuous on E.

Proof. Let f be uniformly continuous on E. To prove that f is ward continuous on E, take any quasi-Cauchy sequence (x_n), and $\varepsilon > 0$. Uniform continuity of f on E implies that there exists a $\delta > 0$, depending on ε, such that $d(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$ and $x, y \in E$. For this $\delta > 0$, there exists an $N = N(\delta) = N_1(\varepsilon)$ such that $\Delta x_n < \delta$ whenever $n > N$. Hence $\Delta f(x_n) < \varepsilon$ if $n > N$. Hence it follows that $(f(x_n))$ is quasi-Cauchy, which completes the proof of that (UC) implies (WC). To prove that uniform continuity of f implies slowly oscillating continuity of f on E, take a slowly oscillating sequence $\mathbf{x} = (x_n)$ of points in E. Let $\varepsilon > 0$. Uniform continuity of f implies that there exists a $\delta > 0$ such that

$d(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$. Since (x_n) is slowly oscillating, for this $\delta > 0$, there exist a $\delta_1 > 0$ and $N = N(\delta) = N_1(\varepsilon)$ such that $d(x_m, x_n) < \delta$ if $n \geq N(\delta)$ and $n \leq m \leq (1+\delta_1)n$. Hence $d(f(x_m), f(x_n)) < \varepsilon$ if $n \geq N(\delta)$ and $n \leq m \leq (1+\delta_1)n$. It follows from this that $(f(x_n))$ is slowly oscillating.

To prove that (SOC) implies (UC) suppose that f is not uniformly continuous on E so that there exists an $\varepsilon > 0$ such that for any $\delta > 0$ $x, y \in E$ with $d(x, y) < \delta$ but $d(f(x), f(y)) \geq \varepsilon$. For each positive integer n, fix $d(x_n, y_n) < \frac{1}{n}$, and $d(f(x_n), f(y_n)) \geq \varepsilon$. As in the proof of Lemma 4, one can construct a slowly oscillating sequence (t_n) which has a subsequence $(x_n) = (t_{k_n})$ such that $\lim_{n\to\infty} \Delta x_n = 0$, but $d(f(x_{n+1}), d(f(f(x_n)) \geq \varepsilon$. Therefore the transformed sequence $(f(x_n))$ is not slowly oscillating. Thus this contradiction yields that (SOC) implies (UC). Now let us prove that (WC) implies (UC). Suppose that f is not uniformly continuous on E. Then since the sequence constructed in the proof that (SOC) implies (UC) is clearly quasi-Cauchy, but $(f(x_n))$ is not, we see that (WC) implies (UC). This completes the proof of the theorem.

□

Corollary 6. Let G be a regular subsequential method. If a function is uniformly continuous on E, then it is G-sequentially continuous on E(see [16]).

5. Conclusion

The present work contains not only an improvement and a generalization of the works of Cakalli [8], Section 1 of the paper of Burton and Coleman [6] (note that in that paper the authors presented the main theorem in the real case, although they studied some other concepts in the metric space setting in Section 2 of their paper) and Section II of the paper of Vallin [19] as it has been presented in a more general setting, i.e. in a metric space which is more general than the real space, but also an investigation of some further results for real functions, which are also new for the real case. So that one may expect it to be more useful tool in the field of metric space theory in modeling various problems occurring in many areas of science, computer science, information theory, and biological science. For further study, we suggest to investigate quasi-Cauchy sequences, statistically-quasi-Cauchy-sequences, and lacunary statistically-quasi-Cauchy-sequences of fuzzy points, and characterizations of uniform continuity for the fuzzy functions defined on a fuzzy metric space. However due to the change in settings, the definitions and methods of proofs will not always be analogous to those of the present work (for example see [20]).

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Huseyin Cakalli, Department of Mathematics, Faculty of Science and Letters, Maltepe University, Marmara Eğİtİm Köyü, TR 34857, Maltepe, İstanbul-Turkey
Phone:(+90216)6261050 ext:1206, fax:(+90216)6261113

E-mail address: hcakalli@maltepe.edu.tr; hcakalli@gmail.com