arXiv:1012.5672v1 [math.AP] 27 Dec 2010

Positive solutions for singularly perturbed nonlinear elliptic problem on manifolds via Morse theory

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Dedicated to Franco Nicolosi

Abstract

Given (M, g_0) we consider the problem $-\varepsilon^2\Delta_{g_0+h}u + u = (u^+)^{p-1}$ with $(\varepsilon, h) \in (0, \bar\varepsilon) \times \mathscr{B}_\rho$. Here $\mathscr{B}_\rho$ is a ball centered at 0 with radius ρ in the Banach space of all C^k symmetric covariant 2-tensors on M. Using the Poincaré polynomial of M, we give an estimate on the number of nonconstant solutions with low energy for (ε, h) belonging to an residual subset of $(0, \bar\varepsilon) \times \mathscr{B}_\rho$, for $\bar\varepsilon$, ρ small enough.

Keywords: singular perturbation, nondegenerate critical points, Morse theory

AMS subject classification: 58G03, 58E30

1 Introduction

Let (M, g) be a smooth compact connected Riemannian manifold of dimension $n \geq 2$ without boundary, endowed with the metric tensor g. We are interested in the following problem

$$\begin{cases} -\varepsilon^2\Delta_g u + u = |u|^{p-2}u & \text{in } M \\ u \in H^1_g(M), & u > 0. \end{cases} \tag{1.1}$$

where $2 < p < \frac{2n}{n-2}$ with $n \geq 3$, $p > 2$ if $n = 2$, and ε is a positive parameter. Here $H^1_g(M)$ is the completion of $C^\infty(M)$ with respect to the norm $||u||^2_g = \displaystyle\int_M |\nabla_g u|^2 + u^2 d\mu_g$.

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It is well known that any critical point of the energy functional $J_{\varepsilon,g}$: $H^1_g \to \mathbb{R}$ constrained to the Nehari manifold $N_{\varepsilon,g}$ is a solution of (1.1). Here

$$J_{\varepsilon,g}(u) = \frac{1}{\varepsilon^n}\int_M \left(\frac{\varepsilon^2}{2}|\nabla_g u|^2 + \frac{1}{2}u^2 - \frac{1}{p}(u^+)^p\right) d\mu_g$$
$$N_{\varepsilon,g} = \{u \in H^1_g(M) \smallsetminus 0 \ : \ J'_{\varepsilon,g}(u)[u] = 0\}.$$

A lot of work has been devoted to the problem (1.1) in various kinds of subsets of $\mathbb{R}^n$. We limit ourselves to citing the pioneering papers [1, 2, 3, 6, 7, 9].

In [8] the authors shows that the least energy solution of (1.1), i.e. the minimum of $J_{\varepsilon,g}$ on $N_{\varepsilon,g}$, is a positive solution with a spike layer, whose peak converges to the maximum point of the scalar curvature S_g of (M, g) as ε goes to zero. Both topology and geometry influence the multiplicity of positive solution of problem (1.1). Recently in [10, 14, 15] it has been proved that the existence of positive solutions is strongly related to the geometry of M, that is stable critical points of the scalar curvature S_g generate positive solutions with one ore more peaks as ε goes to zero. Previously in [5] (see also [12, 16]) the authors point out that the topology of M has effect on the number of solutions of (1.1), that is (1.1) has at least $\operatorname{cat} M$ nonconstant solutions for ε small enough. Here $\operatorname{cat} M$ is the Lusternik Schnirelmann category of M. Moreover in [5] the Poincaré polynomial is considered (see Definition 2.1) and the authors assume that

$$\text{all the solution of the problem (1.1) are nondegenerate.} \tag{1.2}$$

Then they prove that problem (1.1) has at least $2P_1(M) - 1$ solutions.

Our main result reads as following.

Theorem 1.1. *Given $g_0 \in \mathscr{M}^k$, the set*

$$D = \left\{ \begin{array}{c} (\varepsilon, h) \in (0, \tilde{\varepsilon}) \times \mathscr{B}_{\tilde{\rho}} : \ \textit{the problem } -\varepsilon^2\Delta_{g_0+h}u + u = (u^+)^{p-1} \\ \textit{has at least } P_1(M) \textit{ nonconstant solutions } u \textit{ with } J_{\varepsilon,g_0+h}(u) < 2m_\infty \end{array} \right\}$$

is an residual subset in $(0, \tilde{\varepsilon}) \times \mathscr{B}_{\tilde{\rho}}$, for $\tilde{\varepsilon}$ and $\tilde{\rho}$ chosen small enough.

Here $\mathscr{S}^k$ is the space of all C^k symmetric covariant 2-tensors on M and $\mathscr{M}^k$ is the set of all C^k Riemannian metrics on M with $k \geq 2$. The set $\mathscr{B}_\rho$ is the ball centered at 0 with radius ρ in the Banach space $\mathscr{S}^k$. The number m_∞ is defined by

$$m_\infty = \inf\{J_\infty(v) \ : \ J'_\infty(v)v = 0 \text{ and } v \neq 0\}$$

where $J_\infty(v)=\displaystyle\int_{\mathbb{R}^n}\frac{1}{2}(|\nabla v|^2+v^2)-\frac{1}{p}|v|^p dx$.

The paper is organized as follows. In Section 2 we fix some notations and we recall some results which will be crucial in the proof of main result. In Section 3 we prove the main result, using some technical results showed in sections 4, 5.

2 Notation, definition, known results

Through this paper we will use the following notations

- $B(0,R)$ is the ball in $\mathbb{R}^n$ of center 0 and radius R.
- $B_g(q,R)$ is the geodesic ball in M of center q and radius R with the distance given by the metric g.
- $\mathscr{B}_\rho$ is the ball in the Banach space $\mathscr{S}^k$ of center 0 and radius ρ.
- $I(u,r)$ is the ball in H^1_g of center u and radius r
- For $u\in H^1_g(M)$ we use the norms

$$||u||_g^2=\int_M(|\nabla_g u|^2+|u|^2)d\mu_g \qquad |u|_{p,g}^p=\int_M|u|^p d\mu_g$$
$$|||u|||_{g,\varepsilon}^2=\frac{1}{\varepsilon^n}\int_M(\varepsilon^2|\nabla_g u|^2+|u|^2)d\mu_g \qquad |u|_{p,g,\varepsilon}^p=\frac{1}{\varepsilon^n}\int_M|u|^p d\mu_g$$

- For $u\in H^1(\mathbb{R}^n)$ we use the norms

$$||u||^2=\int_{\mathbb{R}^n}(|\nabla u|^2+|u|^2)dx \qquad |u|_p^p=\int_{\mathbb{R}^n}|u|^p dx$$

- $m_{\varepsilon,g}=\inf\{J_{\varepsilon,g}(v)\ :\ v\in N_{\varepsilon,g}\}$

It is know that there exists a unique positive spherically symmetric function $U\in H^1(\mathbb{R}^n)$ such that $J_\infty(U)=m_\infty$. Obviously we have

$$-\Delta U+U=U^{p-1}. \tag{2.1}$$

For $\varepsilon>0$ we set $U_\varepsilon(x)=U(x/\varepsilon)$ and we get $-\varepsilon^2\Delta U_\varepsilon+U_\varepsilon=U_\varepsilon^{p-1}$.

Now we shall recall some topological tools which are used in the paper

Definition 2.1. *If (X, Y) is a couple of topological spaces, the Poincaré polynomial $P_t(X, Y)$ is defined as the following power series in t*

$$P_t(X, Y) = \sum_k \dim H_k(X, Y)t^k \tag{2.2}$$

where $H_k(X, Y)$ is the k-th homology group of the couple (X, Y) with coefficient in some field. Moreover we set

$$P_t(X) = P_t(X, \emptyset) = \sum_k \dim H_k(X)t^k \tag{2.3}$$

If X is a compact manifold there only a finite number of nontrivial $H_k(X)$ and $\dim H_k(X) < \infty$. In this case $P_t(X)$ is a polynomial and not a formal series.

Definition 2.2. *Let J be a C^2 functional on a Banach space X and let $u \in X$ be an isolated critical point of J with $J(u) = c$. If $J^c := \{v \in X \ : \ J(v) \leq c\}$, then the (polynomial) Morse index $i_t(u)$ is the series*

$$i_t(u) = \sum_k \dim H_k(J^c, J^c \smallsetminus \{u\})t^k, \tag{2.4}$$

If u is a nondegenerate critical point of J then $i_t(u) = t^{\mu(u)}$ where $\mu(u)$ is the (numerical) Morse index of u, and it is given by the dimension of the maximal subspace on which the bilinear form $J''(u)[\cdot, \cdot]$ is negative definite.

It is useful to recall the following result (see [7])

Remark 2.3. Let X and Y be topological spaces. If $f : X \to Y$ and $g : Y \to X$ are continuous maps such that $g \circ f$ is homotopic to the identity map on X, then

$$P_t(Y) = P_t(X) + Z(t) \tag{2.5}$$

where $Z(t)$ is a polynomial with non negative coefficients.

Definition 2.4. *Let J be a C^1 functional on a Banach space X. We say that J satisfies the Palais Smale condition if any sequence $\{x_n\}_n \subset X$ for which $J(x_n)$ is bounded and $J'(x_n) \to 0$ has a convergent subsequence.*

We now introduce the Banach space $\mathscr{S}^k$ which will be the parameter space. We denote by $\mathscr{S}^k$ the Banach space of all C^k symmetric covariant symmetric 2-tensors on M. The norm $||\cdot||_k$ is defined in the following way. We fix a finite covering $\{V_\alpha\}_{\alpha \in L}$ of M such that the closure of V_α is contained in U_α where $\{U_\alpha, \psi_\alpha\}$ is an open coordinate neighborhood. If $h \in \mathscr{S}^k$ we denote by $h_{i,j}$ the components of h with respect to the coordinates $(x_1, \dots, x_n)$ on V_α. We define

$$||h||_k = \sum_{\alpha \in L} \sum_{|\beta| \le k} \sum_{i,j=1}^{n} \sup_{\psi_\alpha(V_\alpha)} \frac{\partial^\beta h_{i,j}}{\partial x_1^{\beta_1} \cdots \partial x_k^{\beta_k}} \tag{2.6}$$

The set $\mathscr{M}^k$ of all C^k Riemannian metrics on M is an open subset of $\mathscr{S}^k$.

On the tangent bundle of any compact connected Riemannian manifold M, it is defined the exponential map $\exp : TM \to M$ which is a C^∞ map. Then, for ρ small enough, the manifold M has a special set of charts given by $\exp_x : B(0, R) \to B_g(x, R)$ (where T_xM is identified with $\mathbb{R}^n$) for $x \in M$. The system of coordinates corresponding to these charts are called normal coordinates.

Remark 2.5. Let g_0 a fixed C^k Riemannian metric on the manifold M. By the compactness of M there exist two positive constant c, C such that

$$\begin{aligned} &\forall x \in M, \forall \xi \in T_xM \quad c||\xi||^2 \le g_0(x)(\xi, \xi) \le C||\xi||^2; \\ &\forall x \in M \quad c^n \le |g_0(x)| \le C^n. \end{aligned}$$

By definition of the norms $|||u|||_{g,\varepsilon}$ and $||h||_k$ we have that there exists $\rho_1 > 0$ such that, if $h \in \mathscr{B}_{\rho_1}$, the two sets $H^1_g(M)$, $H^1_{g_0}(M)$ are the same and the two norms $|||u|||_{g,\varepsilon}$, $|||u|||_{g_0,\varepsilon}$ (as well as $|||u|||_g$, $|||u|||_{g_0}$) are equivalent, and the positive constants for the equivalence do not depend on ε, for $0 < \varepsilon < 1$.

Remark 2.6. It is trivial that there exists ρ_2 such that, for any $h \in \mathscr{B}_{\rho_2}$, we have

$$J_{\varepsilon, g_0+h}(1) = \left(\frac{1}{2} - \frac{1}{p}\right) \frac{1}{\varepsilon^n} \int_M 1 d\mu_{g_0+h} > \frac{p-2}{2p} \frac{1}{\varepsilon^n} \frac{\mu_{g_0}(M)}{2}. \tag{2.7}$$

Then $J_{\varepsilon,g_0+h}(1) > 2m_\infty$ for $\varepsilon < \left[\frac{p-2}{8pm_\infty} \mu_{g_0}(M)\right]^{1/n}$ and $h \in \mathscr{B}_{\rho_2}$.

In the following we consider $g = g_0 + h$ with $h \in \mathscr{B}_{\hat\rho}$ where $\hat\rho = \min\{\rho_1, \rho_2\}$.

Lemma 2.7. *There exists $\varepsilon_1 \in (0, 1)$ such that, for any $\varepsilon < \varepsilon_1$ and for any $h \in \mathscr{B}_{\hat\rho}$, we have*

$$A(h, \varepsilon) := \{u \in N_{\varepsilon, g_0+h} \ : \ J_{\varepsilon, g_0+h}(u) \le 2m_\infty\} \subset I(0, \alpha) \setminus 1 \tag{2.8}$$

for some $\alpha > 0$.

Proof. If $u \in A(h,\varepsilon)$ we have

$$J_{\varepsilon,g}(u) = \left(\frac{1}{2} - \frac{1}{p}\right) |||u|||^2_{\varepsilon,g} \leq 2m_\infty, \tag{2.9}$$

where $g = g_0 + h$ with $h \in \mathscr{B}_{\hat{\rho}}$. By definition of $\hat{\rho}$, there exists $c_1 > 0$ such that

$$c_1||u||^2_{g_0} \leq c_1|||u|||^2_{\varepsilon_1,g_0} \leq c_1|||u|||^2_{\varepsilon,g_0} \leq |||u|||^2_{\varepsilon,g} \tag{2.10}$$

for $0 < \varepsilon < \varepsilon_1$ and $h \in \mathscr{B}_{\hat{\rho}}$. Then by (2.9) and (2.10) we have

$$||u||^2_{g_0} \leq \frac{2p}{p-2}\frac{2m_\infty}{c_1}. \tag{2.11}$$

By Remark 2.6 we have $u \neq 1$ for ε_1 small enough. By the following Remark 5.6, we have that $A(h,\varepsilon) \neq \emptyset$ for ε_1, and $\hat{\rho}$ small enough. $\square$

Now we recall a result about the nondegeneracy of positive solutions of (1.1) with respect to the pair of parameters (ε, g), where ε is a positive number and g is a Riemannian metric (see [13]).

Theorem 2.8. *Given $g_0 \in \mathscr{M}^k$, and an open ball of $H^1_{g_0}(M)$ without the constant 1 $A = I(0,\alpha) \smallsetminus \{1\}$, the set*

$$D = \left\{ \begin{array}{c} (\varepsilon,h) \in (0,1) \times \mathscr{B}_\rho \text{ s.t. any } u \in A \text{ solution of the equation} \\ -\varepsilon^2\Delta_{g_0+h}u + u = (u^+)^{p-1} \text{is nondegenerate} \end{array} \right\}$$

is an residual subset in $(0,1) \times \mathscr{B}_\rho$ for ρ small enough.

3 The main ingredient of the proof

Let us sketch the proof of our main result. We are going to find an estimate of the number of nonconstant critical points of the functional J_{ε,g_0+h} with energy close to m_∞, with respect to the parameters $(\varepsilon,h) \in (0,\tilde{\varepsilon}) \times \mathscr{B}_{\tilde{\rho}}$.

First of all we apply Theorem 2.8 choosing the positive numbers $\tilde{\varepsilon}, \tilde{\rho}$ small enough and the open bounded set A equal to $I(0,\alpha) \smallsetminus \{1\}$, where α is given by Lemma 2.7. So we get that the set

$$D(\tilde{\varepsilon},\tilde{\rho}) = \left\{ \begin{array}{l} (\varepsilon,h) \in (0,\tilde{\varepsilon}) \times \mathscr{B}_{\tilde{\rho}} : \text{ any } u \in I(0,\alpha) \smallsetminus \{1\} \text{ solution of} \\ -\varepsilon^2\Delta_{g_0+h}u + u = (u^+)^{p-1} \text{ is non degenerate} \end{array} \right\} \supset$$

$$\supset \left\{ \begin{array}{c} (\varepsilon,h) \in (0,\tilde{\varepsilon}) \times \mathscr{B}_{\tilde{\rho}} : \text{any solutions of } -\varepsilon^2\Delta_{g_0+h}u + u = (u^+)^{p-1} \\ \text{nonconstant, such that } J_{\varepsilon,g_0+h}(u) < 2m_\infty \text{ is non degenerate} \end{array} \right\}$$

is an residual subset in $(0,\tilde{\varepsilon}) \times \mathscr{B}_{\tilde{\rho}}$, for $\tilde{\varepsilon}$ and $\tilde{\rho}$ small enough.

Since $\lim\limits_{(\varepsilon,h)\to(0,0)} m_{\varepsilon,g_0+h} = m_\infty$ (see following Remark 5.6), given $\delta \in (0, m_\infty/4)$, for $(\varepsilon,h) \in \mathbb{R}^+ \times \mathscr{S}^k$ small enough, we have

$$0 < m_\infty - \delta < m_{\varepsilon,g_0+h} < m_\infty + \delta < 2m_\infty.$$

Thus $m_\infty - \delta$ is not a critical value of J_{ε,g_0+h}.

By the compactness of M, it holds Palais Smale condition for the functional J_{ε,g_0+h}. At this point we take $(\varepsilon,h) \in D(\tilde{\varepsilon},\tilde{\rho})$ with the positive numbers $\tilde{\varepsilon}$, $\tilde{\rho}$ small enough. Thus we have that the critical points u of J_{ε,g_0+h} with $J_{\varepsilon,g_0+h}(u) < 2m_\infty$ are in a finite number, then we can assume that $m_\infty + \delta$ is not a critical value for J_{ε,g_0+h}. It holds the following relation proved in [4, 7] (see [7, Lemma 5.2])

$$P_t(J_{\varepsilon,g_0+h}^{m_\infty+\delta}, J_{\varepsilon,g_0+h}^{m_\infty-\delta}) = tP_t(J_{\varepsilon,g_0+h}^{m_\infty+\delta} \cap N_{\varepsilon,g_0+h}). \tag{3.1}$$

On the other hand, by proposition 4.1 and 5.1 and Lemma 5.5, we can build two maps Φ_{ε,g_0+h} and β_{g_0+h} such that

$$M \stackrel{\Phi_{\varepsilon,g_0+h}}{\longrightarrow} N_{\varepsilon,g_0+h} \cap J_{\varepsilon,g_0+h}^{m_\infty+\delta} \stackrel{\beta_{g_0+h}}{\longrightarrow} M_{r(M)}, \tag{3.2}$$

where $\beta_{g_0+h} \circ \Phi_{\varepsilon,g_0+h}$ is homotopic to the identity map and $M_{r(M)}$ is homotopically equivalent to M. Therefore, by Remark 2.3 we have

$$P_t(J_{\varepsilon,g_0+h}^{m_\infty+\delta} \cap N_{\varepsilon,g_0+h}) = P_t(M) + Z(t) \tag{3.3}$$

where $Z(t)$ is a polynomial with nonnegative integer coefficients.

Since the functional J_{ε,g_0+h} satisfies the Palais Smale condition and the critical points u of J_{ε,g_0+h} such that $J_{\varepsilon,g_0+h}(u) < m_\infty + \delta$ are nondegenerate, by Morse theory we have

$$\sum_{u\in C} i_t(u) = \sum_{u\in C} t^{\mu(u)} = P_t(J_{\varepsilon,g_0+h}^{m_\infty+\delta}, J_{\varepsilon,g_0+h}^{m_\infty-\delta}). \tag{3.4}$$

Here $\mu(u)$ is the dimension of the maximal subspace on which the bilinear form $J''_{\varepsilon,g_0+h}(u)[\cdot,\cdot]$ is negative definite and the set C is defined by

$$C = \{u \ : \ J'_{\varepsilon,g_0+h}(u) = 0 \text{ and } m_\infty - \delta < J_{\varepsilon,g_0+h}(u) < m_\infty + \delta\}. \tag{3.5}$$

Then by (3.1), (3.4) and (3.5), for any $(\varepsilon,h) \in D(\tilde{\varepsilon},\tilde{\rho})$ with the positive numbers $\tilde{\varepsilon}$, $\tilde{\rho}$ small enough, we get that the functional J_{ε,g_0+h} has at least $P_1(M)$ nonconstant critical points u such that $J_{\varepsilon,g_0+h}(u) < m_\infty + \delta < 2m_\infty$.

4 The function $\Phi_{\varepsilon,g}$

Let us define a smooth real cut off function χ_R such that $\chi_R(t) = 1$ if $0 \leq t \leq R/2$, $\chi_R(t) = 0$ if $t \geq R$, and $|\chi'(t)| \leq 2/R$. Fixed $q \in M$ and $\varepsilon > 0$, we define on M the function

$$w^g_{q,\varepsilon}(x) = \begin{cases} U_\varepsilon(\exp_q^{-1}(x))\chi_R(|\exp_q^{-1}(x)|) & \text{if } x \in B_g(q,R); \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

For any $u \in H^1_{g_0+h}(M)$ with $u^+ \neq 0$ we define $t(u) \in \mathbb{R}$ as

$$t^{p-2}(u) = \frac{\displaystyle\int_M (\varepsilon^2|\nabla_{g_0+h}u|^2 + u^2)d\mu_{g_0+h}}{\displaystyle\int_M |u^+|^p d\mu_{g_0+h}}, \tag{4.2}$$

so $t(u)$ is the unique number such that $t(u)u \in N_{\varepsilon,g_0+h}$.

Thus we can define a map $\Phi_{\varepsilon,g} : M \to N_{\varepsilon,g}$ by

$$\Phi_{\varepsilon,g}(q) = t(w^g_{q,\varepsilon})w^g_{q,\varepsilon}. \tag{4.3}$$

Proposition 4.1. *Given $g_0 \in \mathscr{M}^k$, for any $\varepsilon > 0$ and for any $h \in \mathscr{B}_{\tilde{\rho}} \subset \mathscr{S}^k$ the operator $\Phi_{\varepsilon,g_0+h} : M \to N_{\varepsilon,g_0+h}$ is continuous. Moreover, given g_0, for any $\delta > 0$ there exists $\varepsilon_2 = \varepsilon_2(\delta)$ such that, if $\varepsilon < \varepsilon_2$, then*

$$\Phi_{\varepsilon,g_0+h}(q) \in N_{\varepsilon,g_0+h} \cap J^{m_\infty+\delta}_{\varepsilon,g_0+h} \qquad \forall q \in M, \forall h \in \mathscr{B}_{\hat{\rho}}. \tag{4.4}$$

Proof. It easy to prove the continuity of $q \to \Phi_{\varepsilon,g_0+h}(q) \in N_{\varepsilon,g_0+h}$ from M to $H^1_{g_0+h}(M)$. To obtain the second statement we recall that

$$J_{\varepsilon,g_0+h}(\Phi_{\varepsilon,g_0+h}(w^{g_0+h}_{q,\varepsilon})) = \frac{1}{\varepsilon^n}\left(\frac{1}{2} - \frac{1}{p}\right)[t(w^{g_0+h}_{q,\varepsilon})]^p |w^{g_0+h}_{q,\varepsilon}|^p_{p,g_0+h}. \tag{4.5}$$

Moreover the following limits hold

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^n}|w^{g_0+h}_{q,\varepsilon}|^2_{2,g_0+h} = |U|^2_2 \tag{4.6}$$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^n}|w^{g_0+h}_{q,\varepsilon}|^p_{p,g_0+h} = |U|^p_p \tag{4.7}$$

$$\lim_{\varepsilon\to 0} \frac{\varepsilon^2}{\varepsilon^n}|\nabla_{g_0+h}w^{g_0+h}_{q,\varepsilon}|^2_{2,g_0+h} = |\nabla U|^2_2 \tag{4.8}$$

uniformly with respect to $q \in M$ and $h \in \mathscr{B}_\rho \subset \mathscr{S}^k$, $k \geq 2$. We prove only the first limit, the others follow in a similar way. Using the normal coordinates with respect to g_0+h at the point $q\in M$ we have

$$\begin{aligned}\frac{1}{\varepsilon^n}|w^{g_0+h}_{q,\varepsilon}|^2_{2,g_0+h}-|U|_2^2=&\left|\int_{\mathbb{R}^n}U^2(z)[|\chi^2(\varepsilon|z|)|\ |g_{0,q}(\varepsilon z)+h_q(\varepsilon z)|^{1/2}-1]dz\right|\leq\\ \leq&\left|\int_{B(0,T)}U^2(z)[|\chi^2(\varepsilon|z|)|\ |g_{0,q}(\varepsilon z)+h_q(\varepsilon z)|^{1/2}-1]dz\right|+\\ &+\left|\int_{\mathbb{R}^n\smallsetminus B(0,T)}U^2(z)[|\chi^2(\varepsilon|z|)|\ |g_{0,q}(\varepsilon z)+h_q(\varepsilon z)|^{1/2}-1]dz\right|.\end{aligned}\tag{4.9}$$

We point out that

$$(g_{0,q}(0)+h_q(0))^{ij}=\delta_{ij}\text{ and}$$

$$\begin{aligned}|(g_{0,q}(y)+h_q(y))^{ij}-\delta_{ij}|&=|\nabla[(g_{0,q}(\theta y))^{ij}]\cdot y+\nabla[(h_q(\theta y))^{ij}]\cdot y|\leq\\ &\leq[|\nabla(g_{0,q}(\theta y))^{ij}|+|\nabla(h_q(\theta y))^{ij}|]\cdot|y|.\end{aligned}$$

Because M is compact, we have $[|\nabla(g_{0,q}(\theta y))^{ij}|+|\nabla(h_q(\theta y))^{ij}|]$ is bounded independently on $y\in B(0,r)$, $q\in M$, and $h\in\mathscr{B}_\rho$.

At this point it is clear that the second addendum of formula (4.9) vanishes as $T\to+\infty$. Moreover, fixed T large enough, the first addendum of (4.9) vanishes as $\varepsilon\to 0$.

By the previous limits we get that $\lim\limits_{\varepsilon\to 0}t(w^{g_0+h}_{\varepsilon,q})=1$ uniformly with respect to $q\in M$ and $h\in\mathscr{B}_\rho$. Then we have

$$\lim_{\varepsilon\to 0}J_{\varepsilon,g_0+h}\big(t(w^{g_0+h}_{\varepsilon,q})w^{g_0+h}_{\varepsilon,q}\big)=m_\infty\tag{4.10}$$

uniformly with respect to $q\in M$ and $h\in\mathscr{B}_\rho$. □

Remark 4.2. By Proposition 4.1 we get

$$\limsup_{\varepsilon\to 0}m_{\varepsilon,g_0+h}\leq m_\infty,\tag{4.11}$$

uniformly with respect to $h\in\mathscr{B}_\rho$. Here $m_{\varepsilon,g}=\inf\limits_{N_{\varepsilon,g}}J_{\varepsilon,g}$ and $m_\infty=\inf\limits_{N_\infty}J_\infty$.

5 The operator β_g

For any function $u\in N_{\varepsilon,g}$ we can define its centre of mass as a point $\beta_g(u)\in\mathbb{R}^N$ by

$$\beta_g(u)=\frac{\displaystyle\int_M x(u^+)^p d\mu_g}{\displaystyle\int_M (u^+)^p d\mu_g}.\tag{5.1}$$

The function β_g is well defined on $N_{\varepsilon,g}$ since, if $u \in N_{\varepsilon,g}$ then $u^+ \neq 0$. We will prove that, if $u \in N_{\varepsilon,g} \cap J_{\varepsilon,g}^{m_\infty+\delta}$, then $\beta_g(u) \in M_{r(M)}$, using the concentration properties of the functions in $N_{\varepsilon,g} \cap J_{\varepsilon,g}^{m_\infty+\delta}$ as ε and δ are suitably small. In the following we use the same arguments of Section 5 of [BBM], but here we have to take in account the dependence of the metric g, hence some calculations are different.

Our aim is to get the following statement

Proposition 5.1. *Given $g_0 \in \mathscr{M}^k$, there exist δ_0, ρ_0 and ε_0 such that, for any $\delta \in (0,\delta_0)$, for any $\varepsilon \in (0,\varepsilon_0)$, for any $h \in \mathscr{B}_{\rho_0}$, and for any $u \in N_{\varepsilon,g_0+h} \cap J_{\varepsilon,g_0+h}^{m_\infty+\delta}$, it holds $\beta_{g_0+h}(u) \in M_{r(M)}$ where $r(M)$ is the radius of topological invariance of M and $M_{r(M)} = \{x \in \mathbb{R}^N \ : \ d(x,M) < r(M)\}$.*

To prove Proposition 5.1 we need some technical results.

First of all we consider partitions of the compact manifold M. Given $\varepsilon > 0$ and a metric g_0, a partition $\mathscr{P} = \mathscr{P}(\varepsilon) = \{P_j = P_J(\varepsilon)\}_{j\in\Lambda(\varepsilon)}$ is called a "good" partition if:

- for any $j \in \Lambda(\varepsilon)$ the set P_j is closed
- $P_j \cap P_i \subset \partial P_j \cap \partial P_i$ for $i \neq j$
- there exists $\rho > 0$ such that, for any j, there exists $q_j \in P_j$ such that $B_g(q_j,\varepsilon) \subset P_j \subset B_g(q_j,(1+1/a)\varepsilon)$ with a constant a independent on ε and $g = g_0 + h$ with $h \in \mathscr{B}_\rho$
- any point $x \in M$ is contained in at most ν_M balls $B_g(q_j,(1+1/a)\varepsilon)$ where ν_M does not depend on ε and g, $g = g_0 + h$ and $h \in \mathscr{B}_\rho$.

Lemma 5.2. *There exist $\gamma > 0$ and $\rho > 0$ such that, for any $\delta > 0$ and $\varepsilon > 0$, given any "good" partition $\mathscr{P}(\varepsilon)$ and any $u \in N_{\varepsilon,g} \cap J_{\varepsilon,g}^{m_\infty+\delta}$, where $g = g_0 + h$ with $h \in \mathscr{B}_\rho$, there exists a set $P \in \mathscr{P}(\varepsilon)$ such that*

$$\frac{1}{\varepsilon^n}\int_P (u^+)^p d\mu_{g_0+h} \geq \gamma, \ \textit{with } h \in \mathscr{B}_\rho \tag{5.2}$$

The proof of this Lemma can be obtained following the same argument of the proof of [5, Lemma 5.3]. Indeed, every constant appearing in [5, Lemma 5.3] can be chosen independently on $h \in \mathscr{B}_\rho$ with ρ small enough.

Proposition 5.3. *For any $\eta \in (0,1)$ there exist δ_0, ρ_0 and ε_0 such that, for any $\delta < \delta_0$, for any $\varepsilon < \varepsilon_0$, for any $h \in \mathscr{B}_{\rho_0}$ and for any $u \in N_{\varepsilon,g_0+h} \cap J_{\varepsilon,g_0+h}^{m_\infty+\delta}$, there exists $q = q(u)$ for that*

$$\left(\frac{1}{2} - \frac{1}{p}\right)\frac{1}{\varepsilon^n}\int_{B_{g_0+h}(q,r(M)/2)} (u^+)^p d\mu_{g_0+h} > (1-\eta)m_\infty \tag{5.3}$$

Proof. We only prove the proposition for any $u \in N_{\varepsilon,g_0+h} \cap J_{\varepsilon,g_0+h}^{m_{\varepsilon,g_0+h}+2\delta}$. Indeed, by this result and by Remark 4.2 we get

$$\lim_{(\varepsilon,h)\to(0,0)} m_{\varepsilon,g_0+h} = m_\infty. \tag{5.4}$$

Hence it holds $J_{\varepsilon,g_0+h}^{m_\infty+\delta} \subset J_{\varepsilon,g_0+h}^{m_{\varepsilon,g_0+h}+2\delta}$ for δ, ε and $||h||_k$ small enough. So the thesis holds.

We argue by contradiction. Suppose that there exists $\eta \in (0,1)$ such that we can find sequences of vanishing numbers $\{\delta_k\}_k$, $\{\varepsilon_k\}_k$, a sequence $h_k \to 0$ in $\mathscr{S}^k$ and a sequence $u_k \in N_{\varepsilon_k,g_0+h_k} \cap J_{\varepsilon_k,g_0+h_k}^{m_{\varepsilon,g_0+h_k}+2\delta_k}$ such that, for all $q \in M$,

$$\left(\frac{1}{2} - \frac{1}{p}\right) \frac{1}{\varepsilon_k^n} \int_{B_{g_0+h_k}(q,r(M)/2)} (u_k^+)^p d\mu_{g_0+h_k} \leq (1-\eta)m_\infty. \tag{5.5}$$

By Ekeland variational principle (see [11]), and by definition of the Nehari manifold, we can assume that

$$|J'_{\varepsilon_k,g_0+h_k}(u_k)(\xi)| \leq \sqrt{\delta_k}|||\xi|||_{\varepsilon_k} \quad \forall \xi \in H^1_{g_0+h_k}(M). \tag{5.6}$$

By Lemma 5.2 there exists a set $P_k \in \mathscr{P}_{\varepsilon_k}$ such that

$$\frac{1}{\varepsilon_k^n} \int_{P_k} (u_k^+)^p d\mu_{g_0+h_k} \geq \gamma. \tag{5.7}$$

We choose a point q_k interior to P_k and we consider the function $w_k : \mathbb{R}^n \to \mathbb{R}$ defined by

$$u_k(x)\chi_R(\exp_{q_k}^{-1}(x)) = u_k(\exp_{q_k}(\varepsilon_k z)\chi_R(\varepsilon_k|z|) = w_k(z) \tag{5.8}$$

where $x \in M$ and χ_R is a smooth cut off function $\chi_R(t) \equiv 1$ for $0 < t < R/2$, $\chi_R(t) \equiv 0$ for $t > R$ and R small enough. It easily follows that $w_k \in H_0^1(B(0,R/\varepsilon_k)) \subset H^1(\mathbb{R}^n)$.

We now establish some properties of the functions w_k by some lemmas. The proof of these lemmas are in Section 6

Lemma 5.4. *By considering a subsequence, there exists $w \in H^1(\mathbb{R}^n)$ such that $\lim_k w_k = w$ as a weak limit in $H^1(\mathbb{R}^n)$ and a strong limit in $L^p_{loc}(\mathbb{R}^n)$.*

Lemma 5.5. *The limit function $w \in H^1(\mathbb{R}^n)$ is a weak solution of*

$$-\Delta w + w = |w|^{p-2}w, \quad w > 0 \tag{5.9}$$

At this point we observe that, by definition of w_k and by (5.5), for any $\sigma \in (0,\eta)$, and for any $T>0$ we have, for k large enough,

$$\left(\frac{1}{2}-\frac{1}{p}\right)\int_{B(0,T)}|w_k^+(x)|^p dx < \frac{1-\eta}{1-\sigma}m_\infty.$$

On the other hand by Lemma 5.5 we have

$$\left(\frac{1}{2}-\frac{1}{p}\right)||w||^2 = \left(\frac{1}{2}-\frac{1}{p}\right)|w|_p^p \geq m_\infty.$$

By Lemma 5.4, for T and k large enough

$$\left(\frac{1}{2}-\frac{1}{p}\right)\int_{B(0,T)}|w_k^+(x)|^p dx > \frac{1-\eta}{1-\sigma}m_\infty,$$

and this leads to a contradiction. □

Remark 5.6. By Proposition 5.3 and by Remark 4.2, it holds

$$\lim_{(\varepsilon,h)\to(0,0)} m_{\varepsilon,g_0+h} = m_\infty$$

uniformly with respect to $h \in \mathscr{B}_\rho$ (here ρ is given as in Proposition 5.3)

Proof of Proposition 5.1. By Proposition 5.3 for any $\eta \in (0,1)$ and for any $u \in N_{\varepsilon,g_0+h} \cap J_{\varepsilon,g_0+h}^{m_\infty+\delta}$ with ε, δ, and h small enough, there exists $q \in M$ such that

$$(1-\eta)m_\infty < \left(\frac{1}{2}-\frac{1}{p}\right)\frac{1}{\varepsilon^n}\int_{B_{g_0+h}(q,r(M)/2)}(u^+)^p d\mu_{g_0+h}. \tag{5.10}$$

Moreover, since $u \in N_{\varepsilon,g_0+h} \cap J_{\varepsilon,g_0+h}^{m_\infty+\delta}$, it holds

$$\left(\frac{1}{2}-\frac{1}{p}\right)\frac{1}{\varepsilon^n}\int_M (u^+)^p d\mu_{g_0+h} < m_\infty + \delta. \tag{5.11}$$

Then, by (5.10) and (5.11), we have

$$\begin{aligned}
|\beta_g(u)-q| \leq & \left|\frac{\int_M (x-q)(u^+(x))^p d\mu_{g_0+h}}{\int_M (u^+(x))^p d\mu_{g_0+h}}\right| \leq \\
\leq & \left|\frac{\int_{B_{g_0+h}(q,r(M)/2)} (x-q)(u^+(x))^p d\mu_{g_0+h}}{\int_M (u^+(x))^p d\mu_{g_0+h}}\right| + \\
& + \left|\frac{\int_{M\smallsetminus B_{g_0+h}(q,r(M)/2)} (x-q)(u^+(x))^p d\mu_{g_0+h}}{\int_M (u^+(x))^p d\mu_{g_0+h}}\right| \leq \\
\leq & \frac{r(M)}{2} + 2D\left(1-\frac{(1-\eta)m_\infty}{m_\infty+\delta}\right)
\end{aligned} \tag{5.12}$$

where D is the diameter of M as a compact subset of $\mathbb{R}^N$. Choosing η and δ suitably small we get the claim. $\square$

The last result of this section is that the composition $I^g_\varepsilon := \beta_g \circ \Phi_{\varepsilon,g}$ is homotopic to the identity on M.

Lemma 5.7. *There exists $\varepsilon_2 < \varepsilon_0$ such that, for any $\varepsilon \in (0, \varepsilon_2)$ and for any $h \in \mathscr{B}_{\hat{\rho}}$*

$$I^{g_0+h}_\varepsilon := \beta_{g_0+h} \circ \Phi_{\varepsilon,g_0+h} : M \to M_{r(M)} \tag{5.13}$$

is well defined and it is homotopic to the identity on M.

Proof. By Proposition 4.1 and (5.7) the map $I^{g_0+h}_\varepsilon$ is well defined. To prove that $I^{g_0+h}_\varepsilon$ is homotopic to the identity on M it is enough to evaluate the map. Here $g = g_0 + h$.

$$I^g_\varepsilon(q) - q = \frac{\varepsilon \int_{B(0,R/\varepsilon)} z|U(z)\chi_R(\varepsilon|z|)|^p |g_q(\varepsilon z)|^{\frac{1}{2}} dz}{\int_{B(0,R/\varepsilon)} |U(z)\chi_R(\varepsilon|z|)|^p |g_q(\varepsilon z)|^{\frac{1}{2}} dz}, \tag{5.14}$$

hence $|I^g_\varepsilon(q) - q| \le c \cdot \varepsilon$ for a constant $c = c(M, g_0, \hat{\rho})$ that does not depend on q and on $h \in \mathscr{B}_{\hat{\rho}}$ $\square$

6 Proof of technical lemmas

Proof of Lemma 5.4. Here $g_k = g_0 + h_k$ with $h_k \in \mathscr{B}_\rho$, $h_k \to 0$ and $\tilde{u}_k(y) = u_k(\exp_{q_k}(y))$. We recall that $\left(\frac{1}{2} - \frac{1}{p}\right) |||u_k|||^2_{\varepsilon_k,g_k} \le m_{\varepsilon_k,g_k} + 2\delta_k$. By Remark 4.2, for k large we get

$$\begin{aligned}
2\frac{2p}{p-2} m_\infty &\ge \frac{1}{\varepsilon_k^n} \int_M u_k^2(x) d\mu_{g_k} \ge \frac{1}{\varepsilon_k^n} \int_{B_{g_k}(q_k,R)} \chi_R^2(|\exp_{q_k}^{-1}(x)|) u_k^2(x) d\mu_{g_k} = \\
&= \frac{1}{\varepsilon_k^n} \int_{B(0,R)} \chi_R^2(|y|) u_k^2(\exp_{q_k} y) |g_{k,q_k}(y)|^{1/2} dy = \\
&= \int_{B(0,R/\varepsilon_k)} \chi_R^2(\varepsilon_k |z_k|) \tilde{u}_k^2(\varepsilon_k z) |g_{k,q_k}(\varepsilon_k z)|^{1/2} dz \ge \text{const} \int_{\mathbb{R}^n} w_k^2(z) dz.
\end{aligned}$$

Let us now estimate the L^2 norm for ∇w_k

$$\int_{B(0,R/\varepsilon_k)} \sum_i \left(\frac{\partial w_k}{\partial z_i}(z) \right)^2 dz = I_1 + I_2 + I_3, \text{ where}$$

$$
\begin{aligned}
I_1 &= \int_{B(0,R/\varepsilon_k)} \chi_R^2(\varepsilon_k|z|) \sum_i \left(\frac{\partial \tilde{u}_k}{\partial z_i}(\varepsilon_k z) \right)^2 dz \\
I_2 &= \int_{B(0,R/\varepsilon_k)} \tilde{u}_k^2(\varepsilon_k|z|) \sum_i \left(\frac{\partial \chi_R}{\partial z_i}(\varepsilon_k z) \right)^2 dz \\
I_3 &= 2 \int_{B(0,R/\varepsilon_k)} \chi_R(\varepsilon_k|z|) \tilde{u}_k(\varepsilon_k z) \sum_i \frac{\partial \chi_R}{\partial z_i}(\varepsilon_k|z|) \frac{\partial \tilde{u}_k}{\partial z_i}(\varepsilon_k|z|)
\end{aligned}
$$

By Remark 2.5 we have

$$
\begin{aligned}
2\frac{2p}{p-2} m_\infty \geq \frac{\varepsilon_k^2}{\varepsilon_k^n} \int_M |\nabla_{g_k} u_k(x)|^2 d\mu_{g_k} \geq \\
\geq \int_{B(0,R/\varepsilon_k)} \left(\sum_{ij} g_{k,q_k}^{ij}(\varepsilon_k z) \frac{\partial \tilde{u}_k}{\partial z_i}(\varepsilon_k z) \frac{\partial \tilde{u}_k}{\partial z_j}(\varepsilon_k z) \right) |g_{k,q_k}(\varepsilon_k z)|^{1/2} dz \geq \\
\geq \text{const} \int_{B(0,R/\varepsilon_k)} |\nabla \tilde{u}_k(\varepsilon_k z)|^2 dz.
\end{aligned}
\tag{6.1}
$$

Thus I_1 is bounded. Analogously for addenda I_2 and I_3 $\square$

Proof of Lemma 5.5. For any $\varphi \in C_0^\infty(\mathbb{R}^n)$ we have that, for k big enough, $w_k(z) = u_k(\exp_{q_k}^{-1}(\varepsilon_k z))$ for $z \in \operatorname{supt} \varphi$, because $\operatorname{supt} \varphi \subset \{z \ : \ \chi_R(\varepsilon_k|z|) = 1\}$ for k big enough. We put $\hat{\varphi}_k(x) = \varphi_k \left(\frac{1}{\varepsilon_k} \exp_{q_k}^{-1}(x) \right)$ for $x \in M$. If $\operatorname{supt} \varphi \subset B(0,T)$, then $\operatorname{supt} \hat{\varphi}_k \subset B_{g_k}(q_k, \varepsilon_k T)$ so we have

$$
\begin{aligned}
J'_{\varepsilon_k}(u_k)[\hat{\varphi}_k] = \frac{1}{\varepsilon^n} \int_M \varepsilon_k^2 \nabla_{g_k} u_k \nabla_{g_k} \hat{\varphi}_k + u_k \hat{\varphi}_k - (u_k^+)^{p-1} \hat{\varphi}_k d\mu_{g_k} = \\
= \int_{B(0,T)} |g_{k,q_k}(\varepsilon_k z)|^{1/2} \left[\sum_{ij} g_{k,q_k}^{ij}(\varepsilon_k z) \frac{\partial w_k}{\partial z_i} \frac{\partial \hat{\varphi}_k}{\partial z_j} + w_k \hat{\varphi}_k - (w_k^+)^{p-1} \hat{\varphi}_k dz. \right]
\end{aligned}
\tag{6.2}
$$

By the Ekeland principle we have

$$
|J'_{\varepsilon_k}(u_k)[\hat{\varphi}_k]| \leq \sqrt{\delta_k} |||\hat{\varphi}_k|||_{\varepsilon_k}. \tag{6.3}
$$

It is sufficient to prove that $|||\hat{\varphi}_k|||_{\varepsilon_k}$ is bounded to obtain that

$$
J'_{\varepsilon_k}(u_k)[\hat{\varphi}_k] \to 0 \quad \text{as } k \to \infty. \tag{6.4}
$$

In fact we have

$$
|||\hat{\varphi}_k|||_{\varepsilon_k}^2 = \int_{B(0,T)} \left[\sum_{ij} g_{k,q_k}^{ij}(\varepsilon_k z) \frac{\partial \hat{\varphi}_k}{\partial z_i} \frac{\partial \hat{\varphi}_k}{\partial z_j} + \hat{\varphi}_k^2 \right] |g_{k,q_k}(\varepsilon_k z)|^{1/2} dz \tag{6.5}
$$

and

$$
\begin{aligned}
g_{k,q_k}^{ij}(\varepsilon_k z) &= g_{0,q_k}^{ij}(\varepsilon_k z) + h_k^{ij}(\varepsilon_k z) = \\
&= \delta_{ij} + \varepsilon_k d g_{0,q_k}(\theta \varepsilon_k z)(z) + h_k^{ij}(\varepsilon_k z).
\end{aligned} \tag{6.6}
$$

Because $h_k \to 0$ as $k \to \infty$ in the Banach space $\mathscr{S}^k$, by (6.4) and (6.5) we get the boundness of $|||\hat{\varphi}_k|||_{\varepsilon_k}^2$.

By (6.2) and (6.6) we have

$$
J'_{\varepsilon_k}(u_k)[\hat{\varphi}_k] \to J'_\infty(w)[\varphi] \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n). \tag{6.7}
$$

So by (6.7) and (6.4) we have $-\Delta w + w = |w|^{p-2} w$ with $w \geq 0$. Now we show that $w \neq 0$. By the definition of "good" partition, by Lemma 5.2 and by (5.7) we can choose a number $T > 0$ and $q_k \in M$ such that, for k big enough $q_k \in P_k \subset B_{g_k}(q_k, \varepsilon_k T)$. By Remark 2.5 and Lemma 5.2 we get, for k large enough

$$
\begin{aligned}
\int_{B(0,T)} (w_k^+)^p &= \int_{B(0,T)} (u_k^+(\exp_{q_k}(\varepsilon_k z)))^p \frac{|g_{k,q_k}(\varepsilon_k z)|^{1/2}}{|g_{k,q_k}(\varepsilon_k z)|^{1/2}} dz \geq \\
&\geq \text{const} \frac{1}{\varepsilon^n} \int_{B_{g_k}(q_k, \varepsilon_k T)} |u^+(x)|^p d\mu_g \geq \text{const} \cdot \gamma.
\end{aligned} \tag{6.8}
$$

So we get $w \neq 0$ because w_k converges strongly to $w \in L^p(B(0,T))$ by Lemma 5.4, hence w_k^+ converge strongly to $w^+ = w$ in $L^p(B(0,T))$. $\square$

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