On the number of nodal solutions for a nonlinear elliptic problem on symmetric Riemannian manifolds

M.Ghimenti*, A.M.Micheletti*

Abstract

We consider the problem $-\varepsilon^2\Delta_g u+u=|u|^{p-2}u$ in M, where (M,g) is a symmetric Riemannian manifold. We give a multiplicity result for antisymmetric changing sign solutions.

Keywords: Riemannian Manifolds, Nodal Solutions, Topological Methods

Mathematics Subject Classification: 35J60, 58G03

1 Introduction

Let (M,g) be a smooth connected compact Riemannian manifold of finite dimension $n\geq 2$ embedded in $\mathbb{R}^N$. We consider the problem

$$-\varepsilon^2\Delta_g u+u=|u|^{p-2}u \text{ in } M,\quad u\in H^1_g(M) \tag{$\mathscr{P}$}$$

where $2<p<2*=\frac{2N}{N-2}$, if $N\geq 3$.

Here $H^1_g(M)$ is the completion of $C^\infty(M)$ with respect to

$$||u||^2:g=\int_M|\nabla_g u|^2+u^2d\mu_g \tag{1}$$

It is well known that the problem ($\mathscr{P}$) has a mountain pass solution u_ε. In [3] the authors showed that u_ε has a spike layer and its peak point converges to the maximum point of the scalar curvature of M as ε goes to 0.

Recently there have been some results on the influence of the topology and the geometry of M on the number of solutions of the problem. In

*Dipartimento di Matematica Applicata, Università di Pisa, via Buonarroti 1c, 56100, Pisa, Italy, e-mail `ghimenti@mail.dm.unipi.it, a.micheletti@dma.it`

[1] the authors proved that, if M has a rich topology, problem ($\mathscr{P}$) has multiple solutions. More precisely they show that problem ($\mathscr{P}$) has at least $\mathrm{cat}(M)+1$ positive nontrivial solutions for ε small enough. Here $\mathrm{cat}(M)$ is the Lusternik-Schnirelmann category of M. In [17] there is the same result for a more general nonlinearity. Furthermore in [9] it was shown that the number of solution is influenced by the topology of a suitable subset of M depending on the geometry of M. To point out the role of the geometry in finding solutions of problem ($\mathscr{P}$), in [13] it was shown that for any stable critical point of the scalar curvature it is possible to build positive single peak solutions. The peak of these solutions approaches such a critical point as ε goes to zero.

Successively in [6] the authors build positive k-peak solutions whose peaks collapse to an isolated local minimum point of the scalar curvature as ε goes to zero.

The first result on sign changing solution is in [12] where it is showed the existence of a solution with one positive peak η_1^ε and one negative peak η_2^ε such that, as ε goes to zero, the scalar curvature $S_g(\eta_1^\varepsilon)$ (respectively $S_g(\eta_2^\varepsilon)$) goes to the minimum (resp. maximum) of the scalar curvature when the scalar curvature of (M,g) is non constant. Here we give a multiplicity result for changing sign solutions when the Riemannian manifold (M,g) is symmetric.

We look for solutions of the problem

$$\begin{cases} -\varepsilon^2\Delta_g u + u = |u|^{p-2}u & u \in H_g^1(M); \\ \\ u(\tau x) = -u(x) & \forall x \in M, \end{cases} \qquad (\mathscr{P}_\tau)$$

where $\tau : \mathbb{R}^N \to \mathbb{R}^N$ is an orthogonal linear transformation such that $\tau \neq \mathrm{Id}$, $\tau^2 = \mathrm{Id}$, Id being the identity of $\mathbb{R}^N$. Here M is a compact connected Riemannian manifold of dimension $n \geq 2$ and M is a regular submanifold of $\mathbb{R}^N$ which is invariant with respect to τ. Let $M_\tau := \{x \in M \ : \ \tau x = x\}$ be the set of the fixed points with respect to the involution τ; in the case $M_\tau \neq \emptyset$ we assume that M_τ is a regular submanifold of M.

We obtain the following result.

Theorem 1. *The problem $\mathscr{P}_\tau$ has at least $G_\tau{-}\mathrm{cat}(M-M_\tau)$ pairs of solutions $(u,-u)$ which change sign (exactly once) for ε small enough*

Here $G_\tau - \mathrm{cat}$ is the G_τ-equivariant Lusternik Schnirelmann category for the group $G_\tau = \{\mathrm{Id}, \tau\}$.

In [4] the authors prove a result of this type for the Dirichlet problem

$$\begin{cases} -\Delta u - \lambda u - |u|^{2^*-2}u = 0 \quad u \in H^1_0(\Omega); \\ u(\tau x) = -u(x). \end{cases} \tag{$\mathscr{P}_\lambda$}$$

Here Ω is a bounded smooth domain invariant with respect to τ and λ is a positive parameter.

We point out that in the case of the unit sphere $S^{N-1} \subset \mathbb{R}^N$ (with the metric g induced by the metric of $\mathbb{R}^N$) the theorem of existence of changing sign solutions of [12] can not be used because it holds for manifold of non constant curvature. Instead, we can apply Theorem 1 to obtain sign changing solutions because we can consider $\tau = -\operatorname{Id}$, and we have $G_\tau - \operatorname{cat} S^{N-1} = N$.

Equation like ($\mathscr{P}$) has been extensively studied in a flat bounded domain $\Omega \subset \mathbb{R}^N$. In particular, we would like to compare problem ($\mathscr{P}$) with the following Neumann problem

$$\begin{cases} -\varepsilon^2 \Delta u + u = |u|^{p-2}u & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = 0 & \text{in } \partial\Omega. \end{cases} \tag{$\mathscr{P}_N$}$$

Here Ω is a smooth bounded domain of $\mathbb{R}^N$ and ν is the unit outer normal to Ω. Problems ($\mathscr{P}$) and ($\mathscr{P}_N$) present many similarities. We recall some classical results about the Neumann problem.

In the fundamental papers [11, 14, 15], Lin, Ni and Takagi established the existence of least-energy solution to ($\mathscr{P}_N$) and showed that for ε small enough the least energy solution has a boundary spike, which approaches the maximum point of the mean curvature H of $\partial\Omega$, as ε goes to zero. Later, in [16, 18] it was proved that for any stable critical point of the mean curvature of the boundary it is possible to construct single boundary spike layer solutions, while in [7, 19, 10] the authors construct multiple boundary spike solutions at multiple stable critical points of H. Finally, in [5, 8] the authors proved that for any integer K there exists a boundary K-peaks solutions, whose peaks collapse to a local minimum point of H.

2 Setting

We consider the functional defined on $H^1_g(M)$

$$J_\varepsilon(u) = \frac{1}{\varepsilon^N} \int_M \left(\frac{1}{2}\varepsilon^2 |\nabla_g u|^2 + \frac{1}{2}|u|^2 - \frac{1}{p}|u|^p \right) d\mu_g. \tag{2}$$

It is well known that the critical points of $J_\varepsilon(u)$ constrained on the Nehari manifold

$$\mathcal{N}_\varepsilon = \left\{ u \in H^1_g \smallsetminus \{0\} \ : \ J'_\varepsilon(u)u = 0 \right\} \tag{3}$$

are non trivial solution of problem ($\mathscr{P}$).

The transformation $\tau : M \to M$ induces a transformation on H^1_g we define the linear operator τ^* as

$$\begin{aligned}\tau^* \ &: \ H^1_g(M) \to H^1_g(M)\\ &\quad \tau^*(u(x)) = -u(\tau(x))\end{aligned}$$

and τ^* is a selfadjoint operator with respect to the scalar product on $H^1_g(M)$

$$\langle u, v\rangle_\varepsilon = \frac{1}{\varepsilon^N}\int_M \big(\varepsilon^2 \nabla_g u \cdot \nabla_g v + u \cdot v\big) d\mu_g. \tag{4}$$

Moreover $||\tau^* u||_{L^p(M)} = ||u||_{L^p(M)}$, and $||\tau^* u||_\varepsilon = ||u||_\varepsilon$, thus $J_\varepsilon(\tau^* u) = J_\varepsilon(u)$. Then, for the Palais principle, the nontrivial solutions of ($\mathscr{P}_\tau$) are the critical points of the restriction of J_ε to the τ-invariant Nehari manifold

$$\mathcal{N}^\tau_\varepsilon = \{u \in \mathcal{N}_\varepsilon \ : \ \tau^* u = u\} = \mathcal{N}_\varepsilon \cap H^\tau. \tag{5}$$

Here $H^\tau = \{u \in H^1_g \ : \ \tau^* u = u\}$.

In fact, since $J_\varepsilon(\tau^* u) = J_\varepsilon(u)$ and τ^* is a selfadjoint operator we have

$$\langle \nabla J_\varepsilon(\tau^* u), \tau^* \varphi\rangle_\varepsilon = \langle \nabla J_\varepsilon(u), \varphi\rangle_\varepsilon \ \ \forall \varphi \in H^1_g(M). \tag{6}$$

Then $\nabla J_\varepsilon(u) = \tau^* \nabla J_\varepsilon(\tau^* u) = \tau^* \nabla J_\varepsilon(u)$ if $\tau^* u = u$.

We set

$$m_\infty = \inf_{\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 = \int_{\mathbb{R}^N} |u|^p} \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p; \tag{7}$$

$$m_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} J_\varepsilon; \tag{8}$$

$$m^\tau_\varepsilon = \inf_{u \in \mathcal{N}^\tau_\varepsilon} J_\varepsilon. \tag{9}$$

Remark 2. It is easy to verify that J_ε satisfies the Palais Smale condition on $\mathcal{N}^\tau_\varepsilon$. Then there exists v_ε minimizer of m^τ_ε and v_ε is a critical point for J_ε on $H^1_g(M)$. Thus v^+_ε and v^-_ε belong to $\mathcal{N}_\varepsilon$, then $J_\varepsilon(v_\varepsilon) \geq 2m_\varepsilon$.

We recall some facts about equivariant Lusternik-Schnirelmann theory. If G is a compact Lie group, then a G-space is a topological space X with a continuous G-action $G \times X \to X$, $(g, x) \mapsto gx$. A G-map is a continuous function $f : X \to Y$ between G-spaces X and Y which is compatible with the G-actions, i.e. $f(gx) = gf(x)$ for all $x \in X$, $g \in G$. Two G-maps f_0, $f_1 : X \to Y$ are G-homotopic if there is a homotopy $\theta : X \times [0, 1] \to Y$ such that $\theta(x, 0) = f_0(x)$, $\theta(x, 1) = f_1(x)$ and $\theta(gx, t) = g\theta(x, t)$ for all$x \in X$, $g \in G$, $t \in [0, 1]$. A subset A of a X is G-invariant if $ga \in A$ for every $a \in A$, $g \in G$. The G-orbit of a point $x \in X$ is the set $Gx = \{gx : g \in G\}$.

Definition 3. *The G-category of a G-map $f : X \to Y$ is the smallest number $k = G - \operatorname{cat}(f)$ of open G-invariant subsets $X_1, \dots, X_k$ of X which cover X and which have the property that, for each $i = 1, \dots, k$, there is a point $y_i \in Y$ and a G-map $\alpha_i : X_i \to Gy_i \subset Y$ such that the restriction of f to X_i is G-homotopic to α_i. If no such covering exists we define $G - \operatorname{cat}(f) = \infty$.*

In our applications, G will be the group with two elements, acting as $G_\tau = \{\mathrm{Id}, \tau\}$ on Ω, and as $\mathbb{Z}/2 = \{1, -1\}$ by multiplication on the Nehari manifold $\mathcal{N}_\varepsilon^\tau$. We remark the following result on the equivariant category.

Theorem 4. *Let $\phi : M \to \mathbb{R}$ be an even C^1 functional on a complete $C^{1,1}$ submanifold M of a Banach space which is symmetric with respect to the origin. Assume that ϕ is bounded below and satisfies the Palais Smale condition $(PS)_c$ for every $c \leq d$. Then ϕ has at least $\mathbb{Z}/2 - \operatorname{cat}(\phi^d)$ antipodal pairs $\{u, -u\}$ of critical points with critical values $\phi(\pm u) \leq d$.*

3 Sketch of the proof

In our case we consider the even positive C^2 functional J_ε on the C^2 Nehari manifold $\mathcal{N}_\varepsilon^\tau$ which is symmetric with respect to the origin. As claimed in Remark 2, J_ε satisfies Palais Smale condition on $\mathcal{N}_\varepsilon^\tau$. Then we can apply Theorem 4 and our aim is to get an estimate of this lower bound for the number of solutions. For $d > 0$ we consider

$$M_d = \{x \in \mathbb{R}^N \ : \ \operatorname{dist}(x, M) \leq d\}; \tag{10}$$

$$M_d^- = \{x \in M \ : \ \operatorname{dist}(x, M_\tau) \geq d\}. \tag{11}$$

We choose d small enough such that

$$G_\tau - \operatorname{cat}_{M_d} M_d = G_\tau - \operatorname{cat}_M M \tag{12}$$

$$G_\tau - \operatorname{cat}_M M_d^- = G_\tau - \operatorname{cat}_M (M - M_\tau) \tag{13}$$

Now we build two continuous operator

$$\Phi_\varepsilon^\tau \ : \ M_d^- \to \mathcal{N}_\varepsilon^\tau \cap J_\varepsilon^{2(m_\infty + \delta)}; \tag{14}$$

$$\beta \ : \ \mathcal{N}_\varepsilon^\tau \cap J_\varepsilon^{2(m_\infty + \delta)} \to M_d, \tag{15}$$

such that $\Phi_\varepsilon^\tau(\tau q) = -\Phi_\varepsilon^\tau(q)$, $\tau\beta(u) = \beta(-u)$ and $\beta \circ \Phi_\varepsilon^\tau$ is G_τ homotopic to the inclusion $M_d^- \to M_d$.

By equivariant category theory we obtain

$$G_\tau - \operatorname{cat}_M(M - M_\tau) = G_\tau - \operatorname{cat}(M_d^- \hookrightarrow M_d) = \\ = G_\tau - \operatorname{cat} \beta \circ \Phi_\varepsilon^\tau \leq \mathbb{Z}_2 - \operatorname{cat} \mathcal{N}_\varepsilon^\tau \cap J_\varepsilon^{2(m_\infty + \delta)} \tag{16}$$

4 Technical lemmas

First of all, we recall that there exists a unique positive spherically symmetric function $U \in H^1(\mathbb{R}^n)$ such that

$$-\Delta U + U = U^{p-1} \text{ in } \mathbb{R}^n \tag{17}$$

It is well known that $U_\varepsilon(x) = U\left(\frac{x}{\varepsilon}\right)$ is a solution of

$$-\varepsilon^2 \Delta U_\varepsilon + U_\varepsilon = U_\varepsilon^{p-1} \text{ in } \mathbb{R}^n. \tag{18}$$

Secondly, let us introduce the exponential map $\exp : TM \to M$ defined on the tangent bundle TM of M which is a C^∞ map. Then, for ρ sufficiently small (smaller than the injectivity radius of M and smaller than $d/2$), the Riemannian manifold M has a special set of charts $\{\exp_x : B(0,\rho) \to M\}$. Throughout the paper we will use the following notation: $B_g(x,\rho)$ is the open ball in M centered in x with radius ρ with respect to the distance given by the metric g. Corresponding to this chart, by choosing an orthogonal coordinate system $(x_1, \ldots, x_n) \subset \mathbb{R}^n$ and identifying T_xM with $\mathbb{R}^n$ for $x \in M$, we can define a system of coordinates called *normal coordinates*.

Let χ_ρ be a smooth cut off function such that

$$\begin{aligned} \chi_\rho(z) = 1 \qquad & \text{if } z \in B(0,\rho/2); \\ \chi_\rho(z) = 0 \qquad & \text{if } z \in \mathbb{R}^n \smallsetminus B(0,\rho); \\ |\nabla \chi_\rho(z)| \le 2 \qquad & \text{for all } x. \end{aligned}$$

Fixed a point $q \in M$ and $\varepsilon > 0$, let us define the function $w_{\varepsilon,q}(x)$ on M as

$$w_{\varepsilon,q}(x) = \begin{cases} U_\varepsilon(\exp_q^{-1}(x))\chi_\rho(\exp_q^{-1}(x)) & \text{if } x \in B_g(q,\rho) \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

For each $\varepsilon > 0$ we can define a positive number $t(w_{\varepsilon,q})$ such that

$$\Phi_\varepsilon(q) = t(w_{\varepsilon,q}) w_{\varepsilon,q} \in H_g^1(M) \cap \mathcal{N}_\varepsilon \text{ for } q \in M. \tag{20}$$

Namely, $t(w_{\varepsilon,q})$ turns out to verify

$$t(w_{\varepsilon,q})^{p-2} = \frac{\displaystyle\int_M \varepsilon^2 |\nabla_g w_{\varepsilon,q}|^2 + |w_{\varepsilon,q}|^2 d\mu_g}{\displaystyle\int_M |w_{\varepsilon,q}|^p d\mu_g} \tag{21}$$

Lemma 1. *Given $\varepsilon > 0$ the application $\Phi_\varepsilon(q) : M \to H^1_g(M) \cap \mathcal{N}_\varepsilon$ is continuous. Moreover, given $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\delta)$ such that, if $\varepsilon < \varepsilon_0(\delta)$ then $\Phi_\varepsilon(q) \in \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_\infty+\delta}$.*

For the proof see [1, Proposition 4.2].

At this point, fixed a point $q \in M_d^-$, let us define the function

$$\Phi^\tau_\varepsilon(q) = t(w_{\varepsilon,q})w_{\varepsilon,q} - t(w_{\varepsilon,\tau q})w_{\varepsilon,\tau q} \tag{22}$$

Lemma 2. *Given $\varepsilon > 0$ the application $\Phi^\tau_\varepsilon(q) : M_d^- \to H^1_g(M) \cap \mathcal{N}^\tau_\varepsilon$ is continuous. Moreover, given $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\delta)$ such that, if $\varepsilon < \varepsilon_0(\delta)$ then $\Phi^\tau_\varepsilon(q) \in \mathcal{N}^\tau_\varepsilon \cap J_\varepsilon^{2(m_\infty+\delta)}$.*

Proof. Since $U_\varepsilon(z)\chi_\rho(z)$ is radially symmetric we set $U_\varepsilon(z)\chi_\rho(z) = \tilde{U}_\varepsilon(|z|)$. We recall that

$$\begin{aligned} |\exp^{-1}_{\tau q} \tau x| &= d_g(\tau x, \tau q) = d_g(x,q) = |\exp^{-1}_q x|; \\ |\exp^{-1}_q \tau x| &= d_g(\tau x, q) = d_g(x, \tau q). \end{aligned}$$

We have

$$\begin{aligned} \tau^*\Phi^\tau_\varepsilon(q)(x) &= -t(w_{\varepsilon,q})w_{\varepsilon,q}(\tau x) + t(w_{\varepsilon,\tau q})w_{\varepsilon,\tau q}(\tau x) = \\ &= -t(w_{\varepsilon,q})\tilde{U}_\varepsilon(|\exp^{-1}_q(\tau x)|) + t(w_{\varepsilon,\tau q})\tilde{U}_\varepsilon(|\exp^{-1}_{\tau q}(\tau x)|) = \\ &= t(w_{\varepsilon,\tau q})\tilde{U}_\varepsilon(|\exp^{-1}_q(x)|) - t(w_{\varepsilon,q})\tilde{U}_\varepsilon(|\exp^{-1}_q(\tau x)|) = \\ &= t(w_{\varepsilon,q})\tilde{U}_\varepsilon(|\exp^{-1}_q(x)|) - t(w_{\varepsilon,q})\tilde{U}_\varepsilon(|\exp^{-1}_{\tau q}(x)|), \end{aligned}$$

because by the definition we have $t(w_{\varepsilon,q}) = t(w_{\varepsilon,\tau q})$.

Moreover by definition the support of the function Φ^τ_ε is $B_g(q,\rho) \cup B_g(\tau q, \rho)$, and $B_g(q,\rho) \cap B_g(\tau q,\rho) = \emptyset$ because $\rho < d/2$ and $q \in M_d^-$. Finally, because

$$\int_M |w_{\varepsilon,q}|^\alpha d\mu_g = \int_M |w_{\varepsilon,\tau q}|^\alpha d\mu_g \quad \text{for } \alpha = 2, p; \tag{23}$$

$$\int_M |\nabla w_{\varepsilon,q}|^2 d\mu_g = \int_M |\nabla w_{\varepsilon,\tau q}|^2 d\mu_g, \tag{24}$$

we have

$$J_\varepsilon(\Phi^\tau_\varepsilon(q)) = \left(\frac{1}{2} - \frac{1}{p}\right)\frac{1}{\varepsilon^n}\int_M |\Phi^\tau_\varepsilon(q)|^p d\mu_g = 2J_\varepsilon(\Phi_\varepsilon(q)). \tag{25}$$

Then by previous lemma we have the claim. □

Lemma 3. *We have $\lim_{\varepsilon\to 0} m^\tau_\varepsilon = 2m_\infty$*

Proof. By the previous lemma and by Remark 2 we have that for any $\delta > 0$ there exists $\varepsilon_0(\delta)$ such that, for $\varepsilon < \varepsilon_0(\delta)$

$$2m_\varepsilon \leq m_\varepsilon^\tau \leq 2J_\varepsilon(\Phi_\varepsilon(q)) \leq 2(m_\infty + \delta). \tag{26}$$

Since $\lim_{\varepsilon\to 0} m_\varepsilon = m_\infty$ (see [1, Remark 5.9]) we get the claim. □

For any function $u \in \mathcal{N}_\varepsilon^\tau$ we can define a point $\beta(u) \in \mathbb{R}^N$ by

$$\beta(u) = \frac{\displaystyle\int_M x|u^+(x)|^p d\mu_g}{\displaystyle\int_M |u^+(x)|^p d\mu_g} \tag{27}$$

Lemma 4. *There exists δ_0 such that, for any $0 < \delta < \delta_0$ and any $0 < \varepsilon < \varepsilon_0(\delta)$ (as in Lemma 2) and for any function $u \in \mathcal{N}_\varepsilon^\tau \cap J_\varepsilon^{2(m_\infty+\delta)}$, it holds $\beta(u) \in M_d$.*

Proof. Since $\tau^* u = u$ we set

$$M^+ = \{x \in M \ : \ u(x) > 0\} \qquad M^- = \{x \in M \ : \ u(x) < 0\}.$$

It is easy to see that $\tau M^+ = M^-$. Then we have

$$\begin{aligned} J_\varepsilon(u) &= \left(\frac{1}{2} - \frac{1}{p}\right)\frac{1}{\varepsilon^n}\int_M |u|^p d\mu_g = \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\frac{1}{\varepsilon^n}\left[\int_{M^+} |u^+|^p d\mu_g + \int_{M^-} |u^-|^p d\mu_g\right] = 2J_\varepsilon(u^+) \end{aligned} \tag{28}$$

By the assumption $J_\varepsilon(u) \leq 2(m_\infty + \delta)$ we have $J_\varepsilon(u^+) \leq m_\infty + \delta$ then by Proposition 5.10 of [1] we get the claim. □

Lemma 5. *There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ the composition*

$$I_\varepsilon = \beta \circ \Phi_\varepsilon^\tau : M_d^- \to M_d \subset \mathbb{R}^N \tag{29}$$

is well defined, continuous, homotopic to the identity and $I_\varepsilon(\tau q) = \tau I_\varepsilon(q)$.

Proof. It is easy to check that

$$\Phi_\varepsilon^\tau(\tau q) = -\Phi_\varepsilon^\tau(q) \qquad \beta(-u) = \tau\beta(u). \tag{30}$$

Moreover, by Lemma 2 and by Lemma 4, for any $q \in M_d^-$ we have $\beta \circ \Phi_\varepsilon^\tau(q) = \beta(\Phi_\varepsilon(q)) \in M_d$, and I_ε is well defined.

In order to show that I_ε is homotopic to identity, we evaluate the difference between I_ε and the identity as follows.

$$I_\varepsilon(q) - q = \frac{\displaystyle\int_M (x-q)|u^+|^p d\mu_g}{\displaystyle\int_M |u^+|^p d\mu_g} = \frac{\displaystyle\int_{B(0,\rho)} z \left| U\left(\frac{z}{\varepsilon}\right)\chi_\rho(|z|)\right|^p \left|g_q(z)\right|^{\frac{1}{2}}}{\displaystyle\int_{B(0,\rho)} \left| U\left(\frac{z}{\varepsilon}\right)\chi_\rho(|z|)\right|^p \left|g_q(z)\right|^{\frac{1}{2}}} =$$

$$= \frac{\varepsilon \displaystyle\int_{B(0,\frac{\rho}{\varepsilon})} z\left|U(z)\chi_\rho(|\varepsilon z|)\right|^p\left|g_q(\varepsilon z)\right|^{\frac{1}{2}}}{\displaystyle\int_{B(0,\frac{\rho}{\varepsilon})} \left|U(z)\chi_\rho(|\varepsilon z|)\right|^p\left|g_q(\varepsilon z)\right|^{\frac{1}{2}}}, \quad (31)$$

hence $|I_\varepsilon(q) - q| < \varepsilon c(M)$ for a constant $c(M)$ that does not depend on q. □

Now, by previous lemma and by Theorem 4 we can prove Theorem 1.

In fact, we know that, if ε is small enough, there exist $G_\tau - \mathrm{cat}(M - M_\tau)$ minimizers which change sign, because they are antisymmetric. We have only to prove that any minimizer changes sign exactly once. Let us call $\omega = \omega_\varepsilon$ one of these minimizers. Suppose that the set $\{x \in M \ : \ \omega_\varepsilon(x) > 0\}$ has k connected components $M_1, \ldots, M_k$. Set

$$\omega_i = \begin{cases} \omega_\varepsilon(x) & x \in M_i \cup \tau M_i; \\ 0 & \text{elsewhere} \end{cases} \quad (32)$$

For all i, $\omega_i \in \mathcal{N}_\varepsilon^\tau$. Furthermore we have

$$J_\varepsilon(\omega) = \sum_i J_\varepsilon(\omega_i), \quad (33)$$

thus

$$m_\varepsilon^\tau = J_\varepsilon(\omega) = \sum_{i=1}^k J_\varepsilon(\omega_i) \geq k \cdot m_\varepsilon^\tau, \quad (34)$$

so $k = 1$, that concludes the proof.

References

[1] V. Benci, C. Bonanno, and A.M. Micheletti, *On the multiplicity of solutions of a nonlinear elliptic problem on Riemannian manifolds*, J. Funct. Anal. **252** (2007), no. 2, 464–489.

[2] V. Benci and G. Cerami, *The effect of the domain topology on the number of positive solutions of nonlinear elliptic problems*, Arch. Rational Mech. Anal. **114** (1991), no. 1, 79–93.

[3] J. Byeon and J. Park, *Singularly perturbed nonlinear elliptic problems on manifolds*, Calc. Var. Partial Differential Equations **24** (2005), no. 4, 459–477.

[4] A. Castro and M. Clapp, *The effect of the domain topology on the number of minimal nodal solutions of an elliptic equation at critical growth in a symmetric domain*, Nonlinearity **16** (2003), no. 2, 579–590.

[5] E. Dancer and S. Yan, *Multipeak solutions for a singularly perturbed Neumann problem*, Pacific J. Math **189** (1999), no. 2, 241–262.

[6] E. Dancer, A.M. Micheletti, and Angela Pistoia, *Multipeak solutions for some singularly perturbed nonlinear elliptic problems in a Riemannian manifold*, to appear on Manus. Math.

[7] C. Gui, *Multipeak solutions for a semilinear Neumann problem*, Duke Math J. **84** (1996), no. 3, 739–769.

[8] C. Gui, J. Wei, and M. Winter, *Multiple boundary peak solutions for some singularly perturbed Neumann problems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **17** (2000), no. 1, 47–82.

[9] N. Hirano, *Multiple existence of solutions for a nonlinear elliptic problem on a Riemannian manifold*, Nonlinear Anal., **70** (2009), no. 2, 671–692.

[10] Y.Y. Li, *On a singularly perturbed equation with Neumann boundary condition*, Comm. Partial Differential Equations **23** (1998), no. 3-4, 487–545.

[11] C.S. Lin, W.M. Ni, and I. Takagi, *Large amplitude stationary solutions to a chemiotaxis system*, J. Differential Equations **72** (1988), no. 1, 1–27.

[12] A.M. Micheletti and A. Pistoia, *Nodal solutions for a singularly perturbed nonlinear elliptic problem in a Riemannian manifold*, to appear on Adv. Nonlinear Stud.

[13] A.M. Micheletti and A. Pistoia, *The role of the scalar curvature in a nonlinear elliptic problem in a Riemannian manifold*, Calc. Var. Partial Differential Equation, **34** (2009), 233–265.

[14] W.M. Ni and I. Takagi, *On the shape of least-energy solutions to a semilinear Neumann problem*, Comm. Pure Appl. Math. **44** (1991), no. 7, 819–851.

[15] W.M. Ni and I. Takagi, *Locating the peaks of least-energy solutions to a semilinear Neumann problem*, Duke Math. J. **70** (1993), no. 2, 247–281.

[16] M. Del Pino, P. Felmer, and J. Wei, *On the role of mean curvature in some singularly perturbed Neumann problems*, SIAM J. Math. Anal. **31** (1999), no. 1, 63–79.

[17] D. Visetti, *Multiplicity of solutions of a zero-mass nonlinear equation in a Riemannian manifold*, J. Differential Equations, **245** (2008), no. 9, 2397–2439.

[18] J. Wei, *On the boundary spike layer solutions to a singularly perturbed Neumann problem*, J. Differential Equations **134** (1997), no. 1, 104–133.

[19] J. Wei and M. Winter, *Multipeak solutions for a wide class of singular perturbation problems*, J. London Math. Soc. **59** (1999), no. 2, 585–606.