COMPLETE REDUCIBILITY AND STEINBERG ENDOMORPHISMS

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Abstract. Let G be a connected reductive algebraic group defined over an algebraically closed field of positive characteristic. We study a generalization of the notion of G-complete reducibility in the context of Steinberg endomorphisms of G. Our main theorem extends a special case of a rationality result in this setting.

1. Introduction

Let p be a prime number and let $k = \overline{\mathbb{F}_p}$ be the algebraic closure of the field of p elements. Let G be a connected reductive linear algebraic group defined over k and let H be a closed subgroup of G. Let $\mathbb{F}_p \subseteq k' \subseteq k$ be a field extension of $\mathbb{F}_p$. Following Serre [12], we say that a k'-defined subgroup H of G is *G-completely reducible over k'* provided that whenever H is contained in a k'-defined parabolic subgroup P of G, it is contained in a k'-defined Levi subgroup of P. If $k' = k$, then H is G-completely reducible over k' if and only if H is G-completely reducible (or G-cr for short). For an overview of this concept see for instance [11] and [12].

The starting point for our discussion is the following special case of the rationality result [1, Thm. 5.8]. Let q be a power of p and let $\mathbb{F}_q$ be the field of q elements.

Theorem 1.1. *Suppose that both G and H are defined over $\mathbb{F}_q$. Then H is G-completely reducible if and only if it is G-completely reducible over $\mathbb{F}_q$.*

Let $\sigma : G \to G$ be a *Steinberg endomorphism* of G, i.e. a surjective endomorphism of G that fixes only finitely many points, see Steinberg [14] for a detailed discussion (for this terminology, see [6, Def. 1.15.1b]). The set of all Steinberg endomorphisms of G is a subset of all isogenies $G \to G$ (see [14, 7.1(a)]) that encompasses in particular all (generalized) Frobenius endomorphisms, i.e. endomorphisms of G some power of which are Frobenius endomorphisms corresponding to some $\mathbb{F}_q$-rational structure on G.

Example 1.2. Let F_1, F_2 be the Frobenius maps of $G = \mathrm{SL}_2$ given by raising coefficients to the pth and p^2th powers, respectively. Then the map $\sigma = F_1 \times F_2 : G \times G \to G \times G$ is a Steinberg morphism of $G \times G$ that is not a Frobenius morphism, cf. the remark following [6, Thm. 2.1.11].

If G is almost simple, then σ is a (generalized) Frobenius map (e.g. see [6, Thm. 2.1.11]), and the possibilities for σ are well known ([14, §11], e.g. see [7, Thm. 1.4]): σ is conjugate to either σ_q, $\tau\sigma_q$, $\tau'\sigma_q$ or τ', where σ_q is a standard Frobenius morphism, τ is an automorphism of algebraic groups coming from a graph automorphism of types A_n, D_n or E_6, and τ' is a bijective endomorphism coming from a graph automorphism of type B_2 ($p = 2$), F_4 ($p = 2$) or G_2 ($p = 3$).

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Example 1.3. If G is not simple, then a generalized Frobenius map may fail to factor into a field and a graph automorphism as stated above. For example, let $p = 2$ and let H_1, H_2 be simple, simply connected groups of type B_n and C_n ($n \geq 3$), respectively. Then there are special isogenies $\phi_1 : H_1 \to H_2$ and $\phi_2 : H_2 \to H_1$ whose composites $\phi_1 \circ \phi_2$ and $\phi_2 \circ \phi_1$ are standard Frobenius maps with respect to p on H_2, respectively H_1, see [4, p 5 of Exp. 24]. Let $G = H_1 \times H_2$ and define $\sigma : G \to G$ by $\sigma(h_1, h_2) = (\phi_2(h_2), \phi_1(h_1))$. Then σ is an example of such a more complicated generalized Frobenius map.

We now give an extension of Serre's notion of G-complete reducibility in this setting of Steinberg endomorphisms: Let σ be a Steinberg endomorphism of G and let H be a subgroup of G. We say that H is *σ-completely reducible* (or σ-cr for short), provided that whenever H lies in a σ-stable parabolic subgroup P of G, it lies in a σ-stable Levi subgroup of P. This notion is motivated as follows: If σ_q is a standard Frobenius morphism of G, then a subgroup H of G is defined over $\mathbb{F}_q$ if and only if it is σ_q-stable and if so, H is G-completely reducible over $\mathbb{F}_q$ if and only if it is σ_q-completely reducible. In view of this new notion, the goal of this note is the following generalization of Theorem 1.1 to arbitrary Steinberg endomorphisms of G (the special case of Theorem 1.4 when $\sigma = \sigma_q$ gives Theorem 1.1).

Theorem 1.4. *Let σ be a Steinberg endomorphism of G. Let H be a σ-stable subgroup of G. Then H is G-completely reducible if and only if H is σ-completely reducible.*

Theorem 1.4 follows from Theorems 2.4 and 2.5 proved in the next section.

Example 1.5. Theorem 1.4 is false without the σ-stability condition on H. For instance, a maximal torus T of G is always G-cr, cf. [1, Lem. 2.6]. But it may happen that T is contained in a σ-stable Borel subgroup of G, without being itself σ-stable. Then T clearly fails to be σ-cr. In the other direction, G may contain a maximal parabolic subgroup P of G that is not σ-stable. The only σ-stable parabolic subgroup of G containing P is G itself. Then P is σ-cr for trivial reasons, whereas a proper parabolic subgroup of G is not G-cr.

Remark 1.6. Even if H is not σ-stable, Theorem 1.4 gives some information about the notion of σ-complete reducibility, as follows. Let $\overline{H}^{\sigma}$ be the algebraic subgroup of G generated by all translates $\sigma^i H$, $i \geq 0$. Then $\overline{H}^{\sigma}$ is σ-stable and contained in the same σ-stable subgroups of G as H. In particular, H is σ-cr if and only if $\overline{H}^{\sigma}$ is σ-cr. Thus, by Theorem 1.4, this is equivalent to $\overline{H}^{\sigma}$ being G-cr.

2. Proof of Theorem 1.4

In addition to the notation already fixed in the Introduction, $\sigma : G \to G$ is always a Steinberg endomorphism of G and from now on the subgroup H of G is assumed to be σ-stable. We begin with a generalization of (a special case of) [8, Prop. 2.2 and Rem. 2.4].

Proposition 2.1. *If H is not G-completely reducible, then there exists a proper σ-stable parabolic subgroup of G containing H.*

Proof. First we assume that G is almost simple. We want to reduce to the case where H is a finite, σ-stable subgroup of G, and then apply [8, Prop. 2.2 and Rem. 2.4]. Since G is almost simple, we can assume that $\sigma^m = \sigma_q$ is a standard Frobenius map for some positive integer m. We choose a closed embedding $G \to \mathrm{GL}_n(k)$ so that σ_q is the restriction of the standard Frobenius map of $\mathrm{GL}_n(k)$ that raises coefficients to the qth power (see [5, Prop.

4.1.11]). For $r \in \mathbb{Z}, r \geq 1$, let $\tilde{H}(r) = H \cap \mathrm{GL}_n(\mathbb{F}_{q^{r!}})$. Then we can write H as the directed union of finite subgroups $H = \bigcup_{r \geq 1} \tilde{H}(r)$. Note that the union is indeed directed, that is

$$\tilde{H}(r) \subseteq \tilde{H}(r+1) \ \forall r \geq 1. \tag{2.2}$$

We wish to construct a similar, but σ-stable filtration of H. For this purpose we set $H(r) = \bigcap_{l=0}^{m-1} \sigma^l \tilde{H}(r)$. Then each $H(r)$ is a finite, σ-stable subgroup of H (for the σ-stability, we use that each $\tilde{H}(r)$ is stable under $\sigma^m = \sigma_q$). Moreover, we claim that H is the directed union $H = \bigcup_{r \geq 1} H(r)$. Indeed, if $h \in H$, then the identities $H = \sigma H$ and $H = \bigcup_{r \geq 1} \tilde{H}(r)$ imply that for each $l = 0, \ldots, m-1$ we can find some r_l such that $h \in \sigma^l \tilde{H}(r_l)$. But then (2.2) implies that $h \in H(r)$ for $r \geq \max\{r_0, \ldots, r_{m-1}\}$. It follows from the argument in the proof of [1, Lem. 2.10] that there is an integer r' so that $H(r')$ has the following property: H is contained in a parabolic subgroup P of G (respectively a Levi subgroup L of G) if and only if $H(r')$ is contained in P (respectively in L). Therefore, if H is not G-cr, then neither is $H(r')$, and we can apply [8, Prop. 2.2 and Rem. 2.4] to obtain a proper σ-stable parabolic subgroup P of G that contains $H(r')$. But then P also contains H.

Next we drop the simplicity assumption on G. Then we can use the almost simple components of G to reduce to the almost simple case: Let $\pi : G' := Z(G)^\circ \times G_1 \times \cdots \times G_r \to G$ be the product map, where $G_1, \ldots, G_r$ are the almost simple components of the semisimple group $[G, G]$ and let $\pi_i : G' \to G_i$ be the projection ($1 \leq i \leq r$). Then π is an isogeny. Let $H' = \pi^{-1}(H)$. Using [1, Lem. 2.12] and the fact that $Z(G)^\circ$ is a torus, we find that there is some index i such that $H_i := \pi_i(H') \subseteq G_i$ is not G_i-cr. We can assume that $i = 1$. We are now in the situation of the first part of the proof (for $H_1 \subseteq G_1$), except that we have yet to specify a Steinberg endomorphism of G_1 that stabilizes H_1. Since σ stabilizes $[G, G]$ and maps components to components ([4, Exp. 18, Prop. 2]), we can assume that σ permutes $G_1, \ldots, G_s$ cyclically for some $s \leq r$. Moreover, σ stabilizes $Z(G)^\circ = R(G)$ (because σ is an isogeny). Using the restrictions $\sigma|_{Z(G)^\circ}$ and $\sigma|_{[G,G]}$, we can define a Steinberg endomorphism $\sigma' : G' \to G'$ of G' such that $\pi \circ \sigma' = \sigma \circ \pi$. We denote by H'' the image (under the projection) of H' in $G'' := G_1 \times \cdots \times G_s$. Now let $\tau = \sigma^s|_{G_1} : G_1 \to G_1$ denote the generalized Frobenius map on G_1 induced by σ ([6, Thms. 2.1.2(g) and 2.1.11]). Then H_1 is τ-stable, since H is σ^s-stable. We apply the first part of the proof to $H_1 \subseteq G_1$ to obtain a proper τ-stable parabolic subgroup P_1 of G_1 containing H_1. Then $P'' := P_1 \times \sigma P_1 \times \cdots \times \sigma^{s-1} P_1 \subseteq G''$ is a proper $\sigma'|_{G''}$-stable parabolic subgroup of G'' ([13, Cor. 6.2.8]). The bijectivity of $\sigma^s|_{H_i} : H_i \to H_i$ for $1 \leq i \leq s$ implies that $H_i = \sigma^{i-1} H_1$ for $1 \leq i \leq s$. We get that P'' contains H'', since we have $H'' \subseteq H_1 \times H_2 \times \cdots \times H_s$ and $H_1 \subseteq P_1$. Consequently, $P' = Z(G)^\circ \times P'' \times G_{s+1} \times \cdots \times G_r$ is a proper σ'-stable parabolic subgroup of G' containing H'. Finally, $P = \pi(P')$ is a proper σ-stable parabolic subgroup of G containing H, as desired. □

Remark 2.3. In [8, Prop. 2.2 and Rem. 2.4], Liebeck, Martin and Shalev prove the following: Let G be an almost simple algebraic group over k as above. Let $\mathrm{Aut}^{\#}(G)$ denote the group generated by inner automorphisms of G, together with p^i-power field morphisms ($i \geq 1$), and graph automorphisms (which may include the bijective endomorphisms coming from a graph automorphism of type B_2 ($p = 2$), F_4 ($p = 2$) or G_2 ($p = 3$)). (Note that $\mathrm{Aut}^{\#}(G)$ is an extension of the group $\mathrm{Aut}^{+}(G)$ from [8].) Let S be a subgroup of $\mathrm{Aut}^{\#}(G)$ and suppose that $H \subseteq G$ is a finite, S-stable subgroup that is not G-cr. Then H is contained in a proper S-invariant parabolic subgroup of G (note that the notion of strongly reductive subgroups in G is equivalent to the notion of G-completely reducible subgroups, cf. [1, Thm. 3.1]). If we take S to be generated by a (generalized) Frobenius endomorphism σ of G, then we get the assertion of Proposition 2.1 for G almost simple and H finite.

Theorem 2.4. *If H is σ-completely reducible, then it is G-completely reducible.*

Proof. If H is not contained in any proper σ-stable parabolic subgroup of G, then it is G-cr according to Proposition 2.1. So we can assume that there is a proper σ-stable parabolic subgroup P of G containing H. We choose P minimal with these properties. Since H is σ-cr, it is contained in a σ-stable Levi subgroup L of P. Suppose there is a proper σ-stable parabolic subgroup P_L of L containing H. Then $P' = P_L R_u(P) \subsetneq P$ is another parabolic subgroup of G (see [3, Prop. 4.4(c)]) containing H, and P' is σ-stable (σ stabilizes $R_u(P)$ as any isogeny does). But this contradicts our choice of P. So we can use Proposition 2.1 again to deduce that H is L-cr, which in turn implies that H is G-cr ([1, Cor. 3.22]). □

For the converse of Theorem 2.4 we argue as in the last part of the proof of [9, Thm. 9]. But first we recall a parametrization of the parabolic and Levi subgroups of G in terms of cocharacters of G, e.g. see [1, Lem. 2.4]: Given a parabolic subgroup P of G and any Levi subgroup L of P, there exists some cocharacter λ of G such that P and L are of the form $P = P_\lambda = \{g \in G \mid \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} \text{ exists}\}$ and $L = L_\lambda = C_G(\lambda(k^*))$, respectively. The unipotent radical of P_λ is then given by $R_u(P_\lambda) = \{g \in G \mid \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} = 1\}$.

Theorem 2.5. *If H is G-completely reducible, then it is σ-completely reducible.*

Proof. Suppose that P is a σ-stable parabolic subgroup of G containing H. Since H is G-cr, there is some Levi subgroup L of P that contains H. Let $U = R_u(P)$. Then $\Lambda = \{uLu^{-1} \mid u \in U, H \subseteq uLu^{-1}\}$ is the set of all Levi subgroups of P that contain H. Clearly, Λ is σ-stable, since H and P are. We need to prove that Λ contains an element fixed by σ.

If uLu^{-1} is in Λ, then $u^{-1}Hu \subseteq L \cap UH = H$, so that u normalizes H. In fact, u centralizes H, since $[N_U(H), H] \subseteq H \cap U = \{1\}$. So the group $C = C_U(H)$ acts transitively on Λ. We claim that C is connected. In order to prove this, write $P = P_\lambda$, $L = L_\lambda$ and $U = R_u(P_\lambda)$ for some suitable cocharacter λ of G. The torus $\lambda(k^*)$ normalizes $C_G(H)$ (because H is contained in L) and U, hence it normalizes C. Whence, for any fixed $c \in C$, the map $\phi_c : k^* \to C$, given by $t \mapsto \lambda(t)c\lambda(t)^{-1}$, is well-defined. Moreover, $C \subseteq U$ implies that ϕ_c extends to a morphism $\hat{\phi}_c : k \to C$ that maps 0 to 1 and 1 to c. Since the image of $\hat{\phi}_c$ is connected, we get $c \in C^\circ$. It follows that $C = C^\circ$. But now we can apply the Lang-Steinberg theorem (see [14, Thm. 10.1]) to conclude that Λ contains an element fixed by σ. □

Remark 2.6. We conclude by outlining a short alternative approach to Proposition 2.1; the latter was crucial in the proof of Theorem 2.4. This variant utilizes the so called *Centre Conjecture* for spherical buildings due to J. Tits from the 1950s. This deep conjecture has recently been established by work of Leeb and Ramos-Cuevas, e.g. see [2, §2] and the references therein for further details. This conjecture states that in the building $\Delta = \Delta(G)$ of G any convex contractible subcomplex Σ has a simplex which is fixed under any building automorphism of Δ which stabilizes Σ as a subcomplex. Such a fixed simplex is often referred to as a *centre* giving this conjecture its name. Here is a sketch of a building theoretic alternative to the proof of Proposition 2.1: Let H be a σ-stable subgroup of G which is not G-cr. Consider the subcomplex Δ^H of H-fixed points of the building Δ, i.e., Δ^H corresponds to the set of all parabolic subgroups of G that contain H. Note that Δ^H is always convex ([12, Prop. 3.1]) and since H is not G-cr, Δ^H is also contractible ([10, Thm. 2]). The Steinberg morphism σ of G affords a building automorphism of Δ, also denoted by σ. Since H is σ-stable, so is Δ^H. Now since Δ^H is convex and contractible, the Centre Conjecture asserts the existence of a centre of Δ^H with respect to the action of σ which corresponds to a proper parabolic subgroup of G which is σ-stable and contains H. This is precisely the conclusion of Proposition 2.1.

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References

[1] M. Bate, B. Martin, and G. Röhrle. A geometric approach to complete reducibility. *Invent. Math.*, 161(1):177–218, 2005.

[2] ———, Complete reducibility and separable field extensions. *C. R. Math. Acad. Sci. Paris*, 348(9-10):495–497, 2010.

[3] A. Borel and J. Tits. Groupes réductifs. *Inst. Hautes Études Sci. Publ. Math.*, (27):55–150, 1965.

[4] C. Chevalley. *Classification des groupes algébriques semi-simples.* Collected works. Vol. 3. Springer-Verlag, Berlin, 2005.

[5] M. Geck. *An introduction to algebraic geometry and algebraic groups*, volume 10 of *Oxford Graduate Texts in Mathematics.* Oxford University Press, Oxford, 2003.

[6] D. Gorenstein, R. Lyons, and R. Solomon. *The classification of the finite simple groups.* Part I. Chapter A: Almost simple $\mathcal{K}$-groups. vol. 40 No. 3 *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1998.

[7] M. W. Liebeck. Subgroups of simple algebraic groups and of related finite and locally finite groups of Lie type. In *Finite and locally finite groups (Istanbul, 1994)*, volume 471 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 71–96. Kluwer Acad. Publ., Dordrecht, 1995.

[8] M. W. Liebeck, B. M. S. Martin, and A. Shalev. On conjugacy classes of maximal subgroups of finite simple groups, and a related zeta function. *Duke Math. J.*, 128(3):541–557, 2005.

[9] M. W. Liebeck and G. M. Seitz. On the subgroup structure of exceptional groups of Lie type. *Trans. Amer. Math. Soc.*, 350(9):3409–3482, 1998.

[10] J-P. Serre. *La notion de complète réductibilité dans les immeubles sphériques et les groupes réductifs*, Séminaire au Collège de France, résumé dans [15, pp. 93–98], (1997).

[11] ———, *The notion of complete reducibility in group theory*, Moursund Lectures, Part II, University of Oregon, 1998, `arXiv:math/0305257v1 [math.GR]`.

[12] ———, *Complète réductibilité*, Séminaire Bourbaki, 56ème année, 2003–2004, n° 932.

[13] T. A. Springer. *Linear algebraic groups*, volume 9 of *Progress in Mathematics.* Birkhäuser Boston Inc., Boston, MA, second edition, 1998.

[14] R. Steinberg. *Endomorphisms of linear algebraic groups.* Memoirs of the American Mathematical Society, No. 80. American Mathematical Society, Providence, R.I., 1968.

[15] J. Tits. *Théorie des groupes*, Résumé des Cours et Travaux, Annuaire du Collège de France, 97e année, (1996–1997), 89–102.

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