" "Jede" endliche freie Auflösung ist freie Auflösung eines von drei Elementen erzeugten Ideals"

Winfried Bruns

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A translation into English:

'Every' finite free resolution is a free resolution of an ideal generated by three elements.

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1 Introduction

In [4] and [7], Burch and Kohn proved that when R is a commutative Noetherian ring, and n a natural number that is the homological dimension [3] of a torsionless R-module, then there exists an ideal in R generated by at most 3 elements, whose homological dimension is precisely n. Buchsbaum and Eisenbud [3, pg. 135, Conjecture; see also Theorem 7.2] have further conjectured that 'every' finite free resolution is a free resolution of such an ideal. We prove the following generalisation of their conjecture:

Let:

$$0 \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_{m+1} \xrightarrow{f_{m+1}} F_m \longrightarrow M \longrightarrow 0$$

be a projective resolution of M, where F_m is a free R-module and M is an m-torsionless R-module (or equivalently, M is an m-th syzygy module) having a well defined rank. Let r be the rank of the submodule $\mathrm{Im}(f_{m+1})$. Then there exist homomorphisms:

$$c : F_m \to R^{r+m}; f_m : R^{r+m} \to R^{2m-1}; f_j : R^{2j+1} \to R^{2j-1}, j = 1, \ldots, m-1$$

such that if $f'_{m+1} := c \circ f_{m+1}$, the sequence:

$$0 \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_{m+2} \xrightarrow{f_{m+2}} F_{m+1} \xrightarrow{f'_{m+1}} R^{r+m}$$

$$\xrightarrow{f_m} R^{2m-1} \longrightarrow \cdots \longrightarrow R^5 \xrightarrow{f_2} R^3 \xrightarrow{f_1} R$$

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[3] Translator's note: The homological dimension of a module is now more commonly known as its projective dimension.

is exact. (Theorem 3). When $m = 2$ and the F_k, $k = 2, \ldots n$ are free modules, we obtain the conjecture of Buchsbaum and Eisenbud.

Our result also shows that any natural number that is the homological dimension of an m-torsionless R-module is the homological dimension of an m-torsionless R-module of rank m generated by at most $2m + 1$ elements.

The maps f_i and c above are constructed in Theorem 2, which states that if firstly M is an m-torsionless R-module of rank $r > m$ and of finite homological dimension, and if secondly $g : R^n \to M$ is an epimorphism, then there exists a basis element x of R^n such that $M/Rg(x)$ is m-torsionless and of rank $r - 1$. This result requires that $g(x)$ can be included as part of a basis for the free $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ of R having the property that $\text{depth}(R_{\mathfrak{p}}) \leq m$. Buchsbaum and Eisenbud have shown the existence of such elements $x \in R^n$, assuming certain hypotheses on the Krull dimension of the ring $R/\mathfrak{p}$. We are able to prove a similar statement, in which we replace their hypotheses on the Krull dimension of $R/\mathfrak{p}$ with conditions on the difference between m and the depth of $R_{\mathfrak{p}}$ (Theorem 1).

One may replace the condition on the homological dimension of M in the previous paragraph with a condition on the regularity of the rings $R_{\mathfrak{p}}$ whose depth is at most m and still have the result hold. For rings satisfying this condition, we are able to generalize a theorem of Bourbaki [2, pg. 76, Theorème 6], which states that any m-torsionless R-module of rank at least m is an extension of an m-torsionless R-module of rank m by a free R-module (Corollary 2 to Theorem 2).

In our paper, R will throughout denote a commutative Noetherian ring and all R-modules $M, N, \ldots$ shall be finitely generated. We denote by $Q(R)$ the total quotient ring of R. If $M \otimes Q(R)$ is a free $Q(R)$-module, then the rank of M (denoted rank M) is defined to be the number of elements in a basis of $M \otimes Q(R)$. We remind the reader that if any two modules in a short exact sequence have well defined ranks, then so does the third ([9, chap 6]). We denote the homological dimension of M over R by $\text{dh}(M)$, and the homological codimension (depth) by $\text{codh}(M)$. When R is a local ring we define the (well-defined) number of elements in a minimal generating set of M by $\mu(M)$. Ass M denotes the (finite) set of associated primes of M. We denote the grade of an ideal $\mathfrak{a}$ (with respect to elements in R) by $\text{grad}(\mathfrak{a})$; thus $\text{grad}(\mathfrak{a}) = \min\{\text{codh}(R_{\mathfrak{p}}) \mid \mathfrak{p} \supset \mathfrak{a}\}$. For notational convenience, we shall define $\mathfrak{C}_n$ to be the set of prime ideals $\mathfrak{p}$ of R such that $\text{codh}(R_{\mathfrak{p}}) \leq n$.

We briefly explain the notion of 'm-torsionless' for a module. Suppose $F_1 \to F_0 \to M \to 0$ is a presentation of a module M, and let $D(M)$ be the cokernel of the dual of the homomorphism $F_1 \to F_0$ [4]. The $\text{Ext}^i_R(D(M), R)$ modules for $i \geq 1$ do not depend on the choice of presentation of M. Thus we may say that M is m-torsionless provided that $\text{Ext}^i_R(D(M), R) = 0$ for $i = 1, \ldots m$ [1]. The nomenclature derives from the fact that $\text{Ext}^1_R(D(M), R)$ and $\text{Ext}^2_R(D(M), R)$ are the kernel and the cokernel respectively of the natural homomorphism of M into its bidual. Hence M is 1-torsionless (resp. 2-torsionless) if M is torsionless (resp. reflexive) in the usual sense. If $\text{dh}(M) < \infty$ there are the following equivalent [1, Theorem 4.25] characterisations of m-torsionlessness:

1. (a_m) Every R-regular sequence of at most m elements is M-regular (thus if $m = 1$ then M is torsion-free).
2. (b_m) $\text{codh}(M_{\mathfrak{p}}) \geq \min(m, \text{codh}(R_{\mathfrak{p}}))$ for every prime ideal $\mathfrak{p}$ of R.
3. (s_m) M is an m-th syzygy module of a projective resolution.
4. (t_m) M is m-torsionless.

When the localization $R_{\mathfrak{p}}$ is a Gorenstein ring for all prime ideals $\mathfrak{p} \in \mathfrak{C}_{m-1}$, the above equivalences also hold even if $\text{dh}(M)$ is not finite [6, Theorem 4.6]. We call rings R satisfying this property on its localisations m-Gorenstein. In this paper we will only consider m-Gorenstein

[4] Translator's note: this is the 'transpose' of M in the sense of Auslander, as remarked in "Linear Free Resolutions and Minimal Multiplicity", by D. Eisenbud and S. Goto; Journal of Algebra **88**, 89-133 (1984).

rings; in fact, the rings $R_{\mathfrak{p}} : \mathfrak{p} \in \mathfrak{C}_{m-1}$, shall be regular local rings.

In formulating Theorem 1, we have used the term 's-basic'. A submodule M' of M is called s-basic in M at $\mathfrak{p}$ if $\mu(M/M')_{\mathfrak{p}} \leq \mu(M_{\mathfrak{p}}) - s$. An element $x \in M$ is called *basic* at $\mathfrak{p}$ if Rx is 1-basic. In the proof of Theorem 1 we consider the set of prime ideals of R such that $\mu(M_{\mathfrak{p}}) > n$. This set is a variety in $\mathrm{Spec}(R)$: if $I_n(M)$ is the sum of colon ideals $(N : M)$ where N is generated by at most n elements, then $\mu(M_{\mathfrak{p}}) > n$ if and only if $\mathfrak{p} \supset I_n(M)$.

2 On the existence of basic elements

In this section we shall prove Theorem 1. The theorem asserts, under certain conditions, the existence of elements $y \in M$ that are basic at all $\mathfrak{p} \in \mathfrak{C}_m$. The result will allow us to pass from an m-torsionless R-module M to an m-torsionless R-module M/Ry. Theorem 1 is largely analogous to [5, Theorem A]; we replace their condition on the Krull dimension of $R/\mathfrak{p}$ by a different condition on $\mathfrak{p}$ and an additional condition on M. (Remark 1 following the proof of Theorem 1 indicates a special case of this result).

Theorem 1. *Let $n \geq 0$ be a natural number and M be an R-module such that for all prime ideals $\mathfrak{p}, \mathfrak{q} \in \mathfrak{C}_n$ we have that $\mathrm{codh}(R_{\mathfrak{p}}) > \mathrm{codh}(R_{\mathfrak{q}})$ implies $\mu(M_{\mathfrak{p}}) \geq \mu(M_{\mathfrak{q}})$. Then:*

1. *If $\mu(M_{\mathfrak{p}}) > n$ for every prime ideal $\mathfrak{p} \in \mathfrak{C}_n$, then there exists some $x \in M$ that is basic in M at all $\mathfrak{p} \in \mathfrak{C}_n$.*
2. *Let $x_1, \ldots, x_k \in M$, for $k \geq 1$ and define $M' := \Sigma_{i=1}^{k} Rx_i$. If M' is $\min(k, n+1-\mathrm{codh}(R_{\mathfrak{p}}))$-basic in M at all $\mathfrak{p} \in \mathfrak{C}_n$, then there exists $x' \in \Sigma_{i=2}^{k} Rx_i$ such that $x_1 + x'$ is basic in M at all $\mathfrak{p} \in \mathfrak{C}_n$.*

We largely follow the proof of [5, Theorem A], although we use the following lemma instead of [5, Lemma 1]:

Lemma. *Let $\mathfrak{a}$ be an ideal of R. Then there exist only finitely many prime ideals $\mathfrak{p} \supset \mathfrak{a}$ such that $\mathrm{codh}(R_{\mathfrak{p}}) = \mathrm{grad}(\mathfrak{a})$.*

Proof. The case $\mathfrak{a} = R$ is trivial. So suppose that $\mathfrak{a} \subsetneq R$ and let $x_1, \ldots, x_n$, where $n = \mathrm{grad}(\mathfrak{a})$, be an R-regular sequence contained in $\mathfrak{a}$. Then if $\mathfrak{p} \supset \mathfrak{a}$, $x_1, \ldots, x_n$ is also an $R_{\mathfrak{p}}$-regular sequence; thus when $\mathrm{codh}(R_{\mathfrak{p}}) = n$, we have:

$$\mathrm{codh}((R/Rx_1 + \cdots + Rx_n)_{\mathfrak{p}}) = \mathrm{codh}(R_{\mathfrak{p}}/(R_{\mathfrak{p}}x_1 + \cdots + R_{\mathfrak{p}}x_n)) = 0.$$

Thus $\mathfrak{p} \in \mathrm{Ass}\,(R/(Rx_1 + \cdots + Rx_n))$. But there are only finitely many primes in $\mathrm{Ass}\,(R/Rx_1 + \cdots + Rx_n)$. □

Proof of Theorem 1. (1) follows from (2) by taking $x_1, \ldots x_k$ to be members of a system of generators of M. We will prove (2) by induction on k. The case $k = 1$ is trivial. Suppose then that $k > 1$ and let $a_1, \ldots, a_{k-1}$ be given such that $M'' = R(x_1 + a_1 x_k) + \Sigma_{i=2}^{k-1} R(x_i + a_i x_k)$ is $\min(k-1, n+1-\mathrm{codh}(R_{\mathfrak{p}}))$-basic at all $\mathfrak{p} \in \mathfrak{C}_n$.

We begin by proving that there are only finitely many primes $\mathfrak{p} \in \mathfrak{C}_n$ at which M' is not $\min(k, n+2-\mathrm{codh}(R_{\mathfrak{p}}))$-basic, and hence only finitely many $\mathfrak{p} \in \mathfrak{C}_n$ such that

$$\mu(M/M')_{\mathfrak{p}} > \mu(M_{\mathfrak{p}}) - \min(k, n+2-\mathrm{codh}(R_{\mathfrak{p}})).$$

We note that it is enough to prove this for fixed $t := \mathrm{codh}(R_{\mathfrak{p}})$ and fixed $s := \mu(M_{\mathfrak{p}}) - \min(l, n+2-t)$, since s and t will take only finitely many values. For any $\mathfrak{q} \in \mathfrak{C}_n$ with $\mathrm{codh}(R_{\mathfrak{q}}) < t$, we obtain the following bound

$$\mu((M/M')_{\mathfrak{q}}) \leq \mu(M_{\mathfrak{q}}) - \min(k, n+1-\mathrm{codh}(R_{\mathfrak{q}})) \leq \mu(M_{\mathfrak{q}}) - \min(k, n+2-t) \leq s,$$

since for every $\mathfrak{p} \in \mathfrak{C}_n$ such that $\mathrm{codh}(R_{\mathfrak{p}}) = t$, we have $\mu(M_{\mathfrak{q}}) \leq \mu(M_{\mathfrak{p}})$ by hypothesis. Thus $\mathfrak{q}$ does not contain $I_s(M/M')$, whence $\mathrm{grad}(I_s(M/M')) \geq t$. In the case that $\mathrm{grad}(I_s(M/M')) > t$, there are no prime ideals $\mathfrak{p}$ with $\mathrm{codh}(R_{\mathfrak{p}}) = t$ that contain $I_s(M/M')$, and in the case that $\mathrm{grad}(I_s(M/M')) = t$ there are only finitely many such ideals, as follows from the previous lemma.

Now let E be the finite set of primes $\mathfrak{p} \in \mathfrak{C}_n$ at which M' is not $\min(k, n+2-\mathrm{codh}(R_{\mathfrak{p}}))$-basic in M. By [5, Lemma 3] there exist $a_1, \ldots, a_{k-1} \in R$ such that $M'' = R(x_1 + a_1 x_k) + \Sigma_{i=2}^{k-1} R(x_i + a_i x_k)$ is $\min(k-1, n+1-\mathrm{codh}(R_{\mathfrak{p}}))$-basic at all $\mathfrak{p} \in E$. Note that this latter condition also holds at all $\mathfrak{p} \in \mathfrak{C}_n \setminus E$.

□

Remark.

1. The statement and proof of Theorem 1 are valid more generally (for instance, as in [5, Theorem A]), provided that R is a commutative Noetherian ring, A a (not necessarily commutative) R-algebra that is finitely generated as an R-module, and M a finitely generated A-module. For given A-modules N the definitions of $\mu(N_{\mathfrak{p}})$ and $I_s(N)$ correspond to $\mu(A_{\mathfrak{p}}, N_{\mathfrak{p}})$ and $I_s(A, N)$ respectively, as in [5]. However, we may not relax the condition that R be Noetherian. Furthermore, Part 2 of Theorem 1 (which is analogous to [5, Theorem A(iib)]) can be generalised in the required manner by assuming (a, x_1) is basic in $A \oplus M$ at all $\mathfrak{p} \in \mathfrak{C}_n$ for every $a \in A$, thus obtaining basic elements of the form $x_1 + ax', x' \in \Sigma_{i=2}^{k} Ax_i$.
2. Several conclusions follow immediately from Theorem 1, each analogous to the collorarie s of [5, Theorem A]. We give the following example, corresponding to a theorem of Serre, that indicates how to formulate these conclusions along the lines of [5]: if P is a projective R-module with a well defined rank greater than $\max\{\mathrm{codh}(R_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathrm{Spec}(R)\}$, then P splits as a direct sum of rank 1 free modules. (For a proof, see [5, Corollary 1]; the hypotheses of Theorem 1 are satisfied since P has a well defined rank).

3 The reduction of ranks of m-torsionless modules

In this section we show that we may 'replace', in an appropriate sense, an m-torsionless R-module of rank greater than m by an m-torsionless R-module of rank m.

Theorem 2. *Let M be an m-torsionless R-module of rank $r \geq m$, and suppose that either $\mathrm{dh}(M) < \infty$ or $R_{\mathfrak{p}}$ is a regular local ring for all $\mathfrak{p} \in \mathfrak{C}_m$. Suppose further that $g : R^n \to M$ is an epimorphism, that $e_1, \ldots, e_n$ constitute a basis for R^n and $s := n - r + m + 1$. Then there exist elements $x_s, \ldots, x_n \in R^n$ satisfying the following conditions:*

1. *$e_1, \ldots, e_{s-1}, x_s, \ldots, x_n$ is a basis for R^n.*
2. *$g(x_s), \ldots, g(x_n)$ are linearly independent elements of M.*
3. *$M' := M/\Sigma_{i=s}^{n} Rg(x_i)$ is m-torsionless and of rank m.*
4. *The natural epimorphism $p : R^n \to F := R^n/\Sigma_{i=s}^{n} Rx_i$ gives an isomorphism between $\mathrm{Ker}(g)$ and the kernel of the induced epimorphism $g' : F \to M'$.*

Proof. We proceed by induction on r. If $r = m$, there is nothing to prove. If $r > m$, it is enough to show the existence of an element $x \in R^n$ satisfying the following properties:

1. $e_1, \ldots, e_{n-1}, x$ is a basis for R^n,
2. $g(x)$ is linearly independent,
3. $M'' := M/Rg(x)$ is m-torsionless and of rank $r-1$,

4. the canonical projection $p' := R^n \to R^n/Rx$ gives an isomorphism between $\mathrm{Ker}(g)$ and the kernel of the induced epimorphism $g'' := R^n/Rx \to M''$.

The inequality $r > m$ implies $\mu(M_{\mathfrak{p}}) > m$ for all prime ideals $\mathfrak{p}$ in R. Our hypotheses on M and R imply that $\mathrm{dh}(M_{\mathfrak{p}}) < \infty$ for all $\mathfrak{p} \in \mathfrak{C}_m$ (in fact, $\mathrm{dh}(M_{\mathfrak{p}}) = 0$), since if $\mathfrak{p} \in \mathfrak{C}_m$, one obtains by the characterization (b_m) of m-torsionless R-modules in the introduction that $\mathrm{codh}(M_{\mathfrak{p}}) = \mathrm{codh}(R_{\mathfrak{p}})$. Hence $M_{\mathfrak{p}}$ is free for every $\mathfrak{p} \in \mathfrak{C}_m$. As M has rank r, we have that $\mu(M_{\mathfrak{p}}) = \mu(M_{\mathfrak{q}}) = r$ for all $\mathfrak{p}, \mathfrak{q} \in \mathfrak{C}_m$. The elements $g(e_1), \ldots g(e_n)$ generate M, and so by Theorem 1, (2), there exist $a_1, \ldots, a_{n-1} \in R$ such that if $x := e_n + \Sigma_{i=1}^{n-1} a_i e_i$, then $g(x)$ is not in $\mathfrak{p}M_{\mathfrak{p}}$, at all $\mathfrak{p} \in \mathfrak{C}_m$.

It is clear that for this choice of x, condition (1') is satisfied. Since $g(x)$ can be included as part of a basis of the free $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ for any $\mathfrak{p} \in \mathfrak{C}_m$ (and in particular, for any $\mathfrak{p} \in$Ass R), we see that $M''_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for these primes $\mathfrak{p}$ and that $g(x)$ is a linearly independent element of M. Thus (2') follows.

If $\mathfrak{p} \notin \mathfrak{C}_m$ (i.e., $\mathrm{codh}(R_{\mathfrak{p}}) > m$), it follows from the exact sequence

$$0 \longrightarrow R_{\mathfrak{p}} g(x) \longrightarrow M_{\mathfrak{p}} \longrightarrow M''_{\mathfrak{p}} \longrightarrow 0$$

that $\mathrm{codh}(R_{\mathfrak{p}} g(x)) = \mathrm{codh}(R_{\mathfrak{p}}) > m$, $\mathrm{codh}(M_{\mathfrak{p}}) \geq m$ and $\mathrm{codh}(M''_{\mathfrak{p}}) \geq m$. Altogether, this gives $\mathrm{codh}(M''_{\mathfrak{p}}) \geq \min(m, \mathrm{codh}(R_{\mathfrak{p}}))$ for all primes $\mathfrak{p}$ in R. Since either M'' has finite homological dimension or R is m-Gorenstein, it follows that M'' is m-torsionless. From (2') it follows that $\mathrm{rank}(M'') = r - 1$.

It only remains to verify (4'). It is elementary to show that p' maps the kernel of g onto the kernel of g''. But $\mathrm{rank}(\mathrm{Ker}(g)) = \mathrm{rank}(\mathrm{Ker}(g''))$, whence $\mathrm{Ker}(p' \mid_{Ker(g)})$ is a torsion-free R-module of rank 0, and thus equal to the zero module.

□

Remark. We may relax the hypotheses of Theorem 2 while ensuring that Theorem 1 is still applicable to get the following statement:

Suppose M is an R-module such that $M_{\mathfrak{p}}$ is free for $\mathfrak{p} \in \mathfrak{C}_m$ and such that $\mu(M_{\mathfrak{p}}) \geq \mu(M_{\mathfrak{q}})$ whenever $\mathrm{codh}(R_{\mathfrak{p}}) > \mathrm{codh}(R_{\mathfrak{q}})$ for $\mathfrak{p}, \mathfrak{q} \in \mathfrak{C}_m$. Further, let

$$r := \min\{\mu(M_{\mathfrak{q}}) \mid \mathfrak{q} \in \text{Ass } R\} \geq m,$$

and $g, e_1, \ldots, e_n, s$ be as in Theorem 2. Then there exist elements $x_s, \ldots, x_n \in R^n$ satisfying conditions (1), (2) and (4) of Theorem 2 and such that the following condition (in place of (3)) holds: $(M/(\Sigma_{i=s}^n Rg(x_i)))_{\mathfrak{p}}$ is free for $\mathfrak{p} \in \mathfrak{C}_m$ and there is $\mathfrak{q} \in$Ass R with

$$\langle M/(\Sigma_{i=s}^n Rg(x_i))\rangle_{\mathfrak{q}} = m.$$

(The proof is similar to that of Theorem 2 and proceeds by induction on r, in which the conclusion of Theorem 1 serves as the inductive step. One requires only a slightly more subtle argument for (4): an R-homomorphism $f : N \to N'$, where N is torsionfree, is injective if and only if for all $\mathfrak{q} \in$Ass$R$, $f \otimes_R R_{\mathfrak{q}}$ is injective.)

The following corollary shows that under the slightly weaker hypotheses of Theorem 2 on M (respectively R), the projective modules in an exact sequence:

$$0 \longrightarrow M \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_1$$

can be chosen in the following prescribed manner:

Corollary 1. *Let M be an m-torsionless R-module of well defined rank r, where $m \geq 2$. Let either* $\mathrm{dh}(M) < \infty$ *or $R_{\mathfrak{p}}$ be a regular local ring for all $\mathfrak{p} \in \mathfrak{C}_{m-1}$. Then there exists an exact sequence*

$$0 \longrightarrow M \xrightarrow{f} R^{r+m-1} \longrightarrow R^{2m-3} \longrightarrow R^{2m-5} \longrightarrow \cdots \longrightarrow R^3 \longrightarrow R^1.$$

In particular, M admits an embedding $f : M \to R^{r+m-1}$ such that $\mathrm{Coker}(f)$ *is $(m-1)$-torsionless and M is the $(m-1)$-th syzygy module of an ideal of R. When $m > 2$ this ideal is generated by three elements, and when $m = 2$ it is generated by $r+1$ elements.*

Proof. Projective modules in the above exact sequence that satisfy property (s_m) and are m-torsionless can without loss of generality be assumed to be free, so that there exists an exact sequence

$$0 \longrightarrow M \longrightarrow F \longrightarrow N \longrightarrow 0$$

where F is free and N is $(m-1)$-torsionless. N has a well defined rank and is of finite homological dimension if M is. Applying Theorem 2 to the epimorphism $F \to N$ gives in the case that $\mathrm{rank}(F) \geq r+m-1$ an exact sequence

$$0 \to M \to F' \to N' \to 0,$$

where N' is $(m-1)$-torsionless and of rank $m-1$, whence F' is isomorphic to R^{r+m-1}. If $\mathrm{rank}(F) < r+m-1$ one obtains this exact sequence by adding to F a free direct summand of the appropriate rank.

□

The observation just made about M yields for N' an exact sequence

$$0 \to N' \to R^{2m-3} \to N'' \to 0,$$

such that N'' is $(m-2)$-torsionless, etc.

A theorem of Bourbaki [2, pg. 76, Theorème 6] is obtained as a special case of Theorem 2. Bourbaki proves the special case $m = 1$ in the following corollary (under the additional assumption that R does not contain any zero-divisors).

Corollary 2. *Let the localizations $R_{\mathfrak{p}}$ be regular local rings for $\mathfrak{p} \in \mathfrak{C}_m$ and let M be an m-torsionless R-module having a well-defined rank $\geq m$. Then there exists a free submodule F of M, such that M/F is m-torsionless and of rank m.*

4 On the extension of certain projective resolutions to resolutions of ideals generated by three elements

We are now prepared to give an easy proof of the following result that was highlighted in the introduction:

Theorem 3. *Let M be an m-torsionless R-module having a well-defined rank, and let*

$$0 \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_{m+1} \xrightarrow{f_{m+1}} F_m \xrightarrow{g} M \longrightarrow 0$$

be a projective resolution of M, with $n > m \geq 1$, F_m a free R-module and $r :=$ $\mathrm{rank}(\mathrm{Im}(f_{m+1}))$. *Then there exist homomorphisms $c : F_m \to R^{r+m}, f_m : R^{r+m} \to R^{2m-1}, f_j : R^{2j+1} \to R^{2j-1}, j = 1, \ldots, m-1$, such that if $f'_{m+1} := c \circ f_{m+1}$, the sequence:*

$$0 \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_{m+2} \xrightarrow{f_{m+2}} F_{m+1} \xrightarrow{f'_{m+1}} R^{r+m}$$

$$\xrightarrow{f_m} R^{2m-1} \xrightarrow{f_{m-1}} R^{2m-3} \longrightarrow \cdots \longrightarrow R^3 \xrightarrow{f_1} R$$

is exact.

Proof. If $u := rank(F_m) \leq r + m$, we may choose c to be the embedding $F_m \to F_m \oplus R^{r+m-u}$. Then

$$0 \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_{m+2} \xrightarrow{f_{m+2}} F_{m+1} \xrightarrow{f'_{m+1}} R^{r+m}$$

is exact, $\mathrm{Coker}(f'_{m+1})$ is m-torsionless and of rank m.

If $u > r + m$, then by Theorem 2 there exist basis elements $x_s, \ldots, x_u, s := r + m + 1$ of F_m such that $M' := M/(Rg(x_s) + \cdots + Rg(x_u))$ is m-torsionless and of rank m. Further, the natural epimorphism $c : F_m \to F_m/(Rx_s + Rx_u) \cong R^{r+m}$ induces an exact sequence: $0 \to \mathrm{Im}(f_{m+1}) \cong \mathrm{Coker}(f_{m+2}) \to R^{r+m} \to M' \to 0$. Thus f'_{m+1} has the required propery in this case as well. Finally, Corollary 1 to Theorem 2 gives the homomorphisms $f_i, i = 1, \ldots, m$.

$\square$

Remark. 1. When the localizations $R_{\mathfrak{p}}$ are regular local rings, Theorem 3 applies to infinite resolutions also. (See Theorem 2 and Corollary 1.)

2. It would suffice to prove Theorem 3 in the case that the modules $F_{n-1}, \ldots, F_m$ are free and F_n is projective and of well-defined rank, since M has such a resolution. Conversely, if M has such a resolution, then M automatically has a well-defined rank.

3. If all the F_i are free modules, then M is m-torsionless if and only if $\mathrm{grade}(I(f_j)) \geq j$ for all $j = m + 1, \ldots, n$. (The following definition of $I(f_j)$ is found in [3]: $I(f_j)$ is the $(\mathrm{rank}(\mathrm{Coker}(f_j)))$-th Fitting ideal of $\mathrm{Coker}(f_j)$). This follows immediately from the characterization of m-torsionless R-modules of finite homological dimension (b_m) in the introduction.

4. In Theorem 2 (and its corollaries) m is always a lower bound for the rank of the constructed m-torsionless R-modules. We conjecture that this is not a consequence of our method of proof, but is in fact because there are no non-projective m-torsionless R-modules of finite homological dimension and of rank $< m$. This is always true if either $m \leq 2$ or $\mathrm{dh}(M) \leq 1$. To see this it is enough to consider the case when R is local. The case $m = 1$ is trivial. A 1-torsionless, or even more generally, a 2-torsionless R-module of rank 1 is isomorphic to an ideal $\mathfrak{a}$ of R, which under the given hypotheses has a finite free resolution and has $\mathrm{grad}(\mathfrak{a}) \geq 1$. But then $\mathfrak{a}$ is isomorphic to an ideal $\mathfrak{a}'$ with $\mathrm{grade}(\mathfrak{a}') \geq 2$ ([8, Corollary 5.6] or [3, Corollary 5.2]). The ideal $\mathfrak{a}'$ is not 2-torsionless, unless $\mathfrak{a}' = R$. When $\mathrm{dh}(M) \leq 1$, our claim follows from known estimates on the degree of determinantal ideals: the degree of a $(\mathrm{rank}(M))$-th Fitting ideal of M is, for M free, less than or equal to $\mathrm{rank}(M) + 1$ (see also [10, Theorem 2]).

5. [3, Theorem 8.1] derives a particular case of Theorem 3, where $n = 3$, $m = 2$, $\mathrm{rank}(F_2) = \mathrm{rank}(F_1) + 2$. The proof given there uses the structure theory of free resolutions developed in [3].

The following corollary says that every natural number that is the homological dimension of an m-torsionless R-module is also the homological dimension of a $(2m + 1)$-generated m-torsionless R-module of rank m. The special case $m = 1$ was proved using entirely different methods by Burch [4] and Kohn [7].

Corollary. *Let $s, m \geq 1$ be natural numbers such that there exists an R-module N with $\mathrm{dh}(N) = s + m$. Then there is an m-torsionless R-module M with $\mathrm{dh}(M) = s$, having rank m, and generated by at most $2m + 1$ elements.*

Proof. Let $n := s + m$. Since $\mathrm{dh}(N) = n$, there exists an R-regular sequence $x_1, \ldots, x_n$. When $s = 1$, we let $M = R^n / R(x_1, \ldots x_n)$. If $s > 1$, consider the following section of the Koszul complex associated to $(x_1, \ldots, x_n)$:

$$0 \longrightarrow R \xrightarrow{f_n} R^n \xrightarrow{f_{n-1}} \wedge^2 R^n \longrightarrow \cdots \longrightarrow \wedge^{s-2} R^n \xrightarrow{f_{m+2}} \wedge^{s-1} R^n.$$

$\mathrm{Coker}(f_{m+2})$ is $(m+1)$-torsionless and has a well-defined rank; let $r := \mathrm{rank}(\mathrm{Im}(f_{m+2}))$. By Theorem 3, there are homomorphisms f'_{m+2}, f_{m+1} such that the complex

$$0 \longrightarrow R \xrightarrow{f_n} R^n \longrightarrow \cdots \longrightarrow \wedge^{s-3} R^n \xrightarrow{f_{m+3}} \wedge^{s-2} R^n$$

$$\xrightarrow{f'_{m+2}} R^{r+m+1} \xrightarrow{f_{m+1}} R^{2m+1}$$

is exact and $\mathrm{Coker}(f_{m+1})$ is m-torsionless. It is clear that the claims pertaining to the rank and the number of generators of M hold. When $s = 2$, we have $f'_{m+2} = f_n$ (indeed, $\mathrm{rank}(\mathrm{Coker}(f_n)) = m + 1$) and when $s > 2$, f_n is independent of the construction of f'_{m+2}. Since f_n does not split, we have $\mathrm{dh}(M) = s$.

If the conjecture made in Remark 4 to Theorem 3 holds, then there is an m-torsionless R-module of finite homological dimension (in fact of homological dimension ≤ 1), generated by less than $2m + 1$ elements. [5] (It is sufficient, again, to consider only local rings R; then M has a well-defined rank whenever $\mathrm{dh}(M) < \infty$.)

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[5] *Addendum & Correction.* D. Eisenbud has informed us that the conjecture stated in Remark 4 to Theorem 3 was first made by P. Hackman in a currently unpublished article "Exterior Powers and Homology". The referee requested that we make a reference to the article "Tout ideal premier d'un anneau noethérien est associé à un ideal engendré par trois éléments" by T. Gulliksen (*C. R. Acad. Sci. Paris Ser. A* **271** (1970), 1207-1208). The main result of the paper stated in its title follows immediately from the corollary to Theorem 3.