

Generating functions for the Bernstein polynomials: A unified approach to deriving identities for the Bernstein basis functions

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Abstract

The main aim of this paper is to provide a unified approach to deriving identities for the Bernstein polynomials using a novel generating function. We derive various functional equations and differential equations using this generating function. Using these equations, we give new proofs both for a recursive definition of the Bernstein basis functions and for derivatives of the n th degree Bernstein polynomials. We also find some new identities and properties for the Bernstein basis functions. Furthermore, we discuss analytic representations for the generalized Bernstein polynomials through the binomial or Newton distribution and Poisson distribution with mean and variance. Using this novel generating function, we also derive an identity which represents a pointwise orthogonality relation for the Bernstein basis functions. Finally, by using the mean and the variance, we generalize Szasz-Mirakjan type basis functions.

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1. INTRODUCTION AND MAIN DEFINITION

The Bernstein polynomials have many applications in approximations of functions, in statistics, in numerical analysis, in p -adic analysis and in the solution of differential equations. It is also well-known that in Computer Aided Geometric Design polynomials are often expressed in terms of the Bernstein basis functions.

Many of the known identities for the Bernstein basis functions are currently derived in an *ad hoc* fashion, using either the binomial theorem, the binomial distribution, tricky algebraic manipulations or blossoming. The main purpose of this work is to construct novel generating functions for the Bernstein polynomials. Using these novel generating functions, we develop a unified approach both to standard and to new identities for the Bernstein polynomials.

The following definition gives us generating functions for the Bernstein basis functions:

Definition 1. Let a and b be nonnegative real parameters with $a \neq b$. Let m a be positive integer and let $x \in [a, b]$. Then the Bernstein basis functions $\mathbb{Y}_k^n(x; a, b, m)$ are defined by means of the following generating function:

$$\begin{aligned} f_{\mathbb{Y},k}(x, t; a, b, m) &= \sum_{j=0}^{\infty} \sum_{l=0}^k \binom{j+m-1}{j} (-1)^{k-l} \frac{t^k x^l a^{j+k-l} b^{-m-j} e^{(b-x)t}}{l!(k-l)!} \\ &= \sum_{n=0}^{\infty} \mathbb{Y}_k^n(x; a, b, m) \frac{t^n}{n!}, \end{aligned} \quad (1.1)$$

where $t \in \mathbb{C}$ and $0^j = \begin{cases} 0 & \text{if } j \neq 0, \\ 1 & \text{if } j = 0. \end{cases}$

The remainder of this study is organized as follows:

Section 2: We find many *functional equations* and *differential equations* of this novel generating function. Using these equations, many properties of the Bernstein basis functions can be determined. For instance, we give new proofs of the recursive definition of the Bernstein basis functions as well as a novel derivation for the two term formula for the derivatives of the n th degree Bernstein basis functions. We also prove many other properties of the Bernstein basis functions via functional equations.

Jetter and Stöckler [9] proved an identity for multivariate Bernstein polynomials on a simplex, which is considered a pointwise orthogonality relation. The integral version of this identity provides a new representation for the polynomial basis dual to the Bernstein basis. An identity for the reproducing kernel is used to define quasi-interpolants of arbitrary order. As an application of the identity of Jetter and Stöckler, Abel and Li [1] gave Proposition 1, in Section 3. Their method is based on generating functions, which reveals the general structure of the identity. As an applications of Proposition 1 they derive generating functions for the Baskakov basis functions and the Szasz-Mirakjan basis functions. Using Eq-(2.6) in Section 2, they exhibit a special case of the identity of Jetter and Stöckler for the Bernstein basis functions. In Section 3; we give relations between the Bernstein basis functions, the binomial distribution and the Poisson distribution. Using the Poisson distribution, we give generating functions for the Szasz-Mirakjan type basis functions. By using Abel and Li's [1] method, and applying our generating functions to Proposition 1, we derive identities which give pointwise orthogonality relations for the Bernstein polynomials and the Szasz-Mirakjan type basis functions.

2. UNIFIED APPROACH TO DERIVING NEW PROOFS OF THE IDENTITIES AND PROPERTIES FOR THE BERNSTEIN POLYNOMIALS

The Bernstein polynomials and related polynomials have been studied and defined in many different ways, for examples by q -series, complex functions, p -adic Volkenborn integrals and many algorithms.

In this section, we provide fundamental properties of the Bernstein basis functions and their generating functions. We introduce some functional equations and differential equations of the novel generating functions for the Bernstein basis functions. We also give new proofs

of some well known properties of the Bernstein basis functions via functional equations and differential equations.

2.1. Generating Functions. We now modify (1.1) as follows:

By the negative binomial theorem, we have

$$\frac{1}{b^m(1-\frac{a}{b})^m} = \frac{1}{b^m} \sum_{j=0}^{\infty} \binom{j+m-1}{j} a^j b^{-m-j}. \quad (2.1)$$

Substituting (2.1) into (1.1), we get

$$\begin{aligned} f_{\mathbb{Y},k}(x,t;a,b,m) &= \frac{t^k e^{(b-x)t}}{(b-a)^m k!} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} x^l a^{k-l} \\ &= \sum_{n=0}^{\infty} \mathbb{Y}_k^n(x;a,b,m) \frac{t^n}{n!}. \end{aligned}$$

Thus we obtain the following novel generating function, which is a modification of (1.1):

$$\begin{aligned} f_{\mathbb{Y},k}(x,t;a,b,m) &= \frac{t^k (x-a)^k e^{(b-x)t}}{(b-a)^m k!} \\ &= \sum_{n=0}^{\infty} \mathbb{Y}_k^n(x;a,b,m) \frac{t^n}{n!}. \end{aligned} \quad (2.2)$$

Remark 1. If we set $a = 0$ and $b = 1$ in (2.2), we obtain a result given by Simsek and Acikgoz [13] and Acikgoz and Arici [2]:

$$\frac{(xt)^k}{k!} e^{(1-x)t} = \sum_{n=0}^{\infty} B_k^n(x) \frac{t^n}{n!},$$

so that, obviously;

$$\mathbb{Y}_k^n(x;0,1,m) = B_k^n(x),$$

where $B_k^n(x)$ denote the Bernstein polynomials.

By using the Taylor series for $e^{(b-x)t}$ in (2.2), we get

$$\frac{(x-a)^k}{(b-a)^m k!} \sum_{n=0}^{\infty} (b-x)^n \frac{t^{n+k}}{n!} = \sum_{n=0}^{\infty} \mathbb{Y}_k^n(x;a,b,m) \frac{t^n}{n!}.$$

Comparing the coefficients of t^k on the both sides of the above equation, we arrive at the following theorem:

Theorem 1. Let a and b be nonnegative real parameters with $a \neq b$. Let m be a positive integer and let $x \in [a, b]$. Let k and n be non-negative integers with $n \geq k$. Then

$$\mathbb{Y}_k^n(x;a,b,m) = \binom{n}{k} \frac{(x-a)^k (b-x)^{n-k}}{(b-a)^m}, \quad (2.3)$$

where $k = 0, 1, \dots, n$, and $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Remark 2. For $m = n$, the Bernstein basis functions of degree n are defined by (2.3).

Remark 3. In the special case when $m = n$, Theorem 1 immediately yields the corresponding well known results concerning the Bernstein basis functions $B_k^n(x)$ that appears for example in Goldman [5, p. 384, Eq.(24.6)] and cf. [3]:

$$\mathbb{Y}_k^n(x; a, b, n) = B_k^n(x; a, b) = \binom{n}{k} \frac{(x-a)^k (b-x)^{n-k}}{(b-a)^n},$$

where $k = 0, 1, \dots, n$ and $x \in [a, b]$. One can easily see that

$$B_k^n(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad (2.4)$$

where $k = 0, 1, \dots, n$ and $x \in [0, 1]$ cf. [1]-[13]. In [5], Goldman gives many properties of the Bernstein polynomials $B_k^n(x, a, b)$. The functions $B_0^n(x, a, b), \dots, B_n^n(x, a, b)$ are called the Bernstein basis functions. Goldman [5], in Chapter 26, shows that the Bernstein basis functions form a basis for the polynomials of degree n . The **Bezier curve** $B(t)$ with control points P_0, \dots, P_n is defined as follows:

$$B(t) = \sum_{k=0}^n P_k B_k^n(x, a, b) \text{ cf. [5].}$$

Remark 4. By using (2.3), we have

$$\sum_{n=0}^{\infty} \mathbb{Y}_k^n(x; a, b, m) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \binom{n}{k} \frac{(x-a)^k (b-x)^{n-k} t^n}{(b-a)^m n!}.$$

From this equation, we obtain

$$\sum_{n=0}^{\infty} \mathbb{Y}_k^n(x; a, b, m) \frac{t^n}{n!} = \frac{(x-a)^k t^k}{k!(b-a)^m} \sum_{n=k}^{\infty} (b-x)^{n-k} \frac{t^{n-k}}{(n-k)!}.$$

The series on the right hand side is the Taylor series for $e^{(b-x)t}$; thus we arrive at (2.2).

Substituting $m = n$ in (2.3), we now give another well-known generating function for the Bernstein basis functions:

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^n \mathbb{Y}_k^n(x; a, b, n) t^k \right) \frac{z^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k} t^k \left(\frac{x-a}{b-a} \right)^k \left(\frac{b-x}{b-a} \right)^{n-k} \right) \frac{z^n}{n!}.$$

By using the Cauchy product in the above equation, we have

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^n \mathbb{Y}_k^n(x; a, b, n) t^k \right) \frac{z^n}{n!} = \sum_{n=0}^{\infty} \left(t \frac{x-a}{b-a} \right) \frac{z^n}{n!} \sum_{n=0}^{\infty} \left(\frac{b-x}{b-a} \right) \frac{z^n}{n!}.$$

From this equation, we find that

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^n \mathbb{Y}_k^n(x; a, b, n) t^k \right) \frac{z^n}{n!} = e^{z \left(\frac{b-x}{b-a} + t \frac{x-a}{b-a} \right)}.$$

After some elementary calculations in the above relation, we arrive at the following *generating function* for the Bernstein basis functions:

$$\sum_{k=0}^n \mathbb{Y}_k^n(x; a, b, n) t^k = \left(\frac{b-x}{b-a} + t \frac{x-a}{b-a} \right)^n. \quad (2.5)$$

Remark 5. If we set $a = 0$, $b = 1$ and $m = n$ in (2.5), then we have

$$\sum_{k=0}^n B_k^n(x) t^k = ((1-x) + tx)^n. \quad (2.6)$$

This generating functions is given by Goldman [7]-[6, Chapter 5, pages 299-306]. Goldman [7]-[6, Chapter 5, pages 299-306] also constructs the following generating functions the univariate and bivariate Bernstein basis functions:

$$\begin{aligned} \sum_{k=0}^n B_k^n(x) e^{ky} &= ((1-x) + te^y)^n, \\ \sum_{i+j+k=n} B_{i,j,k}^n(s, t) x^i y^j &= ((1-s-t) + sx + ty)^n, \end{aligned}$$

where

$$B_{i,j,k}^n(s, t) = \binom{n}{ijk} s^i t^j (1-s-t)^k \quad \text{and} \quad \binom{n}{ijk} = \frac{n!}{i!j!k!}$$

and

$$\sum_{i+j+k=n} B_{i,j,k}^n(s, t) e^{ix} e^{jy} = ((1-s-t) + se^x + te^y)^n.$$

Below are some well-known properties of the Bernstein basis functions:

Non-negative property:

$$\mathbb{Y}_k^n(x; a, b, m) \geq 0, \quad \text{for } 0 \leq a \leq x \leq b. \quad (2.7)$$

Symmetry property:

$$\mathbb{Y}_k^n(x; a, b, m) = \mathbb{Y}_{n-k}^n(b+a-x; a, b, m). \quad (2.8)$$

Corner values:

$$\mathbb{Y}_k^n(a; a, b, m) = \begin{cases} 0 & \text{if } k \neq 0, \\ 1 & \text{if } k = 0, \end{cases} \quad (2.9)$$

and

$$\mathbb{Y}_k^n(b; a, b, m) = \begin{cases} 0 & \text{if } k \neq n, \\ 1 & \text{if } k = n. \end{cases} \quad (2.10)$$

Alternating sum:

Substituting $m = n$ in (2.3), we get

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^n (-1)^k \mathbb{Y}_k^n(x; a, b, n) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \frac{\left(\frac{a-x}{b-a}\right)^k \left(\frac{b-x}{b-a}\right)^{n-k}}{k!(n-k)!} \right) t^n.$$

By using the Cauchy product in the above equation, we have

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^n (-1)^k \mathbb{Y}_k^n(x; a, b, n) \right) \frac{t^n}{n!} = e^{\left(\frac{a+b-2x}{b-a}\right)t}.$$

From this relation, we arrive at the following formula for the alternating sum.

$$\sum_{k=0}^n (-1)^k \mathbb{Y}_k^n(x; a, b, n) = \left(\frac{a+b-2x}{b-a} \right)^n. \quad (2.11)$$

Remark 6. *If we set $a = 0$, $b = 1$ and $m = n$, then Eq-(2.7)-Eq-(2.11) reduce to Goldman's results [7]-[6, Chapter 5, pages 299-306]. In [7] and [6, Chapter 5, pages 299-306], Goldman also gives many identities and properties for the univariate and bivariate Bernstein basis functions, for example boundary values, maximum values, partitions of unity, representation of monomials, representation in terms of monomials, conversion to monomial form, linear independence, Descartes' law of sign, discrete convolution, unimodality, subdivision, directional derivatives, integrals, Marsden identities, De Boor-Fix formulas, and the other properties.*

A Bernstein polynomial $\mathcal{P}(x, a, b, m)$ is a polynomial represented in the Bernstein basis functions:

$$\mathcal{P}(x, a, b, m) = \sum_{k=0}^m c_k^n \mathbb{Y}_k^n(x; a, b, m). \quad (2.12)$$

Remark 7. *If we set $a = 0$, $b = 1$ and $m = n$ (2.12), then we have*

$$P(x) = \sum_{k=0}^n c_k^n B_k^n(x)$$

cf. [4].

By using (2.2), we obtain the following functional equation:

$$f_{\mathbb{Y},k_1}(x, t; a, b, m_1) f_{\mathbb{Y},k_2}(x, t; a, b, m_2) = \frac{\binom{k_1 + k_2}{k_1}}{2^{k_1+k_2}} f_{\mathbb{Y},k_1+k_2}(x, 2t; a, b, m_1 + m_2),$$

where

$$\binom{k_1 + k_2}{k_1} = \binom{k_1 + k_2}{k_2} = \frac{(k_1 + k_2)!}{k_1! k_2!}.$$

By using the definition of the novel generating function $f_{\mathbb{Y},k}(x, t; a, b, m)$ in the preceding equation, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \mathbb{Y}_{k_1}^n(x; a, b, m_1) \frac{t^n}{n!} \sum_{n=0}^{\infty} \mathbb{Y}_{k_2}^n(x; a, b, m_2) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \mathbb{Y}_{k_1+k_2}^n(x; a, b, m_1 + m_2) \frac{2^{n-k_1-k_2} (k_1 + k_2)! t^n}{n! k_1! k_2!}. \end{aligned}$$

And using the Cauchy product in this equation, we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \left(\sum_{j=0}^n \binom{n}{j} \mathbb{Y}_{k_1}^j(x; a, b, m_1) \mathbb{Y}_{k_2}^{n-j}(x; a, b, m_2) \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \mathbb{Y}_{k_1+k_2}^n(x; a, b, m_1 + m_2) \frac{2^{n-k_1-k_2} (k_1 + k_2)! t^n}{n! k_1! k_2!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, we arrive at the following theorem:

Theorem 2. *Let m_1 and m_2 be integers. Then the following identity holds:*

$$\mathbb{Y}_{k_1+k_2}^n(x; a, b, m_1 + m_2) = \frac{2^{k_1+k_2-n} k_1! k_2!}{(k_1 + k_2)!} \sum_{j=0}^n \binom{n}{j} \mathbb{Y}_{k_1}^j(x; a, b, m_1) \mathbb{Y}_{k_2}^{n-j}(x; a, b, m_2).$$

Observe that if we set $a = 0$ and $b = 1$, then we have

$$B_{k_1+k_2}^n(x) = \frac{2^{k_1+k_2-n} k_1! k_2!}{(k_1 + k_2)!} \sum_{j=0}^n \binom{n}{j} B_{k_1}^j(x) B_{k_2}^{n-j}(x).$$

Note that many new identities can be found via functional equations for the novel generating functions of the Bernstein basis functions. We derive some functional equations and identities related to the generating functions and the Bernstein basis functions in the remainder of this section.

2.2. Subdivision property. The following functional equation of the novel generating functions is fundamental to driving the subdivision property for the Bernstein basis functions.

Let us define

$$f_{\mathbb{Y},j}(xy, t; a, b, n) = f_{\mathbb{Y},j} \left(x, t \left(\frac{y-a}{b-a} \right); a, b, n \right) e^{t \left(\frac{b-y}{b-a} \right)}. \quad (2.13)$$

From this generating function, we have the following theorem:

Theorem 3. *Let $a \leq yx \leq b$. Then the following identity holds:*

$$\mathbb{Y}_j^n(xy; a, b, n) = \sum_{k=j}^n \mathbb{Y}_j^k(x; a, b, k) \mathbb{Y}_k^n(y; a, b, n-k).$$

Proof. By equations (2.2) and (2.13), we obtain

$$\begin{aligned} & \sum_{n=j}^{\infty} \mathbb{Y}_j^n(xy; a, b, n) \frac{t^n}{n!} \\ &= \left(\sum_{n=0}^{\infty} \mathbb{Y}_j^n(x; a, b, n) \left(\frac{y-a}{b-a} \right)^n \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{\left(\frac{b-y}{b-a} \right)^n t^n}{n!} \right). \end{aligned}$$

Using the Cauchy product in this equation, we get

$$\sum_{n=j}^{\infty} \mathbb{Y}_j^n(xy; a, b, m) \frac{t^n}{n!} = \sum_{n=j}^{\infty} \left(\sum_{k=j}^n \mathbb{Y}_j^n(x; a, b, k) \frac{\left(\frac{y-a}{b-a}\right)^k \left(\frac{b-y}{b-a}\right)^{n-k}}{k! (n-k)!} \right) t^n.$$

Substituting (2.3) into the above equation then after some elementary manipulations, we arrive at the desired result. \square

Remark 8. *Substituting $a = 0$, $b = 1$ and $m = n$ into Theorem 3, we have*

$$B_j^n(xy) = \sum_{k=j}^n B_j^k(x) B_k^n(y). \quad (2.14)$$

The above identity is essentially the subdivision property for the Bernstein basis functions. This identity is a bit tricky to prove with algebraic manipulations.

Remark 9. *Goldman [7]-[6, Chapter 5, pages 299-306] proves equation (2.14) with algebraic manipulations. He also proves the following subdivision properties:*

$$B_j^n((1-y)x + y) = \sum_{k=0}^j B_{j-k}^{n-k}(x) B_k^n(y),$$

and

$$B_j^n((1-y)x + yz) = \sum_{k=0}^n \left(\sum_{p+q=j} B_p^{n-k}(x) B_q^k(z) \right) B_k^n(y)$$

for the others see cf. [7]-[6, Chapter 5, pages 299-306].

2.3. Differentiating the generating function. In this section we give higher order derivatives of the Bernstein basis functions by differentiating the generating function in (2.2) with respect to x . Using Leibnitz's formula for the l th derivative, with respect to x , of the product $f_{\mathbb{Y},k}(x, t; a, b, m)$ of two functions $g(t, x; a, b) = \frac{t^k (x-a)^k}{(b-a)^m k!}$ with $a \neq b$ and $h(t, x; b) = e^{(b-x)t}$, we obtain the following *higher order partial derivative equation*:

$$\frac{\partial^l f_{\mathbb{Y},k}(x, t; a, b, m)}{\partial x^l} = \sum_{j=0}^l \binom{l}{j} \left(\frac{\partial^j g(t, x; a, b)}{\partial x^j} \right) \left(\frac{\partial^{l-j} h(t, x; b)}{\partial x^{l-j}} \right).$$

From this equation, we arrive at the following theorem:

Theorem 4. *Let l be a non-negative integer. Then*

$$\frac{\partial^l f_{\mathbb{Y},k}(x, t; a, b, m)}{\partial x^l} = \sum_{j=0}^l \binom{l}{j} (-1)^{l-j} \frac{t^l}{(b-a)^j} f_{\mathbb{Y},k-j}(x, t; a, b, m-j).$$

By using Theorem 4, we obtain higher order derivatives of the Bernstein basis functions by the following theorem:

Theorem 5. Let a and b be nonnegative real parameters with $a \neq b$. Let m be a positive integer and let $x \in [a, b]$. Let k, l and n be nonnegative integers with $n \geq k$. Then

$$\frac{d^l \mathbb{Y}_k^n(x; a, b, m)}{dx^l} = \sum_{j=0}^l (-1)^{l-j} \binom{n}{n-l, l-j, j} \frac{l!}{(b-a)^j} \mathbb{Y}_{k-j}^{n-l}(x; a, b, m-j),$$

where

$$\binom{n}{x, y, z} = \frac{n!}{x!y!z!}, \text{ with } n = x + y + z.$$

Remark 10. Substituting $a = 0, b = 1$ and $m = n$ into Theorem 5, we have

$$\frac{d^l B_k^n(x)}{dx^l} = \sum_{j=0}^l (-1)^{l-j} \binom{n}{n-l, l-j, j} l! B_{k-j}^{n-l}(x),$$

or

$$\frac{d^l B_k^n(x)}{dx^l} = \frac{n!}{(n-l)!} \sum_{j=0}^l (-1)^{l-j} \binom{n}{j} B_{k-j}^{n-l}(x),$$

cf. ([7], [6, Chapter 5, pages 299-306]).

Substituting $l = 1$ into Theorem 5, we arrive at the following corollary:

Corollary 1. Let a and b be nonnegative real parameters with $a \neq b$. Let m be a positive integer and let $x \in [a, b]$. Let k and n be nonnegative integers with $n \geq k$. Then

$$\frac{d}{dx} \mathbb{Y}_k^n(x; a, b, m) = n \left(\frac{\mathbb{Y}_{k-1}^{n-1}(x; a, b, m-1) - \mathbb{Y}_k^{n-1}(x; a, b, m-1)}{b-a} \right).$$

Remark 11. By setting $m = n$ in Corollary 1, we arrive at the known result recorded by Goldman [5]:

$$\frac{d}{dx} B_k^n(x; a, b) = n \left(\frac{B_{k-1}^{n-1}(x; a, b) - B_k^{n-1}(x; a, b)}{b-a} \right).$$

Remark 12. One can also see the following special case of Theorem 1 when $a = 0$ and $b = 1$:

$$\frac{d}{dx} B_k^n(x) = n (B_{k-1}^{n-1}(x) - B_k^{n-1}(x))$$

cf. [1]-[13].

2.4. Recurrence Relation. In this section by using higher order derivatives of the novel generating function with respect to t , we derive a partial differential equation. Using this equation, we shall give a new proof of the recurrence relation for the Bernstein basis functions.

Differentiating Eq-(1.1) with respect to t , we prove a recurrence relation for the polynomials $\mathbb{Y}_k^n(x; a, b, m)$. This recurrence relation can also be obtained from Eq-(2.3). By using Leibnitz's formula for the v th derivative, with respect to t , of the product $f_{\mathbb{Y},k}(x, t; a, b, m)$

of two function $g(t, x; a, b) = \frac{t^k(x-a)^k}{(b-a)^m k!}$ with $a \neq b$ and $h(t, x; b) = e^{(b-x)t}$, we obtain another higher order partial differential equation as follows:

$$\frac{\partial^v f_{\mathbb{Y},k}(x, t; a, b, m)}{\partial t^v} = \sum_{j=0}^v \binom{v}{j} \left(\frac{\partial^j g(t, x; a, b)}{\partial t^j} \right) \left(\frac{\partial^{v-j} h(t, x; b)}{\partial t^{v-j}} \right).$$

From the above equation, we have the following theorem:

Theorem 6. *Let v be an integer number. Then*

$$\frac{\partial^v f_{\mathbb{Y},k}(x, t; a, b, m)}{\partial t^v} = \sum_{j=0}^v (b-a)^{v-j} \mathbb{Y}_j^v(x; a, b, v) f_{\mathbb{Y},k-j}(x, t; a, b, m-j),$$

where $f_{\mathbb{Y},k}(x, t; a, b, m)$ and $\mathbb{Y}_j^v(x; a, b, v)$ are defined in (2.2) and (2.3), respectively.

Using definition (2.2) and (2.3) in Theorem 6, we obtain a recurrence relation for the Bernstein basis functions by the following theorem:

Theorem 7. *Let a and b be nonnegative real parameters with $a \neq b$. Let m be a positive integer and let $x \in [a, b]$. Let k, v and n be nonnegative integers with $n \geq k$. Then*

$$\mathbb{Y}_k^n(x; a, b, m) = \sum_{j=0}^v (b-a)^{v-j} \mathbb{Y}_j^v(x; a, b, v) \mathbb{Y}_{k-j}^{n-v}(x; a, b, m-j).$$

Remark 13. *Substituting $a = 0$ and $b = 1$ into Theorem 7, we obtain the following result:*

$$B_k^n(x) = \sum_{j=0}^v B_j^v(x) B_{k-j}^{n-v}(x).$$

Substituting $v = 1$ into Theorem 7, we arrive at the following corollary:

Corollary 2. (Recurrence Relation) *Let a and b be nonnegative real parameters with $a \neq b$. Let m be a positive integer and let $x \in [a, b]$. Let k and n be nonnegative integers with $n \geq k$. Then*

$$\begin{aligned} \mathbb{Y}_k^n(x; a, b, m) &= \frac{x-a}{b-a} \mathbb{Y}_{k-1}^{n-1}(x; a, b, m-1) \\ &\quad + \frac{b-x}{b-a} \mathbb{Y}_k^{n-1}(x; a, b, m-1). \end{aligned} \tag{2.15}$$

Remark 14. *Differentiating equation (1.1) with respect to t , we also get*

$$\begin{aligned} &\frac{x-a}{b-a} f_{\mathbb{Y},k-1}(x, t; a, b, m-1) + \frac{b-x}{b-a} f_{\mathbb{Y},k}(x, t; a, b, m-1) \\ &= \sum_{n=1}^{\infty} \mathbb{Y}_k^n(x; a, b, m) \frac{t^{n-1}}{(n-1)!}. \end{aligned}$$

From this equation, one can also obtain Corollary 2.

Remark 15. *By setting $a = 0$ and $b = 1$ in (2.15), one obtains the following relation:*

$$B_k^n(x) = (1-x)B_k^{n-1}(x) + xB_{k-1}^{n-1}(x).$$

2.5. **Multiplication and division by powers of $(\frac{x-a}{b-a})^d$ and $(\frac{b-x}{b-a})^d$.** In [4], Buse and Goldman present much background material on computations with Bernstein polynomials. They provide formulas for multiplication and division of Bernstein polynomials by powers of x and $1-x$ and for degree elevation of Bernstein polynomials. Our method is similar to that of Buse and Goldman's [4]. In this section we find two functional equations. Using these equations, we also give new proofs of both the multiplication and division properties for the Bernstein polynomials.

By using the generating function in (1.1), we provide formulas for multiplying Bernstein polynomials by powers of $(\frac{x-a}{b-a})^d$ and $(\frac{b-x}{b-a})^d$ and for degree elevation of the Bernstein polynomials.

Using (2.2), we obtain the following **functional equation**:

$$\left(\frac{x-a}{b-a}\right)^d f_{\mathbb{Y},k}(x, t; a, b, n) = \frac{(k+d)!}{k!t^d} f_{\mathbb{Y},k}(x, t; a, b, n).$$

After elementary manipulations in this equation, we get

$$\left(\frac{x-a}{b-a}\right)^d \mathbb{Y}_k^n(x; a, b, n) = \frac{n!(k+d)!}{k!(n+d)!} \mathbb{Y}_{k+d}^{n+d}(x; a, b, n+d). \quad (2.16)$$

Substituting $d = 1$, we have

$$\left(\frac{x-a}{b-a}\right) \mathbb{Y}_k^n(x; a, b, n) = \frac{k+1}{n+1} \mathbb{Y}_{k+1}^{n+1}(x; a, b, n+1). \quad (2.17)$$

Remark 16. *Substituting $a = 0$ and $b = 1$ into (2.17), we have*

$$xB_k^n(x) = \frac{k+1}{n+1} B_{k+1}^{n+1}(x).$$

The above relation can also be proved by (2.4) cf. [4].

Similarly, using (2.3), we obtain

$$\left(\frac{b-x}{b-a}\right)^d \mathbb{Y}_k^n(x; a, b, n) = \frac{n!(n+d-k)!}{(n+d)!(n-k)!} \mathbb{Y}_k^{n+d}(x; a, b, n+d).$$

Substituting $d = 1$ into the above equation, we have

$$\left(\frac{b-x}{b-a}\right) \mathbb{Y}_k^n(x; a, b, n) = \frac{n+1-k}{n+1} \mathbb{Y}_k^{n+1}(x; a, b, n+1). \quad (2.18)$$

Consequently, by the same method as in [4], if we have (2.12), then

$$\left(\frac{x-a}{b-a}\right)^d \mathcal{P}(x, a, b) = \sum_{k=0}^n c_k^n \frac{n!(k+d)!}{k!(n+d)!} \mathbb{Y}_{k+d}^{n+d}(x; a, b, n+d), \quad (2.19)$$

and

$$\left(\frac{b-x}{b-a}\right)^d \mathcal{P}(x, a, b) = \sum_{k=0}^n c_k^n \frac{n!(n+d-k)!}{(n+d)!(n-k)!} \mathbb{Y}_k^{n+d}(x; a, b, n+d). \quad (2.20)$$

We now consider *division* properties. We assume that (2.12) holds and that we are given an integer $j > 0$. Since $(\frac{x-a}{b-a})^j$ divides $\mathbb{Y}_k^n(x; a, b, n)$ for all $k \geq j$, it follows that $(\frac{x-a}{b-a})^j$ divides $\mathcal{P}(x, a, b)$. Similarly, using (2.2), we obtain the following **functional equation**:

$$\frac{f_{\mathbb{Y},k}(x, t; a, b, n)}{(\frac{x-a}{b-a})^j} = \frac{(k-f)!t^j}{k!} f_{\mathbb{Y},k-j}(x, t; a, b, n-j).$$

For $k \geq j$, from the above equation, we have

$$\frac{\mathbb{Y}_k^n(x; a, b, n)}{(\frac{x-a}{b-a})^j} = \frac{n!(k-j)!}{k!(n-j)!} \mathbb{Y}_{k-j}^{n-j}(x; a, b, n-j).$$

By a calculation similar to the calculation in [4], for $j \leq n-k$, we have

$$\frac{\mathbb{Y}_k^n(x; a, b, n)}{(\frac{b-x}{b-a})^j} = \frac{n!(n-j-k)!}{(n-k)!(n-j)!} \mathbb{Y}_k^{n-j}(x; a, b, n-j).$$

Therefore

$$\frac{\mathcal{P}(x, a, b)}{(\frac{x-a}{b-a})^j} = \sum_{k=j}^n c_k^n \frac{n!(k-j)!}{k!(n-j)!} \mathbb{Y}_{k-j}^{n-j}(x; a, b, n-j), \quad (2.21)$$

and

$$\frac{\mathcal{P}(x, a, b)}{(\frac{b-x}{b-a})^j} = \sum_{k=0}^{n-j} c_k^n \frac{n!(n-j-k)!}{(n-k)!(n-j)!} \mathbb{Y}_k^{n-j}(x; a, b, n-j). \quad (2.22)$$

2.6. Degree elevation. According to Buse and Goldman [4], given a polynomial represented in the univariate Bernstein basis of degree n , degree elevation computes representations of the same polynomial in the univariate Bernstein bases of degree greater than n . Degree elevation allows us to add two or more Bernstein polynomials which are not represented in the same degree Bernstein basis functions.

Adding (2.17) and (2.18), we obtain the degree elevation formula for the Bernstein basis functions:

$$\mathbb{Y}_k^n(x; a, b, n) = \frac{k+1}{n+1} \mathbb{Y}_{k+1}^{n+1}(x; a, b, n+1) + \frac{n+1-k}{n+1} \mathbb{Y}_k^{n+1}(x; a, b, n+1).$$

Substituting $d = 1$ into (2.20), and adding these two equations gives the following degree elevation formula for the Bernstein polynomials:

$$\mathcal{P}(x, a, b) = \sum_{k=0}^n \left(\frac{k}{n+1} c_{k-1}^n + \frac{n+1-k}{(n+1)} c_k^n \right) \mathbb{Y}_k^{n+1}(x; a, b, n+1), \quad (2.23)$$

where

$$c_k^{n+1} = \frac{k}{n+1} c_{k-1}^n + \frac{n+1-k}{(n+1)} c_k^n.$$

Remark 17. If we set $a = 0$ and $b = 1$, then (2.23) reduces to Eq-(2.5) in [4, p. 853].

3. RELATION BETWEEN THE GENERATING FUNCTIONS $f_{\mathbb{Y},k}(x, t; a, b, m)$, POISSON DISTRIBUTION AND SZASZ-MIRAKJAN TYPE BASIS FUNCTIONS

The identity of Jetter and Stöckler represents a pointwise orthogonality relation for the multivariate Bernstein polynomials on a simplex. This identity give us a new representation for the dual basis which can be used to construct general quasi-interpolant operators cf. (See, for details, [9], [1]). As an application of the generating functions for the basis functions to the identity of Jetter and Stöckler, Abel and Li [1] proved Proposition 1, which is given in this section. Applying our generating functions to Proposition 1, we give pointwise orthogonality relations for the Bernstein polynomials and the Szasz-Mirakjan basis functions.

In this section, we give relations between the Bernstein basis functions, the binomial distribution and the Poisson distribution. First we we consider the generalized binomial or Newton distribution (probability function). Suppose that $0 \leq \frac{x-a}{b-a} \leq 1$ and $0 \leq \frac{b-x}{b-a} \leq 1$. Set

$$\mathbb{Y}_k^n(x; a, b, n) = \binom{n}{k} \left(\frac{x-a}{b-a} \right)^k \left(\frac{b-x}{b-a} \right)^{n-k}. \quad (3.1)$$

From the above definition, one can see that

$$\sum_{k=0}^n \mathbb{Y}_k^n(x; a, b, n) = 1.$$

Remark 18. If we set $a = 0$ and $b = 1$, then (3.1) reduces to

$$\mathbb{Y}_k^n(x; 0, 1, n) = \binom{n}{k} x^k (1-x)^{n-k}$$

which is the binomial or Newton distribution (probabilities) function. If $0 \leq x \leq 1$ is the probability of an event E , then $\mathbb{Y}_k^n(x; 0, 1, n)$ is the probability that E will occur exactly k times in n independent trials cf. [11].

Expected value or mean and variance of $\mathbb{Y}_k^n(x; a, b, n)$ are given by

$$\mu = \sum_{k=0}^n k \mathbb{Y}_k^n(x; a, b, n) = n \left(\frac{x-a}{b-a} \right),$$

and

$$\sigma^2 = \sum_{k=0}^n k^2 \mathbb{Y}_k^n(x; a, b, n) - \mu^2 = \frac{n(x-a)(b-x)}{(b-a)^2}.$$

If we let $n \rightarrow \infty$ in (3.1), then we arrive at the well-known Poisson distribution function:

$$\mathbb{Y}_k^n\left(\frac{b-a}{n}\mu + a; a, b, n\right) \rightarrow \frac{\mu^k e^{-\mu}}{k!}. \quad (3.2)$$

The following proposition is proved by Abel and Li [1, p. 300, Proposition 3]:

Proposition 1. Let the system $\{f_n(x)\}$ of functions be defined by the generating function

$$A_t(x) = \sum_{n=0}^{\infty} f_n(x) t^n.$$

If there exists a sequence $w_k = w_k(x)$ such that

$$\sum_{k=0}^{\infty} w_k \mathcal{D}^k A_t(x) \mathcal{D}^k A_z(x) = A_{tz}(x)$$

with $\mathcal{D} = \frac{d}{dx}$, then for $i, j = 0, 1, \dots$,

$$\sum_{k=0}^{\infty} w_k \mathcal{D}^k f_i(x) \mathcal{D}^k f_j(x) = \delta_{i,j} f_i(x).$$

As an application of Proposition 1, Abel and Li [1] use the generating function in Eq-(2.6) for the Bernstein basis functions. They also use generating functions for the Szasz-Mirakjan basis functions and Baskakov basis functions.

In this section, we apply our novel generating functions to Proposition 1, which give pointwise orthogonality relations for the Bernstein polynomials and the Szasz-Mirakjan type basis functions, respectively.

As applications of Proposition 1, we give the following examples:

Example 1. For given n and k , the Bernstein basis functions

$$f_i(x, n; a, b) = \mathbb{Y}_i^n(x; a, b, n) = \binom{n}{i} \left(\frac{x-a}{b-a} \right)^i \left(\frac{b-x}{b-a} \right)^{n-i}$$

are generated by the function in (2.2), that is

$$A_t(x) = \frac{t^k (x-a)^k e^{(b-x)t}}{(b-a)^n k!} = \sum_{i=0}^{\infty} \frac{f_i(x, n; a, b)}{i!} t^i.$$

It is easy to check that Proposition 1 holds with $w_k = w_k(x) = \mathbb{Y}_k^n(x; a, b, n)$.

Example 2. Using (3.2), for $i \geq 0$, we generalize the Szasz-Mirakjan type basis functions as follows

$$f_i(x, n; a, b) = \frac{\left(n \frac{x-a}{b-a} \right)^i e^{-n \frac{x-a}{b-a}}}{i!},$$

where a and b are nonnegative real parameters with $a \neq b$, n is a positive integer and $x \in [a, b]$. The functions $f_i(x, n; a, b)$ are generated by

$$A_t(x) = \exp \left((t-1)n \left(\frac{x-a}{b-a} \right) \right) = \sum_{i=0}^{\infty} f_i(x, n; a, b) t^i,$$

where $\exp(x) = e^x$. In this case, Proposition 1 holds with $w_k = w_k(x) = \frac{\left(\frac{x-a}{b-a} \right)^k}{n^k k!}$. Therefore, we have

$$\sum_{k=0}^{\infty} \frac{\left(\frac{x-a}{b-a} \right)^k}{n^k k!} \mathcal{D}^k f_i(x, n; a, b) \mathcal{D}^k f_j(x, n; a, b) = \delta_{i,j} f_i(x, n; a, b).$$

Remark 19. If $a = 0$ and $b = 1$ in Example 2, then we arrive at the Szasz-Mirakjan basis functions which are given in [1, p. 300, Example 2].

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