The equivalence relationship between Li-Yorke δ-chaos and distributional δ-chaos in a sequence *

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Abstract: In this paper, we discuss the relationship between Li-Yorke chaos and distributional chaos in a sequence. We point out the set of all distributional δ-scramble pairs in the sequence Q is a G_δ set, and prove that Li-Yorke δ-chaos is equivalent to distributional δ-chaos in a sequence, a uniformly chaotic set is a distributional scramble set in some sequence and a class of transitive system implies distributional chaos in a sequence.
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1 Introduction

Throughout this paper a *topological dynamical system* (TDS for short) is a pair (X, f), where X is a non-vacuous compact metric space with metric d and f is a continuous map for X to itself. If $Y \subset X$ is a closed invariant set (i.e. $f(Y) \subset Y$), then we call $(Y, f|_Y)$ is a *subsystem* of (X, f).

Let A be a subset of X, denote the *closure* of A by $\overline{A}$. For a given positive number δ, put $[A]_\delta = \{x \in X \mid \inf_{y\in A} d(x, y) < \delta\}$. Denote the *diagonal* of the product space $X \times X$ by $\Delta = \{(x, x) \in X \times X \mid x \in X\}$.

In [1], Li and Yorke first introduced the word "chaos" to describe the complexity of the orbits.

Definition 1. Let (X, f) be a TDS. A pair $(x, y) \in X \times X$ is called a *Li-Yorke scrambled pair*, if

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0, \quad \limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0.$$

A subset C of X is called a *Li-Yorke chaotic set*, if every $(x, y) \in C \times C \setminus \Delta$ is a Li-Yorke scrambled pair. The system (X, f) is called *Li-Yorke chaotic*, if there exists some uncountable Li-Yorke chaotic set.

For a give positive number δ, a pair $(x, y) \in X \times X$ is called a *Li-Yorke δ-scrambled pair*, if

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0, \quad \limsup_{n\to\infty} d(f^n(x), f^n(y)) > \delta.$$

A subset C of X is called a *Li-Yorke δ-chaotic set*, if every $(x, y) \in C \times C \setminus \Delta$ is a Li-Yorke δ-scrambled pair. The system (X, f) is called *Li-Yorke δ-chaotic*, if there exists some uncountable Li-Yorke δ-chaotic set.

In [2], Schweizer and Smital introduced a new kind of chaos, which is usually called distribution chaos. Later, the authors in [3] introduced the conception of distribution chaos in a sequence. See [4] and [5] for recent results.

Let (X, f) be a TDS and $Q = \{m_i\}_{i=1}^\infty$ be a strictly increasing sequence of positive integers. For $x, y \in X$, $t > 0$ and $n \geq 1$, put

$$\Phi^n_{(xy,Q)}(t) = \frac{1}{n}\#\{1 \leq i \leq n \mid d(f^{m_i}(x), f^{m_i}(y)) \leq t\},$$

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where $\#\{\cdot\}$ denote the cardinal number of a set. Let

$$\Phi_{(xy,Q)}(t) = \liminf_{n\to\infty} \Phi^n_{(xy,Q)}(t), \quad \Phi^\star_{(xy,Q)}(t) = \limsup_{n\to\infty} \Phi^n_{(xy,Q)}(t).$$

then $\Phi_{(xy,Q)}$ and $\Phi^\star_{(xy,Q)}$ are called the *lower and upper distribution function of* (x, y) *with respect to the sequence* Q. Clearly, for every $t > 0$, $\Phi_{(xy,Q)}(t) \le \Phi^\star_{(xy,Q)}(t)$.

Definition 2. Let (X, f) be a TDS and Q be a strictly increasing sequence of positive integers. A pair $(x, y) \in X \times X$ is called a *distributional scrambled pair in the sequence* Q, if

(1) for every $t > 0$, $\Phi^\star_{(xy,Q)}(t) = 1$;

(2) there exists some $s > 0$ such that $\Phi_{(xy,Q)}(s) = 0$.

A subset D of X is called a *distributional chaotic set in the sequence* Q, if every $(x, y) \in D \times D \setminus \Delta$ is a distributional scrambled pair in the sequence Q. The system (X, f) is called *distributional chaotic in a sequence*, if there exists some strictly increasing sequence of positive integers P such that there is an uncountable distributional chaotic set in the sequence P.

For a give positive number δ, a pair $(x, y) \in X \times X$ is called a *distributional δ-scrambled pair in the sequence* Q, if

(1) for every $t > 0$, $\Phi^\star_{(xy,Q)}(t) = 1$;

(2) $\Phi_{(xy,Q)}(\delta) = 0$.

A subset D of X is called a *distributional δ-chaotic set in the sequence* Q, if every $(x, y) \in D \times D \setminus \Delta$ is a distributional δ-scrambled pair in the sequence Q. The system (X, f) is called *distributional δ-chaotic in a sequence*, if there exists some strictly increasing sequence of positive integers P such that there is an uncountable distributional δ-chaotic set in the sequence P.

Recently, the Li-Yorke and distributional chaos have aroused great interest. For a continuous map f from the unit closed interval $[0, 1]$ to itself, if f has a period point with periodic 3, then the system $([0, 1], f)$ is Li-Yorke chaotic [1]. The system $([0, 1], f)$ is Li-Yorke chaotic if and only if it is distributional chaotic in a sequence [3]. If a system (X, f) has positive entropy then it is Li-Yorke chaotic [7]. A weakly mixing system is distributional chaotic in a sequence [4].

In this paper, we discuss the relationship between Li-Yorke chaos and distributional chaos in a sequence and chaos properties in transitive systems. The organization of the paper is as follows. In section 2, we give a equivalent definition of distributional scrambled pair and point out the set of all distributional δ-scramble pairs in the sequence Q is a G_δ set. In section 3, we first show that a countable Li-Yorke chaotic set (resp. Li-Yorke δ-chaotic set) is also a distributional chaotic set in some sequence (resp. distributional δ-chaotic set in some sequence). Then using the well-known Mycielski theorem we show that Li-Yorke δ-chaos is equivalent to distributional δ-chaos in a sequence. In section 4, we focus on the transitive systems. We prove that a uniformly chaotic set is a distributional scramble set in some sequence and a class of transitive system implies distributional chaos in a sequence.

2 The structure of the set of all distributional δ-scrambled pairs in the sequence Q

Let $Q = \{m_i\}_{i=1}^\infty$ be a strictly increasing sequence of positive integers and P be a subsequence of Q. The upper limit

$$\limsup_{k\to\infty} \frac{\#\{P \cap \{m_1, \cdots, m_k\}\}}{k}$$

is called the *upper density of P with respect to* Q, denoted by $\overline{d}(P \mid Q)$

For every $a \in [0, 1]$, put $\overline{\mathcal{M}}_Q(a) = \{P \subset Q \mid P \text{ is infinite and } \overline{d}(P \mid Q) \ge a\}$

Let (X, f) be a TDS and $U \subset X$ be a nonempty open set. For every $a \in [0, 1]$, define

$$\mathcal{F}(U, Q, \overline{\mathcal{M}}_Q(a)) = \{x \in X \mid N(x, U, Q) \in \overline{\mathcal{M}}_Q(a)\},$$

where $N(x, U, Q) = \{m \in Q \mid f^m(x) \in U\}$.

Following the proof of Theorem 3.2 in [8], we have

Proposition 1. Let (X,f) be a TDS and Q be a strictly increasing sequence of positive integers. Then for every $a \in [0,1]$ and nonempty open set $U \subset X$, $\mathcal{F}(U,Q,\overline{\mathcal{M}}_Q(a))$ is a G_δ set.

Note that the metric in the product space $X \times X$ is denoted by d^2, i.e. for every $(x_1,x_2),(y_1,y_2) \in X \times X$, $d^2((x_1,x_2),(y_1,y_2)) = \max\{d(x_1,y_1), d(x_2,y_2)\}$. Now we can easily get a equivalent definition of distributional scrambled pair in the sequence Q.

Proposition 2. Let (X,f) be a TDS and Q be a strictly increasing sequence of positive integers. Then $(x,y) \in X \times X$ is a distributional scrambled pair in the sequence Q if and only if

(1) for every $\epsilon > 0$, $(x,y) \in \mathcal{F}([\Delta]_\epsilon, Q, \overline{\mathcal{M}}_Q(1))$;

(2) there exists some $t > 0$, such that $(x,y) \in \mathcal{F}(X \times X \setminus \overline{[\Delta]_t}, Q, \overline{\mathcal{M}}_Q(1))$.

For a given $\delta > 0$, $(x,y) \in X \times X$ is a distributional δ-scrambled pair in the sequence Q if and only if

(1) for every $\epsilon > 0$, $(x,y) \in \mathcal{F}([\Delta]_\epsilon, Q, \overline{\mathcal{M}}_Q(1))$;

(2) $(x,y) \in \mathcal{F}(X \times X \setminus \overline{[\Delta]_\delta}, Q, \overline{\mathcal{M}}_Q(1))$.

Combining Proposition 1 and 2, we have

Proposition 3. Let (X,f) be a TDS, Q be a strictly increasing sequence of positive integers and $\delta > 0$. Then the set of all distributional δ-scrambled pairs in the sequence Q is a G_δ subset of $X \times X$.

3 Li-Yorke δ-chaos is equivalent to distributional δ-chaos in a sequence

Let (X,f) be a TDS. A subset C of X is called a *Cantor set*, if it is homeomorphic to the standard Cantor ternary set. A subset A of X is called a *Mycielski set*, if it is a union of countable Cantor sets.

Theorem 1 (Mycielski[9])**.** Let a X be a complete second countable metric space without isolated points. If R is a dense G_δ subset of $X \times X$, then there exists some dense Mycielski subset $K \subset X$ such that $K \times K \setminus \Delta \subset R$.

Lemma 1. [5] Let $\{S_i\}_{i=1}^\infty$ be a sequence of strictly increasing sequences of positive integers. Then there exists a strictly increasing sequence Q of positive integers such that $\overline{d}(S_i \cap Q \mid Q) = 1$ for every $i \geq 1$.

Theorem 2. Let (X,f) be a TDS and $\delta > 0$. If $C \subset X$ is a countable Li-Yorke chaotic set (resp. Li-Yorke δ-chaotic set), then there exists a strictly increasing sequence Q of positive integers such that C is a distributional chaotic set in the sequence Q (resp. distributional δ-chaotic set in the sequence Q).

Proof. Let $C = \{x_i \in X : i = 1,2,\ldots\}$. By the definition of Li-Yorke scrambled pair, for every $i \neq j$, there exists $\delta_{ij} > 0$ such that

$$\liminf_{n\to\infty} d(f^n(x_i), f^n(x_j)) = 0, \quad \limsup_{n\to\infty} d(f^n(x_i), f^n(x_j)) > \delta_{ij}.$$

i.e., there are two strictly increasing sequences $P_{i,j} = \{n_k^{i,j}\}_{k=1}^\infty$ and $S_{i,j} = \{m_k^{i,j}\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty} d(f^{n_k^{i,j}}(x_i), f^{n_k^{i,j}}(x_j)) = 0, \quad \lim_{k\to\infty} d(f^{m_k^{i,j}}(x_i), f^{m_k^{i,j}}(x_j)) > \delta_{ij}.$$

By Lemma 1, there exists a strictly increasing sequence Q such that

$$\overline{d}(P_{i,j} \cap Q|Q) = \overline{d}(S_{i,j} \cap Q|Q) = 1, \ \forall i \neq j.$$

Then it is easy to see that for every $t > 0$

$$(x_i,x_j) \in \mathcal{F}([\Delta]_t, Q, \overline{\mathcal{M}}_Q(1)), \quad (x_i,x_j) \in \mathcal{F}(X \times X \setminus \overline{[\Delta]_{\delta_{ij}}}, Q, \overline{\mathcal{M}}_Q(1)), \ \forall i \neq j.$$

By Proposition 2, C is a distributional chaotic set in the sequence Q.

For a given $\delta > 0$, if C is a Li-Yorke δ-chaotic set, then we can choose all the above δ_{ij} being δ, therefore, C is a distributional δ-chaotic set in the sequence Q. □

Remark 1. In [10], the authors constructed a countable compact space X and a homeomorphism on X with the whole space being a Li-Yorke chaotic set, then there exists a strictly increasing sequence Q of positive integers such that X is a distributional chaotic set in the sequence Q. As the method of Theorem 4.3 in [10], we can construct a homeomorphism of the Cantor set with the whole space being a distributional chaotic set in some sequence.

Theorem 3. Let (X, f) be a TDS and $\delta > 0$. Then (X, f) is Li-Yorke δ-chaos if and only if it is distributional chaos in a sequence.

Proof. Let D be an uncountable Li-Yorke δ-chaotic set. Since X is a complete separable metric space, without lose of generality, assume that D has no isolated points. Put $X_0 = \overline{D}$, then D is a complete separable metric subspace without isolated points. Choose a countable dense subset C of D. By Theorem 2, there exists a strictly increasing sequence Q of positive integers such that C is a distributional δ-chaotic set in the sequence Q.

Denote E be the collection of all distributional δ-scrambled pairs in the sequence Q. By Proposition 3, E is a G_δ subset of $X \times X$. Since $C \times C \setminus \Delta \subset E$ and C is dense in X_0, By Mycielski Theorem, There exists an uncountable Mycielski set $D_0 \subset X_0$ such that $D_0 \times D_0 \setminus \Delta \subset E$. Thus, (X, f) is distributional chaos in a sequence. □

Remark 2. (1) For the system (I, f) on the unit closed $I = [0, 1]$, [11] proved that if (I, f) has a Li-Yorke pair, then there exists some $\delta > 0$ such that it is Li-Yorke δ-chaotic. Therefore, it is also distributional δ-chaotic in a sequence.

(2) [7] proved that positive entropy implies Li-Yorke chaos. By the proof of Theorem 2.3 in [7], if (X, f) has positive entropy, then there exists some $\delta > 0$ such that it is Li-Yorke δ-chaotic. Therefore, it is also distributional δ-chaotic in a sequence.

(3) Let (X, f) be a TDS. The system (X, f) is called *topologically transitive*, if for every two nonempty open subsets $U, V \subset X$, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. The system (X, f) is called *Devaney chaotic*, if it is transitive and has dense periodic points. [12] proved that if (X, f) is Devaney chaotic, then there exists some $\delta > 0$ such that it is Li-Yorke δ-chaotic. Therefore, it is also distributional δ-chaotic in a sequence.

(4) Let (X, f) be a TDS. The system (X, f) is called *topologically weakly mixing*, if the product system $(X \times X, f \times f)$ is transitive. By the main result of [13], if (X, f) is weakly mixing, then there exists some $\delta > 0$ such that it is Li-Yorke δ-chaotic. Therefore, it is also distributional δ-chaotic in a sequence.

4 Transitive systems

Definition 3. [14] Let (X, f) be a TDS.

1. A subset A of X is called *uniformly proximal*, if for every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $d(f^k(x), f^k(y)) < \epsilon$ for every $x, y \in A$.

2. A subset A of X is called *uniformly rigid*, if for every $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that $d(f^k(x), x) < \epsilon$ for every $x \in A$.

3. A subset A of X is called *uniformly chaotic*, if there exists a sequence of Cantor sets $C_1 \subset C_2 \subset \cdots$ such that $A = \bigcup_{i=1}^{\infty} C_i$ and C_i is both uniformly proximal and uniformly rigid for every $i \geq 1$.

Theorem 4. Let (X, f) be a TDS and $A \subset X$. If A is a uniformly chaotic set, then there exists a strictly increasing sequence Q of positive integers such that A is a distributional chaotic set in the sequence Q.

Proof. By the definition of uniformly chaos, there exists a sequence of Cantor sets $C_1 \subset C_2 \subset \cdots$ such that $A = \bigcup_{i=1}^{\infty} C_i$ and C_i is both uniformly proximal and uniformly rigid for every $i \geq 1$. Then there are two strictly increasing sequences $P_N = \{n_k\}_{k=1}^{\infty}$ and $S_N = \{m_k\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} d(f^{n_k^{i,j}}(x), f^{n_k^{i,j}}(y)) = 0, \quad \lim_{k\to\infty} d(f^{m_k^{i,j}}(x), f^{m_k^{i,j}}(y)) = d(x,y) > 0, \ \forall x \neq y \in A_N.$$

By Lemma 1, there exists a strictly increasing sequence Q such that

$$\overline{d}(P_N \cap Q|Q) = \overline{d}(S_N \cap Q|Q) = 1, \ \forall N \in \mathbb{N}.$$

Similarly to the proof of Theorem 2, we have A is a distributional chaotic set in the sequence Q. □

Theorem 5. [14] Let (X, f) be a transitive system, where X is a compact metric space without isolated points. If there exists a subsystem (Y, f) such that $(X \times Y, f \times f)$ is transitive, then (X, f) has a dense Mycielski uniformly chaotic set.

Corollary 1. [14] Let (X, f) be a transitive system, where X is a compact metric space without isolated points. If the system (X, f) satisfies one of the following conditions:
(1) (X, f) is transitive and has a fixed point;
(2) (X, f) is totally transitive with a periodic point;
(3) (X, f) is scattering;
(4) (X, f) is weakly scattering with an equicontinuous minimal subsystem;
(4) (X, f) is weakly mixing.
Then (X, f) has a dense Mycielski uniformly chaotic set. Moreover, if (X, f) is transitive and has a periodic point of order d, then there is a closed f^d-invariant subset $X_0 \subset X$ such that (X_0, f^d) has a dense uniformly chaotic set and $X = \bigcup_{j=0}^{d-1} f^j X_0$; in particular, (X, f) has a uniformly chaotic set.

Corollary 2. Let (X, f) be a TDS. If the system (X, f) satisfies the condition of Theorem 5 or Corollary 1, then (X, f) is distributional chaotic in a sequence.

Question: is Li-Yorke chaos equivalent to distributional chaos?

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