STAR PRODUCTS WITH SEPARATION OF VARIABLES ADMITTING A SMOOTH EXTENSION

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ABSTRACT. Given a complex manifold M with an open dense subset Ω endowed with a pseudo-Kähler form ω which cannot be smoothly extended to a larger open subset, we consider various examples where the corresponding Kähler-Poisson structure and a star product with separation of variables on (Ω, ω) admit smooth extensions to M. We suggest a simple criterion of the existence of a smooth extension of a star product and apply it to these examples.

1. INTRODUCTION

A formal differential star product on a Poisson manifold $(M, \{\cdot, \cdot\})$ is an associative product on the space $C^\infty(M)[[\nu]]$ of smooth complex-valued formal functions on M given by the formula

$$(1) \qquad f * g = \sum_{r \geq 0} \nu^r C_r(f, g),$$

where C_r are bidifferential operators on M, $C_0(f, g) = fg$ and $C_1(f, g) - C_1(g, f) = i\{f, g\}$ (see [1]). It was proved by Kontsevich in [9] that deformation quantizations exist on any Poisson manifold.

We will assume that the unit constant is the unity with respect to the star product: $f * 1 = 1 * f = f$ for all $f \in C^\infty(M)[[\nu]]$. Given functions $f, g \in C^\infty(M)[[\nu]]$, we will denote by L_f and R_g the left star multiplication operator by f and the right star multiplication operator by g, respectively, so that $f * g = L_f g = R_g f$. The associativity of the star product $*$ is equivalent to the fact that $[L_f, R_g] = 0$ for all $f, g \in C^\infty(M)[[\nu]]$. A star-product on a Poisson manifold M can be restricted to any open subset of M.

We call a Poisson tensor on a complex manifold M a Kähler-Poisson tensor if it is of type (1,1) with respect to the complex structure. If a Kähler-Poisson tensor written in local coordinates $\{z^k, \bar{z}^l\}$ as g^{lk} is

nondegenerate, its inverse is a pseudo-Kähler metric tensor g_{kl}. We call a complex manifiold M endowed with a Kähler-Poisson tensor a Kähler-Poisson manifold. Any pseudo-Kähler manifold is a Kähler-Poisson manifold. In this paper we will give several examples of Kähler-Poisson manifolds with the Kähler-Poisson tensor degenerate on the complement of an open dense subset.

A star product (1) on a Kähler-Poisson manifold defines a deformation quantization with separation of variables if the operators C_r differentiate their first argument in antiholomorphic directions and the second argument in holomorphic ones. If the unit constant is the unity with respect to the star product, the condition of separation of variables can be equivalently stated as follows: for any local holomorphic function a and a local antiholomorphic function b the identities $a * f = af$ and $f * b = bf$ hold. Otherwise speaking, $L_a = a$ and $R_b = b$ are pointwise multiplication operators.

It is not known whether there exists a star product with separation of variables on an arbitrary Kähler-Poisson manifold. However, star products with separation of variables exist on any pseudo-Kähler manifold M (see [2], [5]).

Given a star product with separation of variables $*$ on a Kähler-Poisson manifold M, the formal Berezin transform of the star product $*$ is a formal differential operator $B = 1 + \nu B_1 + \nu^2 B_2 + \dots$ globally defined on M by the condition that

$$B(ab) = b * a$$

for any local holomorphic function a and a local antiholomorphic function b. A star product with separation of variables can be recovered from its Berezin transform.

A deformation quantization with separation of variables on a pseudo-Kähler manifold M equipped with a pseudo-Kähler form ω is called standard if its restriction to any contractible coordinate chart $(U, \{z^k\})$ has the property that

$$L_{\frac{\partial \Phi}{\partial z^k}} = \frac{\partial \Phi}{\partial z^k} + \nu \frac{\partial}{\partial z^k} \text{ and } R_{\frac{\partial \Phi}{\partial \bar{z}^l}} = \frac{\partial \Phi}{\partial \bar{z}^l} + \nu \frac{\partial}{\partial \bar{z}^l},$$

where Φ is a potential of the pseudo-Kähler form ω on U, i.e., $\omega = i\partial\bar{\partial}\Phi$. This property defines the standard deformation quantization with separation of variables uniquely and globally on any pseudo-Kähler manifold M (see [5]).

Let M be a Kähler-Poisson manifold M such that the Kähler-Poisson structure on M given by a tensor g^{lk} is nondegenerate on a dense open subset Ω of M and its inverse on Ω is a metric tensor g_{kl} with the

corresponding pseudo-Kähler form ω. It was shown in [6] that the coefficients of the operators C_1 and C_2 of the standard star product with separation of variables are polynomials in partial derivatives of g^{lk}, while the operator C_3 is the sum of an operator with the same property and the operator

$$S(u,v) = g_{mn} \frac{\partial g^{ls}}{\partial \bar{z}^q} \frac{\partial g^{np}}{\partial z^s} \frac{\partial g^{qm}}{\partial \bar{z}^t} \frac{\partial g^{tk}}{\partial z^p} \frac{\partial u}{\partial \bar{z}^l} \frac{\partial v}{\partial z^k},$$

which depends on the metric tensor g_{kl}. It follows from this observation that a star-product with separation of variables on (Ω, ω) does not necessarily have a smooth extension to M. In this paper we give examples of Kähler-Poisson manifolds with open dense pseudo-Kähler submanifolds such that the standard deformation quantization with separation of variables on these pseudo-Kähler submanifolds admits a smooth extension to the whole manifold.

2. Examples of deformation quantizations with separation of variables on Kähler-Poisson manifolds

In this section we will give two examples of a Kähler-Poisson manifold M with the Kähler-Poisson structure which is nondegenerate on a dense open subset Ω and such that the standard deformation quantization with separation of variables on Ω admits a smooth extension to M.

Example 1.

Let ψ be a defining function of a Levi-nondegenerate hypersurface Σ in an open set $U \subset \mathbb{C}^n$. This means that ψ is a smooth real function on U with the zero set Σ and such that the Monge-Ampère matrix

$$(2) \qquad \Gamma = \begin{pmatrix} \frac{\partial^2 \psi}{\partial z^k \partial \bar{z}^l} & \frac{\partial \psi}{\partial z^k} \\ \frac{\partial \psi}{\partial \bar{z}^l} & \psi \end{pmatrix}$$

is nondegenerate at the points of Σ. Shrinking, if necessary, the neighborhood U around Σ, we may assume that the matrix Γ is nondegenerate on U. On the complement $U \backslash \Sigma$ of Σ the potential $\log |\psi|$ defines a pseudo-Kähler form ω whose inverse is a Kähler-Poisson bivector field which has a smooth extension to U by zero (i.e., vanishing on Σ). The following theorem was proved in [8]:

Theorem 1. *The standard star product with separation of variables on $(U \backslash \Sigma, \omega)$ admits a smooth extension to a star-product on U.*

Similar statements were proved earlier by different methods in [3] and [10].

Example 2.

The following example of a Kähler-Poisson manifold comes from the theory of complex symmetric domains. Denote by $\mathcal{E}$ the set of non-degenerate complex $(p + r) \times r$ matrices with the right action of the group $GL(r, \mathbb{C})$. Then $Gr(r, p+r) = \mathcal{E}/GL(r, \mathbb{C})$ is the Grassmannian of r-dimensional subspaces in $\mathbb{C}^{p+r}$. Consider the indefinite metric

$$(3) \qquad \sum_{k=1}^{p} z_k \bar{w}_k - \sum_{k=p+1}^{p+r} z_k \bar{w}_k$$

on $\mathbb{C}^{p+r}$. The left action of the group $U(p, r)$ on $\mathcal{E}$ induces an action on the Grassmannian $Gr(r, p+r)$. Let Ω be the set of points of the Grassmannian $Gr(r, p+r)$ corresponding to the subspaces of $\mathbb{C}^{p+r}$ such that the restriction of the indefinite metric (3) to them is nondegenerate. It is a dense open $U(p, r)$-invariant subset of $Gr(r, p+r)$. Given a matrix $A \in \mathcal{E}$, denote by U_A and V_A its blocks of size $p \times r$ (top p rows) and $r \times r$ (bottom r rows), respectively. Denote by $\mathcal{E}'$ the set of matrices $A \in \mathcal{E}$ such that the block V_A is nondegenerate. Then $\mathcal{Z} = \mathcal{E}'/GL(r, \mathbb{C})$ is an affine subset of $Gr(r, p + r)$ parametrized by the $p \times r$-matrices Z so that the coset of $A \in \mathcal{E}'$ corresponds to the matrix $Z = U_A/V_A$. The elements $\{z_{k\alpha}\}, 1 \leq k \leq p, 1 \leq \alpha \leq r$, of a $p \times r$-matrix Z are thus holomorphic coordinates on $\mathcal{Z}$. The set $\mathcal{Z} \cap \Omega$ is parametrized by the matrices Z such that $E - Z^{\dagger}Z$ is nondegenerate. The pseudo-Kähler metric on $\mathcal{Z} \cap \Omega$ defined by the potential

$$\Phi(Z, Z^{\dagger}) = \log |\det(E - Z^{\dagger}Z)|$$

extends to a $U(p, r)$-invariant pseudo-Kähler metric on Ω. In particular, the set $\Omega_+ \subset \mathcal{Z} \subset Gr(r, p + r)$ parametrized by the matrices Z such that $E - Z^{\dagger}Z$ is positive definite is an open $U(p, r)$-invariant subset of $Gr(r, p + r)$. It is a bounded symmetric domain and the metric corresponding to the potential Φ is Kähler. The standard deformation quantization with separation of variables on Ω is $U(p, r)$-invariant. The corresponding formal Berezin transform $B = 1 + \nu B_1 + \ldots$ is a $U(p, r)$-invariant formal differential operator on Ω. It is known (see [4]) that all $U(p, r)$-invariant differential operators on Ω_+ are induced by the elements of the center of the universal enveloping algebra of the Lie algebra $u(p, r)$. These elements induce global $U(p, r)$-invariant operators on the Grassmannian $Gr(r, p + r)$. Since the coefficients of $U(p, r)$-invariant differential operators are algebraic functions on $Gr(r, p + r)$,

it implies that the formal Berezin transform B and the corresponding star product smoothly extend to $Gr(r, p + r)$. The corresponding $U(p, r)$-invariant Kähler-Poisson bivector is also globally defined on $Gr(r, p + r)$. In the coordinates $(Z, Z^\dagger)$ it is given by the formula

$$i \left(\delta_{st} - \sum_{\gamma=1}^{r} z_{s\gamma} \bar{z}_{t\gamma} \right) \left(\delta_{\alpha\beta} - \sum_{k=1}^{p} \bar{z}_{k\alpha} z_{k\beta} \right) \frac{\partial}{\partial z_{s\beta}} \wedge \frac{\partial}{\partial \bar{z}_{t\alpha}}.$$

On Ω it is the inverse of the pseudo-Kähler form corresponding to the potential Φ.

3. Smooth extensions of star products

Given an open subset U of a smooth real n-dimensional manifold M, an n-tuple of smooth complex-valued functions $\{f^1, \ldots, f^n\}$ on U is called a frame if for each point $x \in U$ the differentials $df^1(x), \ldots df^n(x)$ form a basis of the complexified cotangent space $T_x^* M \otimes \mathbb{C}$. An n-tuple of smooth formal complex-valued functions $f^k = f_0^k + \nu f_1^k + \nu^2 f_2^k + \ldots, 1 \le k \le n$, on U is called a formal frame if $\{f_0^1, \ldots, f_0^n\}$ is a frame on U.

Lemma 1. *Let $U \subset \mathbb{R}^n$ be an open set with a dense open subset $V \subset U$ and $\{f^1, \ldots, f^n\}$ be a frame on U. If A is a differential operator of finite order on V such that the function $A1$ and the operators $[\ldots[[A, f^{k_1}], f^{k_2}], \ldots, f^{k_N}]$ have smooth extensions to U for any N and any indices $k_i, 1 \le k_i \le n$, then the operator A has a smooth extension to U.*

Proof. The lemma will be proved by induction on the order of the operator A. If A is of order zero, it is the operator of pointwise multiplication by the function $A1$, which has a smooth extension to U. Assume that the statement of the lemma is true for any operator of order less than r and that A is of order r. Then for any indices $k_i, 1 \le i \le r$, the following identity holds:

$$[\ldots[[A, f^{k_1}], f^{k_2}], \ldots, f^{k_r}] = r! p \left(df^{k_1} \otimes \ldots \otimes df^{k_r} \right),$$

where $p : (T^*V)^{\otimes r} \to \mathbb{C}$ is the (polarized) principal symbol of the operator A. Since the functions $\{f^i\}$ form a frame on U, the principal symbol of the operator A has a smooth extension to U. One can construct an operator B of order r on U whose principal symbol is the extension of p to U. Now the operator $A - B$ is of order less than r and satisfies the conditions of the lemma. Therefore $A - B$ has a smooth extension to U, whence the lemma follows. $\square$

Let M be a smooth real n-dimensional manifold with a dense open subset Ω. Assume that π is a Poisson bivector field on Ω and $*$ is a star product on the Poisson manifold (Ω, π).

Theorem 2. *Given a point $a \in M \backslash \Omega$ in a coordinate chart $U \subset M$, let $\{f^1, \ldots, f^n\}$ be a formal frame on U. If the operators of right star-multiplication $R_{f^k}, k = 1, \ldots, n$ (or the operators of left star multiplication $L_{f^k}, k = 1, \ldots, n$) on $\Omega \cap U$ can be extended to smooth formal differential operators on U, then the star-product $*$ has a smooth extension to a star-product on U. In particular, then π extends to a smooth Poisson bivector field on U.*

Proof. Let $u = u_0 + \nu u_1 + \ldots$ be a smooth formal function on U. The left star-multiplication operator L_u on $U \cap \Omega$ commutes with the operators $R_{f^k}, k = 1, \ldots, n$. Writing $L_u = A_0 + \nu A_1 + \ldots$ and $R_{f^k} = B_0^k + \nu B_1^k + \ldots$, we have that the operators $A_0 = u_0$ and $B_r^k, r \geq 0$, have smooth extensions to U. We will prove by induction on r that the operator A_r has a smooth extension to U. This is true for $r = 0$. Assume that this is true for all $r < s$. We have

$$(4) \qquad\qquad [\ldots [[L_u, R_{f^{k_1}}], R_{f^{k_2}}], \ldots, R_{f^{k_N}}] = 0$$

for any N and indices k_i. Consider the coefficient at ν^s of the left-hand side of (4). Since $B_0^k = f_0^k$, this coefficient can be written as

$$(5) \qquad\qquad [\ldots [[A_s, f_0^{k_1}], f_0^{k_2}], \ldots, f_0^{k_N}]$$

plus a sum of commutators of the operators A_i and B_j^k with $i < s$ and $j \leq s$ which all have smooth extensions to U. Thus the operator (5) also has a smooth extension to U. Taking into account that $A_s 1 = u_s$, we get from Lemma 1 that the operator A_s has a smooth extension to U. Therefore the operator L_u has a smooth extension to U for any formal function u on U. This implies that the star product $*$ extends to a smooth formal star-product on U. In particular, π extends to a Poisson bivector field on U. $\qquad\qquad\square$

4. A KÄHLER-POISSON TENSOR VANISHING ON A LEVI-NONDEGENERATE HYPERSURFACE

In this section we want to give yet another proof of Theorem 1 from Example 1 based upon Theorem 2.

Recall that ψ is a defining function of a Levi-nondegenerate hypersurface Σ in an open set $U \subset \mathbb{C}^n$ and Γ is the Monge-Ampère matrix (2) which we assume to be nondegenerate on U. Fix a point $x_0 \in \Sigma$. Then $\psi(x_0) = 0$ and $(\partial \psi)(x_0) \neq 0$, since the matrix $\Gamma(x_0)$ is nondegenerate. Therefore, there exists an index s such that $\frac{\partial \psi}{\partial z^s}(x_0) \neq 0$. Denote by V

the neighborhood of x_0 within U where $\frac{\partial \psi}{\partial z^s}$ does not vanish. We will construct formal functions $\{f^1, \ldots, f^n\}$ on V such that the functions $\{z^1, \ldots, z^n, f^1, \ldots, f^n\}$ form a formal frame on a neighborhood $W \subset V$ of x_0 and the operators $L_{f^k}, k = 1, \ldots, n$, of the star product $*$ on $V \backslash \Sigma$ have a smooth extension to W (this is trivially true for the operators $L_{z^k} = z^k$). Theorem 2 will then imply that the standard star product with separation of variables $*$ on $(U \backslash \Sigma, \omega)$ smoothly extends from $U \backslash \Sigma$ to U. Introduce the following invertible operator on V,

$$Q = 1 + \nu\psi \left(\frac{\partial \psi}{\partial z^s} \right)^{-1} \frac{\partial}{\partial z^s}.$$

On $V \backslash \Sigma$ the operator

$$L_{\frac{\partial \log |\psi|}{\partial z^s}} = \frac{\partial \log |\psi|}{\partial z^s} + \nu \frac{\partial}{\partial z^s} = \psi^{-1} \frac{\partial \psi}{\partial z^s} + \nu \frac{\partial}{\partial z^s} = \psi^{-1} \frac{\partial \psi}{\partial z^s} Q$$

is invertible. The inverse operator

$$\left(L_{\frac{\partial \log |\psi|}{\partial z^s}} \right)^{-1} = Q^{-1} \circ \left(\left(\frac{\partial \psi}{\partial z^s} \right)^{-1} \psi \right)$$

is also a left multiplication operator of the star product $*$ on $V \backslash \Sigma$. It admits a smooth extension to V which we will denote X^s. Then $f^s = X^s 1$ is a smooth formal function on V, $f^s = f_0^s + \nu f_1^s + \ldots$, such that $f^s * \frac{\partial \log |\psi|}{\partial z^s} = 1$ on $V \backslash \Sigma$ and

$$f_0^s = \left(\frac{\partial \psi}{\partial z^s} \right)^{-1} \psi$$

on V. For $k \neq s$ the operator

$$\left(L_{\frac{\partial \log |\psi|}{\partial z^s}} \right)^{-1} L_{\frac{\partial \log |\psi|}{\partial z^k}} = Q^{-1} \circ \left(\left(\frac{\partial \psi}{\partial z^s} \right)^{-1} \left(\frac{\partial \psi}{\partial z^k} + \nu\psi \frac{\partial}{\partial z^k} \right) \right)$$

is a left multiplication operator of the star product $*$ on $V \backslash \Sigma$. It admits a smooth extension to V which we will denote X^k. Then $f^k = X^k 1$ is a smooth formal function on V, $f^k = f_0^k + \nu f_1^k + \ldots$, and

$$f_0^k = \left(\frac{\partial \psi}{\partial z^s} \right)^{-1} \left(\frac{\partial \psi}{\partial z^k} \right).$$

We want to prove that $\{z^1, \ldots, z^n, f^1, \ldots, f^n\}$ is a formal frame on a neighborhood of x_0. It suffices to prove that the covectors

(6)
$$\left(\frac{\partial f_0^k}{\partial \bar{z}^1}(x_0), \ldots, \frac{\partial f_0^k}{\partial \bar{z}^n}(x_0) \right)$$

for $k = 1, \ldots, n$, are linearly independent. Taking into account that $\psi(x_0) = 0$, we see that for $k = s$ the covector (6) is nonzero and proportional to the nonzero covector

$$(7) \qquad \left(\frac{\partial \psi}{\partial \bar{z}^1}(x_0), \ldots, \frac{\partial \psi}{\partial \bar{z}^n}(x_0) \right).$$

For $k \neq s$ the covector (6) is proportional to the covector

$$(8) \qquad \left(\frac{\partial^2 \psi}{\partial z^k \partial \bar{z}^1} \frac{\partial \psi}{\partial z^s} - \frac{\partial^2 \psi}{\partial z^s \partial \bar{z}^1} \frac{\partial \psi}{\partial z^k}, \ldots, \frac{\partial^2 \psi}{\partial z^k \partial \bar{z}^n} \frac{\partial \psi}{\partial z^s} - \frac{\partial^2 \psi}{\partial z^s \partial \bar{z}^n} \frac{\partial \psi}{\partial z^k} \right)$$

at x_0. It is a simple consequence of formulas (7) and (8) that the linear independence of the covectors (6) for $k = 1, \ldots, n$, is equivalent to the nondegeneracy of the matrix $\Gamma(x_0)$.

Thus we have proved Theorem 1 from [8] using a different approach.

5. Roots of formal differential operators

In the next section we will construct a family of star products with separation of variables on the complement of a Levi-nondegenerate hypersurface in an open subset of $\mathbb{C}^n$ which admit a smooth extension to the whole open subset. In order to use Theorem 2 we will have to find a root of a specific formal differential operator. To this end we will now prove several technical statements.

Let $\mathcal{X}$ denote the ring of polynomials in an infinite number of variables,

$$\mathcal{X} = \mathbb{C}[t_0, t_1, \ldots],$$

and $\mathcal{A}$ denote the algebra of differential operators on $\mathcal{X}$ generated by the multiplication operators by the elements of $\mathcal{X}$ and a single differentiation operator

$$\delta = \sum_{k=0}^{\infty} t_{k+1} \frac{\partial}{\partial t_k}.$$

For this operator, $\delta t_k = t_{k+1}$. Clearly, $\mathcal{A}$ is generated by t_0 and δ. Given a manifold X, a function f, and a vector field v on X, denote by D the algebra of differential operators on X generated by f and v. Then there exists a well defined surjective homomorphism $\tau : \mathcal{A} \to D$ such that $\tau(t_0) = f$ and $\tau(\delta) = v$.

Lemma 2. *Let B_r be a differential operator in $\mathcal{A}$ of order not greater than r. Then for any natural number N there exists a unique differential operator $A_r \in \mathcal{A}$ satisfying the equation*

$$(9) \qquad \sum_{i=0}^{N} t_0^{N-i} A_r \circ t_0^i = t_0^{N(r+1)} B_r.$$

The order of A_r is not greater than r.

Proof. We will prove that equation (9) has a unique solution by induction on r. Comparing the principal symbols of the operators on both sides of equation (9) we see that the order of the operator A_r must be equal to the order of B_r. First consider equation (9) with $r = 0$. Both A_0 and B_0 must be multiplication operators by elements of $\mathcal{X}$ and

$$A_0 = \frac{1}{N+1} B_0.$$

Denote the principal symbol of order p of a differential operator X of order not greater than p by $\sigma_p(X)$. If A_r is a solution of equation (9), then

$$(N+1)t_0^N \sigma_r(A_r) = t_0^{N(r+1)} \sigma_r(B_r),$$

which implies that

$$\sigma_r(A_r) = \frac{1}{N+1} t_0^{Nr} \sigma_r(B_r).$$

Therefore the order of the operator

$$A_{r-1} := A_r - \frac{1}{N+1} t_0^{Nr} B_r$$

must be not greater than $r - 1$ and A_{r-1} should satisfy equation (9) with r replaced with $r - 1$ and with

$$B_{r-1} := t_0^N B_r - \frac{1}{N+1} \sum_{i=0}^{N} t_0^{N-i} B_r \circ t_0^i.$$

It is clear that $\sigma_r(B_{r-1}) = 0$, whence the order of B_{r-1} is not greater than $r - 1$. By the induction principle, it implies that equation (9) has a unique solution for any r. $\square$

We introduce a bidegree on the algebra $\mathcal{A}$ such that the operator t_0 has the bidegree $(1, 0)$ and the operator δ has the bidegree $(0, 1)$. Then the operator t_r has the bidegree $(1, r)$. Observe that if the operator B_r from Lemma 2 is a homogeneous element of algebra $\mathcal{A}$ of bidegree (q, r), then the solution A_r of eqn. (9) is homogeneous of bidegree $(Nr + q, r)$.

Lemma 3. *Given the operator*

$$(10) \qquad S := \sum_{k=0}^{\infty} \nu^k \left(t_0^{N+1}\delta\right)^k \circ t_0^{N+1} = t_0^{N+1} + \nu t_0^{N+1}\delta \circ t_0^{N+1} + \ldots$$

in the algebra $\mathcal{A}[[\nu]]$, *there exists a unique operator* $A = A_0 + \nu A_1 + \dots$ *in* $\mathcal{A}[[\nu]]$ *such that* $A_0 = t_0$ *and*

$$(11) \qquad\qquad A^{N+1} = S.$$

Proof. Equating the coefficients at ν^r of the operators on the both sides of equation (11), we obtain the equation

$$\sum_{i_0+\dots i_N=r} A_{i_0} \dots A_{i_N} = \left(t_0^{N+1}\delta\right)^r \circ t_0^{N+1}$$

which can be rewritten as follows:

$$(12) \qquad \sum_{i=0}^{N} t_0^{N-i} A_r \circ t_0^i = \left(t_0^{N+1}\delta\right)^r \circ t_0^{N+1} - \sum_{i_0+\dots i_N=r, i_s<r} A_{i_0} \dots A_{i_N}.$$

Notice that the right-hand side of eqn. (12) depends only on the operators A_i with $i < r$. We can find the components A_r from equation (12) by induction on r using Lemma 2. Applying induction to eqn. (12), we have to show simultaneously that the bidegree of A_r is $((N+1)r+1, r)$ and that the right-hand side of eqn. (12) can be represented in the form $t_0^{N(r+1)} B_r$ for some operator $B_r \in \mathcal{A}$. To justify the latter statement observe that the right-hand side of eqn. (12) being a homogeneous element of the algebra $\mathcal{A}$ of bidegree $((N+1)(r+1), r)$, can be written as a linear combination of operators of the form

$$\left(\prod_{s\geq 0} (t_s)^{i_s}\right) \delta^j,$$

where

$$\sum_{s\geq 0} i_s = (N+1)(r+1) \text{ and } \sum_{s\geq 1} si_s + j = r.$$

It implies that

$$i_0 = \left(\sum_{s\geq 2}(s-1)i_s\right) + j + Nr + N + 1 > N(r+1),$$

which means that any homogeneous element of the algebra $\mathcal{A}$ of bidegree $((N+1)(r+1), r)$ is divisible on the left by $t_0^{N(r+1)}$, which concludes the proof. $\qquad\square$

We will also need the two following lemmas.

Lemma 4. *Given a nonvanishing smooth function f and two formal differential operators $A = A_0 + \nu A_1 + \dots$ and $B = B_0 + \nu B_1 + \dots$ on a manifold M such that $A_0 = B_0 = f$ is the pointwise multiplication*

operator by f and $A^{N+1} = B^{N+1}$ for a nonnegative integer N, then $A = B$.

Proof. Let D be a differential operator on M such that

$$(13) \qquad f^N D + f^{N-1} D \circ f + \ldots + D \circ f^N = 0.$$

Assume that D is a nonzero operator of order r with the nonzero principal symbol $\sigma_r(D)$. Now, the principal symbol of the left-hand side of (13) is $(N+1) f^N \sigma_r(D) = 0$, whence $\sigma_r(D) = 0$. This contradiction implies that $D = 0$. Using this observation, we will prove by induction on r that $A_r = B_r$ for all $r \geq 0$. We have that $A_0 = B_0$. Given $r > 0$, assume that $A_k = B_k$ for all $k < r$. Denote

$$X := A_0 + \nu A_1 + \ldots + \nu^{r-1} A_{r-1} = B_0 + \nu B_1 + \ldots + \nu^{r-1} B_{r-1}.$$

It follows from the condition $A^{N+1} = B^{N+1}$ that

$$\nu^r \left(X^N A_r + X^{N-1} A_r X + \ldots A_r X^N \right) =$$
$$\nu^r \left(X^N B_r + X^{N-1} B_r X + \ldots B_r X^N \right) \qquad (\mathrm{mod}\ \nu^{r+1}),$$

whence $f^N D + f^{N-1} D \circ f + \ldots + D \circ f^N = 0$ for $D = A_r - B_r$. Therefore, $A_r = B_r$, which concludes the proof. $\qquad \square$

Lemma 5. *Given a star product $*$ on a Poison manifold M, a non-vanishing complex-valued function u_0 on M, and a formal function $v = v_0 + \nu v_1 + \ldots$ such that $v_0 = u_0^q$ for some $q \in \mathbb{Z}$, there exists a unique formal function $u = u_0 + \nu u_1 + \ldots$ on M such that $v = u^{*q}$.*

Here u^{*q} is the qth power of u with respect to the star product $*$.

Proof. Assume that $q > 0$. We will show by induction on l the existence and uniqueness of each coefficient $u_l, l > 1$. For $f_1, \ldots f_q \in C^\infty(M)$ set

$$f_1 * \ldots * f_q = \sum_{r=0}^{\infty} \nu^r C_r(f_1, \ldots, f_q).$$

Then, in particular, $C_0(f_1, \ldots, f_q) = f_1 \ldots f_q$. Equating the coefficients at ν^l of u^{*q} and v we get

$$(14) \qquad \sum_{i_0+\ldots+i_q=l} C_{i_0}(u_{i_1}, \ldots, u_{i_q}) = v_l.$$

The terms containing u_l on the left hand side of (14) are

$$C_0(u_l, u_0, \ldots u_0) + C_0(u_0, u_l, \ldots u_0) + \ldots + C_0(u_0, \ldots, u_l) = q u_0^{q-1} u_l,$$

which shows that u_l is uniquely expressed through the coefficients u_j for $j < l$. The statement of the lemma is well known for $q = -1$. Assume that $q < 0$. The equation $v = u^{*q}$ is equivalent to $v^{*(-1)} = u^{*(-q)}$ which reduces the case of $q < 0$ to the case of $q > 0$. $\qquad \square$

6. A family of Kähler-Poisson tensors vanishing on a Levi-nondegenerate hypersurface

Given an open subset $U \subset \mathbb{C}^n$ and a Levi-nondegenerate hypersurface $\Sigma \subset U$ with a defining function ψ, for each positive integer N we will introduce a Kähler-Poisson tensor on a neighborhood U_N of Σ in U vanishing on Σ and nondegenerate on its complement $U_N \backslash \Sigma$ such that the corresponding standard deformation quantization on $U_N \backslash \Sigma$ admits a smooth extension to U_N.

For each nonnegative integer N define a matrix

$$\Gamma_N = \left(\begin{array}{cc} \frac{\partial^2 \psi}{\partial z^k \partial \bar{z}^l} & \frac{\partial \psi}{\partial z^k} \\ \frac{\partial \psi}{\partial \bar{z}^l} & (N+1)^{-1}\psi \end{array} \right)$$

on U. In particular, $\Gamma_0 = \Gamma$. Set $\Omega = U \backslash \Sigma$. For $N > 0$, let ω_N be the closed $(1,1)$-form on Ω whose potential is $\Phi = \frac{1 - \psi^{-N}}{N}$. Set

$$g_{kl} = \frac{\partial^2 \Phi}{\partial z^k \partial \bar{z}^l}.$$

Lemma 6. *The form ω_N is nondegenerate if and only if the matrix Γ_N is nondegenerate.*

Proof. For each $k \leq n$ multiply the last row in the matrix Γ_N by $(N+1)\psi^{-1}\frac{\partial \psi}{\partial z^k}$ and subtract it from the kth row. The resulting matrix is

$$\left(\begin{array}{cc} X_{kl} & 0 \\ \frac{\partial \psi}{\partial \bar{z}^l} & (N+1)^{-1}\psi \end{array} \right),$$

where $X_{kl} = \frac{\partial^2 \psi}{\partial z^k \partial \bar{z}^l} - (N+1)\psi^{-1}\frac{\partial \psi}{\partial z^k}\frac{\partial \psi}{\partial \bar{z}^l}$. The lemma follows from the observation that

$$(15) \qquad\qquad g_{kl} = \psi^{-N-1}X_{kl}.$$

$\square$

Remark. If $N = 0$, the tensor (15) defines the $(1,1)$-form ω with the potential $\log|\psi|$ as in Example 1.

Since ψ is a defining function of the Levi-nondegenerate hypersurface $\Sigma \subset U$, the matrix Γ_N is nondegenerate at every point of Σ. Thus, the matrix Γ_N is nondegenerate on some neighborhood U_N of Σ in U and therefore ω_N is a pseudo-Kähler form on $U_N \backslash \Sigma$.

Consider the inverse matrix

$$\Gamma_N^{-1} = \left(\begin{array}{cc} A^{lm} & B^l \\ C^m & D \end{array} \right)$$

on U_N. A simple calculation shows that the matrix A^{lm} is inverse to X_{kl}, which implies that the inverse g^{lm} of the matrix g_{kl} is

$$g^{lm} = \psi^{N+1} A^{lm}.$$

Taking into account that the matrix A^{lm} is smooth on U_N, we see that the Kähler-Poisson tensor g^{lm} admits a smooth extension to U_N which vanishes on $U_N \cap \Sigma$. We will prove that the standard star-product with separation of variables $*$ on $(U_N \backslash \Sigma, \omega_N)$ also admits a smooth extension to U_N.

As in Section 4, assume that x_0 is an arbitrary point in Σ and s is an index such that $\frac{\partial \psi}{\partial z^s} \neq 0$ on some neighborhood $V \subset U_N$ of x_0. On $U_N \backslash \Sigma$ we have

$$L_{\frac{\partial \Phi}{\partial z^k}} = \frac{\partial \Phi}{\partial z^k} + \nu \frac{\partial}{\partial z^k} = \psi^{-N-1} \frac{\partial \psi}{\partial z^k} + \nu \frac{\partial}{\partial z^k} =$$
$$\psi^{-N-1} \left(\frac{\partial \psi}{\partial z^k} + \nu \psi^{N+1} \frac{\partial}{\partial z^k} \right)$$

and the operator $L_{\frac{\partial \Phi}{\partial z^s}}$ is invertible on $V \backslash \Sigma$. Moreover, its inverse

$$(16) \quad \left(L_{\frac{\partial \Phi}{\partial z^s}} \right)^{-1} = \left(1 + \nu \psi^{N+1} \left(\frac{\partial \psi}{\partial z^s} \right)^{-1} \frac{\partial}{\partial z^s} \right)^{-1} \circ \left(\left(\frac{\partial \psi}{\partial z^s} \right)^{-1} \psi^{N+1} \right)$$

and the operators

$$\left(L_{\frac{\partial \Phi}{\partial z^s}} \right)^{-1} L_{\frac{\partial \Phi}{\partial z^k}} =$$

$$(17)$$

$$\left(1 + \nu \psi^{N+1} \left(\frac{\partial \psi}{\partial z^s} \right)^{-1} \frac{\partial}{\partial z^s} \right)^{-1} \circ \left(\left(\frac{\partial \psi}{\partial z^s} \right)^{-1} \left(\frac{\partial \psi}{\partial z^k} + \nu \psi^{N+1} \frac{\partial}{\partial z^k} \right) \right)$$

for $k \neq s$ admit smooth extensions to V.

As in Section 4, we want to construct a formal frame

$$\{z^1, \ldots, z^n, f^1, \ldots, f^n\}$$

on some neighborhood $W \subset V$ of x_0 such that the operators $L_{f^k}, k = 1, \ldots, n$, of the standard star product with separation of variables on $W \backslash \Sigma$ have smooth extensions to W. For $k \neq s$ denote by X_k the smooth extension of the operator (17) to V and set $f^k = X_k 1$. Then the coefficient f_0^k at the zero degree of the formal parameter ν in f^k is given by the formula

$$f_0^k = \left(\frac{\partial \psi}{\partial z^s} \right)^{-1} \left(\frac{\partial \psi}{\partial z^k} \right).$$

as in Section 4. To define the function f^s in the formal frame we will show that the smooth extension of the operator (16) to V has a smooth root of degree $N+1$ on a neighborhood of x_0. Shrinking, if necessary, the neighborhood V of x_0, denote by χ any root of degree $N+1$ of the function $\left(\frac{\partial \psi}{\partial z^s}\right)^{-1}$ on V. According to Lemma 5, there exists a unique formal function $u = u_0 + \nu u_1 + \ldots$ on $V\backslash\Sigma$ such that $u_0 = \chi\psi$ and $u^{*(-N-1)} = \psi^{-N-1}\frac{\partial \psi}{\partial z^s} = \frac{\partial \Phi}{\partial z^s}$. Therefore,

$$\left(L_u\right)^{N+1} = \left(L_{\frac{\partial \Phi}{\partial z^s}}\right)^{-1}$$

on $V\backslash\Sigma$. On the other hand, the operator $L_{\frac{\partial \Phi}{\partial z^s}}$ can be written in the form

$$L_{\frac{\partial \Phi}{\partial z^s}} = \psi^{-N-1}\frac{\partial \psi}{\partial z^s} + \nu\frac{\partial}{\partial z^s} = (\psi\chi)^{-N-1} + \nu\frac{\partial}{\partial z^s}$$

on $V\backslash\Sigma$, and its inverse,

$$\left(L_{\frac{\partial \Phi}{\partial z^s}}\right)^{-1} = \sum_{k=0}^{\infty} \nu^k \left((\psi\chi)^{N+1}\left(-\frac{\partial}{\partial z^s}\right)\right)^k \circ (\psi\chi)^{N+1},$$

has a smooth extension to V. Consider a homomorphism τ from the algebra $\mathcal{A}$ introduced in Section 5 to the algebra of differential operators on V such that $\tau(t_0) = \psi\chi$ and $\tau(\delta) = -\frac{\partial}{\partial z^s}$. Extend it to the mapping from $\mathcal{A}[[\nu]]$ to the algebra of formal differential operators on V by ν-linearity. Then

$$\tau(S) = \left(L_{\frac{\partial \Phi}{\partial z^s}}\right)^{-1} = (L_u)^{N+1}$$

for $S \in \mathcal{A}[[\nu]]$ given by formula (10). According to Lemma 3, there exists an element $A \in \mathcal{A}[[\nu]]$ such that $A^{N+1} = S$ and $A = t_0 \pmod{\nu}$. Therefore, $\tau(A) = \psi\chi \pmod{\nu}$ and

$$(\tau(A))^{N+1} = \left(L_{\frac{\partial \Phi}{\partial z^s}}\right)^{-1} = (L_u)^{N+1}$$

on $V\backslash\Sigma$. It follows from Lemma 4 that $\tau(A) = L_u$ on $V\backslash\Sigma$. Therefore the operator $\tau(A)$ is a smooth extension of the operator L_u to V. Set $f^s = \tau(A)1$. The coefficient of f^s at the zero degree of ν is $f_0^s = \psi\chi$. Now, taking into account that $\psi(x_0) = 0$, we get

$$\frac{\partial f_0^s}{\partial \bar{z}^l}(x_0) = \left(\frac{\partial \psi}{\partial \bar{z}^l}\chi\right)(x_0).$$

Since $\chi(x_0) \neq 0$, we obtain that the covector $\bar{\partial} f_0^s(x_0)$ is proportional to the nonzero covector $\bar{\partial}\psi(x_0)$ and is nonzero itself. Thus it can be proved as in Section 4 that the formal functions

(18) $\{z^1, \ldots, z^n, f^1, \ldots, f^n\}$

form a formal frame on a neighborhood of the point x_0. This formal frame satisfies the conditions of Theorem 2, which concludes the proof of the following theorem:

Theorem 3. *Let ψ be a defining function of a Levi-nondegenerate hypersurface Σ in an open set $U \subset \mathbb{C}^n$ and N be a natural number. Then there exists a neighborhood $U_N \subset U$ of Σ such that the potential*

$$\frac{1 - \psi^{-N}}{N}$$

defines a pseudo-Kähler form ω_N on $U_N \backslash \Sigma$, and both the corresponding Kähler-Poisson structure and the standard deformation quantization with separation of variables on $(U_N \backslash \Sigma, \omega_N)$ admit smooth extensions to the neighborhood U_N.

7. A smooth extension of a star product on an open subset of a Grassmannian

In this section we will use Theorem 2 to prove that the star product with separation of variables from Example 2 in Section 2 admits a smooth extension. Namely, let M be the set of complex $p \times r$-matrices $Z = (z_{k\alpha})$ and $\mathcal{O}$ be the subset of matrices $Z \in M$ such that the matrix $\Psi = E - Z^\dagger Z$ is nondegenerate. The potential $\Phi = \log |\det \Psi|$ determines a pseudo-Kähler form ω on $\mathcal{O}$.

Theorem 4. *The Kähler-Poisson structure corresponding to the pseudo-Kähler structure on $\mathcal{O}$ given by the form ω and the standard star product with separation of variables $*$ on $(\mathcal{O}, \omega)$ admit smooth extensions to M.*

Proof. The matrix $\Psi = E - Z^\dagger Z$ has the entries

$$\psi_{\alpha\beta} = \delta_{\alpha\beta} - \sum_{k=1}^{p} \bar{z}_{k\alpha} z_{k\beta}$$

and is invertible on $\mathcal{O}$. Denote the inverse matrix by $X = (\chi_{\beta\gamma})$. Then

$$\frac{\partial \Phi}{\partial z_{k\varkappa}} = \frac{\partial}{\partial z_{k\varkappa}} \log |\det \Psi| = \sum_{\alpha,\beta} \chi_{\beta\alpha} \frac{\partial \psi_{\alpha\beta}}{\partial z_{k\varkappa}} = -\sum_{\alpha} \chi_{\varkappa\alpha} \bar{z}_{k\alpha}.$$

Therefore, the left multiplication operator by $u_{\varkappa k} := \sum_{\alpha} \chi_{\varkappa\alpha} \bar{z}_{k\alpha}$ with respect to the star product $*$ is

$$L_{u_{\varkappa k}} = u_{\varkappa k} - \nu \frac{\partial}{\partial z_{k\varkappa}} = \sum_{\alpha} \chi_{\varkappa\alpha} \left(\bar{z}_{k\alpha} - \nu \sum_{\gamma} \psi_{\alpha\gamma} \frac{\partial}{\partial z_{k\gamma}} \right).$$

Taking into account that

$$\sum_k u_{\varkappa k} z_{k\beta} = \sum_{k,\alpha} \chi_{\varkappa\alpha} \bar{z}_{k\alpha} z_{k\beta} = \sum_\alpha \chi_{\varkappa\alpha}(\delta_{\alpha\beta} - \psi_{\alpha\beta}) = \chi_{\varkappa\beta} - \delta_{\varkappa\beta},$$

we obtain that

$$\sum_k L_{z_{k\beta}} L_{u_{\varkappa k}} = \sum_k \left(u_{\varkappa k} z_{k\beta} - \nu z_{k\beta} \frac{\partial}{\partial z_{k\varkappa}} \right) =$$

$$\chi_{\varkappa\beta} - \delta_{\varkappa\beta} - \nu \sum_k z_{k\beta} \frac{\partial}{\partial z_{k\varkappa}}$$

is the left multiplication operator by $\chi_{\varkappa\beta} - \delta_{\varkappa\beta}$. It follows that

$$L_{\chi_{\varkappa\beta}} = \chi_{\varkappa\beta} - \nu \sum_k z_{k\beta} \frac{\partial}{\partial z_{k\varkappa}} =$$

$$\sum_\alpha \chi_{\varkappa\alpha} \left(\delta_{\alpha\beta} - \nu \sum_{k,\lambda} \psi_{\alpha\lambda} z_{k\beta} \frac{\partial}{\partial z_{k\lambda}} \right).$$

Interpreting the operators $L_{\chi_{\varkappa\beta}}, \chi_{\varkappa\alpha}$ and

$$(19) \qquad\qquad \delta_{\alpha\beta} - \nu \sum_{k,\lambda} \psi_{\alpha\lambda} z_{k\beta} \frac{\partial}{\partial z_{k\lambda}}$$

as matrices whose entries are formal differential operators on $\mathcal{O}$, we see that the matrix (19) has a smooth extension to M and is invertible on M. Denote its inverse on M by $Q_{\beta\alpha}$. The inverse matrix of $L_{\chi_{\varkappa\beta}}$ on $\mathcal{O}$ is

$$J_{\beta\varkappa} := \sum_\alpha Q_{\beta\alpha} \circ \psi_{\alpha\varkappa}.$$

Its entries $J_{\beta\varkappa}$ are left multiplication operators with respect to the product $*$. The matrix $J_{\beta\varkappa}$ has a smooth extension to M. Observe that the entries of the matrix

$$K_{\beta k} := \sum_\varkappa J_{\beta\varkappa} L_{u_{\varkappa k}} = \sum_\alpha Q_{\beta\alpha} \circ \left(\bar{z}_{k\alpha} - \nu \sum_\gamma \psi_{\alpha\gamma} \frac{\partial}{\partial z_{k\gamma}} \right)$$

are also left multiplication operators with respect to the product $*$ and the matrix $K_{\beta k}$ also has a smooth extension to M. Denote by $f^{\beta k} = f_0^{\beta k} + \nu f_1^{\beta k} + \dots$ the formal function on M given by the formula $f^{\beta k} = K_{\beta k} 1$. Since $f_0^{\beta k} = \bar{z}_{k\beta}$, the functions $\{z_{j\alpha}, f^{\beta k}\}$ form a formal frame on M. Also, $L_{f^{\beta k}} = K_{\beta k}$ on $\mathcal{O}$. Now, it follows from Theorem 2 that the star product $*$ admits a smooth extension to M. $\square$

References

[1] Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., and Sternheimer, D.: Deformation theory and quantization. I. Deformations of symplectic structures. Ann. Physics **111** (1978), no. 1, 61 – 110.

[2] Bordemann, M. and Waldmann, S.: A Fedosov star product of the Wick type for Kähler manifolds. Lett. Math. Phys. **41** (3) (1997), 243 – 253.

[3] Engliš, M.: Weighted Bergman kernels and quantization, Commun. Math. Phys. **227** (2002), 211–241.

[4] Helgason, S.: Some Results on Invariant Differential Operators on Symmetric Spaces. Amer. J. Math. **114** (1992), 789-811.

[5] Karabegov, A.: Deformation quantizations with separation of variables on a Kähler manifold. Commun. Math. Phys. **180** (1996), 745–755.

[6] Karabegov, A.: A covariant Poisson deformation quantization with separation of variables up to the third order. Lett. Math. Phys. **61** (2002), 255 – 261.

[7] Karabegov, A.: Formal symplectic groupoid of a deformation quantization. Commun. Math. Phys. **258** (2005), 223–256.

[8] Karabegov, A.: Deformation quantization of a Kähler-Poisson structure vanishing on a Levi nondegenerate hypersurface. Contemporary Math. **450** (2008), 163 – 171.

[9] Kontsevich, M.: Deformation quantization of Poisson manifolds, I. Lett. Math. Phys. **66** (2003), 157 – 216.

[10] Leichtnam, E., Tang, X., and Weinstein, A.: Poisson geometry and deformation quantization near a strictly pseudoconvex boundary. Journal of the European Mathematical Society, **9**, (2007), 681–704.

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