Virial Theorem and Hypervirial Theorem in a spherical geometry

Yan Li,[1] Fu-Lin Zhang,[2,*] and Jing-Ling Chen[1,†]

[1] *Theoretical Physics Division, Chern Institute of Mathematics, Nankai University, Tianjin 300071, People's Republic of China*

[2] *Physics Department, School of Science, Tianjin University, Tianjin 300072, People's Republic of China*

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Abstract

In the paper, we obtain the Virial Theorem and Hypervirial Theorem in a spherical geometry. The Hypervirial Theorem and Hellmann-Feynman Theorem are used to formulate a *perturbation theorem without wave functions*. Here, this perturbation theorem is applied to one-dimensional harmonic oscillators and radial Coulomb problem.

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*Email:flzhang@tju.edu.cn

†Email:chenjl@nankai.edu.cn

I. INTRODUCTION

The Virial Theorem (VT) has been known for a long time in both classical mechanics and quantum mechanics. In the classical case, it provides a general equation relating the average over time of the kinetic energy $\langle T\rangle$ with that of the function of potential energy $\langle \vec{r}\cdot\nabla V\rangle$. The VT was given its technical definition by Clausius in 1870 [1]. Mathematically, the theorem states

$$2\langle T\rangle = \langle \vec{r}\cdot\nabla V\rangle. \tag{1}$$

If the potential takes the power function $V(r)=\alpha r^n$ with $r=|\vec{r}|$, the VT adopts a simple form as

$$2\langle T\rangle = n\langle V\rangle. \tag{2}$$

Thus, twice the average kinetic energy equals n times the average potential energy. The VT in quantum mechanics has the same form as the classical one, indicated in Eqs. (1) and (2) with the the average over a eigenstate of the system. It can date back to the old papers of Born, Heisenberg and Jordan [2], and be derived from the fact that the expectation value of the time-independent operator $\vec{r}\cdot\vec{p}$ under a eigenstate is a constant [3],

$$\frac{d}{dt}\langle\psi|\vec{r}\cdot\vec{p}|\psi\rangle = \langle\psi|[\vec{r}\cdot\vec{p},H]|\psi\rangle = 0,$$

where $H=\frac{p^2}{2}+V$ is the Hamiltonian and $|\psi\rangle$ is a eigenket of H.

In 1960, Hirschfelder [4] generalized the relationship by pointing out that $\vec{r}\cdot\vec{p}$ can be replaced by any other operators which are not dependent on time explicitly. In this way, he established the Hypervirial Theorem (HVT). For example, in a one-dimensional system, one can replace $\vec{r}\cdot\vec{p}=xp$ by the *hypervirial operator* x^kp, and obtain the recurrence relation of $\langle x^k\rangle$,

$$2kE\langle x^{k-1}\rangle = 2k\langle x^{k-1}V\rangle + \langle x^k\frac{dV}{dx}\rangle - \frac{1}{4}k(k-1)(k-2)\langle x^{k-3}\rangle, \tag{3}$$

where k is an integer.

The Hellmann-Feynman (HF) Theorem is another important theorem in quantum mechanics. which has been applied to the force concept in molecules by using the internuclear distance as a parameter [5, 6]. Let the Hamiltonian $H(\xi)$ of a system be a time-independent

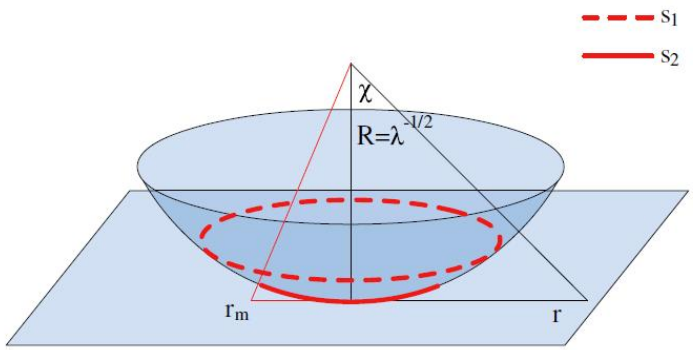

FIG. 1: The gnomonic projection, which is the projection onto the tangent plane from the center of the sphere in the embedding space. Two classical orbits are shown as s_1 (dashed line) and s_2 (solid line), corresponding to the circular motion and the radial one respectively.

operator that depends explicitly upon a continuous parameter ξ, and $|\psi(\xi)\rangle$ be a normalized eigenfunction of $H(\xi)$ with the eigenvalue $E_n(\xi)$, i.e. $H(\xi)|\psi(\xi)\rangle = E_n(\xi)|\psi(\xi)\rangle$, $\langle\psi(\xi)|\psi(\xi)\rangle = 1$. The HF theorem states that

$$\frac{\partial E_n(\xi)}{\partial \xi} = \langle\psi(\xi)|\frac{\partial H(\xi)}{\partial \xi}|\psi(\xi)\rangle. \tag{4}$$

If the potential takes the power function $V(r) = \alpha r^n$, the HF gives an equation representing the relation between eigenenergy E_m and mean value of r^n,

$$\frac{\partial E_m}{\partial \alpha} = \langle r^n\rangle. \tag{5}$$

Based on the relations in Eqs. (3) and (5), the Hypervirial-Hellmann-Feynman Theorem (HVHF) perturbation theorem had been established [7, 8]. It provides a very efficient algorithm for the generation of perturbation expansions to large order, replacing the formal manipulation of Fourier series expansions with recursion relations. This perturbation method just need the energy instead of the wave functions of the system, and it is easy to achieve on the computer.

These results are well known, but have not, to our knowledge, been exploited in a curved space. In the present work, we focus on the one- and two-dimensions spherical geometry. The coordinate systems adopted in this paper are shown in Fig. 1: (i) The three-dimensional cartesian coordinates (q_1, q_2, q_0) and spherical polar coordinates (R, χ, θ). We have $(q_1, q_2, q_0) = (R\sin\chi\cos\theta, R\sin\chi\sin\theta, R\cos\chi)$, where the curvature of the sphere is given by $\lambda = 1/R^2$. (ii) The two-dimensional gnomonic projection, which is the projection onto the tangent plane from the center of the sphere in the embedding space, corresponding to the cartesian coordinates (x_1, x_2) and polar coordinates (r, θ). Here, the radius $r = R\tan\chi$ and $(x_1, x_2) = (r\cos\theta, r\sin\theta)$.

In 1979, Higgs [9] introduced a generalization of the hydrogen atom and harmonic oscillator in a spherical space. He demonstrated that, in the gnomonic projection as shown in Fig. 1, the orbits of the motion on a sphere can be described by

$$\frac{1}{2}L^2[r^{-4}(\frac{dr}{d\theta})^2 + r^{-2}] + V(r) = E - \frac{1}{2}\lambda L^2, \tag{6}$$

where the angular momentum $L = x_1p_2 - x_2p_1$ is an invariant quantity with the potential $V(r)$ being radial symmetric. The Hamiltonian can be written as

$$H = \frac{\pi^2}{2} + \frac{1}{2}\lambda L^2 + V(r), \tag{7}$$

where $\vec{\pi} = \vec{p} + \frac{\lambda}{2}[\vec{x}(\vec{x}\cdot\vec{p}) + (\vec{p}\cdot\vec{x})\vec{x}]$ are the conserved vector in free particle motion on the sphere. Since the curvature appears only in the right combination $E - \frac{1}{2}\lambda L^2$ of Eq. (6), the projected orbits are the same, for a given $V(r)$, as in Euclidean geometry. Consequently, according with the Bertrand Theorem [10, 11], the orbits are closed only if the potential takes the Coulomb or isotropic oscillator form, i.e. $V(r) = -\frac{\kappa}{r}$ or $V(r) = \frac{1}{2}\omega^2 r^2$, with κ and ω being constants. Therefore the systems described by Eq. (53) with the two mentioned potentials are defined as the Kepler problem and isotropic oscillator in a spherical geometry in [9]. The algebraic relations of their conserved quantities reveal the dynamical symmetries of the two systems are described by the $SO(3)$ and $SU(2)$ Lie groups respectively. These results are the beginning of the so called *Higgs Algebra*, which has been studied in a variety of directions [12–16].

Since the symmetries of the Kepler problem and isotropic oscillator on a 2-sphere described by Eq. (53) adhere to the behaviors in two-dimensional Euclidean geometry, our question is: Do there exist more qualities of being homogeneous? This paper is aimed at constructing the VT and the HVT for the spherical geometry and studying their applications. The article is organized as follows. In the Sec. II, the VT in both classical mechanics and quantum mechanics are construced. In the Sec. III, we generalize the VT to HVT, and give the quantum hypervirial relation. Two examples are taken to demonstrate the perturbation method which is combined HVT and HF theorem in the Sec. IV. We end the paper with some relevant discussions in the last section.

II. THE VIRIAL THEOREM

A. Classical Virial Theorem

To obtain the classical TV in the two-dimensional spherical geometry, we start with two special orbits of motion. (i) The first one is the uniform circular motion with $\dot{r} = 0$ as shown by the curve s_1 in Fig. 1. The kinetic energy of this case is given by

$$T = \frac{1}{2}(R\sin\chi_0)^2\dot{\theta}^2,$$

where $R\sin\chi_0$ is the radius of the path. The corresponding centripetal force is

$$F = \frac{2T}{R\sin\chi_0} = \frac{(1+\lambda r^2)\vec{r}\cdot\nabla V}{R\sin\chi_0}. \tag{8}$$

Hence, one can obtain

$$2\langle T\rangle = \langle(1+\lambda r^2)\vec{r}\cdot\nabla V\rangle, \tag{9}$$

which can be considered as the VT under the case of uniform circular motion. (ii) The orbit s_2 in Fig. 1 depicted the case when the angular momentum $L = 0$. In the same way as the first case, the relationship between kinetic energy and potential energy in this case can be obtain as

$$2\langle(1+\lambda r^2)T\rangle = \langle(1+\lambda r^2)\vec{r}\cdot\nabla V\rangle. \tag{10}$$

These serve a good inspiration for us to presume that the VT in a spherical geometry is

$$2\langle(1+\lambda r^2)T_r\rangle + 2\langle T_\theta\rangle = \langle(1+\lambda r^2)\vec{r}\cdot\nabla V\rangle, \tag{11}$$

where T_r and T_θ are the radial and rotational kinetic energy.

Then, we will give the strict proof for the above relation. From the Eq. (6), we know that, when

$$E_s - \frac{1}{2}\lambda L_s^2 = E_p, \quad L_s = L_p, \tag{12}$$

the projected orbits are the same, for a given $V(r)$, as in Euclidean geometry. Here the subscripts p and s represent respectively the dynamic variables on a plane and on a sphere. This notations are is used in this paragraph only. It is easy to find that, for the corresponding points $(r_s, \theta_s) = (r_p, \theta_p) = (r, \theta)$, the velocities satisfy

$$\vec{v}_s = (1 + \lambda r^2)\vec{v}_p, \tag{13}$$

where $\vec{v}_s = (\dot{r}_s, r_s\dot{\theta}_s)$ and $\vec{v}_p = (\dot{r}_p, r_p\dot{\theta}_p)$. For the system in a flat space whose Hamiltonian given by $H = p^2/2 + V$, the two terms in Eq. (1) are

$$\begin{aligned}\langle \vec{r}_p \cdot \nabla V\rangle &= \frac{1}{\tau_p}\int_0^{\tau_p} \vec{r}_p \cdot \nabla V dt_p = \frac{1}{\tau_p}\int_c \vec{r}_p \cdot \nabla V \frac{1}{v_p^2}\vec{v}_p \cdot d\vec{s}_p,\\ \langle T_p\rangle &= \frac{1}{\tau_p}\int_0^{\tau_p} \frac{1}{2}v_p^2 dt_p = \frac{1}{\tau_p}\int_c \frac{1}{2}v_p^2\frac{1}{v_p^2}\vec{v}_p \cdot d\vec{s}_p,\end{aligned} \tag{14}$$

where $d\vec{s}_p = (dr_p, r_p d\theta_p)$, c denotes the orbit of motion, and τ_p is the period (for the aperiodic case $\tau_p \to +\infty$). Suppose the period of the system with the same orbit c in the sphere described by Eq. (53) is τ_s. Then, considering the relations in Eqs. (12) and (13), one can find

$$\begin{aligned}\langle \vec{r}_p \cdot \nabla V\rangle &= \frac{\tau_s}{\tau_p}\langle (1 + \lambda r_s^2)\vec{r}_s \cdot \nabla V\rangle,\\ \langle T_p\rangle &= \frac{\tau_s}{\tau_p}\left[\langle (1 + \lambda r_s^2)T_{sr}\rangle + \langle T_{s\theta}\rangle\right] = \frac{\tau_s}{\tau_p}\langle (1 + \lambda r_s^2)\frac{\pi_s^2}{2}\rangle,\end{aligned} \tag{15}$$

where the radial kinetic energy $T_{sr} = R^2\dot{\chi}_s^2/2 = \dot{r}_s^2/[2(1 + \lambda r_s^2)^2]$ and the rotational kinetic energy $T_{s\theta} = R^2 \sin^2\chi_s\dot{\theta}_s^2/2 = r_s^2\dot{\theta}_2^2/[2(1 + \lambda r_s^2)]$. And here the proof comes to an end.

Therefore, the relation in Eq. (11) is the VT in a spherical geometry, and it equivalents to

$$\langle (1 + \lambda r^2)\vec{r} \cdot \nabla V\rangle = \langle (1 + \lambda r^2)\pi^2\rangle = \langle (1 + \lambda r^2)(2T - \lambda L^2)\rangle. \tag{16}$$

It is easy to find that, when the curvature $\lambda \to 0$, the above result reduces to Eq. (1).

B. Quantum Virial Theorem

In the literature [9], to construct the the conserved quantities on the sphere, Higgs replaced the momentum $\vec{p}$ in the generators on the plan by the vector $\vec{\pi}$. This enlightens us obtain the VT in the sphere by the time independent operator $\vec{r}\cdot\vec{\pi}+\vec{\pi}\cdot\vec{r}$. The expected value of the commutator is

$$\langle[\vec{r}\cdot\vec{\pi}+\vec{\pi}\cdot\vec{r},H]\rangle=0. \tag{17}$$

For the system in the one-dimensional curve, whose Hamiltonian is given by $H=\pi^2/2+V$ with $\pi=p+\lambda(x^2p+px^2)/2$, the above relation leads to

$$\langle(1+\lambda x^2)\frac{\pi^2}{2}+\frac{\pi^2}{2}(1+\lambda x^2)\rangle+\frac{1}{2}\langle\lambda(1+\lambda x^2)(1+3\lambda x^2)\rangle=\langle(1+\lambda x^2)x\frac{dV}{dx}\rangle. \tag{18}$$

And for the two-dimensional case in Eq. (53), we obtain

$$\langle(1+\lambda r^2)\frac{\pi^2}{2}+\frac{\pi^2}{2}(1+\lambda r^2)\rangle+\frac{1}{2}\langle\lambda(1+\lambda r^2)(2+3\lambda r^2)\rangle=\langle(1+\lambda r^2)\vec{r}\cdot\nabla V\rangle. \tag{19}$$

In the polar coordinate, the Hamiltonian (53) can be written as

$$\begin{aligned}
&H_0=T_r+T_\theta+V,\\
&T_r=-\frac{1}{2}\left[3\lambda+\frac{15}{4}\lambda^2r^2+\frac{(1+\lambda r^2)(1+5\lambda r^2)}{r}\frac{\partial}{\partial r}+(1+\lambda r^2)^2\frac{\partial^2}{\partial r^2}\right],\\
&T_\theta=-\frac{1}{2}\left[\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\lambda\frac{\partial^2}{\partial\theta^2}\right],
\end{aligned} \tag{20}$$

where T_r and T_θ denote the radial and rotational kinetic energy. The relation in Eq. (19) equivalents to

$$\langle(1+\lambda r^2)T_r+T_r(1+\lambda r^2)\rangle+2\langle T_\theta\rangle+\frac{1}{2}\langle\lambda(1+\lambda r^2)(2+3\lambda r^2)\rangle=\langle(1+\lambda r^2)\vec{r}\cdot\nabla V\rangle. \tag{21}$$

These results have the same form with the classical mechanical counterparts in Eqs. (11) and (16), but with a term $\frac{1}{2}\langle\lambda(1+\lambda r^2)(2+3\lambda r^2)\rangle$ in addition. And, the term is different with the corresponding one in the one-dimensional case (18), which comes form the commutation relation of $\vec{r}$ and $\vec{\pi}$.

III. THE HYPERVIRIAL THEOREMS

In the above, we have got the VT in both classical mechanics and quantum mechanics. We will discuss the HTV in the present part. A natural candidate of the hypervirial operator is $r^k\pi+\pi r^k$ according to r^kp in the plane we mentioned in Sec. I, with k being integers.

A. One-dimensional

In the one-dimensional case, one can calculate directly the commutation relation in the expected value

$$\langle [x^k\pi + \pi x^k, H]\rangle = 0, \tag{22}$$

and obtain

$$\begin{aligned}[x^k\pi + \pi x^k, H] \;=\; & i\frac{k}{2}\Big\{2(1+\lambda x^2)x^{k-1}\pi^2 + 2\pi^2(1+\lambda x^2)x^{k-1} \\ & +(1+\lambda x^2)\big[(k-1)(k-2)x^{k-3} + 2\lambda k^2 x^{k-1} + \lambda^2(k+1)(k+2)x^{k+1}\big]\Big\} \\ & -2ix^k(1+\lambda x^2)\frac{dV}{dx}.\end{aligned} \tag{23}$$

Because of $\pi^2/2 = H - V$ and $\langle H\rangle = E_n$ being the eigenvalues of the eigenstate, the Eq. (22) turns to

$$\begin{aligned}& 2kE_n\langle x^{k-1}\rangle_\lambda - 2k\langle x^{k-1}V\rangle_\lambda - \langle x^k\frac{dV}{dx}\rangle_\lambda \\ & +\frac{k}{4}\big[(k+1)(k+2)\lambda^2\langle x^{k+1}\rangle_\lambda + 2k^2\lambda\langle x^{k-1}\rangle_\lambda + (k-1)(k-2)\langle x^{k-3}\rangle_\lambda\big] = 0,\end{aligned} \tag{24}$$

in which we denote $\langle f\rangle_\lambda = \langle (1+\lambda x^2)f\rangle$. Hence, we get the recurrence formula of $\langle x^k\rangle_\lambda$, which is the quantum hypervirial relations in the one-dimensional sphere.

B. Two-dimensional

We now consider HVT in the two-dimensional spherical geometry. For a radial potential $V = V(r)$ in the Hamiltonian (20), the eigenfunction of energy can be written as

$$\Psi(r,\theta) = e^{im\theta}\psi(r), \tag{25}$$

with $m = 0, \pm1, \pm2...$ is the eigenvalue of the conserved angular momentum L. The Schrödinger equation

$$H_0\Psi(r,\theta) = E\Psi(r,\theta) \tag{26}$$

reduces to the radial equation as

$$H_1\psi(r) = E\psi(r), \tag{27}$$

where the Hamiltonian H_1 is given by

$$H_1 = -\frac{1}{2}\left[(1+\lambda r^2)^2\frac{d^2}{dr^2} + \frac{(1+\lambda r^2)(1+5\lambda r^2)}{r}\frac{d}{dr} - \frac{1+\lambda r^2}{r^2}m^2 + 3\lambda + \frac{15}{4}\lambda^2 r^2\right] + V. \quad (28)$$

It can be written as

$$H_1 = \frac{\pi_r^2}{2} + V_1, \quad (29)$$

where the radial component of $\vec{\pi}$ is $\pi_r = -i[(1+\lambda r^2)\frac{d}{dr}+\frac{1}{2r}+\frac{3}{2}\lambda r]$ and $V_1 = V - \frac{1}{2}[(\frac{1}{2}-m^2)\lambda - \frac{m^2-1/4}{r^2}]$. Choosing the hypervirial operator as $r^k\pi_r + \pi_r r^k$, one can we get the recurrence relation

$$\begin{aligned}&2kE_n\langle r^{k-1}\rangle_\lambda - 2k\langle r^{k-1}V_1\rangle_\lambda - \langle r^k\frac{dV_1}{dx}\rangle_\lambda \\ &+\frac{k}{4}\left[(k+1)(k+2)\lambda^2\langle r^{k+1}\rangle_\lambda + 2k^2\lambda\langle r^{k-1}\rangle_\lambda + (k-1)(k-2)\langle r^{k-3}\rangle_\lambda\right] = 0, \quad (30)\end{aligned}$$

from

$$\langle[r^k\pi_r + \pi_r r^k, H_1]\rangle = 0. \quad (31)$$

Here, the notation $\langle f\rangle_\lambda = \langle(1+\lambda r^2)f\rangle$. It is the two-dimensional quantum hypervirial relation we will discuss in the present work. And when $\lambda \to 0$, it reduces to the result in the 2-plane case [17].

IV. APPLICATION OF THE HYPERVIRIAL THEOREMS

In this section, we will present the HFHV theorem in the spherical space based on the hypervirial relations in the above. When the perturbation of potential $V(r)$ takes the form as $r^l(1+\lambda r^2)$ with l being integers, we can determine the eigenenergies in the various orders of approximation without calculating the wavefunction, as the the HFHV theorem in the Euclidean geometry. In the following, we will give two sample examples to illustrate this method.

A. One-dimensional Harmonic Oscillator

The Hamiltonian of the one-dimensional harmonic oscillator in the spherical geometry with a perturbation potential we consider is

$$H = \frac{\pi^2}{2} + \frac{1}{2}\alpha x^2 + \beta x^l(1+\lambda x^2), \quad (32)$$

where α, β are real numbers, l is a integer and λ is the curvature of the sphere. The perturbation $\beta x^l(1+\lambda x^2)$ has to be very small, and β is the smallness parameter.

Then, the HVHF recurrence relation in Eq. (24) becomes

$$\left[(k+1)\alpha - \frac{k}{4}(k+1)(k+2)\lambda^2\right]\langle x^{k+1}\rangle_\lambda = 2kE_n\langle x^{k-1}\rangle_\lambda + \frac{k^3}{2}\lambda\langle x^{k-1}\rangle_\lambda - \frac{k}{4}(k-1)(k-2)\langle x^{k-3}\rangle_\lambda$$
$$-\beta(2k+l)\langle x^{k+m-1}\rangle_\lambda - \beta\lambda(2k+l+2)\langle x^{k+l+1}\rangle_\lambda. \quad (33)$$

The above equation establishes precisely regarding the n-th energy level. In order to obtain the approximate solution of the energy eigenvalues E_n, we expand both E_n and desired expectation values $\langle x^k\rangle_\lambda$ in powers of the perturbation parameter β as

$$E_n = E_n^{(0)} + \beta E_n^{(1)} + \beta^2 E_n^{(2)} + \cdots = \sum_{j=0}^{\infty} \beta^j E_n^{(j)}, \quad (34)$$

$$\langle x^k\rangle_\lambda = \langle x^k\rangle_{\lambda,0} + \beta\langle x^k\rangle_{\lambda,1} + \beta^2\langle x^k\rangle_{\lambda,2} + \cdots = \sum_{j=0}^{\infty} \beta^j \mathcal{Q}_j^k, \quad (35)$$

where we introduce the notation $\mathcal{Q}_j^k = \langle x^k\rangle_{\lambda,j}$ for convenience. We now insert the series (34) and (35) into (33) and order in power of β. It is straightforward to get the relation

$$\left[(k+1)\alpha - \frac{k}{4}(k+1)(k+2)\lambda^2\right]\mathcal{Q}_\gamma^{k+1} \quad (36)$$
$$= 2k\sum_{j=0}^{\gamma} E_n^j \mathcal{Q}_{\gamma-j}^{k-1} + \frac{k}{4}2k^2\lambda\mathcal{Q}_\gamma^{k-1} - \frac{k}{4}(k-1)(k-2)\mathcal{Q}_\gamma^{k-3} - (2k+l)\mathcal{Q}_{\gamma-1}^{k+l-1} - \lambda(2k+l+2)\mathcal{Q}_{\gamma-1}^{k+l+1}.$$

In addition, by the HF theorem, we know that

$$\frac{\partial E_n}{\partial\beta} = \langle\frac{\partial H}{\partial\beta}\rangle = \langle x^l\rangle_\lambda, \quad (37)$$

which gives another relationship of the coefficient of β:

$$E_n^{(j)} = \frac{1}{j}\mathcal{Q}_{j-1}^l. \quad (38)$$

In other words, the j-th approximate of energy eigenvalue $E_n^{(j)}$ is determined by the $(j-1)$-th approximate of desired values $\mathcal{Q}_{j-1}^l$.

In the following, we would like to give an explicit example. We let $l=1$ in the Eqs. (36) and (38) and obtain, respectively,

$$\left[(k+1)\alpha - \frac{k}{4}(k+1)(k+2)\lambda^2\right]\mathcal{Q}_\gamma^{k+1} \quad (39)$$
$$= 2k\sum_{j=0}^{\gamma} E_n^j \mathcal{Q}_{\gamma-j}^{k-1} + \frac{k}{4}2k^2\lambda\mathcal{Q}_\gamma^{k-1} + \frac{k}{4}(k-1)(k-2)\mathcal{Q}_\gamma^{k-3} - (2k+1)\mathcal{Q}_{\gamma-1}^{k} - \lambda(2k+3)\mathcal{Q}_{\gamma-1}^{k+2},$$

$$E_n^{(j)} = \frac{1}{j}\mathcal{Q}_{j-1}^1. \tag{40}$$

One can start from

$$\langle x^0\rangle_\lambda = \langle 1+\lambda x^2\rangle = 1+\lambda\langle x^2\rangle \tag{41}$$

to obtain $\mathcal{Q}_j^0$. By the HF theorem

$$\frac{\partial E_n}{\partial \alpha} = \langle\frac{\partial H}{\partial \alpha}\rangle = \frac{1}{2}\langle x^2\rangle, \tag{42}$$

one can find that

$$\frac{1}{2}\langle x^2\rangle = \sum_{j=0}\beta^j\frac{\partial E_n^{(j)}}{\partial \alpha}. \tag{43}$$

Substituting it to Eq. (41), the expectation value $\langle x^0\rangle_\lambda$ expansion will be denoted as

$$\begin{aligned}\langle x^0\rangle_\lambda = 1+\lambda\langle x^2\rangle &= \mathcal{Q}_0^0+\beta\mathcal{Q}_1^0+\beta^2\mathcal{Q}_2^0+\cdots \\ &= 1+2\lambda\left[\frac{\partial E_n^{(0)}}{\partial\alpha}+\beta\frac{\partial E_n^{(1)}}{\partial\alpha}+\beta^2\frac{\partial E_n^{(2)}}{\partial\alpha}+\cdots\right].\end{aligned} \tag{44}$$

Ordering in power of β, it is easy to find the first term of the recursion:

$$\begin{aligned}\mathcal{Q}_0^0 &= 1+2\lambda\frac{\partial E_n^{(0)}}{\partial\alpha} = 1+\frac{(2n+1)\lambda}{\sqrt{\lambda^2+4\alpha}}, \\ \mathcal{Q}_1^0 &= 2\lambda\frac{\partial E_n^{(1)}}{\partial\alpha}, \\ \mathcal{Q}_2^0 &= 2\lambda\frac{\partial E_n^{(2)}}{\partial\alpha}, \\ &\vdots\end{aligned} \tag{45}$$

The eigenenergy of one-dimensional harmonic oscillator in a spherical geometry is $E_n^{(0)} = (n+\frac{1}{2})\frac{\lambda+\sqrt{\lambda^2+4\alpha}}{2}+\frac{n^2}{2}\lambda$ [18, 19].

When $\gamma=0$, one can substitute $\mathcal{Q}_0^0$ into Eq. (39) and obtain the values of $\mathcal{Q}_0^j$,

$$k = 0 \qquad \mathcal{Q}_0^1 = 0, \tag{46}$$

$$k = 1 \qquad \mathcal{Q}_0^2 = \frac{[(2n+1)\lambda+\sqrt{\lambda^2+4\alpha}]}{(4\alpha-3\lambda^2)\sqrt{\lambda^2+4\alpha}}\left[(2n+1)\sqrt{\lambda^2+4\alpha}+(2n^2+2n+3)\lambda\right], \tag{47}$$

$$\vdots$$

Using Eq. (46) and (40), we can get the first-order perturbation of E_n,

$$E_n^{(1)} = \mathcal{Q}_0^1 = 0. \tag{48}$$

And from the Eq. (45) and (48), we have

$$\mathcal{Q}_1^0 = \frac{\partial E_n^{(1)}}{\partial \alpha} = 0. \tag{49}$$

In the case of $\gamma = 1$, using $\mathcal{Q}_1^0$ and Eq. (39), we can derive the values of $\mathcal{Q}_1^j$, and consequently the second approximation of energy level

$$\begin{aligned} E_n^{(2)} &= -\frac{\sqrt{\lambda^2+4\alpha}+(2n+1)\lambda}{2\alpha\sqrt{\lambda^2+4\alpha}} \\ &\quad -\frac{3\lambda[(2n+1)\lambda+\sqrt{\lambda^2+4\alpha}]}{2\alpha(4\alpha-3\lambda^2)\sqrt{\lambda^2+4\alpha}}\left[(2n+1)\sqrt{\lambda^2+4\alpha}+(2n^2+2n+3)\lambda\right]. \end{aligned} \tag{50}$$

In this way, we can obtain the expectation value expansions $\mathcal{Q}_\gamma^j$ and the energy values $E_n^{(j)}$ in the various orders of approximation as

$$E_n^{(3)} = 0, \tag{51}$$

$$\begin{aligned} E_n^{(4)} &= -\frac{1}{4\alpha}\left\{\left(\frac{6\lambda^2(2E_n^{(0)}+\lambda)}{4\alpha-3\lambda}+\lambda\right)\frac{\partial \mathcal{Q}_1^1}{\partial\alpha}+2E_n^{(2)}\mathcal{Q}_0^0-3\mathcal{Q}_1^1\right. \\ &\quad \left.-\frac{5\lambda}{3\alpha-6\lambda^2}\left[(4E_n^{(0)}+4\lambda)\mathcal{Q}_1^1-(5+\frac{21\lambda E_n^{(0)}}{2\alpha-15\lambda^2})\mathcal{Q}_0^2-\frac{21\lambda}{8\alpha-60\lambda}\mathcal{Q}_0^0\right]\right\} \\ &\quad \vdots \end{aligned} \tag{52}$$

In the limit $\lambda \to 0$, $E_n^{(2)}$ is tending to $-1/(2\alpha)$ and the other $E_n^{(j)}$ is tending to zero which are corresponded with the exact result in the Euclidean space.

It is worth to mentioned that, alien from Euclidean space, (i) The HF theorem has been used twice in this HVHF perturbative method. (ii) Only when the exponent l in the perturbation potential is a positive integer, we can get $\mathcal{Q}_\gamma^j$ from Eqs. (45) (39) and (40).

B. Two-dimensional Coulomb System

Here we wish to show that the HVHF perturbation method can be easily applied to treat the coulomb system with a perturbation in the two-dimensional sphere which is described by the Hamiltonian

$$H = \frac{\pi^2}{2} + \frac{1}{2}\lambda L^2 - \frac{\kappa}{r} + \beta r^l(1+\lambda r^2), \tag{53}$$

where κ is a real number, and β is the perturbation parameter. Hence, the potential in the radial Hamiltonian (29) is

$$V_1 = -\frac{\kappa}{r} + \beta r^l(1+\lambda r^2) - \frac{1}{2}[(\frac{1}{2} - m^2)\lambda - \frac{m^2 - 1/4}{r^2}]. \tag{54}$$

The hypervirial relation Eq. (30) turns to

$$\begin{aligned}
&\frac{1}{4}\left[k(k-1)(k-2) - (k-1)(4m^2-1)\right]\langle r^{k-3}\rangle_\lambda + \frac{\lambda k}{2}(k^2+2-4m^2)\langle r^{k-1}\rangle_\lambda \\
&+2kE_n\langle r^{k-1}\rangle_\lambda + 2(k-1)\kappa\langle r^{k-2}\rangle_\lambda + \frac{k}{4}(k+1)(k+2)\langle r^{k+1}\rangle_\lambda \\
&-\beta(2k+l)\langle r^{k+l-1}\rangle_\lambda - \beta\lambda(2k+l+2)\langle r^{k+l+1}\rangle_\lambda = 0.
\end{aligned} \tag{55}$$

Considering the angular quantum number m^2 as a parameter of the potential V_1, one can obtain the expansion coefficients for $\langle r^{-2}\rangle_\lambda$ by using the HF theorem,

$$\langle r^{-2}\rangle_\lambda \;=\; \langle r^{-2}\rangle + \lambda\langle 1\rangle = \lambda + 2\frac{\partial E_n}{\partial m^2}. \tag{56}$$

From this starting point, as we show in the one-dimensional case, we can get any order perturbation on the energy level, with the precondition l being a negative integer

Taking $l = -3$ for example, in the first approximation, the eigenvalue E_n is

$$\begin{aligned}
E_n \;=\; &-\frac{\alpha^2}{2(n+\sqrt{m^2}+\frac{1}{2})^2} + \frac{\lambda}{2}(n+\sqrt{m^2})(n+\sqrt{m^2}+\frac{1}{2}) \\
&-\beta\frac{4\kappa^3}{\sqrt{m^2}(4m^2-1)(n+\sqrt{m^2}+\frac{1}{2})^3} - \beta\frac{\kappa\lambda}{\sqrt{m^2}(4m^2-1)}(4n+4\sqrt{m^2}+1).
\end{aligned} \tag{57}$$

When $\lambda \to 0$, this result is coincided with the literature [20].

V. CONCLUSION AND DISCUSSION

The VT in a spherical geometry has been proved in both classical and quantum conditions. We also have considered the HVT and got the hypervirial relations. The HTV and HF theorems have been shown to provide a powerful method of generating perturbation expansions. We have taken the Coulomb problem and harmonic oscillator for instances to illustrate this method. When the curvature λ is zero, the results reduce to the counterpart of Euclidean space.

In this paper, we only give attention to one- and two-dimensional systems. Since the Higgs' results have extended to the N-dimensional spherical geometry directly, we can foretell our treatment can be generalized to the N-sphere and suggest the VT is given by

$\langle(1+\lambda r^2)\frac{\pi^2}{2}+\frac{\pi^2}{2}(1+\lambda r^2)\rangle+\frac{1}{2}\langle\lambda(1+\lambda r^2)(N+3\lambda r^2)\rangle = n\langle(1+\lambda r^2)V\rangle$. Some researchers have discussed the superintegrable potentials in the the hyperbolic plane [21], it is interesting and possible to study the VT , HVT and HVHF in the situation of the curvature $\lambda < 0$. On the other hand, the systems in the curved space we investigate in this work also can be considered as the problems with position-dependent effective mass, which are widely applied in various areas of material science and condensed matter [19, 22–24]. We hope to find the applications of our results in these directions in the further research.

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