STRATIFIED BUNDLES AND ÉTALE FUNDAMENTAL GROUP

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Abstract. On X projective smooth over an algebraically closed field of characteristic $p > 0$, we show that irreducible stratified bundles have rank 1 if and only if the commutator $[\pi_1^{\text{ét}}, \pi_1^{\text{ét}}]$ of the étale fundamental group is a pro-p-group, and we show that the category of stratified bundles is semi-simple with irreducible objects of rank 1 if and only if $\pi_1^{\text{ét}}$ is abelian without p-power quotient. This answers positively a conjecture by Gieseker [3, p. 8].

1. Introduction

Let X be a smooth projective variety defined over an algebraically closed field k of characteristic $p > 0$. In [3], stratified bundles are defined and studied. It is shown that they are a characteristic $p > 0$ analog to complex local systems over smooth complex algebraic varieties. In particular, Gieseker shows [3, Theorem 1.10]

(i) If every stratified bundle is trivial, then π_1 is trivial.
(ii) If all the irreducible stratified bundles have rank 1, then $[\pi_1, \pi_1]$ is a pro-p-group.
(iii) If every stratified bundle is a direct sum of stratified line bundles, then π_1 is abelian without non-trivial p-power quotient.

Here π_1 is the étale fundamental group based at some geometric point. He conjectures that in the three statements, the "if" can be replaced by "if and only if". The aim of this note is to give a positive answer to Gieseker's conjecture (see Theorem 3.6).

The converse to (i) is the main theorem of [2], and is analog to Malcev-Grothendieck theorem ([5], ([4]) asserting that if the étale fundamental group of a smooth complex projective variety is trivial, then there is no non-trivial bundle with flat connection.

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The complex analog to the converse to (iii) relies on the following.

If X is an abelian variety over a field k, Mumford [6, Section 16] showed that a non-trivial line bundle L which is algebraically equivalent to 0 fulfills $H^i(X,L) = 0$ for all $i \geq 0$. If $k = \mathbb{C}$, the field of complex numbers, L carries a unitary flat connection with underlying local system ℓ, and $H^1(X,L) = 0$ implies that $H^1(X,\ell) = 0$. In fact, more generally, if X is any complex manifold with abelian topological fundamental group, and ℓ is a non-trivial rank 1 local system, then $H^1(X,\ell) = 0$ (see Remark 3.4). The key point to show the converse to (iii) is the corresponding statement, replacing the rank 1 complex local system ℓ by a rank 1 stratified bundle on X smooth projective (see Lemma 3.2).

Finally the complex analog to the converse to (ii) and (iii) together says that if X is a smooth complex variety, then the only irreducible complex local systems on X have rank 1 if and only if $\pi^{\text{ét}}$ is abelian. However, the semi-simplicity statement has a different phrasing: the only irreducible complex local systems on X have rank 1 and the category is semi-simple if and only if $\pi^{\text{ét}}$ is a finite abelian group (Remark 3.7).

The proof of the converse to (ii) is done in Section 2. It relies on a consequence of the proof of the main theorem in [2], which is formulated in Theorem 2.3. It is a replacement for the finite generation of the topological fundamental group in complex geometry (see the analog statement in Remark 3.7). The proof of the converse to (iii) is done in Section 3. It relies on the Tannaka formalism which allows us to compute extensions of rank 1 stratified bundles by group scheme cohomology, and on the smoothness ([1]) of the group schemes considered.

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2. Stratified bundles

Let k be an algebraically closed field of characteristic $p > 0$, and X a smooth connected projective variety over k. A stratified bundle on X is by definition a coherent $\mathcal{O}_X$-module $\mathcal{E}$ with a homomorphism

$$\nabla : \mathcal{D}_X \to \mathcal{E}nd_k(\mathcal{E})$$

of $\mathcal{O}_X$-algebras, where $\mathcal{D}_X$ is the sheaf of differential operators acting on the structure sheaf of X. By a theorem of Katz (cf. [3, Theorem 1.3]), it is equivalent to the following definition. (The terminology is not unique: stratified bundles from the following definition are called flat bundles in [3], F-divided sheaves in [1]).

Definition 2.1. A stratified bundle on X is a sequence of bundles

$$E = \{E_0, E_1, E_2, \cdots, \sigma_0, \sigma_1, \ldots\} = \{E_i, \sigma_i\}_{i \in \mathbb{N}}$$

where $\sigma_i : F^*E_{i+1} \to E_i$ is a $\mathcal{O}_X$-linear isomorphism, and $F : X \to X$ is the absolute Frobenius.

A morphism $\alpha = \{\alpha_i\} : \{E_i, \sigma_i\} \to \{F_i, \tau_i\}$ between two stratified bundles is a sequence of morphisms $\alpha_i : E_i \to F_i$ of $\mathcal{O}_X$-modules such that

$$\begin{array}{ccc} F^*E_{i+1} & \xrightarrow{F^*\alpha_{i+1}} & F^*F_{i+1} \\ {\scriptstyle \sigma_i}\downarrow & & \downarrow{\scriptstyle \tau_i} \\ E_i & \xrightarrow{\alpha_i} & F_i \end{array}$$

is commutative. The category $\mathbf{str}(X)$ of stratified bundles is abelian, rigid, monoidal. To see it is k-linear, it is better to define stratified bundles and morphisms in the relative version: objects consist of

$$E' = \{E_0, E'_1, E'_2, \cdots, \sigma'_0, \sigma'_1, \ldots\} = \{E'_i, \sigma'_i\}_{i \in \mathbb{N}}$$

where E'_i is a bundle on the i-th Frobenius twist $X^{(i)}$ of X,

$$\sigma'_i : F^*_{i,i+1}E_{i+1} \to E_i$$

is a $\mathcal{O}_{X^{(i)}}$-linear isomorphism, and $F_{i,i+1} : X^{(i)} \to X^{(i+1)}$ is the relative Frobenius; morphisms are the obvious ones. A rational point $a \in X(k)$ yields a fiber functor $\omega_a : \mathbf{str}(X) \to \mathbf{vec}_k, \quad E \mapsto a^*E_0$ with values in the category of finite dimensional vector spaces. Thus $(\mathbf{str}(X), \omega_a)$ is a Tannaka category, and one has an equivalence of categories

$$\mathbf{str}(X) \xrightarrow{\omega_a\ \cong} \mathbf{rep}_k(\pi^{\mathrm{str}}) \tag{2.1}$$

where

$$\pi^{\mathrm{str}} := \pi^{\mathrm{str}}(X, a) := \mathrm{Aut}^{\otimes}(\omega_a)$$

is the Tannaka group scheme, and $\mathbf{rep}_k(\pi^{\mathrm{str}})$ is the category of finite dimensional k-representations of π^{str}. Let $\pi_1 := \pi_1^{\text{ét}}(X, a)$ be the étale fundamental group of X. In [3, Theorem 1.10], D. Gieseker proved the following theorem.

Theorem 2.2. *(Gieseker): Let X be a smooth projective variety over an algebraically closed field k, then*

(i) *If every stratified bundle is trivial, then π_1 is trivial.*
(ii) *If all the irreducible stratified bundles have rank 1, then $[\pi_1, \pi_1]$ is a pro-p-group.*

(iii) *If every stratified bundle is a direct sum of stratified line bundles, then* π_1 *is abelian with no p-power order quotient.*

Then Gieseker conjectured that the converse of the above statements might be true. In [2], it is proven that the converse of statement (i) is true. The aim of this section is to prove the converse of (ii). We prove the converse of (iii) in Section 3.

The proof relies on the following theorem extracted from [2], and which plays a similar rôle as the finite generation of the topological fundamental group in complex geometry (see Remark 3.7). Fixing an ample line bundle $\mathcal{O}_X(1)$ on X, recall that E is said to be μ-stable if for all coherent subsheaves $U \subset E$ one has $\mu(U) < \mu(E)$.

Theorem 2.3. *Let* X *be a smooth projective variety over an algebraically closed field* k. *If there is a stratified bundle* $E = (E_n, \sigma_n)_{n\in\mathbb{N}}$ *of rank* $r \geq 2$ *on* X, *where* $\{E_n\}_{n\in\mathbb{N}}$ *are* μ*-stable, then there exists an irreducible representation* $\rho : \pi_1 \to GL_r(V)$ *with finite monodromy, where* V *is a* r*-dimensional vector space over* $\bar{\mathbb{F}}_p$.

Proof. By the proof of [2, Theorem 3.15], there is a good model $X_S \to S$ of $X \to \operatorname{Spec} k$, with model $a_S \in X_S(S)$ of $a \in X(k)$, where S is smooth affine over $\mathbb{F}_p$, together with a good model $M_S \to S$ of the moduli of stable vector bundles considered in [2, Section 3], there is a closed point $u \to M$ corresponding to a stable bundle E over $X_{\bar{s}}$, where s is the closed point u viewed as a closed point of S, and $\bar{s}$ is a geometric closed point above it, such that $(F^m)^*E \cong E$ where F is the absolute Frobenius on $X_{\bar{s}}$. Thus E yields an irreducible representation $\rho' : \pi_1^{\text{ét}}(X_{\bar{s}}, b) \to GL(E_b)$, where $b = a_S \otimes \bar{s}$. As the specialization homomorphism $\mathrm{sp} : \pi_1^{\text{ét}} \to \pi_1^{\text{ét}}(X_{\bar{s}}, b)$ is surjective ([8, Exposé X, Théorème 3.8]), the composite $\rho = \mathrm{sp} \circ \rho'$ is a representation of π_1 with the same finite irreducible monodromy. This is the solution to the problem. □

We now use the following two elementary lemmas.

Lemma 2.4 (See Chapter 8, Proposition 26 of [7])**.** *Let* G *be a finite* p*-group and* $\rho : G \to GL(V)$ *be a representation on a vector space* V *over a field* k *of characteristic* p. *Then*

$$V^G = \{\, v \in V \mid \rho(g)v = v,\ \forall\, g \in G \,\} \neq 0.$$

Lemma 2.5. *Let* G *be a finite group and* $\rho : G \to GL(V)$ *be an irreducible representation on a finite dimensional vector space* V *over an algebraically closed field* k *of characteristic* $p > 0$. *If the commutator* $[G, G]$ *of* G *is a* p*-group, then* $\dim(V) = 1$.

Proof. If $[G,G]$ is a p-group, by Lemma 2.4, there exists a $0 \neq v \in V$ such that $\rho(g_1)\rho(g_2)v = \rho(g_2)\rho(g_1)v$ for any g_1, $g_2 \in G$. Since V is an irreducible G-module, the sub-vector space spanned by the orbit $\rho(G)v$ is V itself. Thus for all $w \in V$, all $g \in G$, there are $a_g(w) \in k$, such that

$$w = \sum_{g \in G} a_g(w)\rho(g)v.$$

On the other hand, for all $g_1, g_2, g \in G$, one has $g_1g_2 = g_2g_1c$ for some $c \in [G,G]$, thus $g_1g_2g = g_2g_1cg = g_2g_1gcc'$, for some $c' \in [G,G]$. We conclude that $\rho(g_1)\rho(g_2)w = \rho(g_2)\rho(g_1)w$ for any g_1, $g_2 \in G$, that is $\rho(G)$ is abelian. As ρ is irreducible, V must have dimension 1. □

Theorem 2.6. *Let X be a smooth projective variety over an algebraically closed field k. Then the only irreducible stratified bundles on X have rank 1 if and only if the commutator $[\pi_1, \pi_1]$ of π_1 is a pro-p-group.*

Proof. One direction is the (ii) in Theorem 2.2. We prove the converse. Assume that $[\pi_1, \pi_1]$ is a pro-p-group. Let $E = (E_n, \sigma_n)_{n \in \mathbb{N}}$ be an irreducible stratified bundle of rank $r \geq 1$ on X. Assume $r \geq 2$. Then by [2, Proposition 2.3], there is a $n_0 \in \mathbb{N}$ such that the stratified bundle $E(n_0) := (E_n, \sigma_n)_{n \geq n_0}$ is a successive extension of stratified bundles $U = (U_n, \tau_n)$ with underlying bundles U_n being μ-stable. But E being irreducible implies that $E(n_0)$ is irreducible as well. Hence all the E_n are μ-stable for $n \geq n_0$. By Theorem 2.3, there is an irreducible representation $\rho : \pi_1 \to GL(V)$ of dimension $r \geq 2$ with finite monodromy over an algebraically closed field of characteristic $p > 0$. By Lemma 2.5, this is impossible. Thus $r = 1$. □

3. Extensions of stratified line bundles

In this section, we prove the following theorem.

Theorem 3.1. *Let X be a smooth projective connected variety over an algebraically closed field k of characteristic $p > 0$. If the étale fundamental group π_1 of X has no non-trivial p-power order quotient, then any extension*

$$0 \to L \to E \to L' \to 0$$

in $\textbf{str}(X)$ is split when L and L' are rank 1 objects.

Proof. By twisting with $(L')^{-1}$, we may assume that $L' = \mathbb{I}$, the trivial object in $\mathbf{str}(X)$. By Tannaka duality, one has

$$\mathrm{Ext}^1_{\mathbf{str}(X)}(\mathbb{I}, L) = \mathrm{Ext}^1_{\mathbf{rep}_k(\pi^{\mathrm{str}})}(k, \omega_a(L)) = H^1(\pi^{\mathrm{str}}, \omega_a(L))$$

where k is the trivial representation of $\pi^{\rm str}$.

By Tannaka duality again, one has $H^1(\pi^{\rm str}, k) = {\rm Ext}^1_{\bf str}(X)(\mathbb{I}, \mathbb{I})$, and this group vanishes by [2, Proposition 2.4], since π_1 has no non-trivial p-power quotient. So we want to show vanishing of $H^1(\pi^{\rm str}, \ell)$ under the assumption that π_1 has no non-trivial p-power order quotient, and that ℓ is a non-trivial character of the group scheme $\pi^{\rm str}$.

Let $C(\pi^{\rm str}) = [\pi^{\rm str}, \pi^{\rm str}]$ be the commutator of $\pi^{\rm str}$. It is a normal subgroup scheme of $\pi^{\rm str}$. As ℓ is a non-trivial character of $\pi^{\rm str}$, it is induced by a non-trivial character of $G = \pi^{\rm str}/C(\pi^{\rm str})$. One has an exact sequence

$$H^1(G, \ell) \to H^1(\pi^{\rm str}, \ell) \to H^1(C(\pi^{\rm str}), \ell). \tag{3.1}$$

We show that $H^1(\pi^{\rm str}, \ell) = 0$ in a number of lemmas. □

Lemma 3.2. *Let G be a commutative group scheme over an algebraically closed field k such that all its quotients in $\mathbb{G}_m$ are smooth. Let ℓ be a non-trivial character of G. Then $H^1(G, \ell) = 0$.*

Proof. Let $\chi : G \to \mathbb{G}_m = {\rm Aut}(\ell)$ be the non-trivial character, and let $\sigma : G \to \ell$ be a cocycle representing a class in $H^1(G, \ell)$. By definition of a cocycle, one has $\sigma(gh) = \chi(g)\sigma(h) + \sigma(g)$. The commutativity of G implies

$$\sigma(hg) = \chi(h)\sigma(g) + \sigma(h) = \sigma(gh) = \chi(g)\sigma(h) + \sigma(g). \tag{3.2}$$

As χ is non-trivial, and ${\rm Im}(\chi) \subset \mathbb{G}_m$ is assumed to be smooth, there is a $h \in G(k)$ such that

$$0 \neq (\chi(h) - 1) \in {\rm End}(\ell) = k.$$

Thus

$$(\chi(h) - 1) \in k^\times = \mathbb{G}_m(k) = {\rm Aut}(\ell).$$

Set $v = (\chi(h)-1)^{-1}\sigma(h) \in \ell$. Then, by (3.2), one has $\sigma(g) = \chi(g)v - v$, which means that σ is a coboundary. Thus $H^1(G, \ell) = 0$. □

Corollary 3.3. *Let ℓ is a non-trivial character of $\pi^{\rm str}$, and still denote by ℓ the induced character of $G = \pi^{\rm str}/C(\pi^{\rm str})$. Then $H^1(G, \ell) = 0$.*

Proof. By [1, Corollary 12], we know that the monodromy group ${\rm Im}(\chi) \subset {\rm Aut}(\ell) = \mathbb{G}_m$ is smooth, so we can apply Lemma 3.2. □

Remark 3.4. The same proof shows that if X is a smooth complex variety, with abelian topological fundamental group, and if ℓ is a non-trivial rank 1 local system, then $H^1(X, \ell) = 0$. Indeed, G is now $\pi_1^{\rm top}(X, a)$, the topological fundamental group based at a complex point $a \in X(\mathbb{C})$, and non-triviality implies the existence of $h \in G$ with $(\chi(h) - 1) \in \mathbb{C}^\times = {\rm Aut}(\ell)$. One then concludes identically.

This fact ought to be well known, but we could not find a reference in the literature.

Lemma 3.5. *Under the assumption of Theorem 3.1, one has*

$$H^1(C(\pi^{\mathrm{str}}), \ell) = 0.$$

Proof. As ℓ is a character of G, it induces the trivial representation of $C(\pi^{str})$, which we denote by k. We compute

$$H^1(C(\pi^{\mathrm{str}}), \ell) = H^1(C(\pi^{\mathrm{str}}), k)$$

using the exact sequence

$$H^1(\pi^{\mathrm{str}}, k) \to H^1(C(\pi^{\mathrm{str}}), k) \to H^2(G, k). \tag{3.3}$$

Recall that $H^1(\pi^{\mathrm{str}}, k) = \mathrm{Ext}^1_{\mathbf{str}(X)}(\mathbb{I}, \mathbb{I}) = 0$ under the assumption of Theorem 3.1. To show $H^2(G, k) = 0$, let

$$0 \to \mathbb{I} \to E_2 \to E_1 \to \mathbb{I} \to 0 \tag{3.4}$$

be a 2-extension in $\omega_a^{-1}(\mathbf{rep}(G)) \subset \mathbf{str}(X)$ (see (2.1) for the notation). Let $V \subset E_1$ be the image of E_2 in $\omega_a^{-1}(\mathbf{rep}(G)) \subset \mathbf{str}(X)$. This defines an extension $0 \to V \to E_1 \to \mathbb{I} \to 0$ in $\omega_a^{-1}(\mathbf{rep}(G))$, thus a class in $H^1(G, \omega_a(V))$. Since G is abelian, $\omega_a(V)$ has a filtration with rank 1 successive quotients ℓ, thus $H^1(G, \omega_a(V)) = 0$ by Lemma 3.2 if ℓ is non-trivial, and by $H^1(\pi^{\mathrm{str}}, k) = 0$ if $\ell = k$. A fortiori the class of the 2-extension (3.4) is trivial in $H^2(G, k)$. Summarizing, one has

$$H^1(\pi^{\mathrm{str}}, k) = H^2(G, k) = 0. \tag{3.5}$$

We conclude via (3.3) that

$$H^1(C(\pi^{\mathrm{str}}), \ell) = H^1(C(\pi^{\mathrm{str}}), k) = 0 \tag{3.6}$$

as well. □

Theorem 3.6. *Let X be a smooth projective variety over an algebraically closed field k of characteristic $p > 0$, then:*

(i) *Every stratified bundle on X is trivial if and only if π_1 is trivial.*
(ii) *All the irreducible stratified bundles have rank 1 if and only if $[\pi_1, \pi_1]$ is a pro-p-group.*
(iii) *Every stratified bundle is a direct sum of stratified line bundles, that is $\mathbf{str}(X)$ is a semi-simple category with irreducible objects of rank 1, if and only if π_1 is abelian with no non-trivial p-power quotient.*

Proof. (i) is the main theorem of [2]. (ii) is Theorem 2.6. To show (iii), assume that π_1 is abelian with no p-power order quotient. Let $E = (E_n, \sigma_n)_{n \in \mathbb{N}}$ be a stratified bundle on X. By (ii), any irreducible stratified bundle has rank 1. Thus there is a filtration $0 = E^0 \subset E^1 \subset \ldots \subset E^r = E$ in $\mathbf{str}(X)$ such that $L^v = E^v/E^{v-1}$, $1 \le v \le r$, are rank one stratified bundles. Then, by Theorem 3.1, the filtration splits and E is a direct sum of rank 1 objects. □

Remark 3.7. The analog of (ii) and (iii) in complex geometry is more subtle to describe. Let X be a smooth complex variety. If the only irreducible complex local systems have rank 1, then the algebraic completion of the topological fundamental group is abelian, thus so is π_1. Vice-versa, if π_1 is abelian, then the irreducible complex local systems have rank 1. Indeed, let V be a representation of the topological fundamental group $\pi_1^{\rm top}$. As it is finitely generated, it has values in $GL(A)$, for A a ring of finite type over a localization of $\mathbb{Z}$. To say that the representation is irreducible is to say that the finitely many matrices $M_i \in GL(A)$, images of the finitely many generators of $\pi_1^{\rm top}$ chosen, have no common invariant subspace. So there is a maximal ideal $u \subset A$, with finite residue field $\kappa(u)$, such that the $M_i \otimes \kappa(u) \in GL(A \otimes \kappa(u))$ have no common invariant subspace as well, that is the induced representation in $GL(A \otimes \kappa(u))$ is irreducible. Since this is induced by a representation of π_1, it has to have rank 1. Remark 3.4 kills extensions of the trivial complex local system by a non-trivial one, but does not kill the extension of the trivial complex local by itself. Those are classified by the first de Rham cohomology group $H^1_{DR}(X)$, which is equal to 0 if and only if $\pi_1^{\rm top}$, which was supposed to be abelian, is in addition finite.

So we conclude: the only irreducible complex local systems on X have rank 1 if and only if $\pi^{\text{ét}}$ is abelian. In addition, the only irreducible complex local systems have rank 1 and the category of complex local systems is semi-simple if and only of $\pi^{\text{ét}}$ is a finite abelian group.

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