On a periodic 2-component μ-Hunter-Saxton equation

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Abstract

In this paper, we study the Cauchy problem of a periodic 2-component μ-Hunter-Saxton system. We first establish the local well-posedness for the periodic 2-component μ-Hunter-Saxton system by Kato's semigroup theory. Then, we derive precise blow-up scenarios for strong solutions to the system. Moreover, we present a blow-up result for strong solutions to the system. Finally, we give a global existence result to the system.

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1 Introduction

Recently, a new 2-component system was introduced by Zuo in [20] as follows:

$$\begin{cases} \mu(u)_t - u_{txx} = 2\mu(u)u_x - 2u_x u_{xx} - uu_{xxx} + \rho\rho_x & -\gamma_1 u_{xxx}, \\ & t>0,\ x\in\mathbb{R}, \\ \rho_t = (\rho u)_x + 2\gamma_2\rho_x, & t>0,\ x\in\mathbb{R}, \\ u(0,x) = u_0(x), & x\in\mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x\in\mathbb{R}, \\ u(t,x+1) = u(t,x), & t\geq 0, x\in\mathbb{R}, \\ \rho(t,x+1) = \rho(t,x), & t\geq 0, x\in\mathbb{R}, \end{cases} \tag{1.1}$$

where $\mu(u) = \int_{\mathbb{S}} u dx$ with $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ and $\gamma_i \in \mathbb{R}$, $i = 1, 2$. By integrating both sides of the first equation in the system (1.1) over the circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ and using the periodicity of u, one obtain

$$\mu(u_t) = \mu(u)_t = 0.$$

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This yields the following periodic 2-component μ-Hunter-Saxton system:

$$\begin{cases} -u_{txx} = 2\mu(u)u_x - 2u_x u_{xx} - uu_{xxx} + \rho\rho_x & -\gamma_1 u_{xxx}, \\ & t > 0,\ x \in \mathbb{R}, \\ \rho_t = (\rho u)_x + 2\gamma_2 \rho_x, & t > 0,\ x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t, x+1) = u(t, x), & t \geq 0, x \in \mathbb{R}, \\ \rho(t, x+1) = \rho(t, x), & t \geq 0, x \in \mathbb{R}, \end{cases} \tag{1.2}$$

with $\gamma_i \in \mathbb{R}$, $i = 1, 2$. This system is a 2-component generalization of the generalized Hunter-Saxton equation obtained in [15]. The author [20] shows that this system is a bihamiltonian Euler equation, and also can be viewed as a bivariational equation.

Obviously, (1.1) is equivalent to (1.2) under the condition $\mu(u_t) = \mu(u)_t = 0$. In this paper, we will study the system (1.2) under the assumption $\mu(u_t) = \mu(u)_t = 0$.

For $\rho \equiv 0$ and $\gamma = 0$, and replacing t by $-t$, the system (1.2) reduces to the generalized Hunter-Saxton equation (named μ-Hunter-Saxton equation) as follows:

$$-u_{txx} = -2\mu(u)u_x + 2u_x u_{xx} + uu_{xxx}, \tag{1.3}$$

which is obtained and studied in [15]. The μ-Hunter-Saxton equation lies mid-way between the periodic Hunter-Saxton and Camassa-Holm equations with $u = u(t, x)$ being a time-dependent function on the circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ and $\mu(u) = \int_{\mathbb{S}} u dx$ denotes its mean. Recently, the periodic μ-Hunter-Saxton equation and the periodic μ-Degasperis-Procesi equation have also been studied in [9]. For $\mu(u) = 0$, the equation (1.3) reduces to the Hunter-Saxton equation [10]

$$u_{txx} + 2u_x u_{xx} + uu_{xxx} = 0, \tag{1.4}$$

modeling the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal. In the Hunter-Saxton equation [10], x is the space variable in a reference frame moving with the linearized wave velocity, t is a slow-time variable and $u(t, x)$ is a measure of the average orientation of the medium locally around x at time t. More precisely, the orientation of the molecules is described by the field of unit vectors $(\cos u(t, x), \sin u(t, x))$ [19]. The single-component model also arises in a different physical context as the high-frequency limit [6, 11] of the Camassa-Holm equation for shallow water waves [2, 12] and a re-expression of the geodesic flow on the diffeomorphism group of the circle [4] with a bi-Hamiltonian structure [8] which is completely integrable [5]. The Hunter-Saxton equation also has a bi-Hamiltonian structure [12, 17] and is completely integrable [1, 11]. The initial value problem for the Hunter-Saxton equation (1.4) on the line (nonperiodic case) and on the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ were studied by Hunter and Saxton in [10] using the method of characteristics and by Yin in [19] using Kato semigroup method, respectively.

For $\rho \not\equiv 0$, $\gamma_i = 0, i = 1, 2$ $\mu(u) = 0$ and replacing t by $-t$, peakon solutions of the Cauchy problem of the system (1.2) have been analysed in [3]. Moreover, the Cauchy problem of 2-component periodic Hunter-Saxton system has been discussed in [16]. However, the Cauchy problem of the system (1.2) has not been studied yet. The aim of this paper is to establish the local well-posedness for the system (1.2), to derive precise blow-up scenarios, to prove that the system (1.2) has global strong solutions and also finite time blow-up solutions.

The paper is organized as follows. In Section 2, we establish the local well-posedness of the initial value problem associated with the system (1.2). In Section 3, we derive two precise blow-up scenarios. In Section 4, we present a explosion criteria of strong solutions to the system (1.2) with general initial data. In Section 5, we give a new global existence result of strong solutions to the system (1.2).

Notation Given a Banach space Z, we denote its norm by $\|\cdot\|_Z$. Since all space of functions are over $\mathbb{S} = \mathbb{R}/\mathbb{Z}$, for simplicity, we drop $\mathbb{S}$ in our notations of function spaces if there is no ambiguity. We let $[A, B]$ denote the commutator of linear operator A and B. For convenience, we let $(\cdot|\cdot)_{s\times r}$ and $(\cdot|\cdot)_s$ denote the inner products of $H^s \times H^r$, $s, r \in \mathbb{R}_+$ and H^s, $s \in \mathbb{R}_+$, respectively.

2 Local well-posedness

In this section, we will establish the local well-posedness for the Cauchy problem of the system (1.2) in $H^s \times H^{s-1}$, $s \geq 2$, by applying Kato's theory [13].

The condition $\mu(u_t) = 0$ ensures that the first equation in (1.2) can be recast in the form

$$u_t - (u + \gamma_1)u_x = \partial_x(\mu - \partial_x^2)^{-1}(2\mu u + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2),$$

where $A = \mu - \partial_x^2$ is an isomorphism between H^s and H^{s-2}. Using this identity, the system (1.2) takes the form of a quasi-linear evolution equation of hyperbolic type:

$$\begin{cases} u_t - (u + \gamma_1)u_x = \partial_x(\mu - \partial_x^2)^{-1} & (2\mu u + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2), \\ & t > 0,\ x \in \mathbb{R}, \\ \rho_t - (u + 2\gamma_2)\rho_x = u_x\rho, & t > 0,\ x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \\ \rho(t, x + 1) = \rho(t, x), & t \geq 0, x \in \mathbb{R}, \\ u(t, x + 1) = u(t, x), & t \geq 0, x \in \mathbb{R}. \end{cases} \tag{2.1}$$

Let $z := \begin{pmatrix} u \\ \rho \end{pmatrix}$, $A(z) = \begin{pmatrix} -(u + \gamma_1)\partial_x & 0 \\ 0 & -(u + 2\gamma_2)\partial_x \end{pmatrix}$ and

$$f(z) = \begin{pmatrix} \partial_x(\mu - \partial_x^2)^{-1}(2\mu u + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2) \\ u_x\rho \end{pmatrix}.$$

Set $Y = H^s \times H^{s-1}$, $X = H^{s-1} \times H^{s-2}$, $\Lambda = (\mu - \partial_x^2)^{\frac{1}{2}}$ and $Q = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}$. Obviously, Q is an isomorphism of $H^s \times H^{s-1}$ onto $H^{s-1} \times H^{s-2}$.

Similar to the proof of Theorem 2.2 in [7], we get the following conclusion.

Theorem 2.1 *Given $z_0 = (u_0, \rho_0) \in H^s \times H^{s-1}$, $s \geq 2$, then there exists a maximal $T = T(\| z_0 \|_{H^s\times H^{s-1}}) > 0$, and a unique solution $z = (u, \rho)$ to (2.1) such that*

$$z = z(\cdot, z_0) \in C([0, T); H^s \times H^{s-1}) \cap C^1([0, T); H^{s-1} \times H^{s-2}).$$

Moreover, the solution depends continuously on the initial data, i.e., the mapping

$$z_0 \to z(\cdot, z_0) : H^s \times H^{s-1} \to C([0,T); H^s \times H^{s-1}) \cap C^1([0,T); H^{s-1} \times H^{s-2})$$

is continuous.

Recall that the periodic 2-component Hunter-Saxton system discussed in [16] only has local existence but not local well-posedness because of the lack of uniqueness. The ambiguity disappears in the case of the periodic 2-component μ-Hunter-Saxton system from the Theorem 2.1. This is a very important difference between the 2-component Hunter-Saxton system and the 2-component μ-Hunter-Saxton system.

Consequently, we will give another equivalent form of (1.2). Integrating both sides of the first equation of (1.2) with respect to x, we obtain

$$u_{tx} = -2\mu(u)u + \frac{1}{2}u_x^2 + uu_{xx} - \frac{1}{2}\rho^2 + \gamma_1 u_{xx} + a(t),$$

where $a(t)$ is determined by the periodicity of u to be

$$a(t) = 2\mu(u)^2 + \frac{1}{2}\int_{\mathbb{S}} (u_x^2 + \rho^2)dx.$$

Using the system (1.2), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}} (u_x^2 + \rho^2)dx \qquad (2.2)\\ =&\int_{\mathbb{S}} (u_x u_{xt} + \rho\rho_t)dx \\ =&-\int_{\mathbb{S}} uu_{txx}dx + \int_{\mathbb{S}} \rho\rho_t dx \\ =&\int_{\mathbb{S}} 2\mu(u)uu_x dx - 2\int_{\mathbb{S}} uu_x u_{xx}dx - \int_{\mathbb{S}} u^2 u_{xxx}dx + \int_{\mathbb{S}} u\rho_x\rho dx \\ &-\gamma_1 \int_{\mathbb{S}} uu_{xxx}dx + \int_{\mathbb{S}} \rho(u\rho)_x dx + 2\gamma_2 \int_{\mathbb{S}} \rho\rho_x dx \\ =&\int_{\mathbb{S}} u\rho_x\rho dx + \int_{\mathbb{S}} \rho(u\rho)_x dx = 0. \end{aligned}$$

Combing $\mu(u)_t = \mu(u_t) = 0$, we have

$$\frac{d}{dt}a(t) = 0.$$

For the sake of convenience, let

$$\mu_0 := \mu(u_0) = \mu(u) = \int_{\mathbb{S}} u(t,x)dx,$$

$$\mu_1 := \left(\int_{\mathbb{S}} (u_x^2 + \rho^2)dx\right)^{\frac{1}{2}} = \left(\int_{\mathbb{S}} (u_{0,x}^2 + \rho_0^2)dx\right)^{\frac{1}{2}}$$

and write $a := a(0)$ henceforth. Therefore,

$$u_{tx} = -2\mu_0 u + \frac{1}{2}u_x^2 + uu_{xx} - \frac{1}{2}\rho^2 + \gamma_1 u_{xx} + a \qquad (2.3)$$

is a valid reformulation of the first equation in (1.2). Integrating once more in x, we get

$$u_t - (u+\gamma_1)u_x = \partial_x^{-1}(-2\mu_0 u - \frac{1}{2}u_x^2 - \frac{1}{2}\rho^2 + a) + h(t),$$

where $\partial_x^{-1}f(x) := \int_0^x f(y)dy$ and $h(t) : [0,\infty) \to \mathbb{R}$ is a continuous function. The same procedure in the case of the 2-component Hunter-Saxton system leads to the arbitrary continuous function $h(t)$, which is the reason for non-uniqueness. This time, the condition $\mu(u_t) = 0$ implies that the mean value of the expression on the right-hand side above must be zero. This fact and the uniqueness of the solution of (1.2) imply that the continuous function $h(t)$ is unique.

Thus we get another equivalent form of (1.2)

$$\begin{cases} u_t - (u+\gamma_1)u_x = \partial_x^{-1}(-2\mu_0 u - \frac{1}{2}u_x^2 & \\ -\frac{1}{2}\rho^2 + a) + h(t), & t > 0,\ x \in \mathbb{R}, \\ \rho_t - (u+2\gamma_2)\rho_x = u_x\rho, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geq 0, x \in \mathbb{R}, \\ \rho(t,x+1) = \rho(t,x), & t \geq 0, x \in \mathbb{R}, \end{cases} \tag{2.4}$$

where $\partial_x^{-1}f(x) := \int_0^x f(y)dy$ and $h(t) : [0,\infty) \to \mathbb{R}$ is a continuous function.

3 The precise blow-up scenario

In this section, we present the precise blow-up scenarios for strong solutions to the system (1.2).

We first recall the following lemmas.

Lemma 3.1 *[14] If $r > 0$, then $H^r \cap L^\infty$ is an algebra. Moreover*

$$\| fg \|_{H^r} \leq c(\| f \|_{L^\infty} \| g \|_{H^r} + \| f \|_{H^r} \| g \|_{L^\infty}),$$

where c is a constant depending only on r.

Lemma 3.2 *[14] If $r > 0$, then*

$$\| [\Lambda^r, f]g \|_{L^2} \leq c(\| \partial_x f \|_{L^\infty} \| \Lambda^{r-1} g \|_{L^2} + \| \Lambda^r f \|_{L^2} \| g \|_{L^\infty}),$$

where c is a constant depending only on r.

Next we prove the following useful result on global existence of solutions to (1.2).

Theorem 3.1 *Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}$, $s \geq 2$, be given and assume that T is the maximal existence time of the corresponding solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (2.4) with the initial data z_0. If there exists $M > 0$ such that*

$$\|u_x(t,\cdot)\|_{L^\infty} + \|\rho(t,\cdot)\|_{L^\infty} + \|\rho_x(t,\cdot)\|_{L^\infty} \leq M, \quad t \in [0,T),$$

then the $H^s \times H^{s-1}$-norm of $z(t,\cdot)$ does not blow up on [0,T).

Proof Let $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ be the solution to (2.4) with the initial data $z_0 \in H^s \times H^{s-1},\ s \geq 2$, and let T be the maximal existence time of the corresponding solution z, which is guaranteed by Theorem 2.1. Throughout this proof, $c > 0$ stands for a generic constant depending only on s.

Applying the operator Λ^s to the first equation in (2.4), multiplying by $\Lambda^s u$, and integrating over $\mathbb{S}$, we obtain

$$\frac{d}{dt}\|u\|_{H^s}^2 = 2(uu_x, u)_s + 2(u, \partial_x^{-1}(-2\mu_0 u - \frac{1}{2}u_x^2 - \frac{1}{2}\rho^2 + a) + h(t))_s. \tag{3.1}$$

Let us estimate the first term of the right-hand side of (3.1).

$$\begin{aligned} |(uu_x, u)_s| &= |(\Lambda^s(u\partial_x u), \Lambda^s u)_0| \\ &= |([\Lambda^s, u]\partial_x u, \Lambda^s u)_0 + (u\Lambda^s \partial_x u, \Lambda^s u)_0| \\ &\leq \|[\Lambda^s, u]\partial_x u\|_{L^2}\|\Lambda^s u\|_{L^2} + \frac{1}{2}|(u_x \Lambda^s u, \Lambda^s u)_0| \\ &\leq (c\|u_x\|_{L^\infty} + \frac{1}{2}\|u_x\|_{L^\infty})\|u\|_{H^s}^2 \\ &\leq c\|u_x\|_{L^\infty}\|u\|_{H^s}^2, \end{aligned} \tag{3.2}$$

where we used Lemma 3.2 with $r = s$. Let $f \in H^{s-1}, s \geq 2$. We have

$$|\partial_x^{-1} f| = |\int_0^x f dx| \leq \int_{\mathbb{S}} |f| dx \leq \|f\|_{L^2}$$

and

$$\|\partial_x^{-1} f\|_{L^2} = \left(\int_0^1 (\partial_x^{-1} f)^2 dx\right)^{1/2} \leq \left(\int_0^1 \|f\|_{L^2}^2 dx\right)^{1/2} = \|f\|_{L^2}.$$

Thus

$$\|\partial_x^{-1} f\|_{H^s} \leq \|\partial_x^{-1} f\|_{L^2} + \|f\|_{H^{s-1}} \leq 2\|f\|_{H^{s-1}}.$$

Then, we estimate the second term of the right-hand side of (3.1) in the following way:

$$\begin{aligned} &|(\partial_x^{-1}(-2\mu_0 u - \frac{1}{2}u_x^2 - \frac{1}{2}\rho^2 + a) + h(t), u)_s| \\ &\leq \|\partial_x^{-1}(-2\mu_0 u - \frac{1}{2}u_x^2 - \frac{1}{2}\rho^2 + a) + h(t)\|_{H^s}\|u\|_{H^s} \\ &\leq (\|\partial_x^{-1}(-2\mu_0 u - \frac{1}{2}u_x^2 - \frac{1}{2}\rho^2 + a)\|_{H^s} + \|h(t)\|_{H^s})\|u\|_{H^s} \\ &\leq (2\| - 2\mu_0 u - \frac{1}{2}u_x^2 - \frac{1}{2}\rho^2 + a\|_{H^{s-1}} + \|h(t)\|_{H^s})\|u\|_{H^s} \\ &\leq (4|\mu_0|\|u\|_{H^s} + \|u_x^2\|_{H^{s-1}} + \|\rho^2\|_{H^{s-1}} + 2\|a\|_{H^{s-1}} + \|h(t)\|_{H^s})\|u\|_{H^s} \\ &\leq c(\|u\|_{H^s} + \|u_x\|_{L^\infty}\|u_x\|_{H^{s-1}} + \|\rho\|_{L^\infty}\|\rho\|_{H^{s-1}} + |a| + \max_{t\in[0,T)}|h(t)|)\|u\|_{H^s} \\ &\leq c(\|u_x\|_{L^\infty} + \|\rho\|_{L^\infty} + 1)(\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2 + 1), \end{aligned} \tag{3.3}$$

where we used Lemma 3.1 with $r = s - 1$. Combining (3.2) and (3.3) with (3.1), we get

$$\frac{d}{dt}\|u\|_{H^s}^2 \leq c(\|\rho\|_{L^\infty} + \|u_x\|_{L^\infty} + 1)(\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2 + 1). \tag{3.4}$$

In order to derive a similar estimate for the second component ρ, we apply the operator Λ^{s-1} to the second equation in (2.4), multiply by $\Lambda^{s-1}\rho$, and integrate over $\mathbb{S}$, to obtain

$$\frac{d}{dt}\|\rho\|_{H^{s-1}}^2 = 2(u\rho_x,\rho)_{s-1} + 2(u_x\rho,\rho)_{s-1}. \tag{3.5}$$

Let us estimate the first term of the right hand side of (3.5)

$$\begin{aligned}
&|(u\rho_x,\rho)_{s-1}| \\
&= |(\Lambda^{s-1}(u\partial_x\rho),\Lambda^{s-1}\rho)_0| \\
&= |([\Lambda^{s-1},u]\partial_x\rho,\Lambda^{s-1}\rho)_0 + (u\Lambda^{s-1}\partial_x\rho,\Lambda^{s-1}\rho)_0| \\
&\le \|[\Lambda^{s-1},u]\partial_x\rho\|_{L^2}\|\Lambda^{s-1}\rho\|_{L^2} + \frac{1}{2}|(u_x\Lambda^{s-1}\rho,\Lambda^{s-1}\rho)_0| \\
&\le c(\|u_x\|_{L^\infty}\|\rho\|_{H^{s-1}} + \|\rho_x\|_{L^\infty}\|u\|_{H^{s-1}})\|\rho\|_{H^{s-1}} + \frac{1}{2}\|u_x\|_{L^\infty}\|\rho\|_{H^{s-1}}^2 \\
&\le c(\|u_x\|_{L^\infty} + \|\rho_x\|_{L^\infty})(\|\rho\|_{H^{s-1}}^2 + \|u\|_{H^s}^2),
\end{aligned}$$

here we applied Lemma 3.2 with $r = s-1$. Then we estimate the second term of the right hand side of (3.5). Based on Lemma 3.1 with $r = s-1$, we get

$$\begin{aligned}
|(u_x\rho,\rho)_{s-1}| &\le \|u_x\rho\|_{H^{s-1}}\|\rho\|_{H^{s-1}} \\
&\le c(\|u_x\|_{L^\infty}\|\rho\|_{H^{s-1}} + \|\rho\|_{L^\infty}\|u_x\|_{H^{s-1}})\|\rho\|_{H^{s-1}} \\
&\le c(\|u_x\|_{L^\infty} + \|\rho_x\|_{L^\infty})(\|\rho\|_{H^{s-1}}^2 + \|u\|_{H^s}^2).
\end{aligned}$$

Combining the above two inequalities with (3.5), we get

$$\frac{d}{dt}\|\rho\|_{H^{s-1}}^2 \le c(\|u_x\|_{L^\infty} + \|\rho\|_{L^\infty} + \|\rho_x\|_{L^\infty})(\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2 + 1). \tag{3.6}$$

By (3.4) and (3.6), we have

$$\begin{aligned}
&\frac{d}{dt}(\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2 + 1) \\
&\le c(\|u_x\|_{L^\infty} + \|\rho\|_{L^\infty} + \|\rho_x\|_{L^\infty} + 1)(\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2 + 1).
\end{aligned}$$

An application of Gronwall's inequality and the assumption of the theorem yield

$$(\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2 + 1) \le \exp(c(M+1)t)(\|u_0\|_{H^s}^2 + \|\rho_0\|_{H^{s-1}}^2 + 1).$$

This completes the proof of the theorem.

Given $z_0 \in H^s \times H^{s-1}$ with $s \ge 2$. Theorem 2.1 ensures the existence of a maximal $T > 0$ and a solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (2.4) such that

$$z = z(\cdot, z_0) \in C([0,T); H^s \times H^{s-1}) \cap C^1([0,T); H^{s-1} \times H^{s-2}).$$

Consider now the following initial value problem

$$\begin{cases} q_t = u(t,-q) + 2\gamma_2, & t \in [0,T), \\ q(0,x) = x, & x \in \mathbb{R}, \end{cases} \tag{3.7}$$

where u denotes the first component of the solution z to (2.4). Then we have the following two useful lemmas.

Similar to the proof of Lemma 4.1 in [18], applying classical results in the theory of ordinary differential equations, one can obtain the following result on q which is crucial in the proof of blow-up scenarios.

Lemma 3.3 *Let $u \in C([0,T);H^s)\bigcap C^1([0,T);H^{s-1}), s \geq 2$. Then Eq.(3.7) has a unique solution $q \in C^1([0,T)\times\mathbb{R};\mathbb{R})$. Moreover, the map $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with*

$$q_x(t,x) = exp\left(-\int_0^t u_x(s,-q(s,x))ds\right) > 0, \quad (t,x)\in[0,T)\times\mathbb{R}.$$

Lemma 3.4 *Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s\times H^{s-1}$, $s \geq 2$ and let $T>0$ be the maximal existence time of the corresponding solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (1.2). Then we have*

$$\rho(t,-q(t,x))q_x(t,x) = \rho_0(-x), \quad \forall\ (t,x)\in[0,T)\times\mathbb{S}. \tag{3.8}$$

Moreover, if there exists $M>0$ such that $u_x \leq M$ for all $(t,x)\in[0,T)\times\mathbb{S}$, then

$$\|\rho(t,\cdot)\|_{L^\infty} \leq e^{MT}\|\rho_0(\cdot)\|_{L^\infty}, \quad \forall\ t\in[0,T).$$

Proof Differentiating the left-hand side of the equation (3.8) with respect to the time variable t, and applying the relations (2.4) and (3.7), we obtain

$$\begin{aligned}
&\frac{d}{dt}\rho(t,-q(t,x))q_x(t,x)\\
=&(\rho_t(t,-q)-\rho_x(t,-q)q_t(t,x))q_x(t,x)+\rho(t,-q(t,x))q_{xt}(t,x)\\
=&(\rho_t-(u(t,-q)+2\gamma_2)\rho_x)q_x(t,x)-u_x\rho q_x(t,x)\\
=&(\rho_t-(u+2\gamma_2)\rho_x-u_x\rho)q_x(t,x)=0
\end{aligned}$$

This proves (3.8). By Lemma 3.3, in view of (3.8) and the assumption of the lemma, we obtain

$$\begin{aligned}
\|\rho(t,\cdot)\|_{L^\infty(\mathbb{S})} &= \|\rho(t,\cdot)\|_{L^\infty(\mathbb{R})}\\
&= \|\rho(t,-q(t,\cdot))\|_{L^\infty(\mathbb{R})}\\
&= \|exp\left(\int_0^t u_x(s,-q(s,x))ds\right)\rho_0(-x)\|_{L^\infty(\mathbb{R})}\\
&\leq e^{MT}\|\rho_0(\cdot)\|_{L^\infty(\mathbb{R})} = e^{MT}\|\rho_0(\cdot)\|_{L^\infty(\mathbb{S})}, \quad \forall\ t\in[0,T).
\end{aligned}$$

Our next result describes the precise blow-up scenarios for sufficiently regular solutions to (1.2).

Theorem 3.2 *Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s\times H^{s-1}$, $s > \frac{5}{2}$ be given and let T be the maximal existence time of the corresponding solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (2.4) with the initial data z_0. Then the corresponding solution blows up in finite time if and only if*

$$\limsup_{t\to T}\sup_{x\in\mathbb{S}}\{u_x(t,x)\} = +\infty \quad or \quad \limsup_{t\to T}\{\|\rho_x(t,\cdot)\|_{L^\infty}\} = +\infty.$$

Proof By Theorem 2.1 and Sobolev's imbedding theorem it is clear that if

$$\limsup_{t\to T}\sup_{x\in\mathbb{S}}\{u_x(t,x)\}=+\infty \quad\text{or}\quad \limsup_{t\to T}\{\|\rho_x(t,\cdot)\|_{L^\infty}\}=+\infty,$$

then $T<\infty$.

Let $T<\infty$. Assume that there exists $M_1>0$ and $M_2>0$ such that

$$u_x(t,x)\le M_1,\quad \forall\ (t,x)\in[0,T)\times\mathbb{S},$$

and

$$\|\rho_x(t,\cdot)\|_{L^\infty}\le M_2,\quad \forall\ t\in[0,T).$$

By Lemma 3.4, we have

$$\|\rho(t,\cdot)\|_{L^\infty}\le e^{M_1T}\|\rho_0\|_{L^\infty},\quad \forall\ t\in[0,T).$$

By (2.2) and the first equation in (2.4), a direct computation implies the following inequality

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{S}}u(t,x)^2dx \qquad (3.9)\\
&=2\int_{\mathbb{S}}u\left((u+\gamma_1)u_x+\partial_x^{-1}(-2\mu_0u-\frac{1}{2}u_x^2-\frac{1}{2}\rho^2+a)+h(t)\right)dx\\
&\le \int_{\mathbb{S}}u^2dx+\int_{\mathbb{S}}\left(\int_0^x(-2\mu_0u-\frac{1}{2}u_y^2-\frac{1}{2}\rho^2+a)dy\right)^2dx+2|h(t)|\int_{\mathbb{S}}|u(t,x)|dx\\
&\le \int_{\mathbb{S}}u^2dx+8\mu_0^2(\int_{\mathbb{S}}|u|dx)^2+2\left(\int_{\mathbb{S}}(\frac{1}{2}u_x^2+\frac{1}{2}\rho^2+a)dx\right)^2\\
&\quad+\max_{t\in[0,T)}|h(t)|+\max_{t\in[0,T)}|h(t)|\int_{\mathbb{S}}u(t,x)^2dx\\
&=(1+8\mu_0^2+\max_{t\in[0,T)}|h(t)|)\int_{\mathbb{S}}u^2dx+\frac{1}{2}\left[\int_0^1(u_{0,x}^2+\rho_0^2+2a)dx\right]^2+\max_{t\in[0,T)}|h(t)|
\end{aligned}$$

for $t\in(0,T)$.

Multiplying the first equation in (1.2) by $m=u_{xx}$ and integrating by parts, we find

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{S}}m^2dx=&-4\mu\int_{\mathbb{S}}mu_xdx+4\int_{\mathbb{S}}u_xm^2dx+2\int_{\mathbb{S}}umm_xdx \qquad (3.10)\\
&-2\int_{\mathbb{S}}m\rho\rho_xdx+2\gamma_1\int_{\mathbb{S}}mm_xdx\\
=&\,3\int_{\mathbb{S}}u_xm^2dx-2\int_{\mathbb{S}}m\rho\rho_xdx\\
\le&\,3M_1\int_{\mathbb{S}}m^2dx+\|\rho\|_{L^\infty}\int_{\mathbb{S}}m^2+\rho_x^2dx\\
\le&\,(3M_1+\|\rho\|_{L^\infty})\int_{\mathbb{S}}m^2dx+\|\rho\|_{L^\infty}\int_{\mathbb{S}}\rho_x^2dx.
\end{aligned}$$

Differentiating the first equation in (1.2) with respect to x, multiplying the obtained equation by $m_x = u_{xxx}$, integrating by parts and using Lemma 3.4, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{S}} m_x^2 dx \qquad (3.11)\\
= &-4\mu\int_{\mathbb{S}} m m_x + 4\int_{\mathbb{S}} m^2 m_x dx + 6\int_{\mathbb{S}} u_x m_x^2 + 2\int_{\mathbb{S}} u m_{xx} m_x\\
&-2\int_{\mathbb{S}} \rho_x^2 m_x - 2\int_{\mathbb{S}} \rho\rho_{xx} m_x dx + 2\gamma_1\int_{\mathbb{S}} m_x m_{xx} dx\\
&= 5\int_{\mathbb{S}} u_x m_x^2 dx - 2\int_{\mathbb{S}} \rho_x^2 m_x dx - 2\int_{\mathbb{S}} \rho\rho_{xx} m_x dx\\
&\le 5M_1\int_{\mathbb{S}} m_x^2 dx + 2\|\rho_x\|_{L^\infty}^2\int_{\mathbb{S}} |m_x| dx + \|\rho\|_{L^\infty}\int_{\mathbb{S}} (\rho_{xx}^2 + m_x^2) dx\\
&\le 5M_1\int_{\mathbb{S}} m_x^2 dx + \|\rho\|_{L^\infty}\int_{\mathbb{S}} (\rho_{xx}^2 + m_x^2) dx + 2\|\rho_x\|_{L^\infty}^2 + 2\|\rho_x\|_{L^\infty}^2\int_{\mathbb{S}} m_x^2 dx\\
&\le (5M_1 + \|\rho\|_{L^\infty} + 2M_2^2)\int_{\mathbb{S}} m_x^2 dx + \|\rho\|_{L^\infty}\int_{\mathbb{S}} \rho_{xx}^2 dx + 2M_2^2.
\end{aligned}$$

Differentiating the second equation in (1.2) with respect to x, multiplying the obtained equation by ρ_x and integrating by parts, we obtain

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{S}} \rho_x^2 dx &= 3\int_{\mathbb{S}} u_x \rho_x^2 dx + 2\int_{\mathbb{S}} m\rho\rho_x dx \qquad (3.12)\\
&\le 3M_1\int_{\mathbb{S}} \rho_x^2 dx + \|\rho\|_{L^\infty}\int_{\mathbb{S}} (m^2 + \rho_x^2) dx\\
&= (3M_1 + \|\rho\|_{L^\infty})\int_{\mathbb{S}} \rho_x^2 dx + \|\rho\|_{L^\infty}\int_{\mathbb{S}} m^2 dx.
\end{aligned}$$

Differentiating the second equation in (1.2) with respect to x twice, multiplying the obtained equation by ρ_{xx}, integrating by parts and using Lemma 3.4, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{S}} \rho_{xx}^2 dx \qquad (3.13)\\
&= 5\int_{\mathbb{S}} u_x \rho_{xx}^2 dx + \int_{\mathbb{S}} u_{xxx}(2\rho\rho_{xx} - 3\rho_x^2) dx\\
&\le 5M_1\int_{\mathbb{S}} \rho_{xx}^2 dx + \int_{\mathbb{S}} m_x(2\rho\rho_{xx} - 3\rho_x^2) dx\\
&\le 5M_1\int_{\mathbb{S}} \rho_{xx}^2 dx + 3\|\rho_x\|_{L^\infty}^2\int_{\mathbb{S}} |m_x| dx + \|\rho\|_{L^\infty}\int_{\mathbb{S}} 2m_x\rho_{xx} dx\\
&\le (5M_1 + \|\rho\|_{L^\infty})\int_{\mathbb{S}} \rho_{xx}^2 dx + (3M_2^2 + \|\rho\|_{L^\infty})\int_{\mathbb{S}} m_x^2 dx + 3M_2^2.
\end{aligned}$$

Summing (2.2), (3.9)-(3.13), we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{S}} (u^2 + u_x^2 + m^2 + m_x^2 + \rho^2 + \rho_x^2 + \rho_{xx}^2) dx\\
&\le K_1\int_{\mathbb{S}} (u^2 + u_x^2 + m^2 + m_x^2 + \rho^2 + \rho_x^2 + \rho_{xx}^2) dx + K_2,
\end{aligned}$$

where

$$K_1 = 1 + 8\mu_0^2 + \max_{t\in[0,T)} |h(t)| + 8e^{M_1T}\|\rho_0\|_{L^\infty} + 16M_1 + 5M_2^2,$$

$$K_2 = \frac{1}{2}\left[\int_{\mathbb{S}}(u_{0,x}^2 + \rho_0^2 + 2a)dx\right]^2 + \max_{t\in[0,T)} |h(t)| + 5M_2^2.$$

By means of Gronwall's inequality and the above inequality, we deduce that

$$\begin{aligned}&\|u(t,\cdot)\|_{H^3}^2 + \|\rho(t,\cdot)\|_{H^2}^2\\ &\le e^{K_1t}(\|u_0\|_{H^3}^2 + \|\rho_0\|_{H^2}^2 + \frac{K_2}{K_1}), \quad \forall\ t \in [0,T).\end{aligned}$$

The above inequality, Sobolev's imbedding theorem and Theorem 3.1 ensure that the solution z does not blow-up in finite time. This completes the proof of the theorem.

For initial data $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^2 \times H^1$, we have the following precise blow-up scenario.

Theorem 3.3 *Let* $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^2 \times H^1$*, and let* T *be the maximal existence time of the corresponding solution* $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ *to (2.4) with the initial data* z_0*. Then the corresponding solution blows up in finite time if and only if*

$$\limsup_{t\to T}\sup_{x\in\mathbb{S}} u_x(t,x) = +\infty.$$

Proof Let $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ be the solution to (2.4) with the initial data $z_0 \in H^2 \times H^1$, and let T be the maximal existence time of the solution z, which is guaranteed by Theorem 2.1.

Let $T < \infty$. Assume that there exists $M_1 > 0$ such that

$$u_x(t,x) \le M_1, \quad \forall\ (t,x) \in [0,T) \times \mathbb{S}.$$

By Lemma 3.4, we have

$$\|\rho(t,\cdot)\|_{L^\infty} \le e^{M_1T}\|\rho_0\|_{L^\infty}, \quad \forall\ t \in [0,T).$$

Combining (2.2), (3.9)-(3.10) and (3.12), we obtain

$$\frac{d}{dt}\int_{\mathbb{S}}(u^2 + u_x^2 + m^2 + \rho^2 + \rho_x^2)dx \le K_3\int_{\mathbb{S}}(u^2 + u_x^2 + m^2 + \rho^2 + \rho_x^2)dx + K_4,$$

where

$$K_3 = 1 + 8\mu_0^2 + \max_{t\in[0,T)} |h(t)| + 6M_1 + 4e^{M_1T}\|\rho_0\|_{L^\infty},$$

$$K_4 = \frac{1}{2}\left[\int_0^1(u_{0,x}^2 + \rho_0^2 + 2a)dx\right]^2 + \max_{t\in[0,T)} |h(t)|.$$

By means of Gronwall's inequality and the above inequality, we get

$$\|u(t,\cdot)\|_{H^2}^2 + \|\rho(t,\cdot)\|_{H^1}^2 \leq e^{K_3 t}(\|u_0\|_{H^2}^2 + \|\rho_0\|_{H^1}^2 + \frac{K_4}{K_3}).$$

The above inequality ensures that the solution z does not blow-up in finite time.

On the other hand, by Sobolev's imbedding theorem, we see that if

$$\limsup_{t\to T} \sup_{x\in\mathbb{S}} u_x(t,x) = +\infty,$$

then the solution will blow up in finite time. This completes the proof of the theorem.

Remark 3.1 *Note that Theorem 3.2 shows that*

$$T(\|z_0\|_{H^s\times H^{s-1}}) = T(\|z_0\|_{H^{s'}\times H^{s'-1}}), \qquad \forall s, s' > \frac{5}{2},$$

while Theorem 3.3 implies that

$$T(\|z_0\|_{H^s\times H^{s-1}}) \leq T(\|z_0\|_{H^2\times H^1}), \qquad \forall s, s' \geq 2.$$

4 Blow-up

In this section, we discuss the blow-up phenomena of the system (1.2) and prove that there exist strong solutions to (1.2) which do not exist globally in time.

Lemma 4.1 *([9])If $f \in H^1(\mathbb{S})$ is such that $\int_{\mathbb{S}} f(x)dx = 0$, then we have*

$$\max_{x\in\mathbb{S}} f^2(x) \leq \frac{1}{12}\int_{\mathbb{S}} f_x^2(x)dx.$$

Note that $\int_{\mathbb{S}}(u(t,x) - \mu_0)dx = \mu_0 - \mu_0 = 0$. By Lemma 4.1, we find that

$$\max_{x\in\mathbb{S}}[u(t,x) - \mu_0]^2 \leq \frac{1}{12}\int_{\mathbb{S}} u_x^2(t,x)dx \leq \frac{1}{12}\mu_1^2.$$

So we have

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{S})} \leq |\mu_0| + \frac{\sqrt{3}}{6}\mu_1. \tag{4.1}$$

Theorem 4.1 *Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}, s \geq 2$, and T be the maximal time of the solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (1.2) with the initial data z_0. If $\gamma_1 = 2\gamma_2$, $\mu_0 = 0$ and there exists a point $x_0 \in \mathbb{S}$, such that $\rho_0(-x_0) = 0$, then the corresponding solutions to (1.2) blow up in finite time.*

Proof Let $m(t) = u_x(t, -q(t, x_0))$, $\gamma(t) = \rho(t, -q(t, x_0))$, where $q(t, x)$ is the solution of Eq.(3.7). By Eq.(3.7) we can obtain

$$\frac{dm}{dt} = (u_{tx} - (u + \gamma_1)u_{xx})(t, -q(t, x_0)).$$

Evaluating the integrated representation (2.3) at $(t, -q(t, x_0))$ with the assumption $\mu = 0$ we get

$$\frac{d}{dt}m(t) = \frac{1}{2}m(t)^2 - \frac{1}{2}\gamma(t)^2 + a.$$

Since $\gamma(0) = 0$, we infer from Lemmas 3.3-3.4 that $\gamma(t) = 0$ for all $t \in [0, T)$. Note that $a = 2\mu(u)^2 + \frac{1}{2}\int_{\mathbb{S}}(u_x^2 + \rho^2)dx > 0$. (Indeed, if $a(t) = 0$, then $(u, \rho) = (0, 0)$. This is a trivial case, we do not consider it.) Then we have $\frac{d}{dt}m(t) \geq a > 0$. Thus, it follows that $m(t_0) > 0$ for some $t_0 \in (0, T)$. Solving the following inequality yields

$$\frac{d}{dt}m(t) \geq \frac{1}{2}m(t)^2.$$

Therefore

$$0 < \frac{1}{m(t)} \leq \frac{1}{m(t_0)} - \frac{1}{2}(t - t_0), \quad t \in [t_0, T).$$

The above inequality implies that $T < t_0 + \frac{2}{m(t_0)}$ and $\lim_{t\to T} m(t) = +\infty$. In view of Theorem 3.2, this completes the proof of the theorem.

Theorem 4.2 *Let* $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}, s \geq 2$, *and* T *be the maximal time of the solution* $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ *to (1.2) with the initial data* z_0*. If* $\gamma_1 = 2\gamma_2$, $\mu_0 \neq 0$, $|\mu_0| + \frac{\sqrt{3}}{6}\mu_1 < \frac{a}{2|\mu_0|}$ *and there exists a point* $x_0 \in \mathbb{S}$, *such that* $\rho_0(-x_0) = 0$, *then the corresponding solutions to (1.2) blow up in finite time.*

Proof Let $m(t) = u_x(t, -q(t, x_0))$, $\gamma(t) = \rho(t, -q(t, x_0))$, where $q(t, x)$ is the solution of Eq.(3.7). By Eq.(3.7) we can obtain

$$\frac{dm}{dt} = (u_{tx} - (u + \gamma_1)u_{xx})(t, -q(t, x_0)).$$

Evaluating the integrated representation (2.3) at $(t, -q(t, x_0))$ we have

$$\frac{d}{dt}m(t) = \frac{1}{2}m(t)^2 - \frac{1}{2}\gamma(t)^2 + a - 2\mu_0 u.$$

Since $\gamma(0) = 0$, we infer from Lemmas 3.3-3.4 that $\gamma(t) = 0$ for all $t \in [0, T)$. In view of (4.1) and the condition $|\mu_0| + \frac{\sqrt{3}}{6}\mu_1 < \frac{a}{2|\mu_0|}$, we have $a - 2\mu_0 u \geq a - 2|\mu_0 u| > 0$. Then we have $\frac{d}{dt}m(t) \geq a - 2\mu_0 u > 0$. The left proof is the same as Theorem 4.1, so we omit it here.

5 Global Existence

In this section, we will present a global existence result. Firstly, we give two useful lemmas.

Theorem 5.1 *Let* $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^2 \times H^1$, *and* T *be the maximal time of the solution* $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ *to (1.2) with the initial data* z_0. *If* $\gamma_1 = 2\gamma_2$, $\rho_0(x) \neq 0$ *for all* $x \in \mathbb{S}$, *then the corresponding solution* z *exists globally in time.*

Proof By Lemma 3.3, we know that $q(t, \cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with

$$q_x(t,x) = exp\left(\int_0^t u_x(s, q(s,x))ds\right) > 0, \quad \forall\ (t,x) \in [0,T) \times \mathbb{R}.$$

Moreover,

$$\sup_{y\in\mathbb{S}} u_y(t,y) = \sup_{x\in\mathbb{R}} u_x(t, -q(t,x)),\ \forall\ t \in [0,T). \tag{5.1}$$

Set $M(t,x) = u_x(t, -q(t,x))$ and $\alpha(t,x) = \rho(t, -q(t,x))$ for $t \in [0,T)$ and $x \in \mathbb{R}$. By $\gamma_1 = 2\gamma_2$, (1.2) and Eq.(3.7), we have

$$\frac{\partial M}{\partial t} = (u_{tx} - (u+\gamma_1)u_{xx})(t, -q(t,x)) \ \text{ and } \ \frac{\partial \alpha}{\partial t} = \alpha M. \tag{5.2}$$

Evaluating (2.3) at $(t, -q(t,x))$ we get

$$\partial_t M(t,x) = \frac{1}{2}M(t,x)^2 - \frac{1}{2}\alpha(t,x)^2 + a - 2\mu_0 u(t, -q(t,x)).$$

Write $f(t,x) = a - 2\mu_0 u(t, -q(t,x))$. By (4.1) we have

$$\begin{aligned} |f(t,x)| \le a + 2|\mu_0|\|u\|_{L^\infty} &\le a + 2|\mu_0|(|\mu_0| + \frac{\sqrt{3}}{6}\mu_1) \\ &= 4\mu_0^2 + \frac{1}{2}\mu_1^2 + \frac{\sqrt{3}}{3}|\mu_0|\mu_1 \end{aligned}$$

and

$$\partial_t M(t,x) = \frac{1}{2}M(t,x)^2 - \frac{1}{2}\alpha(t,x)^2 + f(t,x). \tag{5.3}$$

By Lemmas 3.3-3.4, we know that $\alpha(t,x)$ has the same sign with $\alpha(0,x) = \rho_0(-x)$ for every $x \in \mathbb{R}$. Moreover, there is a constant $\beta > 0$ such that $\inf_{x\in\mathbb{R}} |\alpha(0,x)| = \inf_{x\in\mathbb{S}} |\rho_0(-x)| \ge \beta > 0$ since $\rho_0(x) \neq 0$ for all $x \in \mathbb{S}$ and $\mathbb{S}$ is a compact set. Thus,

$$\alpha(t,x)\alpha(0,x) > 0, \quad \forall x \in \mathbb{R}.$$

Next, we consider the following Lyapunov function first introduced in [3].

$$w(t,x) = \alpha(t,x)\alpha(0,x) + \frac{\alpha(0,x)}{\alpha(t,x)}(1 + M^2), \quad (t,x) \in [0,T) \times \mathbb{R}. \tag{5.4}$$

By Sobolev's imbedding theorem, we have

$$\begin{aligned} 0 < w(0,x) &= \alpha(0,x)^2 + 1 + M(0,x)^2 \\ &= \rho_0(x)^2 + 1 + u_{0,x}(x)^2 \\ &\leq 1 + \max_{x\in\mathbb{S}}(\rho_0(x)^2 + u_{0,x}(x)^2) := C_1. \end{aligned} \tag{5.5}$$

Differentiating (5.4) with respect to t and using (5.2)-(5.3), we obtain

$$\begin{aligned} \frac{\partial w}{\partial t}(t,x) &= \frac{\alpha(0,x)}{\alpha(t,x)} M(t,x)(2f-1) \\ &\leq |f - \frac{1}{2}| \frac{\alpha(0,x)}{\alpha(t,x)}(1+M^2) \\ &\leq (4\mu_0^2 + \frac{1}{2}\mu_1^2 + \frac{\sqrt{3}}{3}|\mu_0|\mu_1 + \frac{1}{2})w(t,x). \end{aligned}$$

By Gronwall's inequality, the above inequality and (5.5), we have

$$w(t,x) \leq w(0,x)e^{(4\mu_0^2+\frac{1}{2}\mu_1^2+\frac{\sqrt{3}}{3}|\mu_0|\mu_1+\frac{1}{2})t} \leq C_1 e^{(4\mu_0^2+\frac{1}{2}\mu_1^2+\frac{\sqrt{3}}{3}|\mu_0|\mu_1+\frac{1}{2})t}$$

for all $(t,x) \in [0,T) \times \mathbb{R}$. On the other hand,

$$w(t,x) \geq 2\sqrt{\alpha^2(0,x)(1+M^2)} \geq 2\beta|M(t,x)|, \quad \forall \ \ (t,x) \in [0,T) \times \mathbb{R}.$$

Thus,

$$|M(t,x)| \leq \frac{1}{2\beta}w(t,x) \leq \frac{1}{2\beta}C_1 e^{(4\mu_0^2+\frac{1}{2}\mu_1^2+\frac{\sqrt{3}}{3}|\mu_0|\mu_1+\frac{1}{2})t}$$

for all $(t,x) \in [0,T) \times \mathbb{R}$. Then by (5.1) and the above inequality, we have

$$\limsup_{t\to T}\sup_{y\in\mathbb{S}} u_y(t,y) = \limsup_{t\to T}\sup_{x\in\mathbb{R}} u_x(t,-q(t,x)) \leq \frac{1}{2\beta}C_1 e^{(4\mu_0^2+\frac{1}{2}\mu_1^2+\frac{\sqrt{3}}{3}|\mu_0|\mu_1+\frac{1}{2})t}.$$

This completes the proof by using Theorem 3.3.

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