TOPOLOGICAL EXPANDERS

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Abstract. We describe a natural topological generalization of edge expansion for graphs to arbitrary simplicial complexes and prove that this property holds with high probability for certain random complexes.

1. Introduction

Expander graphs is a vast subject — expanders have been studied from the points of view of number theory, combinatorics, probability, and geometry, and have extensive applications in computer science. For a nice introduction and survey see [7]. We define edge expansion precisely below, but roughly speaking expanders are graphs that are simultaneously sparse and well-connected. The main goal of this article is to give a natural topological generalization of edge expansion to simplicial complexes and to provide examples in every dimension. Our definition of degree k topological expansion is in terms of the coboundary operator d_k which defines simplicial cohomology [6].

There have recently been a few other notions of higher-dimensional expanders discussed. Following Lafforgue's proof of the Ramanujan conjectures, Lubotzky, Samuels, and Vishne defined and constructed Ramanujan complexes [10, 11], which have a spectral property analogous to that of Ramanujan graphs. More recently Fox, Gromov, Lafforgue, Naor, and Pach described a geometric overlap property as an analogue of expansion and showed that some of the Ramanujan examples studied earlier provided explicit examples, and they discussed probabilistic examples as well [5]. Our aim here is to complement these studies with a general topological notion of expansion.

We first review the usual definition of edge expansion.

Definition 1.1. For a graph G on n vertices, the *edge expansion* $h(G)$ is defined as

$$h(G) = \min_{0<|S|\leq n/2} \frac{|\partial(S)|}{|S|},$$

where $\partial(S)$ denotes the set of edges with exactly one vertex in S.

Note that $h(G) > 0$ if and only if G is connected. However, rather than discuss edge expansion of a single graph, one is often more interested in families of graphs, where one takes a sequence of graphs with the number of vertices $n \to \infty$. If one had a sequence of d-regular graphs $S = \{G_i\}$, for example, and $h(G_i)$ stayed bounded away from zero as $i \to \infty$, we would consider S to be an *expanding family.*

It is not totally obvious that such families exist. The first known examples of were probabilistic [14], and then Margulis [12] and independently Lubotzky, Philips, and Sarnak [9], gave explicit examples using Deligne's proof of the Weil conjecture. Although the first constructions relied on deep number-theoretic facts, Pinsker's earlier observation was that a sequence of random d-regular graphs already gives an edge-expanding family with probability 1.

Although one often studies expansion for a sequence d-regular graphs, we need not be restricted to graphs of bounded degree, or even to regular graphs. For example, Erdős-Rényi random graphs $G(n,p)$ are also known to have strong edge expansion properties as $n \to \infty$ [2] once $p \gg \log n/n$. (We use $\gg$ and $\ll$ loosely, meaning "much greater than," or "much less than.")

Definition 1.2. The *Erdős-Rényi random graph* $G(n,p)$ is the probability space of all graphs on vertex set $[n] = \{1, 2, \dots, n\}$ with each edge having probability p, independently. In other words for every graph G on vertex set $[n]$,

$$\mathrm{Prob}[G \in G(n,p)] = p^{e(G)}(1-p)^{\binom{n}{2}-e(G)},$$

where $e(G)$ denote the number of edges of G.

We say that $G(n,p)$ asymptotically almost surely (a.a.s.) has property $\mathcal{P}$ if for $G \in G(n,p)$,

$$\mathrm{Prob}[G \in \mathcal{P}] \to 1$$

as $n \to \infty$.

The following theorem is due to Erdős and Rényi [4].

Theorem 1.3. *Let $\omega = \omega(n)$ be any function that tends to infinity with n. If $p = (\log n + \omega)/n$ then $G(n,p)$ is a.a.s. connected, and if $p = (\log n - \omega)/n$ then $G(n,p)$ is a.a.s. disconnected.*

Once p is much larger than $\log n/n$, $G(n,p)$ exhibits strong edge expansion. The following theorem is due to Benjamini, Haber, Krivelevich, and Lubetzky [2]. Let $\delta(G)$ denote the minimum degree of G.

Theorem 1.4. *Let $\epsilon > 0$ and $0 < c < 1/2$ be constants and $p \geq (1+\epsilon)\log n/n$. Then a.a.s. $G \in G(n,p)$ has edge expansion bounded below by*

$$h(G) > c\delta(G).$$

A few comments about Theorem 1.4 are in order.

(1) The importance of the function $\log n/n$ is that if $p \ll \log n/n$ then $G \in G(n,p)$ is a.a.s. disconnected, in which case $h(G) = 0$.

(2) For every graph G we have $h(G) \leq \delta(G)$, so this is the best rate of growth of edge expansion that one might hope for.

(3) The $1/2$ in this statement of Theorem 1.4 can not be improved, since the edge expansion of the complete graph K_n is given by $h(K_n) = (n-1)/2$, and we are getting closer to this case as the edge probability p gets closer to 1.

(4) For every vertex v the expectation of the degree $\deg(v)$ is given by

$$E[\deg(v)] = p(n-1) \approx pn.$$

Once $p \geq (1+\epsilon)\log n/n$, a.a.s. every vertex degree is tightly concentrated around this mean.

Linial and Meshulam defined 2-dimensional analogues of $G(n,p)$ and proved a cohomological analogue of Theorem 1.3 [8], and Meshulam and Wallach extended the result to arbitrary dimension [13].

Let Δ^n denote the $(n-1)$-dimensional simplex and Δ^n_i its i-skeleton.

Definition 1.5. The random simplicial complex $Y_d(n,p)$ is the probability space of all simplicial complexes with complete $(d-1)$-skeleton and each d-dimensional face appearing independently with probability p. In other words for every

$$\Delta^n_{d-1} \subseteq Y \subseteq \Delta^n_d,$$

we have

$$\mathrm{Prob}(Y \in Y_d(n,p)) = p^{f_i(Y)}(1-p)^{\binom{n}{i+1}-f_i(Y)},$$

where f_i denotes the number of i-dimensional faces of Y.

Let R be any (fixed) finite coefficient ring. The main results of [8] and [13] for $Y \in Y(n,p)$ is the following.

Theorem 1.6. *Let $\omega = \omega(n)$ be any function that tends to infinity with n. If $p = \frac{(k+1)\log n+\omega}{n}$ then a.a.s. $H^k(Y,R) = 0$ and if $p = \frac{(k+1)\log n-\omega}{n}$ then a.a.s. $H^k(Y,R) \neq 0$.*

The case $k=1$ and $R = \mathbb{Z}/2$ was proved in [8] and the general case in [13]. Note that $k=0$ is exactly Theorem 1.3. One of the main ideas in proving Theorem 1.6 is an isoperimetric inequality for Δ^n (or Δ^n_{k+1}), which we describe next.

Fix a coefficient ring R, and for a simplicial complex S let $C^k(S)$ denote the set of k-cochains of S, i.e. $C^k(S) = \{f : S(k) \to R\}$, where $S(k)$ is the set of k-dimensional faces of S. Let $d_k : C^k \to C^{k+1}$ denote the simplicial coboundary map (see for example [6]).

Definition 1.7. For a simplicial complex S and $\phi \in C^k(S)$ define

$$b(\phi) = |\mathrm{supp}(d_k(\phi))|,$$

and

$$w(\phi) = \min_{X \in C^{k-1}(S)} |\mathrm{supp}(\phi + d_{k-1}(X)|.$$

Here $\mathrm{supp}(A)$ denotes the support of A, i.e. $\mathrm{supp}(A) = \{\sigma \in S(k) \mid A(\sigma) \neq 0\}$, and $|\cdot|$ denotes the cardinality of a set.

A useful isoperimetric inequality observed by Linial, Meshulam, and Wallach is the following.

Lemma 1.8. *For every $\phi \in C^k(\Delta^n)$,*

$$b(\phi) \geq \frac{n}{k+1} w(\phi).$$

For a short self-contained proof and examples to show that the constant $\frac{1}{k+1}$ is best possible, see [13]. What we are interested in here is whether a similar inequality might hold for much sparser simplicial complexes. This motivates the following definition, which is the first main point of this article.

Definition 1.9. For a simplicial complex S define the *degree k topological expansion* $h_k(S)$ by

$$h_k(S) = \min_{\{\phi \in C^k(S) | w(\phi) \neq 0\}} \frac{b(\phi)}{w(\phi)}.$$

Note that when $k = 0$ this agrees exactly with the usual definition of edge expansion for graphs. First of all, if we talk about sets of vertices and edges, this is really working with $R = \mathbb{Z}/2$ coefficients. Second, if we work in reduced cohomology then the coboundary of the empty set is the set of all n vertices. So for every subset of vertices ϕ, we have $w(\phi) = \min(|\phi|, n - |\phi|)$, or said another way $w(\phi) = |\phi|$ if and only if $|\phi| \leq n/2$. Clearly $b(\phi)$ is the number of edges with one end in ϕ.

We also note that $h_k(S) > 0$ if and only if the kth cohomology $H^k(S, R)$ vanishes. To see this note that $b(\phi) = 0$ if and only if ϕ is a cocycle, and $w(\phi) = 0$ if and only if ϕ is a coboundary. So we only have $h_k(S) = 0$ if there is some k-cocycle which is not a coboundary, i.e. a nontrivial element of kth cohomology.

Continuing the analogy with edge expansions of graphs, we expect this to be most interesting for sequences of simplicial complexes where the number of vertices $n \to \infty$, and where the properly renormalized degree k topological expansion stays bounded away from zero. To discuss the proper renormalization we introduce the following notation. For a simplicial complex S and $k \geq 0$, set

$$\delta_k(S) = \min_{\sigma \in S(k)} \#(k+1)\text{-dimensional faces in S(k+1) containing } \sigma.$$

Just as $h(G) \leq \delta(G)$ for a graph G, it is easy to see more generally that $h_k(S) \leq \delta_k(S)$ for a simplicial complex S. So we might think of a sequence of simplicial complexes $\{S_i\}_{i \in \mathbb{N}}$ as an expanding family if $h_k(S_i)/\delta_k(S_i)$ stays bounded away from zero as $i \to \infty$. The second main point of this article is to show that the random complexes $Y_d(n,p)$ provide such examples once $p \gg \frac{\log n}{n}$.

2. Main result

Fix a finite coefficient ring R and $k \geq 1$. Our main result is the following.

Theorem 2.1. *For every $\epsilon > 0$ there exists $C = C(\epsilon)$ such that for $p \geq \frac{C \log n}{n}$ and $Y \in Y_{k+1}(n,p)$ we have a.a.s.*

$$h_k(Y) > \left(\frac{1}{k+1} - \epsilon \right) \delta_k(Y).$$

The constant $\frac{1}{k+1}$ can not be improved for these complexes, since the bound in Lemma 1.8 is best possible, and this is the limiting case as $p \to 1$. As Theorem 1.6 is a generalization of Theorem 1.3 to higher dimensions, Theorem 2.1 a generalization of Theorem 1.4 (although our constant C is probably not optimal).

Proof of Theorem 2.1. We follow a similar counting argument to that in [8] and [13], but can afford to be much coarser here since we make no effort to optimize the constant C.

First we note that once $p \geq C \log n/n$ with $C > k$ standard large deviation bounds that a.a.s. every k-face σ satisfies $\deg(\sigma) \approx pn$, so in particular $\delta_k(Y) \approx pn$ [3]. A more precise statement is that a.a.s. $\delta(Y)(1 - o(1))pn$, but to ease notation we occasionally just replace $\delta_k(Y)$ by pn.

Given a cochain $\phi \in C^k(Y)$ we desire to put a lower bound on $b(\phi)$ in terms of $w(\phi)$. There is a natural inclusion $i : Y \hookrightarrow \Delta^n_{k+1}$, and we let $\tilde{\phi}$ denote the "image" of ϕ in $C^k(\Delta^n_{k+1})$. Cochains normally pull back rather than push forward, but this makes sense here because Y and Δ^n_{k+1} share the same (complete) k-skeleton, and for the same reason we have $w(\phi) = w(\tilde{\phi})$.

By Lemma 1.8 we have that $b(\tilde{\phi}) \geq \frac{n}{k+1} w(\tilde{\phi})$. It is easy to see that $b(\phi) = |\mathrm{supp}(d_k(\tilde{\phi})) \cap Y|$. By writing $b(\phi)$ as a sum of indicator random variables for the k-faces in $\mathrm{supp}(d_k(\tilde{\phi}))$ we see that has a binomial distribution $S_{N,p}$ where $N = |\mathrm{supp}(d_k(\tilde{\phi})| = b(\tilde{\phi}) \geq \frac{n}{k+1} w(\tilde{\phi})$; see for example the introduction of [3]. Now we can use large deviation bounds for binomial random variables. For example Theorem 1.7 in [3] states that if $0 < p < 1/2$, $p(1-p)N \geq 12$, and $0 < c < \frac{1}{12}$ then we have

$$\mathrm{Prob}(|S_{N,p} - pN| \geq cpN) \leq (cpN)^{-1/2} e^{-c^2 pN/3}$$

Set $c = \frac{\epsilon}{k+1}$. If $w(\phi) = i \geq 1$ then

$$\begin{aligned}
& \mathrm{Prob}\left[\frac{b(\phi)}{w(\phi)} \leq \left(\frac{1}{k+1} - \epsilon\right) \delta_k(Y)\right] \\
= \; & \mathrm{Prob}\left[b(\phi) \leq \left(\frac{1}{k+1} - \epsilon\right) pni\right] \\
\leq \; & \mathrm{Prob}\left[|S_{N,p} - pN| \geq cNp\right] \\
\leq \; & (cpN)^{-1/2} e^{-c^2 pN/3} \\
\leq \; & e^{-c^2 pni/3(k+1)}
\end{aligned}$$

We apply a union bound for the event that there exists a cochain $\phi \in C^k(Y)$ with $\frac{b(\phi)}{w(\phi)} \leq \left(\frac{1}{k+1} - \epsilon\right) \delta_k(Y)$. Note that a simple upper bound on the total number of cochains with $w(\phi) = i$ is given by

$$|\{\phi \in C^k(Y) \mid w(\phi) = i\}| \leq \binom{\binom{n}{k+1}}{i} (r-1)^i,$$

where r is the cardinality of our finite coefficient ring R. So the total probability of a bad cochain is at most

$$\begin{aligned}
& \sum_{i \geq 1} \left|\{\phi \in C^k(Y) \mid w(\phi) = i\}\right| \, \mathrm{Prob}\left[b(\phi) \leq i \left(\frac{1}{k+1} - \epsilon \delta_k(Y)\right]\right) \\
\leq \; & \sum_{i \geq 1} \binom{\binom{n}{k+1}}{i} (r-1)^i e^{-c^2 pni/3(k+1)} \\
\leq \; & \sum_{i \geq 1} \binom{n^{k+1}}{i} (r-1)^i e^{-c^2 pni/3(k+1)} \\
\leq \; & \sum_{i \geq 1} \frac{n^{(k+1)i}}{i!} (r-1)^i e^{-c^2 pni/3(k+1)} \\
\leq \; & \sum_{i \geq 1} \left(\frac{e n^{k+1} (r-1)}{i}\right)^i \left(e^{-c^2 pn/3(k+1)}\right)^i,
\end{aligned}$$

this last step by Stirling's approximation $i! \geq (i/e)^i$.

Now write $p = \frac{C \log n}{n}$. Choose any $C > \frac{3(k+1)^2}{c^2}$ and write $c_1 = k + 1 - \frac{c^2 C}{3(k+1)}$. By choice of C we have that $c_1 > 0$.

Hence the probability that we have any cochain violating the expansion isoperimetric inequality is bounded above by

$$\begin{aligned}
& \sum_{i \geq 1} \left(\frac{e n^{k+1}(r-1)}{i} \right)^i \left(e^{-c^2 pn/3(k+1)} \right)^i \\
= & \sum_{i \geq 1} \left(\frac{e n^{k+1}(r-1)}{i} \right)^i \left(e^{-c^2 C \log n/3(k+1)} \right)^i \\
\leq & \sum_{i \geq 1} \left(\frac{e n^{k+1}(r-1)}{i} \right)^i \left(n^{-c^2 C/3(k+1)} \right)^i \\
\leq & \sum_{i \geq 1}^{\binom{n}{k+1}} \left(\frac{e n^{-c_1}(r-1)}{i} \right)^i \\
\leq & \sum_{i \geq 1}^{\binom{n}{k+1}} \left(e n^{-c_1}(r-1) \right)^i \\
\leq & \frac{e(r-1)n^{-c_1}}{1 - e(r-1)n^{-c_1}} \to 0
\end{aligned}$$

as $n \to \infty$.

This proves the theorem, and in particular we see that since $c = \frac{\epsilon}{k+1}$, choosing $C > \frac{3(k+1)^4}{\epsilon^2}$ and setting $p \geq C \log n / n$ is sufficient to ensure that a.a.s. $h_k(Y) \geq \left(\frac{1}{k+1} - \epsilon \right) \delta_k(Y)$.

$\square$

3. Comments

We have discussed what seems to be the natural topological generalization of edge expansion to higher-dimensional simplicial complexes. Linial, Meshulam, and Wallach had already observed that $(k+1)$-dimensional skeletons of $(n-1)$-dimensional simplexes Δ^n_{k+1} already meet this definition, and we have extended their results for vanishing of cohomology to show that in roughly the same regime where cohomology vanishes we have sparse examples of degree k topological expanders.

Their results and ours depend on the coefficient ring R being finite, and it would be interesting to know what happens, say, for $\mathbb{Z}$ coefficients. On a related note, it was recently showed by Babson, Hoffman, and Kahle that in the $k = 2$ case the threshold for vanishing of $\pi_1(Y)$ is quite different than the threshold for $H^1(Y, R)$, and along the way they establish linear isoperimetric inequalities on $\pi_1(Y)$ [1], showing that area $A(\gamma) \leq cL(\gamma)$ for some constant c and all contractible curves γ. This seems to be a different, but perhaps complementary, kind of isoperimetry to what is discussed here.

As mentioned in the introduction, other examples of "expanding" simplicial complexes have been considered. For example, it is known that if one takes the clique complex of a particular Cayley graph on $PGL_r(\mathbb{F}_{p^m})$, the resulting simplicial complex has interesting spectral properties [10, 11], and also geometric overlap properties [5]. Since there are several closely related notions of expansion for graphs, it would be interesting to know whether these Ramanujan complexes are also topological expanders in the sense described here. At the moment we do not have any explicit examples of topological expanders for $k \geq 1$, and these would seem to be natural candidates.

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