CHARACTERIZATIONS OF PROJECTIVE SPACES AND HYPERQUADRICS

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1. Introduction

Projective spaces and hyperquadrics are the simplest projective algebraic varieties, and they can be characterized in many ways. The aim of this paper is to provide a new characterization of them in terms of positivity properties of the tangent bundle. We refer the reader to the article [ADK08] which reviews these matters. Notice that our results generalize Mori's (see [Mor79]), Wahl's (see [Wah83] and [Dru04]), Andreatta-Wiśniewski's (see [AW01] and [Ara06]), Araujo-Druel-Kovács's (see [ADK08]) and Paris's (see [Par10]) characterizations of projective spaces and hyperquadrics. K. Ross recently posted a somewhat related result (see [ROS10]).

In this paper, we prove the following theorems. Here Q_n denotes a smooth quadric hypersurface in $\mathbf{P}^{n+1}$, and $\mathscr{O}_{Q_n}(1)$ denotes the restriction of $\mathscr{O}_{\mathbf{P}^{n+1}}(1)$ to Q_n. When $n=1$, $(Q_1, \mathscr{O}_{Q_1}(1))$ is just $(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(2))$.

Theorem A. *Let X be a smooth complex projective n-dimensional variety and $\mathscr{E}$ an ample vector bundle on X of rank $r+k$ with $r \geqslant 1$ and $k \geqslant 1$. If $h^0(X, T_X^{\otimes r} \otimes \det(\mathscr{E})^{\otimes -1}) \neq 0$, then $(X, \det(\mathscr{E})) \simeq (\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(l))$ with $r+k \leqslant l \leqslant \frac{r(n+1)}{n}$.*

Theorem B. *Let X be a smooth complex projective n-dimensional variety and $\mathscr{E}$ an ample vector bundle on X of rank $r \geqslant 1$. If $h^0(X, T_X^{\otimes r} \otimes \det(\mathscr{E})^{\otimes -1}) \neq 0$, then either $(X, \det(\mathscr{E})) \simeq (\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(l))$ with $r \leqslant l \leqslant \frac{r(n+1)}{n}$, or $(X, \mathscr{E}) \simeq (Q_n, \mathscr{O}_{Q_n}(1)^{\oplus r})$ and $r = 2i + nj$ with $i \geqslant 0$ and $j \geqslant 0$.*

The line of argumentation follows [AW01] (see also [ADK08] and [Par10]). We first prove Theorem A and Theorem B for Fano manifolds with Picard number $\rho(X) = 1$ (see Proposition 14). Then the argument for the proof of the main Theorem goes as follows. We argue by induction on $\dim(X)$. We may assume $\rho(X) \geqslant 2$. Hence the H-rationally connected quotient of X with respect to an unsplit covering family H of rational curves on X is non-trivial. It can be extended in codimension one so that we can produce a normal variety X_B equipped with a surjective morphism π_B with integral fibers onto a smooth curve B such that either $B \simeq \mathbf{P}^1$, $X_B \to B$ is a $\mathbf{P}^d$-bundle for some $d \geqslant 1$ and $h^0(X_B, T_{X_B/\mathbf{P}^1}^{\otimes i} \otimes \pi^*\mathscr{G}^{\otimes r-i} \otimes \det(\mathscr{E})_{|X_B}^{\otimes -1}) \neq 0$ for some integer $1 \leqslant i \leqslant r$ where $\mathscr{G}$ be a vector bundle on $\mathbf{P}^1$ such that $\mathscr{G}^*(2)$ is nef, or $X_B \to B$ is a $\mathbf{P}^d$-bundle for some $d \geqslant 1$ and $h^0(X_B, T_{X_B/B}^{\otimes r} \otimes \det(\mathscr{E})_{|X_B}^{\otimes -1} \otimes \pi_B^*\mathscr{G}^*) \neq 0$ where $\mathscr{G}$ is a nef vector bundle on C, or the geometric generic fiber of π_B is isomorphic to a smooth hyperquadric and $h^0(X_B, T_{X_B/B}^{[\otimes r]} \otimes \det(\mathscr{E})_{|X_B}^{\otimes -1} \otimes \pi_B^*\mathscr{G}^*) \neq 0$ where $\mathscr{G}$ is a nef vector bundle on C. But this is impossible unless $X \simeq \mathbf{P}^1 \times \mathbf{P}^1$ (see Lemma 3, Lemma 4 and Proposition 6).

Throughout this paper we work over the field of complex numbers.

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2. Proofs

2.1. **Projective spaces and hyperquadrics.** In this section, we gather some properties of the tangent bundle to projective spaces and smooth hyperquadrics.

Lemma 1. *Let n, r and k be integers with $n \geqslant 1$ and $r \geqslant 1$. Then $h^0(\mathbf{P}^n, T_{\mathbf{P}^n}^{\otimes r}(-k)) \neq 0$ if and only if $k \leqslant \frac{r(n+1)}{n}$.*

Proof. It is well-known that $T_{\mathbf{P}^n}$ is stable in the sense of Mumford-Takemoto with slope $\mu(T_{\mathbf{P}^n}) = \frac{n+1}{n}$ with respect to $\mathscr{O}_{\mathbf{P}^n}(1)$. By [HL97, Theorem 3.1.4], $T_{\mathbf{P}^n}^{\otimes r}(-r)$ is semistable with slope $\mu(T_{\mathbf{P}^n}^{\otimes r}(-k)) = \frac{r(n+1)}{n} - k$. It follows that if $h^0(\mathbf{P}^n, T_{\mathbf{P}^n}^{\otimes r}(-k)) \neq 0$ then $k \leqslant \frac{r(n+1)}{n}$. Conversely, let us assume that $k \leqslant \frac{r(n+1)}{n}$. Write $r = an + b$ where a and b are integers with $a \geqslant 0$ and $0 \leqslant b < n$. Then $k - a(n+1) = \llcorner k - a(n+1) \lrcorner \leqslant \llcorner \frac{b(n+1)}{n} \lrcorner = \llcorner b + \frac{b}{n} \lrcorner = b$ and

$$\begin{aligned} h^0(\mathbf{P}^n, T_{\mathbf{P}^n}^{\otimes r}(-k) &= h^0(\mathbf{P}^n, T_{\mathbf{P}^n}^{\otimes an}(-a(n+1)) \otimes T_{\mathbf{P}^n}^{\otimes b}(-k + a(n+1))) \\ &\geqslant h^0(\mathbf{P}^n, [T_{\mathbf{P}^n}^{\otimes n}(-(n+1))]^{\otimes a} \otimes T_{\mathbf{P}^n}^{\otimes b}(-b)) \\ &\geqslant h^0(\mathbf{P}^n, [\det(T_{\mathbf{P}^n})(-(n+1))]^{\otimes a} \otimes T_{\mathbf{P}^n}^{\otimes b}(-b)) \\ &= h^0(\mathbf{P}^n, T_{\mathbf{P}^n}^{\otimes b}(-b)) \geqslant 1, \end{aligned}$$

as claimed. □

Let d be a positive integer. Let $Q \subset \mathbf{P}^{d+1} = \mathbf{P}(W)$ be a smooth hyperquadric defined by a non degenerate quadratic form q on $W := \mathbf{C}^{d+2}$ and let $\mathscr{O}_Q(1)$ denote the restriction of $\mathscr{O}_{\mathbf{P}^{d+1}}(1)$ to Q. Let x be a point of Q and $w \in W \setminus \{0\}$ representing x; then $T_Q(-1)_x$ identifies with $x^{\perp}/ < x >$ and q induces an isomorphism $T_Q(-1) \simeq \Omega_Q^1(1)$ or equivalently a nonzero section in $H^0(Q, (T_Q(-1))^{\otimes 2})$ still denoted by q. Let $V := x^{\perp}/ < x >$. Let $G := SO(W)$ and let $P \subset SO(W)$ be the parabolic subgroup such that $G/P \simeq Q$ corresponding to $x \in Q$. Let $\det \in H^0(Q, \det(T_Q(-1))$ be a nonzero section.

Lemma 2. *Let the notations be as above.*

1. *The vector bundle T_Q is stable in the sense of Mumford-Takemoto; in particular, one has $h^0(Q, T_Q^{\otimes r}(-k)) = 0$ for $k > r \geqslant 1$.*
2. *The space of sections $H^0(Q, (T_Q(-1))^{\otimes r})$ is generated as a $\mathbf{C}$-vector space by the $\sigma \cdot q^{\otimes i} \otimes \det^{\otimes j}$'s where i and j are nonnegative integers such that $r = 2i + dj$ and $\sigma \in \mathfrak{S}_r$ the symmetric group on r letters acting as usual on the vector bundle $(T_Q(-1))^{\otimes r}$.*

Proof. Observe that $T_Q(-1)$ is homogeneous or equivalently that

$$T_Q(-1) \simeq (G \times V)/P$$

over $Q \simeq G/P$ where $g \in P$ acts on $G \times V$ by the formula

$$g \cdot (g', v) = (g'g, \rho(g^{-1}) \cdot v)$$

and

$$\rho : P \to GL(T_Q(-1)_x) = GL(V)$$

is the stabilizer representation. It vanishes on the unipotent radical U of P and can be viewed as the representation of the Levi subgoup $L \simeq \mathbf{C}^* \times SO(V) \subset P$ on V given by the standard representation of $SO(V)$ on V. It is irreducible and therefore $T_Q(-1)$ is indecomposable hence stable by [Ram66] and [Ume78] with slope $\mu(T_Q(-1)) = 0$ with respect to $\mathscr{O}_Q(1)$. By [HL97, Theorem 3.1.4], $(T_Q(-1))^{\otimes k}$ is semistable with slope $\mu((T_Q(-1))^{\otimes r}) = 0$. This ends the proof of the first part of the Lemma.

Observe that $(T_Q(-1))^{\otimes r}$ is homogeneous and that the stabilizer representation

$$P \to GL((T_Q(-1))_x^{\otimes r})$$

is $\rho^{\otimes r}$. In particular, $(T_Q(-1))^{\otimes r}$ decomposes as the direct sum of indecomposable vector bundles hence as the direct sum of stable vector bundles with slope 0. It follows that there is a one-to-one correspondence between the set of nonzero section in $H^0(Q, (T_Q(-1))^{\otimes r})$ and the set of rank one direct summands of $T_Q(-1))^{\otimes r}$. Finally, we obtain an isomorphism

$$H^0(Q, (T_Q(-1))^{\otimes r}) \simeq (V^{\otimes r})^{SO(V)}$$

since $SO(V)$ has no nontrivial character. The result now follows from [Wey39, Theorem 2.9 A]. □

2.2. **Fibrations over curves.** In this section, we prove our main Theorems for fibrations over curves.

Lemma 3. *Let $\mathscr{F}$ be a vector bundle on $\mathbf{P}^1$ of rank $m \geqslant 2$, $X := \mathbf{P}_{\mathbf{P}^1}(\mathscr{F})$ and $\pi : X \to \mathbf{P}^1$ the natural morphism. Let $\mathscr{E}$ be an ample vector bundle on X of rank $r+k$ with $r \geqslant 2$ and $k \geqslant 0$. Let $\mathscr{G}$ be a vector bundle on $\mathbf{P}^1$ such that $\mathscr{G}^*(2)$ is nef. If $h^0(X, T_{X/\mathbf{P}^1}^{\otimes i} \otimes \pi^*\mathscr{G}^{\otimes r-i} \otimes \det(\mathscr{E})^{\otimes -1}) \neq 0$ for some integer $0 \leqslant i \leqslant r$ then $X \simeq \mathbf{P}^1 \times \mathbf{P}^1$, $\mathscr{F} = \mathscr{O}_{\mathbf{P}^1}(a)^{\oplus 2}$ for some integer a, $k = 0$, $2i = r$ and $\det(\mathscr{E}) \simeq \mathscr{O}_{\mathbf{P}^1}(2) \boxtimes \mathscr{O}_{\mathbf{P}^1}(2)$.*

Proof. Write $\mathscr{F} \simeq \mathscr{O}_{\mathbf{P}^1}(a_1) \oplus \cdots \oplus \mathscr{O}_{\mathbf{P}^1}(a_m)$ with $a_1 \leqslant \cdots \leqslant a_m$. Let $b := a_m - a_1 \geqslant 0$. Let σ be a section of π corresponding to a surjective morphism $\mathscr{O}_{\mathbf{P}^1}(a_1) \oplus \cdots \oplus \mathscr{O}_{\mathbf{P}^1}(a_m) \twoheadrightarrow \mathscr{O}_{\mathbf{P}^1}(a_m)$ and let σ_1 the section of π corresponding to the projection map $\mathscr{O}_{\mathbf{P}^1}(a_1) \oplus \cdots \oplus \mathscr{O}_{\mathbf{P}^1}(a_m) \twoheadrightarrow \mathscr{O}_{\mathbf{P}^1}(a_1)$. Then $\sigma \equiv \sigma_1 + b\ell$ where ℓ is vertical line and

$$\det(\mathscr{E}) \cdot \sigma \geqslant r + k + b(r+k) = (r+k)(b+1).$$

We may assume that $h^0(\sigma, (T_{X/\mathbf{P}^1}^{\otimes i} \otimes \pi^*\mathscr{G}^{\otimes r-i} \det(\mathscr{E})^{\otimes -1})_{|\sigma}) \neq 0$ since σ is a free rational curve. But

$$T_{X/\mathbf{P}^1}{}_{|\sigma} \simeq N_{\sigma/X} \simeq \mathscr{O}_{\mathbf{P}^1}(a_m - a_1) \oplus \cdots \oplus \mathscr{O}_{\mathbf{P}^1}(a_m - a_{m-1})$$

and we obtain

$$(r+k)(b+1) \leqslant \det(\mathscr{E}) \cdot \sigma \leqslant ib + 2(r-i). \tag{1}$$

By Lemma 1, we must have

$$r + k \leqslant i\frac{m}{m-1}. \tag{2}$$

We obtain

$$(r+k)(b+1) + 2k \leqslant ib + 2(r+k) - 2i \leqslant ib + 2i\frac{m}{m-1} - 2i = i(b + \frac{2}{m-1}). \tag{3}$$

It follows that $m = 2$, $b = k = 0$ and $r = 2i$. □

Lemma 4. *Let X be a smooth complex projective variety, $\mathscr{E}$ an ample vector bundle on X of rank $r+k$ with $r \geqslant 1$ and $k \geqslant 0$. Let $\pi : X \to B$ be a surjective morphism onto a smooth connected curve with integral fibers. Let $\mathscr{G}$ be a numerically effective vector bundle on B of rank > 0. Assume that the geometric generic fiber is isomorphic to a projective space. Then $h^0(X, T_{X/B}^{\otimes r} \otimes \det(\mathscr{E})^{\otimes -1} \otimes \pi^*\mathscr{G}^*) = 0$.*

Proof. Let η be the generic point of B. Tsen's Theorem implies that $X_\eta \simeq \mathbf{P}^d_{k(\eta)}$. Thus there exists a divisor H on X such that $\mathscr{O}_X(H)_{|X_\eta} \simeq \mathscr{O}_{\mathbf{P}^d_{k(\eta)}}(1)$. Let $\mathscr{L} := \mathscr{O}_X(H)$. Let $r' \geqslant r+k$ be defined by the formula $\det(\mathscr{E})_{|X_\eta} \simeq \mathscr{O}_{\mathbf{P}^d_{k(\eta)}}(r')$. It follows from the semicontinuity Theorem that $h^0(X_b, (\det(\mathscr{E}) \otimes \mathscr{L}^{\otimes -r'})_{|X_b}) \geqslant 1$ and $h^0(X_b, (\mathscr{L}^{\otimes r'} \otimes \det(\mathscr{E})^{\otimes -1})_{|X_b}) \geqslant 1$ for any point b in B. Thus $h^0(X_b, (\det(\mathscr{E}) \otimes \mathscr{L}^{\otimes -r'})_{|X_b}) = 1$ since X_b is integral. By the base change Theorem, $\det(\mathscr{E}) \simeq \mathscr{L}^{\otimes r'} \otimes \pi^*\mathscr{M}$ for some line bundle $\mathscr{M}$ on B. Thus $\mathscr{L}$ is ample/B and by [Fuj75, Corollary 5.4], π is a $\mathbf{P}^d$-bundle. By replacing B with a finite cover $\bar{B} \to B$ and X with $X \times_B \bar{B}$ we may assume that $g(B) \geqslant 1$. Let $\mathscr{M}'$ be a line bundle on B such that $\mathscr{M} \simeq \mathscr{M}'^{\otimes r'}$. Set $\mathscr{L}' := \mathscr{L} \otimes \pi^*\mathscr{M}'^{\otimes -1}$. Then $\mathscr{L}'^{\otimes r'} \simeq \det(\mathscr{E})$ hence $\mathscr{L}'$ is ample. Let $\mathscr{F} := \pi_*(\mathscr{L}')$. Then $\mathscr{F}$ is an ample vector bundle on B and $X := \mathbf{P}_B(\mathscr{F})$. By [CF90], By replacing B with a finite cover $\bar{B} \to B$ and X with $X \times_B \bar{B}$, we may assume that there exist an ample line bundle $\mathscr{M}$ on B, a positive integer m and a surjective map of $\mathscr{O}_B$-modules $\mathscr{M}^{\oplus m} \twoheadrightarrow \mathscr{F}$. Observe that the line bundle $\mathscr{L}' \otimes \pi^*\mathscr{M}^{\otimes -1}$ is generated by its global sections. Let $C = D_1 \cap \cdots \cap D_{\dim(X)-1}$ be general complete intersection curve with $D_i \in |\mathscr{L}' \otimes \pi^*\mathscr{M}^{\otimes -1}|$ (C is a section of π). Then $(T_{X/B})_{|C} \simeq N_{C/X} \simeq (\mathscr{L}' \otimes \pi^*\mathscr{M}^{\otimes -1})_{|C}^{\oplus \dim(X)-1}$. But

$$h^0(C, (\mathscr{L}' \otimes \pi^*\mathscr{M}^{\otimes -1})^{\otimes r} \otimes \det(\mathscr{E})_{|C}^{\otimes -1}) = h^0(C, \mathscr{L}'^{\otimes r - r'} \otimes \pi^*\mathscr{M}_{|C}^{\otimes -r}) = 0$$

and the claim follows. □

When dealing with sheaves that are not necessarily locally free, we use square brackets to indicate taking the reflexive hull.

Notation 5 (Reflexive tensor operations)**.** Let X be a normal variety and $\mathscr{Q}$ a coherent sheaf of $\mathscr{O}_X$-modules. For $n \in \mathbf{N}$, set $\mathscr{Q}^{[\otimes n]} := (\mathscr{Q}^{\otimes n})^{**}$, $S^{[n]}\mathscr{Q} := (S^n\mathscr{Q})^{**}$ and $\det(\mathscr{Q}) := (\wedge^{\operatorname{rank}(\mathscr{Q})}(\mathscr{Q}))^{**}$.

Proposition 6. *Let X be a normal complex projective variety, $\mathscr{E}$ an ample vector bundle on X of rank $r+k$ with $r \geqslant 1$ and $k \geqslant 0$. Let $\pi : X \to B$ be a surjective morphism onto a smooth connected curve with integral fibers. Let $\mathscr{G}$ be a numerically effective vector bundle on B of rank > 0. Assume that the geometric generic fiber is isomorphic to a smooth hyperquadric. Then $h^0(X, T_{X/B}^{[\otimes r]} \otimes \det(\mathscr{E})^{\otimes -1} \otimes \pi^*\mathscr{G}^*) = 0$.*

Proof. Let η be the generic point of B and $k(\bar{\eta})$ be an algebraic closure of $k(\eta)$. Let $q_{\bar{\eta}}$ be a non degenerate quadratic form defining $X_{\bar{\eta}} \subset \mathbf{P}^{d+1}_{k(\bar{\eta})}$ where $d := \dim(X) - 1$. By Lemma 2, $k = 0$ and $\det(\mathscr{E})_{|X_{\bar{\eta}}} \simeq \mathscr{O}_{X_{\bar{\eta}}}(r)$.

Let us assume to the contrary that $h^0(X, T_{X/B}^{[\otimes r]} \otimes \det(\mathscr{E})^{\otimes -1} \otimes \pi^*\mathscr{G}^*) \neq 0$ and let $s \in H^0(X, T_{X/B}^{[\otimes r]} \otimes \det(\mathscr{E})^{\otimes -1} \otimes \pi^*\mathscr{G}^*)$ be a nonzero section. Notice that, for any $\sigma \in \mathfrak{S}_r$ and any non negative integers i and j such that $r = 2i + dj$,

$$\sigma \cdot [(S^{[2]}T_{X/B})^{[\otimes i]} \otimes \det(T_{X/B})^{[\otimes j]}] \otimes \det(\mathscr{E})^{\otimes -1} \otimes \pi^*\mathscr{G}^*$$

is a direct summand of

$$T_{X/B}^{[\otimes r]} \otimes \det(\mathscr{E})^{\otimes -1} \otimes \pi^*\mathscr{G}^*.$$

By Lemma 2, we may assume that

$$s \in H^0(X, (S^{[2]}T_{X/B})^{[\otimes i]} \otimes \det(T_{X/B})^{[\otimes j]} \otimes \det(\mathscr{E})^{\otimes -1} \otimes \pi^*\mathscr{G}^*)$$

and

$$s_{|X_{\bar{\eta}}} = q_{\bar{\eta}}^{\otimes i} \otimes \det{}_{\bar{\eta}}^{\otimes j} \otimes g_{\bar{\eta}}$$

for some non negative integers i and j with $r = 2i + dj$ and some non zero section $g_{\bar{\eta}} \in \pi^* H^0(\bar{\eta}, \mathscr{G}_{|\bar{\eta}})$. It follows that the induced map

$$\mathscr{G} \to \pi_*((S^{[2]}T_{X/B})^{[\otimes i]} \otimes \det(T_{X/B})^{[\otimes j]} \otimes \det(\mathscr{E})^{\otimes -1})$$

has rank one and therefore, we may assume that $\mathscr{G}$ is a line bundle (with $\deg(\mathscr{G}) \geqslant 0$). We obtain a map

$$\varphi_s : {\Omega^1_{X/B}}^{[\otimes i]} \to {T_{X/B}}^{[\otimes i]} \otimes \det(T_{X/B})^{[\otimes j]} \otimes \det(\mathscr{E})^{\otimes -1} \otimes \pi^*\mathscr{G}^*$$

whose restriction to $X_{\bar{\eta}}$ is an isomorphism. Finally, we obtain a nonzero section

$$s' := \det(\varphi_s) \in H^0(X, \det({T_{X/B}}^{[\otimes i]}) \otimes \det({T_{X/B}}^{[\otimes i]} \otimes \det(T_{X/B})^{[\otimes j]} \otimes \det(\mathscr{E})^{\otimes -1} \otimes \pi^*\mathscr{G}^*))$$
$$\simeq H^0(X, \det(T_{X/B})^{[\otimes (2id^{i-1} + d^i j)]} \otimes \det(\mathscr{E})^{\otimes -d^i} \otimes \pi^*\mathscr{G}^{\otimes -d^i}).$$

Observe that s' does not vanish anywhere on a general fiber of π and that any fiber of π is integral. Thus

$$-K_{X/B} \equiv \frac{d^i}{2id^{i-1} + d^i j} c_1(\det(\mathscr{E})) + \pi^*\Delta$$

for some (integral) effective divisor $\Delta \geqslant \frac{d^i}{2id^{i-1}+d^i j} c_1(\mathscr{G})$ and $-K_{X/B}$ is ample. But that contradicts Lemma 7. ☐

Lemma 7 ([ADK08, Theorem 3.1])**.** *Let X be a normal projective variety, $f : X \to C$ a surjective morphism onto a smooth curve, and $\Delta \subseteq X$ a Weil divisor such that (X, Δ) is log canonical over the generic point of C. Then $-(K_{X/C} + \Delta)$ is not ample.*

Lemma 8. *Let S be a smooth projective surface equipped with a surjective morphism $\pi : S \to B$ with connected fibers onto a smooth connected curve. Let $\mathscr{M}$ be a nef and big line bundle on S. Assume that, for a general point b in B, $\mathscr{M} \cdot S_b = 2r$ for some $r \geqslant 1$ and either $g(B) \geqslant 1$ or $B = \mathbf{P}^1$ and S is a ruled surface over B. Then $h^0(S, T_S^{\otimes r} \otimes \mathscr{M}^{\otimes -1}) = 0$.*

Proof. Let $c : S \to \bar{S}$ be a minimal model. Write $\mathscr{M} = c^*\bar{\mathscr{M}}(-E)$ for some divisor E on S supported on the exceptional locus of c. Observe that E is effective and $\bar{\mathscr{M}}$ is nef since $\mathscr{M}$ is nef. Therefore, the natural map $T_S \to c^*T_{\bar{S}}$ induces an inclusion $H^0(S, T_S^{\otimes r} \otimes \mathscr{M}^{\otimes -1}) \subset H^0(\bar{S}, T_{\bar{S}}^{\otimes r} \otimes \bar{\mathscr{M}}^{\otimes -1})$. If $g(B) \geqslant 1$ then π induces a morphism $\bar{S} \to B$ and $\bar{S}$ is a ruled surface over B. We may thus assume that $S \to B$ is smooth. Since $\mathscr{M} \cdot S_b = 2r$ and $T_{S/C} \cdot S_b = 2$ for $b \in B$, we must have

$$H^0(S, T_{S/B}^{\otimes r} \otimes \mathscr{M}^{\otimes -1}) = H^0(S, T_S^{\otimes r} \otimes \mathscr{M}^{\otimes -1}).$$

Let us assume to the contrary that $h^0(S, T_S^{\otimes r} \otimes \mathscr{M}^{\otimes -1}) \neq 0$. Then $r(-K_{S/B}) \sim c_1(\mathscr{M}) + \pi^*\Delta$ where Δ is an effective divisor on C and $K_{S/B}$ is nef and big. But $K_{S/B}^2 = 0$ for any (geometrically) ruled surface, a contradiction. ☐

2.3. **Tools.** The proof of the main Theorem will apply rational curves on X. Our notation is consistent with that of [Kol96].

Let X be a smooth complex projective uniruled variety and H an irreducible component of $\mathrm{RatCurves}(X)$. Recall that only general points in H are in 1:1-correpondence with the associated curves in X. Let ℓ be a rational curve corresponding to a general point in H, with normalization morphism $f : \mathbf{P}^1 \to \ell \subset X$. We denote by $[\ell]$ or $[f]$ the point in H corresponding to ℓ.

We say that H is a *dominating family of rational curves on* X if the corresponding universal family dominates X. A dominating family H of rational curves on X is called *unsplit* if it is proper. It is called *minimal* if, for a general point $x \in X$, the subfamily of H parametrizing curves through x is proper.

Let $H_1, \ldots, H_k$ be minimal dominating families of rational curves on X. For each i, let $\overline{H}_i$ denote the closure of H_i in $\mathrm{Chow}(X)$. We define the following equivalence relation on X, which we call $(H_1, \ldots, H_k)$-equivalence. Two points $x, y \in X$ are $(H_1, \ldots, H_k)$-equivalent if they can be connected by a chain of 1-cycles from $\overline{H}_1 \cup \cdots \cup \overline{H}_k$. By [Cam92] (see also [Kol96, IV.4.16]), there exists a proper surjective morphism $\pi_0 : X_0 \to Y_0$ from a dense open subset of X onto a normal variety whose fibers are $(H_1, \ldots, H_k)$-equivalence classes. We call this map the $(H_1, \ldots, H_k)$*-rationally connected quotient of* X. For more details see [Kol96].

Lemma 9. *Let X be a smooth complex projective variety and $H_1, \ldots, H_k$ unsplit dominating families of rational curves on X. Let $\pi_0 : X_0 \to Y_0$ be the $(H_1, \ldots, H_k)$-rationally connected quotient of X. If the geometric generic fiber is isomorphic to a projective space, then π_0 is a $\mathbf{P}^d$-bundle in codimension one in Y_0 with $d := \dim(X_0) - \dim(Y_0)$.*

Proof. By [ADK08, Lemma 2.2], we may assume that π_0 is a proper surjective equidimensional morphism with integral fibers. Let $C_0 \subset Y_0$ be a general complete intersection curve. Set $X_{C_0} := \pi_0^{-1}(C_0)$. Then X_{C_0} is a smooth variety. Let η be the generic point of C_0 and let $\mathscr{L}_{C_0}$ be a line bundle on X_{C_0} that restricts to $\mathscr{O}_{\mathbf{P}^d_{k(\eta)}}(1)$ on $X_{C_0\eta} \simeq \mathbf{P}^d_{k(\eta)}$ ($d \geqslant 1$) (see the proof of Lemma 4). Let $\mathscr{M}$ be an ample line bundle on X and r a positive integer such that $\mathscr{M}_{|X_{C_0\eta}} \simeq \mathscr{O}_{\mathbf{P}^d_{k(\eta)}}(r)$.

For each i, denote by H_i^j, $1 \leq j \leq n_i$, the unsplit covering families of rational curves on X_{C_0} whose general members correspond to rational curves on X from the family H_i. Then $\pi_{C_0} := \pi_{0|X_{C_0}} : X_{C_0} \to C_0$ is the $(H_1^1, \ldots, H_1^{n_1}, \ldots, H_k^1, \ldots, H_k^{n_k})$-rationally connected quotient of X_{C_0}. Let F be a fiber of π_{C_0}. Let $[H_i^j]$ denote the class of a member of H_i^j in $N_1(F)$ and $\mathcal{H} := \{[H_i^j] \mid i = 1, \ldots, k, j = 1, \ldots, n_i\}$. Then by [Kol96, Proposition IV 3.13.3], $N_1(F)$ is generated by $\mathcal{H}$. Therefore any curve contained in any fiber of π_{C_0} is numerically proportional in $N_1(X_{C_0}/C_0)$ to a linear combination of the $[H_i^j]$'s. Hence $N_1(X_{C_0}/C_0)$ is generated by $\mathcal{H}$ and $c_1(\mathscr{M}_{X_{C_0}}) = rc_1(\mathscr{L}_{C_0}) \in N_1(X_{C_0}/C_0)$. Thus $\mathscr{L}_{X_{C_0}}$ is ample/C_0 and the claim follows from [Fuj75, Corollary 5.4]. □

Notation 10. Let X be a normal variety and $\mathscr{Q}$ be a coherent torsion free sheaf of $\mathscr{O}_X$-modules. Say that a curve $C \subset X$ is a general complete intersection curve for $\mathscr{Q}$ in the sense of Mehta-Ramanathan if $C = H_1 \cap \cdots \cap H_{\dim(X)-1}$, where $H_i \in |m_i H|$ are general, H is an ample line bundle on X and the $m_i \in \mathbf{N}$ are large enough so that the Harder-Narasimhan filtration of $\mathscr{Q}$ commutes with restriction to C.

The following result was established in [Par10, Proposition 4.1].

Lemma 11. *Let X and Y be a smooth complex projective varieties with $\dim(Y) \geqslant 1$, X_0 an open subset of X with $\mathrm{codim}_X(X \setminus X_0) \geq 2$, Y_0 a dense open subset of Y and $\pi_0 : X_0 \to Y_0$ a proper surjective equidimensional morphism. Let $C \subset X_0$ be a general complete intersection curve for $\pi_0^*\Omega^1_{Y_0}$ in the sense of Mehta-Ramanathan. If $(\pi_0^*\Omega^1_{Y_0})_{|C}$ is not nef then Y is uniruled.*

Proof. Let us sketch the proof for the reader's convenience. Fix an ample line bundle H on X, and consider general elements $H_i \in |m_iH|$, for $i \in \{1, \ldots, \dim(X) - 1\}$, where the $m_i \in \mathbf{N}$ are large enough so that the Harder-Narasimhan filtration of $\pi_0^*\Omega^1_{Y_0}$ commutes with restriction to $C := H_1 \cap \cdots \cap H_{\dim(X)-1}$. Setting $Z := H_1 \cap \cdots \cap H_{\dim(X)-\dim(Y)}$ and $Z_0 := Z \cap X_0$, we may assume that Z is a smooth variety of dimension $\dim(Y)$, and that the restriction $\varphi_0 := \pi_0|_{Z_0}$ is a finite morphism.

By the hypothesis $(\varphi_0^*\Omega^1_{Y_0})|_C$ is not nef, therefore $(\varphi_0^*T_{Y_0})|_C$ contains a subsheaf with positive slope. Thus if we denote by $i : Z_0 \hookrightarrow Z$ the inclusion and by $\mathscr{F}$ the reflexive sheaf $i_*(\varphi_0^*T_{Y_0})$, then the maximally destabilizing subsheaf $\mathscr{E}$ of $\mathscr{F}$ has positive slope (with respect to $H_{|Z}$).

Let K be a splitting field of the function field $K(Z_0)$ over $K(Y_0)$, and let $\psi : T \to Z$ be the normalization of Z in K. Consider $T_0 := \psi^{-1}(Z_0)$, and let $j : T_0 \hookrightarrow T$ be the inclusion. If we denote by ψ_0 the restriction of ψ to T_0, then the reflexive sheaf $\mathscr{F}' := (\psi^*\mathscr{F})^{**} = j_*(\psi_0^*\varphi_0^*T_{Y_0})$ contains the sheaf $(\psi^*\mathscr{E})^{**}$. Notice that $(\psi^*\mathscr{E})^{**}$ has positive slope. Consequently the maximally destabilizing subsheaf $\mathscr{E}'$ of $\mathscr{F}'$ has positive slope. Hence by replacing Z_0 with T_0, φ_0 with $\varphi_0 \circ \psi_0$, and $(\mathscr{F}, \mathscr{E})$ with $(\mathscr{F}', \mathscr{E}')$ if necessary, we may assume that $K(Z_0) \supset K(Y_0)$ is a Galois extension with Galois group G.

Because of its uniqueness, the maximally destabilizing subsheaf $\mathscr{E}$ of $\mathscr{F}$ is invariant under the action of G. Thus by replacing Z_0 with another open subset of Z if necessary, we may assume that there exists a saturated subsheaf $\mathscr{G}$ of T_{Y_0} such that $\mathscr{E} = i_*(\varphi_0^*\mathscr{G})$.

As $\mathscr{E}$ has positive slope, it follows from [KSCT07, Proposition 29 and Proposition 30] that the vector bundles $\mathscr{E}_{|C}$ and $(\mathscr{E} \otimes \mathscr{E} \otimes (\mathscr{F}/\mathscr{E})^*)_{|C}$ are ample. The morphism φ_0 being finite, this implies that $\mathscr{G}_{|\varphi_0(C)}$ and $(\mathscr{G} \otimes \mathscr{G} \otimes (T_{Y_0}/\mathscr{G})^*)_{|\varphi_0(C)}$ are ample vector bundles too. In particular we deduce from this that $\mathrm{Hom}(\mathscr{G} \otimes \mathscr{G}, T_{Y_0}/\mathscr{G}) = 0$, because the deformations of the curve $\varphi_0(C)$ dominate the variety Y_0. As a consequence $\mathscr{G}$ is a foliation on Y_0.

Finally, by extending $\mathscr{G}$ to a foliation $\widetilde{\mathscr{G}}$ on the whole variety Y, we can conclude by using [KSCT07, Theorem 1]. Indeed it follows from the fact that $\widetilde{\mathscr{G}}_{|\varphi_0(C)}$ is ample that the leaf of the foliation $\widetilde{\mathscr{G}}$ passing through a general point of $\varphi_0(C)$ is rationally connected; in particular Y is uniruled. □

The proof of our main result is based on the following result wich appears essentially in [Par10].

Corollary 12. *Let X be a smooth complex projective variety, X_0 an open subset of X with $\mathrm{codim}_X(X \setminus X_0) \geq 2$, Y_0 a smooth variety with $\dim(Y_0) \geqslant 1$ and $\pi_0 : X_0 \to Y_0$ a proper surjective equidimensional morphism. Assume that the generic fiber of π_0 is isomorphic to a projective space. Let C be a general complete intersection curve for $\pi_0^*\Omega^1_{Y_0}$ in the sense of Mehta-Ramanathan. If $(\pi_0^*\Omega^1_{Y_0})_{|C}$ is not nef then there exists a minimal free morphism $f : \mathbf{P}^1 \to Y_0$.*

Proof. Let Y be a smooth projective variety containing Y_0 as a dense open subset. By Lemma 11, Y is uniruled. Let H_Y be a minimal dominating family of rational curves on Y. Since the generic fiber of π_0 is isomorphic to a projective space, there exists a dominating family H_X of rational curves on X such that for a general member $[f] \in H_X$, $[\pi_0 \circ f]$ is a general member of H_Y. By

[Kol96, Proposition II 3.7], if $[f] \in H_X$ is a general member then $f(\mathbf{P}^1) \subset X_0$. The claim follows from [Kol96, Corollary IV 2.9]. □

The following Lemma is certainly well known to experts. We include a proof for lack of an adequate reference.

Lemma 13. *Let X be a smooth complex variety and H a minimal dominating family of rational curves on X. Let x be a general point in X and $[\ell] \in H$ with $x \in \ell$. If $T_{\ell,x}$ does not depend on $\ell \ni x$ then there exists a non empty open subset X_0 in X and a proper surjective morphism $\pi_0 : X_0 \to Y_0$ onto a variety Y_0 such that any fiber of π_0 is a rational curve from the family H.*

Proof. Let $[f] \in H$ be a general member. By [Kol96, Corollary IV 2.9], $f^*T_X \simeq \mathscr{O}_{\mathbf{P}^1}(2) \oplus \mathscr{O}_{\mathbf{P}^1}(1)^{\oplus d} \oplus \mathscr{O}_{\mathbf{P}^1}^{\oplus(n-d-1)}$ with $d := -K_X \cdot f_*\mathbf{P}^1 - 2$. Let x be a general point in X with $x \in \ell := f(\mathbf{P}^1)$. By [Hwa01, Proposition 2.3], $d = 0$ using the fact that $T_{\ell,x}$ does not depend on $\ell \ni x$.

Let $\bar{H}$ be the normalization of the closure of H in $\mathrm{Chow}(X)$ and $\bar{U}$ the normalization of the universal family. Let us denote by $\bar{\pi} : \bar{U} \to \bar{H}$ and $\bar{e} : \bar{U} \to X$ the universal morphisms. By shrinking H if necessary, we may assume that H parametrizes free morphisms. Then H is smooth (see [Kol96, Theorem I 2.16]) and $e := \bar{e}_{|U} : U \to X$ is étale where $U := \bar{\pi}^{-1}(H)$ (see [Kol96, Proposition II 3.4]).

It remains to show that there exists a dense open subset H_0 of H such that the restriction of $\bar{e}$ to $\bar{\pi}^{-1}(H_0)$ induces an isomorphism onto the open set $\bar{e}(\bar{\pi}^{-1}(H_0)$. By Zariski's main Theorem, it is enough to prove that $\bar{e}$ is birational. We argue by contradiction. Then there exists a curve $C \subset \bar{U}$ such that $\dim(\bar{\pi}(C)) = 1$ and $\bar{e}(C) = \ell$. Let c be a general point in C. Then $d_c\bar{e}(T_{C,c}) = d_c\bar{e}(T_{\bar{\pi}^{-1}(\bar{\pi}(c),c}) = T_{\ell,\bar{e}(c)}$. But that contradicts the fact that $\bar{e}$ is étale at c. The claim follows. □

2.4. **Characterizations of projective spaces and hyperquadrics.** The proof of the main Theorem stated in the introduction is based on the following result whose proof is similar to that of [ADK08, Theorem 6.3].

Proposition 14. *Let X be a smooth complex projective n-dimensional variety with $\rho(X) = 1$ and $\mathscr{E}$ an ample vector bundle on X of rank $r+k$ with $r \geqslant 1$ and $k \geqslant 0$. If $h^0(X, T_X^{\otimes r} \otimes \det(\mathscr{E})^{\otimes -1}) \neq 0$, then either $X \simeq \mathbf{P}^n$, or $k = 0$ and $X \simeq Q_n$ ($n \neq 2$).*

Proof. Let us give the proof following [ADK08]. First notice that X is uniruled by [Miy87], and hence a Fano manifold with $\rho(X) = 1$. The result is clear if $\dim X = 1$, so we assume that $n \geq 2$. Fix a minimal dominating family H of rational curves on X. Let $\mathscr{L}$ be an ample line bundle on X such that $\mathrm{Pic}(X) = \mathbf{Z}[\mathscr{L}]$.

Let $\mathscr{E}' \subset T_X$ be the maximally destabilizing subsheaf of T_X; $\mathscr{E}'$ is a reflexive sheaf of rank $r' \geqslant 1$. By [ADK08, Lemma 6.2], $\mu_{\mathscr{L}}(\mathscr{E}') \geq \frac{\mu_{\mathscr{L}}(\det(\mathscr{E}))}{r}$. Notice that $\mu_{\mathscr{L}}(\det(\mathscr{E})) \geqslant r+k$ since $\mathscr{E}$ is ample. This implies that $\frac{\deg(f^*\mathscr{E}')}{r'} \geq \frac{\deg(f^*\det(\mathscr{E}))}{r} \geqslant \frac{r+k}{r} \geqslant 1$ for a general member $[f] \in H$. If $r' = 1$, then $\mathscr{E}'$ is ample and we are done by Wahl's Theorem. If $f^*\mathscr{E}'$ is ample, then $X \simeq \mathbf{P}^n$ by [ADK08, Proposition 2.7], using the fact that $\rho(X) = 1$.

Otherwise, as $f^*\mathscr{E}'$ is a subsheaf of $f^*T_X \simeq \mathscr{O}_{\mathbf{P}^1}(2) \oplus \mathscr{O}_{\mathbf{P}^1}(1)^{\oplus d} \oplus \mathscr{O}_{\mathbf{P}^1}^{\oplus(n-d-1)}$ (see [Kol96, Corollary IV 2.9]), we must have $\deg(f^*\det(\mathscr{E}')) = r'$, $\deg(f^*\det(\mathscr{E})) = r$, $k = 0$ and $f^*\mathscr{E}' \simeq \mathscr{O}_{\mathbf{P}^1}(2) \oplus \mathscr{O}_{\mathbf{P}^1}(1)^{\oplus r'-2} \oplus \mathscr{O}_{\mathbf{P}^1}$ for a general $[f] \in H$. Then $\mathscr{O}_{\mathbf{P}^1}(2) \subset f^*\mathscr{E}'$ for general $[f] \in H$. Thus by [Hwa01, Proposition 2.3], $(f^*T_X^+)_p \subset (f^*\mathscr{E}')_p$ for a general $p \in \mathbf{P}^1$ and a general $[f] \in H$. Since $f^*\mathscr{E}'$ is a subbundle of f^*T_X, we have an inclusion of sheaves $f^*T_X^+ \hookrightarrow f^*\mathscr{E}'$, and thus $f^*\det(\mathscr{E}') = f^*\omega_X^{-1}$.

Since $\rho(X) = 1$, this implies that $\det \mathscr{E}' = \omega_X^{-1}$, and thus $0 \neq h^0(X, \wedge^{r'} T_X \otimes \omega_X) = h^{n-r'}(X, \mathscr{O}_X)$. The latter is zero unless $r' = n$ since X is a Fano manifold. Notice that $\deg(f^* \det(\mathscr{E})) = r$. It follows that, for any $[f] \in H$, $f^*\mathscr{E} \simeq \mathscr{O}_{\mathbf{P}^1}(1)^{\oplus r}$. By [AW01, Proposition 1.2] (see also [ROS10, Theorem 4.3]), $\mathscr{E} \simeq \mathscr{L}^{\oplus r}$ and $\deg(f^*\mathscr{L}) = 1$. If $n = r'$, then we must have $\omega_X^{-1} \simeq \det(\mathscr{E}') \simeq \mathscr{L}^{\otimes n}$. Hence $X \simeq Q_n$ by [KO73]. □

We will need the following auxiliary result.

Lemma 15. *Let X be a smooth complex projective variety and $\mathscr{E}$ an ample vector bundle on X of rank $r + k$ with $r \geqslant 2$ and $k \geqslant 0$. Assume that X is uniruled and fix a minimal dominating family H of rational curves on X. If $h^0(X, T_X^{\otimes r} \otimes \det(\mathscr{E})^{\otimes -1}) \neq 0$, then H is unsplit.*

Proof. The proof is similar to that of [Par10, Proposition 4.2]. Let $[f] \in H$ be a general member. Let us assume to the contrary that $h^0(X, T_X^{\otimes r} \otimes \det(\mathscr{E})^{\otimes -1}) \neq 0$ and $f_*(\mathbf{P}^1) \equiv C_1 + C_2$ with C_1 and C_2 nonzero integral effective rational 1-cycles. Notice first that $\det(\mathscr{E}) \cdot C \geqslant r + k$ for all rational curve $C \subset X$. By [Kol96, Corollary IV 2.9], $f^*T_X \simeq \mathscr{O}_{\mathbf{P}^1}(2) \oplus \mathscr{O}_{\mathbf{P}^1}(1)^{\oplus d} \oplus \mathscr{O}_{\mathbf{P}^1}^{\oplus(n-d-1)}$ and we must have $\deg(f^* \det(\mathscr{E})) \leqslant 2r$. Finally, $2(r + k) \leqslant \deg(f^* \det(\mathscr{E})) \leqslant 2r$ and we must have $k = 0$, $\deg(f^* \det(\mathscr{E})) = 2r$ and $f^* \det(\mathscr{E}) \simeq \mathscr{O}_{\mathbf{P}^1}(2r) \subset f^* \wedge^r (T_X) \simeq \wedge^r(\mathscr{O}_{\mathbf{P}^1}(2) \oplus \mathscr{O}_{\mathbf{P}^1}(1)^{\oplus d} \oplus \mathscr{O}_{\mathbf{P}^1}^{\oplus(n-d-1)})$. Hence $T_{\ell,x}^{\otimes r} = \det(\mathscr{E})_x \subset T_{X,x}^{\otimes r}$ for a general point x in ℓ and therefore, $T_{\ell,x}$ does not depend on $\ell \ni x$. Thus, by Lemma 13, there exists a non empty open subset X_0 in X and a proper surjective morphism $\pi_0 : X_0 \to Y_0$ onto a variety Y_0 such that any fiber of π_0 is a rational curve from the family H and $\det(\mathscr{E})_{|X_0} \simeq T_{X_0/Y_0}^{\otimes r}$. Let $\mathscr{L} \subset T_X$ be the saturated line bundle such that $T_{X_0/Y_0} \simeq \mathscr{L}_{|X_0}$. Notice that $\det(\mathscr{E}) \subset \mathscr{L}^{\otimes r}$ with equality on X_0. Let $C \subset X$ be a general complete intersection curve and let S be the normalization of the closure in X of $\pi_0^{-1}(\pi_0(C \cap X_0))$. By [Dru04, Lemme 1.2] (or [ADK08, Proposition 4.5]), the map $\Omega_X^1 \to \mathscr{L}^{\otimes -1}$ induces a map $\Omega_S^1 \to \mathscr{L}_S{}^{\otimes -1}$ where $\mathscr{L}_S$ denotes the pull-back of $\mathscr{L}$ to S. Notice that π_0 induces a surjective morphism $\pi_S : S \to B$ onto a smooth curve. By Lemma 8, $\dim(X_0) \neq 2$. Thus, we may assume $g(B) \geqslant 1$. Let $\tilde{S} \to S$ be a minimal desingularization of S. By [BW74, Proposition 1.2], $\Omega_S^1 \to \mathscr{L}_S{}^{\otimes -1}$ extends to $\Omega_{\tilde{S}}^1 \to \mathscr{L}_{\tilde{S}}{}^{\otimes -1}$. Let $\pi_{\tilde{S}} : \tilde{S} \to B$ be the induced morphism. By replacing $\mathscr{L}_{\tilde{S}}$ with its saturation in $T_{\tilde{S}}$, we may assume $\det(\mathscr{E})_{\tilde{S}} \subset \mathscr{L}_{\tilde{S}}^{\otimes r} \subset T_{\tilde{S}}^{\otimes r}$. Observe also that, for a general point b in B, $\det(\mathscr{E})_{\tilde{S}} \cdot \tilde{S}_b = 2r$. But that contradicts Lemma 8. □

Now we can prove our main theorems.

Theorem 16. *Let X be a smooth complex projective variety and $\mathscr{E}$ an ample vector bundle on X of rank $r + k$ with $r \geqslant 1$ and $k \geqslant 0$ and such that $h^0(X, T_X^{\otimes r} \otimes \det(\mathscr{E})^{\otimes -1}) \neq 0$.*

1. *If $k \geqslant 1$ then $X \simeq \mathbf{P}^n$.*
2. *If $k = 0$ then either $X \simeq \mathbf{P}^n$, or $X \simeq Q_n$.*

Proof. We shall proceed by induction on $n := \dim(X)$. The result is clear if $n = 1$, so we assume that $n \geq 2$. If $r + k = 1$ then we are done by Wahl's Theorem so we assume that $r + k \geqslant 2$. Notice that X is uniruled by [Miy87]. Fix a minimal dominating family H of rational curves on X. By Lemma 15, H is unsplit. Let $\pi_0 : X_0 \to Y_0$ be the H-rationally connected quotient of X. By [ADK08, Lemma 2.2], we may assume $\operatorname{codim}_X(X \setminus X_0) \geqslant 2$ and π_0 is an equidimensional surjective morphism with integral fibers. By shrinking Y_0 if necessary, we may also assume that Y_0 is smooth.

By Proposition 14, we may assume $\rho(X) \geqslant 2$. By [Kol96, Proposition IV 3.13.3], we must have $\dim(Y_0) \geqslant 1$.

Let F be a general fiber of π_0. There exist (see [ADK08, Lemma 5.1] or [Par10, Lemme 2.1]) non negative integers i and j with $i+j=r$ such that $h^0(X, T_{X_0/Y_0}^{[\otimes i]} \otimes \det(\mathscr{E})_{|X_0}^{\otimes -1} \otimes \pi_0^* T_{Y_0}^{\otimes j}) \neq 0$ and $h^0(F, T_F^{\otimes i} \otimes \det(\mathscr{E})_{|F}^{\otimes -1}) \neq 0$. Notice that $i \geqslant 1$ since $\det(\mathscr{E})_{|F}$ is an ample line bundle and $d := \dim(F) \geqslant 1$. The induction hypothesis implies that $F \simeq \mathbf{P}^d$ if $i < r$ or $k \geqslant 1$ and either $F \simeq \mathbf{P}^d$ or $F \simeq Q_d$ if $i = r$ and $k = 0$.

Let $C \subset X_0$ be a general complete intersection curve (with respect to some very ample line bundle on X). Let X_C be the normalization of $\pi_0^{-1}(\pi_0(C))$. Let $\pi_C : X_C \to C$ be the induced map. Notice that X_C is the normalization of $C \times_{Y_0} X_0$ and that $C \times_{Y_0} X_0$ is regular in codimension one. Hence, we must have $h^0(X_C, T_{X_C/C}^{[\otimes i]} \otimes \det(\mathscr{E})_{|X_C}^{\otimes -1} \otimes \pi_C^*(\Omega_{Y_0|C}^{1\ \otimes -j})) \neq 0$. Let us assume that either $(\pi_0^*\Omega_{Y_0}^1)_{|C}$ is a nef vector bundle or $i = r$. If the geometric generic fiber of π_0 is isomorphic to a projective space then π_0 is a $\mathbf{P}^d$-bundle by Lemma 9. But that contradicts Lemma 4. Thus the geometric generic fiber of π_0 is isomorphic to a (smooth) hyperquadric. But that contradicts Proposition 6.

Thus $i < r$, $F \simeq \mathbf{P}^d$ and by Lemma 12, there exists a minimal free morphism $f : \mathbf{P}^1 \to Y_0$. By generic smoothness, we may assume that $X_f := \mathbf{P}^1 \times_{Y_0} X_0$ is smooth. We may also assume that $h^0(X_f, T_{X_f/\mathbf{P}^1}^{[\otimes i]} \otimes \det(\mathscr{E})_{|X_f}^{\otimes -1} \otimes \pi_f^*(T_{Y_0}{}_{|\mathbf{P}^1}^{\otimes j})) \neq 0$. Let $\mathscr{L}_f$ be a line bundle on X_f that restricts to $\mathscr{O}_{\mathbf{P}^d}(1)$ on $F \simeq \mathbf{P}^d$ (see the proof of Lemma 4). By [Fuj75, Corollary 5.4], $\pi_f : X_f \to \mathbf{P}^1$ is a $\mathbf{P}^d$ bundle. It follows from Lemma 3 that $k = 0$, $d = 1$, $(X_f/\mathbf{P}^1) \simeq (\mathbf{P}^1 \times \mathbf{P}^1/\mathbf{P}^1)$ and $\det(\mathscr{E})_{|X_f} \simeq \mathscr{O}_{\mathbf{P}^1}(2) \boxtimes \mathscr{O}_{\mathbf{P}^1}(2)$. Since $\mathscr{E}$ is ample, X admits an unsplit dominating covering family H' of rational curves whose general member corresponds to a ruling of X_f that is not contracted by π. Let $\pi_1 : X_1 \to Y_1$ be the (H, H')-rationally connected quotient of X. By [ADK08, Lemma 2.2], we may assume $\operatorname{codim}_X(X \setminus X_1) \geqslant 2$ and π_1 is an equidimensional surjective morphism with integral fibers. By shrinking Y_1 if necessary, we may also assume that Y_1 is smooth. Replacing $\pi_0 : X_0 \to Y_0$ with $\pi_1 : X_1 \to Y_1$ above, we obtain a contradiction unless $X \simeq \mathbf{P}^1 \times \mathbf{P}^1$. □

Proof of Theorem A. By Theorem 16, $X \simeq \mathbf{P}^n$ and by Lemma 1, $\det(\mathscr{E}) \simeq \mathscr{O}_{\mathbf{P}^n}(l))$ with $r + k \leqslant l \leqslant \frac{r(n+1)}{n}$. □

Proof of Theorem B. By Theorem 16, either $X \simeq \mathbf{P}^n$ or $X \simeq Q_n$. If $X \simeq \mathbf{P}^n$, then the claim follows from Lemma 1. Let us assume $X \simeq Q_n$. By Lemma 2, $\det(\mathscr{E}) \simeq \mathscr{O}_{Q_n}(r)$. Thus, for any line $\mathbf{P}^1 \subset Q_n \subset \mathbf{P}^{n+1}$, $\mathscr{E}_{|\mathbf{P}^1} \simeq \mathscr{O}_{\mathbf{P}^1}(1)^{\oplus r}$, and the claim follows from [AW01, Proposition 1.2] (see also [ROS10, Theorem 4.3]). □

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