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On the limit behavior of metrics in continuity method to Kähler-Einstein problem in toric Fano case

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ABSTRACT: This is a continuation of paper [7]. On any toric Fano manifold, we discuss the behavior of limit metric of a sequence of metrics, which are solutions to a continuity family of complex Monge-Ampère equations in Kähler-Einstein problem. We show that the limit metric satisfies a singular complex Monge-Ampère equation. This shows the conic type singularity for the limit metric. The information of conic type singularities can be read from the geometry of the moment polytope.

1 Introduction

Let (X, J) be a Fano manifold, that is, K_X^{-1} is ample. Fix a reference Kähler metric $\omega \in c_1(X)$. Its Ricci curvature $Ric(\omega)$ also lies in $c_1(X)$. So there exists $h_\omega \in C^\infty(X)$ such that

$$Ric(\omega) - \omega = \partial\bar{\partial}h_\omega, \quad \int_X e^{h_\omega}\omega^n = \int_X \omega^n$$

Consider the following family of Monge-Ampère equations.

$$(\omega + \partial\bar{\partial}\phi_t)^n = e^{h_\omega - t\phi}\omega^n \qquad (*)_t$$

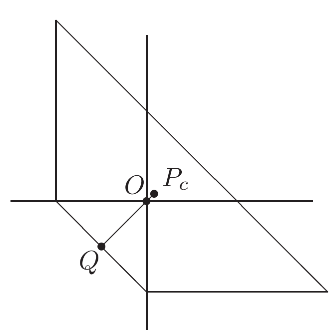
Let $R(X) = \sup\{t : (*)_t \text{ is solvable }\}$. Then Székelyhidi proved that

Proposition 1 ([10]).

$$R(X) = \sup\{t : \exists \textit{ a Kähler metric } \omega \in c_1(X) \textit{ such that } Ric(\omega) > t\omega\}$$

In particular, $R(X)$ is independent of reference metric ω. In [7], we determined $R(X)$ for any toric Fano manifold.

A toric Fano manifold $X_\triangle$ is determined by a reflexive lattice polytope $\triangle$ (For details on toric manifolds, see [8]). For example, let $Bl_p\mathbb{P}^2$ denote the manifold obtained by blowing up one point on $\mathbb{P}^2$. Then $Bl_p\mathbb{P}^2$ is a toric Fano manifold and is determined by the following polytope.



Any such polytope $\triangle$ contains the origin $O \in \mathbb{R}^n$. We denote the barycenter of $\triangle$ by P_c. If $P_c \neq O$, the ray $P_c + \mathbb{R}_{\geq 0} \cdot \overrightarrow{P_cO}$ intersects the boundary $\partial\triangle$ at point Q.

Theorem 1. *[7] If $P_c \neq O$,*

$$R(X_\triangle) = \frac{|\overline{OQ}|}{|\overline{P_cQ}|}$$

Here $|\overline{OQ}|$, $|\overline{P_cQ}|$ are lengths of line segments $\overline{OQ}$ and $\overline{P_cQ}$. If $P_c = O$, then there is Kähler-Einstein metric on $X_\triangle$ and $R(X_\triangle) = 1$.

The next natural problem is what the limit metric looks like as $t \to R(X)$. For the special example $X = Bl_p\mathbb{P}^2$, which is also the projective compactification of total space of line bundle $\mathcal{O}(-1) \to \mathbb{P}^2$. Székelyhidi [10] constructed a sequence of Kähler metric ω_t, with $Ric(\omega_t) \geq t\omega_t$ and ω_t which converge to a metric with conic singularty along the divisor D_∞ of conic angle $2\pi \times 5/7$, where D_∞ is divisor at infinity added in projective compactification. Shi-Zhu [11] proved that rotationally symmetric solutions to the continuity equations $(*)_t$ converge to a metric with conic singularity of conic angle $2\pi \times 5/7$ in Gromov-Hausdorff sense, which seems to be the first strict result on behavior of solutions to $(*)_t$. Note that by the theory of Cheeger-Colding-Tian [2], the limit metric in Gromov-Hausdorff sense should have complex codimension 1 conic type singularities if we only have the positive lower Ricci bounds.

For the more general toric case, if we use a special toric metric, which is just the Fubini-Study metric in the projective embedding given by the vertices of the polytope, then, after transforming by some biholomorphic automorphism, we prove there is a sequence of Kähler metrics which solve the equation $(*)_t$, and converge to a limit metric satisfying a singular complex Monge-Ampère equation (Also see equivalent statement in Theorem 3 and Theorem 4). This generalizes the result of [11] for the special reference Fubini-Study metric.

Theorem 2. *After biholomorphic transformation $\sigma_t : X_\triangle \to X_\triangle$, there is a subsequence $t_i \to R(X)$, such that $\sigma_{t_i}^*\omega_{t_i}$ converge to a Kähler current $\omega_\infty = \omega + \partial\bar{\partial}\psi_\infty$, which satisfy a complex Monge-Ampère equation of the form*

$$(\omega + \partial\bar{\partial}\psi_\infty)^n = e^{-R(X)\psi_\infty} \prod_i \|\tau_i\|_i^{-2a_i(1-R(X))}\Omega \tag{1}$$

Here Ω is a smooth volume form. τ_i is a defining section of the line bundle $[D_i]$ generated by some toric divisor D_i. $a_i > 0$ is some constant. $\|\cdot\|_i$ is a Hermitian metric on $[D_i]$. Further more,

1. *$\psi_\infty \in L^\infty(X_\triangle) \cap C^\infty(X_\triangle \setminus \cup_i D_i)$.*
2. *All $R(X)$, D_i and a_i can be read out from the geometry of polytope. Using the same notation as that in Theorem 1, we have*
   - *$R(X)$ is determined in Theorem 1*
   - *Let $\mathcal{F}$ be the minimal face of $\triangle$ containing $Q = -\frac{R(X)}{1-R(X)}P_c \in \partial\triangle$. Let $\{p_k^{\mathcal{F}}\}$ be all the vertex lattice points of $\mathcal{F}$, and $\{H_r; r = 1, \cdots, K\}$ be all the codimensional one face of $\triangle$ defined by some primitive inward normal vector v_i, i.e.*

     $$\triangle = \bigcap_{r=1}^{K} \{x; \langle x, v_r\rangle \geq -1\}$$

     Then D_r is the toric divisor corresponding to r-th facet H_r, and

     $$a_r = 1 + \min_k \langle p_k^{\mathcal{F}}, v_r\rangle$$

     We always have $a_r \geq 0$. In (1), $\{(\tau_i, D_i, a_i)\}$ are those data with $a_i \neq 0$.

From this theorem we can expect the conic behavior for the limit metric, and we can read out the place of conic singularities and the conic angles from the geometry of the polytope, i.e. conic singularity along D_i with conic angle $2\pi(1-a_i(1-R(X)))$. See the discussion in subsection 3.3 for the reason of toric operations in the above Theorem. In particular, this can give a toric explanation of the special case $Bl_p\mathbb{P}^2$ just mentioned (See example 1).

Note that, although we can prove the limit metric is smooth outside the singular locus, to actually prove it's a conic metric, we need to prove more delicate estimate that we wish to discuss in future.

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2 Consequence of estimates of Wang-Zhu

The proof of Theorem 1 is based on the methods of Wang-Zhu [17].

For a reflexive lattice polytope $\triangle$ in $\mathbb{R}^n = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, we have a Fano toric manifold $(\mathbb{C}^*)^n \subset X_\triangle$ with a $(\mathbb{C}^*)^n$ action. In the following, we will sometimes just write X for $X_\triangle$ for simplicity.

Let $(S^1)^n \subset (\mathbb{C}^*)^n$ be the standard real maximal torus. Let $\{z_i\}$ be the standard coordinates of the dense orbit $(\mathbb{C}^*)^n$, and $x_i = \log|z_i|^2$. We have

Lemma 1. *Any $(S^1)^n$ invariant Kähler metric ω on X has a potential $u = u(x)$ on $(\mathbb{C}^*)^n$, i.e. $\omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u$. u is a proper convex function on $\mathbb{R}^n$, and satisfies the momentum map condition:*

$$Du(\mathbb{R}^n) = \triangle$$

Also,

$$\frac{(\partial\bar{\partial}u)^n/n!}{\frac{dz_1}{z_1}\wedge\frac{d\bar{z}_1}{\bar{z}_1}\cdots\wedge\frac{dz_n}{z_n}\wedge\frac{d\bar{z}_n}{\bar{z}_n}} = \det\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) \tag{2}$$

Let $\{p_\alpha;\ \alpha = 1,\cdots,N\}$ be all the **vertex lattice points** of $\triangle$. Each p_α corresponds to a holomorphic section $s_\alpha \in H^0(X_\triangle, K^{-1}_{X_\triangle})$. We can embed $X_\triangle$ into $\mathbb{P}^N$ using $\{s_\alpha\}$. Define $\tilde{u}_0$ to be the potential on $(\mathbb{C}^*)^n$ for the pull back of Fubini-Study metric (i.e. $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\tilde{u}_0 = \omega_{FS}$):

$$\tilde{u}_0 = \log\left(\sum_{\alpha=1}^{N} e^{<p_\alpha, x>}\right) + C \tag{3}$$

C is some constant determined by normalization condition:

$$\int_{\mathbb{R}^n} e^{-\tilde{u}_0}dx = Vol(\triangle) = \frac{1}{n!}\int_{X_\triangle}\omega^n = \frac{c_1(X_\triangle)^n}{n!} \tag{4}$$

By the normalization of $\tilde{u}_0$, it's easy to see that

$$\frac{e^{h_\omega}\omega^n}{\frac{dz_1}{z_1}\wedge\frac{d\bar{z}_1}{\bar{z}_1}\cdots\wedge\frac{dz_n}{z_n}\wedge\frac{d\bar{z}_n}{\bar{z}_n}} = e^{-\tilde{u}_0} \tag{5}$$

Remark 1. *We only use vertex lattice points because, roughly speaking, later in Lemma 5, vertex lattice points alone helps us to determine which sections become degenerate when doing biholomorphic transformation and taking limit. See remark 3. We expect results similar to Theorem 2 hold for general toric reference Kähler metric.*

So divide both sides of $(*)_t$ by meromorphic volume form $\frac{dz_1}{z_1} \wedge \frac{d\bar{z}_1}{\bar{z}_1} \cdots \wedge \frac{dz_n}{z_n} \wedge \frac{d\bar{z}_n}{\bar{z}_n}$, We can rewrite the equations $(*)_t$ as a family of real Monge-Ampère equations on $\mathbb{R}^n$:

$$\det(u_{ij}) = e^{-(1-t)\tilde{u}_0 - tu} \qquad (**)_t$$

where u is the potential for $\omega + \partial\bar{\partial}\phi$ on $(\mathbb{C}^*)^n$, and is related to ϕ in $(*)_t$ by

$$\phi = u - \tilde{u}_0$$

For simplicity, let

$$w_t(x) = tu(x) + (1-t)\tilde{u}_0$$

Then w_t is also a proper convex function on $\mathbb{R}^n$ satisfying $Dw_t(\mathbb{R}^n) = \triangle$. So it has a unique absolute minimum at point $x_t \in \mathbb{R}^n$. Let

$$m_t = inf\{w_t(x) : x \in \mathbb{R}^n\} = w_t(x_t)$$

Then the main estimate of Wang-Zhu [17] is that

Proposition 2 ([17],See also [3]). *1. there exists a constant C, independent of $t < R(X_\triangle)$, such that*

$$|m_t| < C$$

2. There exists $\kappa > 0$ and a constant C, both independent of $t < R(X_\triangle)$, such that

$$w_t \geq \kappa|x - x_t| - C \qquad (6)$$

Proposition 3 ([17]). *the uniform bound of $|x_t|$ for any $0 \leq t \leq t_0$, is equivalent to that we can solve $(**)_t$, or equivalently solve $(*)_t$, for t up to t_0. More precisely, (by the discussion in introduction,) this condition is equivalent to the uniform C^0-estimates for the solution ϕ_t in $(*)_t$ for $t \in [0, t_0]$.*

By the above proposition, we have

Lemma 2. *If $R(X_\triangle) < 1$, then there exists a subsequence $\{x_{t_i}\}$ of $\{x_t\}$, such that*

$$\lim_{t_i \to R(X_\triangle)} |x_{t_i}| = +\infty$$

By the properness of $\tilde{u}_0$ and compactness of $\triangle$, we ge imediately that

Lemma 3. *If $R(X_\triangle) < 1$, then there exists a subsequence of $\{x_{t_i}\}$ which we still denote by $\{x_{t_i}\}$, and $y_\infty \in \partial\triangle$, such that*

$$\lim_{t_i \to R(X_\triangle)} D\tilde{u}_0(x_{t_i}) = y_\infty \qquad (7)$$

To determine $R(X_\triangle)$ we use the key identity:

$$\frac{1}{Vol(\triangle)} \int_{\mathbb{R}^n} D\tilde{u}_0 e^{-w} dx = -\frac{t}{1-t} P_c \qquad (8)$$

Remark 2. *This identity is a toric form of a general formula for solutions of equations $(*)_t$:*

$$\frac{1}{V} \int_X div_\Omega(v)\omega_t^n = -\frac{t}{1-t} F_{c_1(X)}(v)$$

Here $\Omega = e^{h_\omega}\omega^n$. v is any holomorphic vector field, and $div_\Omega(v) = \frac{\mathcal{L}_v\Omega}{\Omega}$ is the divergence of v with respect to Ω.

$$F_{c_1(X)}(v) = \frac{1}{V} \int_X v(h_\omega)\omega^n$$

is the Futaki invariant in class $c_1(X)$ [5].

By properness of w_t, the left handside of (8) is roughly $D\tilde{u}_0(x_t)$. As long as this is bounded away from the boundary of the polytope, we can control the point x_t. So as t goes to $R(X_\triangle)$, since x_t goes to infinity in $\mathbb{R}^n$, the left handside goes to a point on $\partial\triangle$, which is roughly y_∞. To state a precise statement, assume the reflexive polytope $\triangle$ is defined by inequalities:

$$\lambda_r(y) \geq -1, \ r = 1, \cdots, K \tag{9}$$

$\lambda_r(y) = \langle v_r, y\rangle$ are fixed linear functions. We also identify the minimal face of $\triangle$ where y_∞ lies:

$$\begin{aligned} \lambda_r(y_\infty) &= -1, \ r = 1, \cdots, K_0 \\ \lambda_r(y_\infty) &> -1, \ r = K_0 + 1, \cdots, K \end{aligned} \tag{10}$$

Then Theorem 1 follows from

Proposition 4. *[7] If $P_c \neq O$,*

$$-\frac{R(X_\triangle)}{1 - R(X_\triangle)} P_c \in \partial\triangle$$

Precisely,

$$\lambda_r\left(-\frac{R(X_\triangle)}{1 - R(X_\triangle)} P_c\right) \geq -1 \tag{11}$$

Equality holds if and only if $r = 1, \cdots, K_0$. So $-\frac{R(X_\triangle)}{1-R(X_\triangle)} P_c$ and y_∞ lie on the same faces (10).

3 Discussion of limit conic type metric

3.1 Equation for the limit metric

We first fix the reference metric to be the Fubini-Study metric.

$$\omega = \partial\bar{\partial}\tilde{u}_0 = \partial\bar{\partial}\log(\sum_\alpha |s_\alpha|^2)$$

We want to see what's the limit of ω_t as $t \to R(X)$, where

$$\omega_t = \omega + \partial\bar{\partial}\phi$$

is solution of continuity equation $(*)_t$. Equivalently, under the toric coordinate,

$$\omega_t = \frac{\partial^2 u}{\partial \log z_i \partial \log z_j} d\log z_i \wedge d\log z_j = -\sqrt{-1} u_{ij} dx_i d\theta_j$$

where $u = u_t$ is the solution of real Monge-Ampère equation $(**)_t$.

Let $\sigma = \sigma_t$ be the holomorphic transformation given by

$$\sigma_t(x) = x + x_t$$

Assume $x_t = (x_t^1, \cdots, x_t^n)$, then under complex coordinate, we have

$$\sigma_t(\{z_i\}) = \{e^{x_t^i/2} z_i\}$$

By the analysis of previous section, we do the following transformation.

$$U(x) = \sigma_t^* u(x) - u(x_t) = u(x + x_t) - u(x_t), \quad \tilde{U}_t(x) = \sigma_t^* \tilde{u}_0(x) - \tilde{u}_0(x_t) = \tilde{u}_0(x + x_t) - \tilde{u}_0(x_t) \tag{12}$$

Note that $w_t(x) = tu + (1-t)\tilde{u}_0$. Then $U = U_t(x)$ satisfy the following Monge-Ampère equation

$$\det(U_{ij}) = e^{-tU - (1-t)\tilde{U} - w(x_t)} \tag*{$(**)'_t$}$$

By Proposition 4, we know that $Q = -\frac{R(X_\triangle)}{1-R(X_\triangle)} P_c$ lies on the boundary of $\triangle$. Let $\mathcal{F}$ be the minimal face of $\triangle$ which contains Q. Now the observation is

Proposition 5. *There is a subsequence $t_i \to R(X)$, $\tilde{U}_{t_i}$ converge locally uniformly to a convex function of the form:*

$$\tilde{U}_\infty = \log\left(\sum_{p_\alpha\in\mathcal{F}} b_\alpha e^{\langle p_\alpha, x\rangle}\right) \tag{13}$$

*where $0 < b_\alpha \leq 1$ are some constants. For simplicity, we will use $\sum_\alpha{}' = \sum_{p_\alpha\in\mathcal{F}}$ to denote the sume over all the **vertex lattice points** contained in $\mathcal{F}$.*

Proof. By (3) and (12), we have

$$\tilde{U}(x) = \log(\sum_\alpha e^{\langle p_\alpha, x+x_t\rangle}) - \log(\sum_\alpha e^{\langle p_\alpha, x_t\rangle}) = \log(\sum b(p_\alpha, t)e^{\langle p_\alpha, x\rangle}) \tag{14}$$

where

$$b(p_\alpha, t) = \frac{e^{\langle p_\alpha, x_t\rangle}}{\sum_\beta e^{\langle p_\beta, x_t\rangle}}$$

Since $0 < b(p_\alpha, t) < 1$, we can assume there is a subsequence $t_i \to R(X)$, such that for any vertex lattice point p_α,

$$\lim_{t\to R(X)} b(p_\alpha, t) = b_\alpha \tag{15}$$

We need to prove $b_\alpha \neq 0$ if and only if $p_\alpha \in \mathcal{F}$.

To prove this, we first note that

$$D\tilde{u}_0(x_t) = \frac{\sum_\alpha p_\alpha e^{\langle p_\alpha, x_t\rangle}}{\sum_\beta e^{\langle p_\beta, x_t\rangle}} = \sum_\alpha b(p_\alpha, t)p_\alpha \tag{16}$$

By Lemma 3, $D\tilde{u}_0(x_t) \to y_\infty$. So by letting $t \to R(X)$ in (16) and using (15), we get

$$y_\infty = \sum_\alpha b_\alpha p_\alpha$$

By Proposition 4, $y_\infty \in \partial\triangle$ lies on the same faces as Q does, i.e. $\mathcal{F}$ is also the minimal face containing y_∞, so we must have $b_\alpha = 0$ if $p_\alpha \notin \mathcal{F}$. We only need to show if $p_\alpha \in \mathcal{F}$, then $b_\alpha \neq 0$.

If dim $\mathcal{F}$=k, then there exists k+1 vertex lattice points $\{p_1, \cdots, p_{k+1}\}$ of $\mathcal{F}$, such that the corresponding coefficient $b_i \neq 0$, $i = 1, \cdots, k+1$, i.e. $\lim_{t\to R(X)} b(p_i, t) = b_i > 0$.

Remark 3. *Here is why we need to assume p_α are all vertex lattice points.*

Let p be any vertex point of $\mathcal{F}$, then

$$p = \sum_{i=1}^{k+1} c_i p_i, \quad \text{where} \quad \sum_{i=1}^{k+1} c_i = 1$$

Then

$$b(p,t) = \frac{e^{\langle \sum_{i=1}^{k+1} c_i p_i, x_t\rangle}}{\sum_\beta e^{\langle p_\beta, x_t\rangle}} = \prod_{i=1}^{k+1}\left(\frac{e^{\langle p_i, x_t\rangle}}{\sum_\beta e^{\langle p_\beta, x_t\rangle}}\right)^{c_i} = \prod_{i=1}^{k+1} b(p_i, t)^{c_i} \xrightarrow{t\to R(X)} \prod_{i=1}^{k+1} b_i^{c_i} > 0$$

□

We can state a real version of Theorem 2

Theorem 3. *There is a subsequence $t_i \to R(X)$, $U_{t_i}(x)$ converge to a smooth entire solution of the following equation on $\mathbb{R}^n$*

$$\det(U_{ij}) = e^{-R(X)U(x)-(1-R(X))\tilde{U}_\infty(x)-c} \tag{$(**)'_\infty$}$$

$c = \lim_{t_i\to R(X)} w(x_{t_i})$ *is some constant.*

3.2 Change to Complex Monge-Ampère equation

The proof of Theorem 3 might be done by theory of real Monge-Ampère equation. But here, we will change our view and rewrite $(**)'_t$ as a family of complex Monge-Ampère equations. This will alow us to apply some standard estimates in the theory of complex Monge-Ampère equations.

We rewrite the formula for $\tilde{U}(x)$ (14) as

$$e^{\tilde{U}} = \frac{\sum_\alpha b(p_\alpha,t)e^{\langle p_\alpha,x\rangle}}{\sum_\beta e^{\langle p_\beta,x\rangle}}\sum_\beta e^{\langle p_\beta,x\rangle} = \frac{\sum_\alpha b(p_\alpha,t)|s_\alpha|^2}{\sum_\beta |s_\beta|^2}e^{\tilde{u}_0} = (\sum_\alpha b(p_\alpha,t)\|s_\alpha\|^2)e^{\tilde{u}_0} \tag{17}$$

s_α is the holomorphic section of K_X^{-1} corresponding to lattice point p_α. Here and in the following $\|\cdot\| = \|\cdot\|_{FS}$ is the Fubini-Study metric on K_X^{-1}.

$(**)'_t$ can then be rewritten as

$$\det(U_{ij}) = e^{-t\psi}e^{-\tilde{u}_0}\left(\sum_\alpha b(p_\alpha,t)\|s_\alpha\|^2\right)^{-(1-t)} e^{-w(x_t)}$$

By (2) and (5), $(**)'_t$ can finally be written as the complex Monge-Ampère equation

$$(\omega+\partial\bar{\partial}\psi)^n = e^{-t\psi}\left(\sum_\alpha b(p_\alpha,t)\|s_\alpha\|^2\right)^{-(1-t)} e^{h_\omega - w(x_t)}\omega^n \tag*{$(***)_t$}$$

where

$$\psi = \psi_t = U - \tilde{u}_0 \tag{18}$$

Similarly for $\tilde{U}_\infty$ (13), we write

$$e^{\tilde{U}_\infty} = \frac{\sum_\alpha{}' b_\alpha e^{\langle p_\alpha,x\rangle}}{\sum_\beta e^{\langle p_\beta,x\rangle}}\sum_\beta e^{\langle p_\beta,x\rangle} = (\sum_\alpha{}' b_\alpha\|s_\alpha\|^2)e^{\tilde{u}_0}$$

And the limit equation $(**)'_\infty$ becomes:

$$(\omega+\partial\bar{\partial}\psi)^n = e^{-R(X)\psi}\left(\sum_\alpha{}' b_\alpha\|s_\alpha\|^2\right)^{-(1-R(X))} e^{h_\omega - c}\omega^n \tag*{$(***)_\infty$}$$

For any lattice point $p_\alpha \in \triangle$, let $D_\alpha = \{s_\alpha = 0\}$ be the zero divisor of the corresponding holomorphic section s_α. Let

$$D = \bigcap_{p_\alpha\in\mathcal{F}} D_\alpha = \sum_{i=1}^r a_i D_i \quad \text{with} \quad \mathbb{N}\ni a_i > 0, i = 1,\cdots,r$$

For each i, D_i is a toric divisor and determines a line bundle $[D_i]$. Let τ_i be the defining section of line bundle $[D_i]$ and fix a Hermitian metric $\|\cdot\|_i$ on $[D_i]$, then

$$\sum_\alpha{}' b_\alpha\|s_\alpha\|^2 = f\prod_{i=1}^r \|\tau_i\|_i^{2a_i}$$

f is a smooth nowhere vanishing function on $X_\triangle$. So $(***)_\infty$ becomes the singular complex Monge-Ampère equation (1) appeared in Theorem 2:

$$(\omega+\partial\bar{\partial}\psi)^n = e^{-R(X)\psi}\prod_{i=1}^r \|\tau_i\|_i^{-2(1-R(X))a_i}\Omega \tag{19}$$

Here Ω is a smooth volume form:

$$\Omega = e^{h_\omega - c} f^{-(1-R(X)} \omega^n$$

Define the space of plurisubharmonic potentials with respect to ω

$$\mathcal{PSH}(\omega) = \{\phi \in L^\infty(X); \omega + \partial\bar{\partial}\phi \geq 0\}$$

Now we can reformulate Theorem 3 as

Theorem 4. *There is a subsequence $t_i \to R(X)$, the solution ψ_{t_i} to the equation $(***)_t$ converge to a solution $\psi_\infty \in \mathcal{PSH}(\omega) \cap L^\infty(X) \cap C^\infty(X \backslash D)$ of the equation $(***)_\infty$.*

Note that by (19), this is an equivalent statement to Theorem 2 in Introduction.

3.3 Discussion on the conic behavior of limit metric

From (19), we get

$$Ric(\Omega) - Ric(\omega_\psi) = \partial\bar{\partial}\log\frac{\omega_\psi^n}{\Omega} = -t\partial\bar{\partial}\psi - (1-R(X))a_i\sum_{i=1}^{r}\partial\bar{\partial}\log\|\tau_i\|^2 \tag{20}$$

Since

$$\partial\bar{\partial}\log\|\tau_i\|^2 = -c_1([D_i], \|\cdot\|_i^2) + \{D_i\}$$

where $\{D_i\}$ is the current of integration along divisor D_i.

So when we take cohomology on both sides of (20), we get

$$[Ric(\omega_\psi)] = c_1(X) - (1-R(X))\sum_i a_i(c_1([D_i]) - \{D_i\})$$

So it's reasonable to expect the limit metric is a Kähler metric with conic singularities along the normal crossing divisor $\sum_i D_i$. The conic angle along D_i is

$$\theta_i = 2\pi(1-(1-R(X))a_i) \tag{21}$$

We recall that $R(X)$ is given by Theorem 1. Let $\mathcal{F}$ be the minimal face defined just before Lemma 5. The vertices of $\mathcal{F}$ correspond to a sublinear system $\mathfrak{L}_\mathcal{F}$ of $|K_X^{-1}|$. The base locus of $\mathfrak{L}_\mathcal{F}$ is

$$Bs(\mathfrak{L}_\mathcal{F}) = D = \sum_i a_i D_i$$

All these operations can be realized by standard toric geometry calculations. Indeed, let u be any lattice point, then the holomorphic section s_u corresponding to u has zero locus

$$\{s_u = 0\} = \sum_k (\langle u, v_k\rangle + 1)D_k$$

Here v_k is the primitive inward normal vector to the k-th codimensional one face, and D_k is the toric divisor corresponding to this face. For calculation examples, see Section 5.

4 Proof of Theorem 2

We are now in the general setting of complex Monge-Ampère equations. (19) (or equivalently $(***)_\infty$ is a complex Monge-Ampère equation with poles at righthand side. $(***)_t$ can be seen as regularizations of (19). We ask if the solutions of $(***)_t$ converge to a solution of (19). Starting from Yau's work [18], similar problems has been considered by many people. Due to the large progress made by Kołodziej[6], complex Monge-Ampère equation can be solved with very general, usually singular, righthand side. Kołodziej's result was also proved by first regularizing the singular Monge-Ampère equation, and then taking limit back to get solution of original equation.

We will derive several apriori estimate to prove Theorem 2. For the C^0-estimate, the upper bound follows from how we transform the potential function in (12). The lower bound follows from a Harnack estimate for the transformed potential function which we will prove using Tian's argument in [15]. For the proof of partial C^2-estimate, higher order estimates and convergence of solutions, we use some argument similar to that used by Ruan-Zhang [9], and Demailly and Pali [4].

4.1 C^0-estimate

We first derive the C^0-estimate for $\psi = U - \tilde{u}_0$. Let $\bar{v} = \bar{v}(x)$ be a piecewise linear function defined to be

$$\bar{v}(x) = \max_{p_\alpha} \langle p_\alpha, x \rangle$$

Then u_0 is asymptotic to $\bar{v}$ and it's easy to see that $|\bar{v} - \tilde{u}_0| \leq C$. So we only need to show that $|U(x) - \bar{v}(x)| \leq C$. Here and in the following, C is some constant independent of $t \in [0, R(X))$.

One side is easy. Since $DU(\mathbb{R}^n) = \triangle$ and $U(0) = 0$, we have for any $x \in \mathbb{R}^n$, $U(x) = U(x) - U(0) = DU(\xi) \cdot x \leq \bar{v}(x)$. ξ is some point between 0 and x. So

$$\psi = (U - \bar{v}) + (\bar{v} - \tilde{u}_0) \leq C$$

.

To prove the lower bound for ψ, we only need to prove a Harnack inequality

Proposition 6.

$$\sup_X(-\psi) \leq n \sup_X \psi + C(n)t^{-1} \tag{22}$$

For this we use the same idea of proof in [15]. First we rewrite the $(***)_t$ as

$$(\omega + \partial\bar{\partial}\psi)^n = e^{-t\psi + F - B_t}\omega^n \tag{23}$$

where

$$B_t = (1-t)\log\left(\sum_\alpha b(p_\alpha, t)\|s_\alpha\|^2\right), \quad F = h_\omega - w(x_t)$$

Now consider a new continuous family of equations

$$(\omega + \partial\bar{\partial}\theta_s)^n = e^{-s\theta_s + F - B_t}\omega^n \tag*{$(23)_s$}$$

Define $S = \{s' \in [0,t] | (23)_s \text{ is solvable for } s \in [s', t]\}$. We want to prove $S = [0, t]$. Since (23) has a solution ψ, $t \in S$ and S is nonempty. It is sufficient to show that S is both open and closed.

For openness, we first estimate the first eigenvalue of the metric g_θ associated with the Kähler form $\omega_\theta = \omega + \partial\bar{\partial}\theta$ for the solution θ of $(23)_s$.

$$\begin{aligned} Ric(\omega_\theta) &= s\partial\bar{\partial}\theta - \partial\bar{\partial}F + \partial\bar{\partial}B_t + Ric(\omega) \\ &= s\partial\bar{\partial}\theta + \omega + (1-t)(\sigma^*\omega - \omega) = s(\partial\bar{\partial}\theta + \omega) + (t-s)\omega + (1-t)\sigma^*\omega \\ &= s\omega_\theta + (t-s)\omega + (1-t)\sigma^*\omega > s\omega_\theta \end{aligned} \tag{24}$$

So by Bochner's formula, the first nonzero eigenvalue $\lambda_1(g_{\theta_s}) > s$. This gives the invertibility of linearization operator $(-\Delta_s) - s$ of equation $(23)_s$, so the openness of solution set S follows.

To prove closedness, we need to derive apriori estimate. First define the functional:

$$I(\theta_s) = \frac{1}{V}\int_X \theta_s(\omega^n - \omega_{\theta_s}^n), \quad J(\theta_s) = \int_0^1 \frac{I(x\theta_s)}{x}dx$$

Then we have

Lemma 4. *[1, 15]*

(i)

$$(n+1)J(\theta_s)/n \leq J(\theta_s) \leq (n+1)J(\theta_s)$$

(ii)

$$\frac{d}{ds}(I(\theta_s) - J(\theta_s)) = -\frac{1}{V}\int_X \theta_s(\Delta_s\dot{\theta}_s)\omega_{\theta_s}^n$$

Using $\lambda_1(g_{\theta_s}) > s$, Lemma 4.(ii) gives

Lemma 5. *[1, 15] $I(\theta_s) - J(\theta_s)$ is monotonically increasing.*

Let's recall Bando-Mabuchi's estimate for Green function.

Proposition 7. *[1] For every m-dimensional compact Riemannian manifold (X,g) with $diam(X,g)^2 Ric(g) \geq -(m-1)\alpha^2$, there exists a positive constant $\gamma = \gamma(m,\alpha)$ such that*

$$G_g(x,y) \geq -\gamma(m,\alpha)diam(X,g)^2/V_g \tag{25}$$

Here the Green function $G_g(x,y)$ is normalized to satisfy

$$\int_M G_g(x,y)dV_g(x) = 0$$

Bando-Mabuchi used this estimate to prove the key estimate:

Proposition 8. *[1] Let*

$$\mathcal{H}^s = \{\theta \in C^\infty(X); \omega_\theta = \omega + \partial\bar{\partial}\theta > 0, Ric(\omega_\theta) \geq s\omega_\theta\}$$

then for any $\theta \in \mathcal{H}^s$, we have

(1)

$$\sup_X(-\theta) \leq \frac{1}{V}\int_X(-\theta)\omega_\theta^n + C(n)s^{-1} \tag{26}$$

(2)

$$Osc(\theta) \leq I(\theta) + C(n)s^{-1} \tag{27}$$

Now one can use either Moser iteration or the estimate (27) to get the C^0-estimate for $(23)_s$, see [15, 1] for details. Then we use Yau's estimate to get C^2 and higher order estimate. So we get closedness of solution set S, and conclude that

Proposition 9. $(23)_s$ *is solvable for $0 \leq s \leq t$.*

Then one can use the same argument as in [15] to prove

Proposition 10. *[15]*

$$-\frac{1}{V}\int_X \theta\omega_\theta^n \leq \frac{n}{V}\int_X \theta\omega^n \leq n\sup_X \theta \tag{28}$$

Proof. First by taking derivatives to equation $(23)_s$, we get

$$\Delta_s \dot{\theta} = -\theta - s\dot{\theta}$$

So

$$\begin{aligned} \frac{d}{ds}(I-J)(\theta_s) &= -\int_X \theta \frac{d}{ds}\omega_\theta^n = -\frac{d}{ds}\left(\int_X \theta\omega_\theta^n\right) + \int_X \dot{\theta}\omega_\theta^n \\ &= -\frac{d}{ds}\left(\int_X \theta\omega_\theta^n\right) - \frac{1}{s}\int_X \theta\omega_\theta^n = -\frac{1}{s}\frac{d}{ds}\left(s\int_X \theta\omega_\theta^n\right) \end{aligned}$$

So

$$\frac{d}{ds}(s(I-J)(\theta_s)) - (I-J)(\theta_s) = -\frac{d}{ds}\left(s\int_X \theta\omega_\theta^n\right)$$

Since θ_s can be solved for $s \in [0,t]$, and $\theta_t = \psi = \psi_t$, we can integrate to get

$$t(I-J)(\psi) - \int_0^t (I-J)(\theta_s)ds = -t\int_X \psi\omega_\psi^n$$

Divide both sides by t to get

$$(I-J)(\psi) - \frac{1}{t}\int_0^t (I-J)(\theta_s)ds = -\int_X \psi\omega_\psi^n$$

By lemma 4.(i), we can get

$$\frac{n}{n+1}\int_X \psi(\omega^n - \omega_\psi^n) = \frac{n}{n+1}I(\psi) \geq -\int_X \psi\omega_\psi^n$$

(28) follows from this inequality imediately. □

Combine (28) with Bando-Mabuchi's estimate (26) when $s = t$, we then prove the Harnack estimate (22). So we can derive the lower bound of ψ from the upper bound of ψ and C^0-estimate is obtained.

4.2 Partial C^2-estimate

$(*)_t$ is equivalent to

$$Ric(\omega_\phi) = t\omega_\phi + (1-t)\omega$$

From our transformation (12), we get

$$Ric(\omega_\psi) = t\omega_\psi + (1-t)\sigma^*\omega$$

In particular, $Ric(\omega_\psi) > t\omega_\psi$. We will some argument similar to that used by Ruan-Zhang (see the proof of Lemma 5.2 in [9])

Let $f = tr_{\omega_\psi}\omega$ and Δ' be the complex Laplacian associated with Kähler metric ω_ψ. As in [19], we can calculate

$$\Delta' f = g'^{i\bar{l}}g'^{k\bar{j}}R'_{k\bar{l}}g_{i\bar{j}} + g'^{i\bar{j}}g'^{k\bar{l}}T^\alpha_{i,k}T^\beta_{\bar{j},\bar{l}}g_{\alpha\bar{\beta}} - g'^{i\bar{j}}g'^{k\bar{l}}S_{i\bar{j}k\bar{l}}$$

Here the tensor $T^\alpha_{i,j} = \tilde{\Gamma}^\alpha_{ij} - \Gamma^\alpha_{ij}$ is the difference of Levi-Civita connections $\tilde{\Gamma}$ and Γ associated with g_ω and $g' = g_{\omega_\psi}$ respectively. $R'_{k\bar{j}}$ is the Ricci curvature of ω_ψ and $S_{i\bar{j}k\bar{l}}$ is the curvature of reference metric ω. Let ∇' be the gradient operator associated with g_{ω_ψ}, then

$$\begin{aligned}
\Delta' \log f &= \frac{\Delta' f}{f} - \frac{|\nabla' f|^2_{\omega_\psi}}{f^2} \\
&= \frac{g'^{i\bar{l}} g'^{k\bar{j}} R'_{k\bar{l}} g_{i\bar{j}}}{f} - \frac{g'^{i\bar{j}} g'^{k\bar{l}} S_{i\bar{j}k\bar{l}}}{f} + \frac{g'^{i\bar{j}} g'^{k\bar{l}} T^{\alpha}_{i,k} T^{\beta}_{\bar{j},\bar{l}} g_{\alpha\bar{\beta}}}{f} - \frac{g'^{p\bar{q}} g'^{i\bar{j}} g'^{k\bar{l}} T^{\alpha}_{ip} T^{\bar{\beta}}_{\bar{l}\bar{q}} g_{\alpha\bar{j}} g_{k\bar{\beta}}}{f^2} \\
&= \frac{\sum_i \mu_i^{-2} R_{i\bar{i}}}{\sum_i \mu_i^{-1}} - \frac{\sum_{i,j} \mu_i^{-1}\mu_j^{-1} S_{i\bar{i}j\bar{j}}}{\sum_i \mu_i^{-1}} + \frac{\sum_{i,k,\alpha} \mu_i^{-1}\mu_k^{-1} |T^{\alpha}_{ik}|^2}{\sum_i \mu_i^{-1}} - \frac{\sum_p \mu_p^{-1} |\sum_i \mu_i^{-1} T^i_{ip}|^2}{(\sum_i \mu_i^{-1})^2} \\
&\geq t - C\sum_i \mu_i^{-1} = t - Cf
\end{aligned} \tag{29}$$

In the 3rd equality in (29), for any fixed point $P \in X$, we chose a coordinate near P such that $g_{i\bar{j}} = \delta_{ij}$, $\partial_k g_{i\bar{j}} = 0$. We can assume $g' = g_{\omega_\psi}$ is also diagonalized so that

$$g'_{i\bar{j}} = \mu_i \delta_{ij}, \quad \text{with } \mu_i = 1 + \psi_{i\bar{i}}$$

For the last inequality in (29), we used $Ric(\omega_\psi) > t\omega_\psi$ and the inequality:

$$\begin{aligned}
\sum_p \mu_p^{-1} |\sum_i \mu_i^{-1} T^i_{ip}|^2 &= \sum_p \mu_p^{-1} \left| \sum_i \mu_i^{-1/2} T^i_{ip} \mu_i^{-1/2} \right|^2 \\
&\leq (\sum_{p,i} \mu_p^{-1}\mu_i^{-1} |T^i_{ip}|^2)(\sum_i \mu_i^{-1}) \\
&\leq (\sum_{p,i,\alpha} \mu_p^{-1}\mu_i^{-1} |T^{\alpha}_{ip}|^2)(\sum_i \mu_i^{-1})
\end{aligned}$$

So

$$\Delta'(\log f - \lambda\psi) \geq t - Cf - \lambda tr_{\omega_\psi}(\omega_\psi - \omega) = (\lambda - C)f - (\lambda n - t) = C_1 f - C_2$$

for some constants $C_1 > 0$, $C_2 > 0$, if we choose λ to be sufficiently large. So at the maximum point P of the function $\log f - \lambda\psi$, we have

$$0 \geq \Delta'(\log f - \lambda\psi)(P) \geq C_1 f(P) - C_2$$

So

$$f(P) = tr_{\omega_\psi}(\omega)(P) \leq C_3$$

So for any point $x \in X$, we have

$$tr_{\omega_\psi}\omega(x) \leq C_3 e^{\lambda(\psi(x) - \psi(P))} \leq C_3 e^{\lambda osc(\psi)}$$

By C^0-estimate of ψ, we get the estimate $tr_{\omega_\psi}\omega \leq C_4$. So $\omega_\psi \geq C_4\omega$, i.e. $\mu_i \geq C_4$.

Now by (23),

$$\prod_j \mu_j = \frac{\omega_\psi^n}{\omega^n} = e^{-t\psi + F - B}$$

with $F = h - w(x_t)$ and $B = (1-t)\log\left(\sum_\alpha b(p_\alpha, t)\|s_\alpha\|^2\right)$. So by the C^0-estimate of ψ, we get

$$\mu_i = \frac{\prod_j \mu_j}{\prod_{j\neq i} \mu_j} \leq \frac{e^{-t\psi + F - B}}{C_4^{n-1}} \leq C_5 e^{-B}$$

In conclusion, we get the partial C^2-estimate

$$C_4\omega \leq \omega_\psi \leq C_5 e^{-B}\omega \tag{30}$$

Remark 4. *The partial C^2-upper bound $\omega_\psi \leq C_5 e^{-B}\omega$ can also be proved by maximal principle. In fact, let*

$$\Lambda = \log(n+\Delta\psi) - \lambda\psi + B \tag{31}$$

where $\Delta = \Delta_\omega$ is the complex Laplacian with respect to reference metric ω. Then by standard calculation as in Yau [18], we have

$$\begin{aligned}
\Delta'\Lambda &\geq \left(\inf_{i\neq j} S_{i\bar{i}j\bar{j}} + \lambda\right)\sum_i \frac{1}{1+\psi_{i\bar{i}}} + \left(\Delta F - \Delta B - t\Delta\psi - n^2 \inf_{i\neq j} S_{i\bar{i}j\bar{j}}\right)\frac{1}{n+\Delta\psi} - \lambda n + \Delta' B \\
&= \left(\inf_{i\neq j} S_{i\bar{i}j\bar{j}} + \lambda\right)\sum_i \frac{1}{1+\psi_{i\bar{i}}} + \left(\Delta F + nt - n^2 \inf_{i\neq j} S_{i\bar{i}j\bar{j}}\right)\frac{1}{n+\Delta\psi} + \\
&\quad + \sum_i B_{i\bar{i}}\left(\frac{1}{1+\psi_{i\bar{i}}} - \frac{1}{n+\Delta\psi}\right) - (\lambda n + t)
\end{aligned} \tag{32}$$

Since for each i, $\frac{1}{n+\Delta\psi} \leq \frac{1}{1+\psi_{i\bar{i}}}$, so $\frac{1}{n+\Delta\psi} \leq \frac{1}{n}\sum_i \frac{1}{1+\psi_{i\bar{i}}}$. So the second term on the right of (32) *is bounded below by $-C_0\sum_i \frac{1}{1+\psi_{i\bar{i}}}$ for some positive constant $C_0 > 0$*

For the 3rd term, we observe from (12) *and* (17) *that*

$$\partial\bar{\partial}B = (1-t)(\sigma^*\omega - \omega) \geq -(1-t)\omega$$

So, since again $\frac{1}{n+\Delta\psi} \leq \frac{1}{1+\psi_{i\bar{i}}}$, we have

$$B_{i\bar{i}}\left(\frac{1}{1+\psi_{i\bar{i}}} - \frac{1}{n+\Delta\psi}\right) \geq -(1-t)\left(\frac{1}{1+\psi_{i\bar{i}}} - \frac{1}{n+\Delta\psi}\right) \geq -(1-t)\frac{1}{1+\psi_{i\bar{i}}}$$

By the above discussion, at the maximal point P_t of Λ, we have

$$0 \geq \Delta'\Lambda \geq (\lambda + \inf_{i\neq j} S_{i\bar{i}j\bar{j}} - C_0 - (1-t))\sum_i \frac{1}{1+\psi_{i\bar{i}}} - (\lambda n + t) = C_2\sum_i \frac{1}{1+\psi_{i\bar{i}}} - C_3 \tag{33}$$

for some constants $C_2 > 0$, $C_3 > 0$, by choosing λ sufficiently large.

Now we use the following inequality from [18]

$$\begin{aligned}
\sum_i \frac{1}{1+\psi_{i\bar{i}}} &\geq \left(\frac{\sum_i(1+\psi_{i\bar{i}})}{\prod_j(1+\psi_{j\bar{j}})}\right)^{1/(n-1)} = (n+\Delta\psi)^{1/(n-1)} e^{\frac{B-F+t\psi}{n-1}} \\
&= e^{\frac{\Lambda}{n-1}} e^{\frac{-F+(t+\lambda)\psi}{n-1}}
\end{aligned} \tag{34}$$

By (33) *and* (34)*, we get the bound*

$$e^{\Lambda(P_t)} \leq C_4 e^{-(t+\lambda)\psi(P_t)}$$

So we get estimate that for any $x \in X = X_\triangle$,

$$(n+\Delta\psi)e^{-\lambda\psi}e^B \leq e^{\Lambda(P_t)} \leq C_4 e^{-(t+\lambda)\psi(P_t)}$$

Since we have C^0-estimate for ψ, we get partial C^2-upper estimate:

$$(n+\Delta\psi)(x) \leq C_4 e^{-(t+\lambda)\psi(P_t)} e^{\lambda\psi(x)} e^{-B} \leq C_5\left(\sum_\alpha b(p_\alpha, t)\|s_\alpha\|^2\right)^{-(1-t)} \tag{35}$$

In particular,

$$1 + \psi_{i\bar{i}} \leq C_5 e^{-B}$$

which is same as $\omega_\psi \leq C_5 e^{-B}$.

4.3 Higher order estimate and completion of the proof of Theorem 2

For any compact set $K \subset X\backslash D$, we first get the gradient estimate by interpolation inequality:

$$\max_K |\nabla\psi| \leq C_K(\max_K \Delta\psi + \max_K |\psi|) \tag{36}$$

Next, by the complex version of Evans-Krylov theory [16], we have a uniform $C_{2,K} > 0$, such that $\|\psi\|_{C^{2,\alpha}(K)} \leq C_{2,K}$ sor some $\alpha \in (0,1)$. Now take derivative to the equation:

$$\log\det(g_{i\bar{j}} + \psi_{i\bar{j}}) = \log\det(g_{i\bar{j}}) - t\psi + F - B$$

to get

$$g'^{i\bar{j}}\psi_{i\bar{j},k} = -t\psi_k + F_k - B_k + g^{i\bar{j}}g_{i\bar{j},k} - g'^{i\bar{j}}g_{i\bar{j},k} \tag{37}$$

By (30), (36) and $\|\psi\|_{C^{2,\alpha}(K)} \leq C_{2,K}$, (37) is a linear elliptic equation with C^α coefficients. By Schauder's estimate, we get $\|\psi_k\|_{C^{2,\alpha}} \leq C$, i.e. $\|\psi\|_{C^{3,\alpha}} \leq C$. Then we can iterate in (37) to get $\|\psi\|_{C^{r,\alpha}} \leq C$ for any $r \in \mathbb{N}$. So we see that $(\psi = \psi(t))_{t<R(X)} \subset C^\infty(X\backslash D)$ is precompact in the smooth topology.

Now we can finish the proof of Theorem 2, or equivalently Theorem 4 using argument from [4]

Proof of Theorem 2 and Theorem 4. The uniform estimate $\|\psi\|_{L^\infty}$ implies the existence of a L^1-convergent sequence $(\psi_j = \psi_{t_j})_j$, $t_j \uparrow R(X)$ with limit $\psi_\infty \in \mathcal{PSH}(\omega) \cap L^\infty(X)$. We can assume that a.e.-convergence holds too. The precompactness of the family $(\psi_j) \subset C^\infty(X\backslash D)$ in the smooth topology implies the convergence of the limits over $X\backslash D$:

$$\begin{aligned}
(\omega + \partial\bar{\partial}\psi_\infty)^n &= \lim_{t_j \to R(X)} (\omega + \partial\bar{\partial}\psi_j)^n \\
&= \lim_{t_j \to R(X)} e^{-t_j\psi_{t_j}} \left(\sum_\alpha b(p_\alpha, t_j)\|s_\alpha\|^2\right)^{-(1-t_j)} e^{h_\omega - w(x_{t_j})}\omega^n \\
&= e^{-R(X)\psi_\infty} \left(\sum_\alpha{}' b_\alpha\|s_\alpha\|^2\right)^{-(1-R(X))} e^{h_\omega - c}\omega^n
\end{aligned}$$

The fact that ψ_∞ is a bounded potential implies that the global complex Monge-Ampère measure $(\omega + \partial\bar{\partial}\psi_\infty)^n$ does not carry any mass on complex analytic sets. We conclude that ψ_∞ is a global bounded solution of the complex Monge-Ampère equation $(***)_\infty$ which belongs to the class $\mathcal{PSH}(\omega) \cap L^\infty(X) \cap C^\infty(X\backslash D)$. □

5 Example

Example 1. *$X_\triangle = Bl_p\mathbb{P}^n$. The polytope $\triangle$ is defined by*

$$x_i \geq -1, i = 1, \cdots, n; \quad \sum_i x_i \geq -1; \quad \text{and} \quad -\sum_i x_i \geq -1$$

Using the symmetry of the polytope, we can calculate that

$$Vol(\triangle) = \frac{1}{n!}((n+1)^n - (n-1)^n)$$

$$P_c = \left(x_i = \frac{2(n-1)^n}{(n+1)((n+1)^n - (n-1)^n)}\right), \quad \text{and} \quad Q = \left(x_i = -\frac{1}{n}\right)$$

So

$$R(X_\triangle) = \frac{|\overline{OQ}|}{|\overline{P_cQ}|} = \left(1 + \frac{|\overline{OP_c}|}{|\overline{OQ}|}\right)^{-1} = \frac{(n+1)((n+1)^n - (n-1)^n)}{(n+1)^{(n+1)} + (n-1)^{(n+1)}}$$

$\mathcal{F}$ is the (n-1)-dimensional simplex with vertices

$$P_i = (-1, \cdots, \overset{i-th\ place}{n-2}, \cdots, -1) \quad i = 1, \cdots, n$$

Let e_j be the j-th coordinate unit vector, then $\langle P_i, e_j \rangle = -1$ for $i \neq j$. $\langle P_i, \pm(1, \cdots, 1) \rangle = \mp 1$, so $Bs(\mathfrak{L}_{\mathcal{F}}) = 2D_\infty$. D_∞ is the toric divisor corresponding to the simplex face with vertices $Q_i = (-1, \cdots, n, \cdots, -1)$. If we view $X_\triangle$ as the projective compactification of $\mathcal{O}(-1) \to \mathbb{P}^{n-1}$, then D_∞ is just the divisor added at infinity. So the limit metric should have conic singularity along D_∞ with conic angle

$$\theta = 2\pi \times (1 - (1 - R(X)) \times 2) = 2\pi \frac{(n+1)^{n+1} - (3n+1)(n-1)^n}{(n+1)^{n+1} + (n-1)^{n+1}}$$

In particular, if $n = 2$, i.e. $X_\triangle = Bl_p\mathbb{P}^2$ which is the case of the figure in the Introduction, then

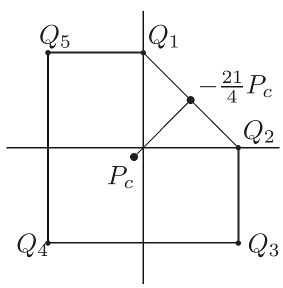
$$R(X_\triangle) = \frac{6}{7}, \quad \theta = 2\pi \times \frac{5}{7}$$

This agrees with the results of [10] and [11]. In fact, the results in [10] and [11] can be easily generalized to $Bl_p\mathbb{P}^n$ which give the same results as here.

Example 2. *$X_\triangle = Bl_{p,q}\mathbb{P}^2$, $P_c = \frac{2}{7}(-\frac{1}{3}, -\frac{1}{3})$, $-\frac{21}{4}P_c \in \partial\triangle$, so $R(X_\triangle) = \frac{21}{25}$.*

$\mathcal{F} = \overline{Q_1Q_2}$. $Bs(\mathfrak{L}_{\mathcal{F}}) = D_1 + D_2$. D_1 and D_2 are the divisors corresponding to the faces $\overline{Q_4Q_3}$ and $\overline{Q_4Q_5}$ respectively. So the conic angle along D_1 or D_2 should be

$$2\pi \times (1 - (1 - \frac{21}{25}) \times 1) = 2\pi \times \frac{21}{25}$$



References

[1] Bando, S., Mabuchi, T.: Uniqueness of Einstein Kähler metrics modulo connected group actions. *Algebraic geometry*, Sendai 1985, Advanced Studies in Pure Mathematics 10, 1987: 11-40

[2] Cheeger, J., Colding, T.H., Tian, G.: On the singularities of spaces with bounded Ricci curvature. Geom. Funct. Anal. Vol.12 (2002) 873-914.

[3] Donaldson, S.K.: Kähler geometry on toric manifolds, and some other manifolds with large symmetry, arXiv:0803.0985

[4] Demailly, J.P., Pali, N.: Degenerate complex Monge-Ampère equations over compact Kähler manifolds, Internat. J. Math. Vol.21, 3(2010) 357-405

[5] Futaki, A.: An obstruction to the existence of Einstein Kähler metrics, Inventiones Mathematicae, 73,437-443 3(1983).

[6] Kołodziej, S.: The complex Monge-Ampère equation, Acta Math. 180 (1998), no. 1, 69-117

[7] Li, C.: Greates lower bounds on the Ricci curvature of toric Fano manifolds. arXiv:0909.3443

[8] Oda, T.: Convex bodies and algebraic geometry-an introduction to the theory of toric varieties, Springer-Vergla, 1988

[9] Ruan, W.D., Zhang, Y.G.: Convergence of Calabi-Yau manifolds. arXiv:0905.3424

[10] Székelyhidi, G.: Greatest lower bounds on the Ricci curvature of Fano manifolds, arXiv:0903.5504

[11] Shi, Y.L., Zhu, X.X.: An example of singular metric arising from the blow-up limit in the continuity approach to Kähler-Einstein metrics, to appear in Pacific Journal of Mathematics

[12] Tian, G.: Canonical Metrics on Kähler Manifolds, Birkhauser, 1999

[13] Tian, G.: On stability of the tangent bundles of Fano varieties. Internat. J. Math. 3, 3(1992), 401-413

[14] Tian, G.: On Kähler-Einstein metrics on certain Kähler manifolds with $c_1(M) > 0$, Invent. Math., 89 (1987) 225-246

[15] Tian, G.: A Harnack type inequality for certain complex Monge-Ampère equations. J. Differential Geom. 29 (1989), no. 3, 481-488

[16] Tian, G.: On the existence of solutions of a class of Monge-Ampère equations. (A Chinese summary appears in Acta Math. Sinica 32 (1989), no.4, 576) Acta Math. Sinica (N.S.) 4 (1988), no 3, 250-265

[17] Wang, X.J. and Zhu, X.H.: Kähler-Ricci solitons on toric manifolds with positive first Chern class. Advances in Math. 188 (2004) 87-103

[18] Yau, S.T.: On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I, Comm. Pure Appl. Math. 31 (1978) 339-441.

[19] Yau, S.T.: A general Schwarz lemma for Kähler manifolds, American Journal of Mathematics, 1978

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