# Rational cohomology of $\overline{R}_2$ (and $\overline{S}_2$)

Sebastian Krug

In this text we compute the rational cohomology ring of $\overline{R}_2$, the moduli space of Prym curves of genus 2, which is, as we also show, isomorphic to the rational Chow ring of this space. G. Bini and C. Fontanari did the same for $\overline{S}_2$, the moduli space of spin curves of genus 2, in [BF09a]. In computing the cohomology of $\overline{R}_2$ we follow their approach in large parts, but also have to apply an idea of Orsola Tommasi, explained below, to compute additional relations in the rational cohomology ring. We also correct some errors made in [BF09a]. Among other things, some of the relations in the cohmology rings computed there are not correct, and we apply the idea just mentioned also to $\overline{S}_2$ in order to replace those relations. We treat the moduli spaces $\overline{R}_2$ of genus 2 Prym curves, $\overline{S}_2^+$ of even genus 2 spin curves, and $\overline{S}_2^-$ of odd spin curves parallely. One fact we make intensive use of in our calculations is that all three moduli spaces are isomorphic to different of what we call moduli spaces of genus 0 curves with 6 *partitioned* marked points (c.f. Lemma 20). Finite surjective morphisms from $\overline{M}_{0,6}$, the moduli space of stable genus 0 curves with 6 *ordered* marked points, to all of the three moduli spaces examined in this text exist, and where introduced in [BF09a]. They factor in a natural way through the mentioned isomorphisms from moduli spaces of curves with partitioned marked points. We show more generally that the normalization of the locus of hyperelliptic curves in every $\overline{R}_g$ and $\overline{S}_g$ is isomorphic to a disjoint union of several moduli spaces of stable genus 0 curves with $2g+2$ partitioned marked points.

I would like to thank Orsola Tommasi, who had the idea of using the morphisms from $\overline{M}_{0,6}$ just mentioned, together with the fact that they factor through the moduli spaces of curves with partitioned points, to compute relations in the cohomology rings of the moduli spaces of our interest.

# Contents



# 1 Preliminaries

In this section we give basic definitions and results, needed in our text, and fix notation. Some notation first:

1. A curve means a projective one dimensional variety (not necessarily smooth or irreducible, but necessarily reduced).

2. By the genus of a curve we will always mean the arithmetic genus.

3. For any ring $B$ and any group $G$ acting on $B$ we denote by $B^G$ the subring of invariants under the action of $G$.

### 1.1 Spin- and Prym curves and their moduli spaces.

**Definition 1** (i) A *stable curve* $C$ (possibly with marked points) is a connected curve having only nodes as singularities and having a finite group of automorphisms (respecting the marked points, if there are any). Having a finite automorphism group is equivalent to the following condition: When we consider as "special points" on a irreducible component of $C$ the marked points as well as the points in which the component meets the rest of $C$, then every component of genus 0 must carry at least three special points, and every component of genus 1 must carry at least one special point.

(ii) A *semistable curve* $X$ is a connected stable curve having only nodes as singularities, and such that every connected component of genus 1 carries at least one special point, and every component of genus 0 carries at least two special points.

(iii) A component of genus 0 of a semistable curve $X$ meeting the rest of $X$ in exactly two points and carrying no marked points is called an *exceptional component* of $X$.

(iv) The *non-exceptional subcurve* $\tilde{X}$ of a semistable curve $X$ is the closure of the complement of all exceptional components of $X$.

(v) A semistable curve $X$ is called *quasistable*, if no two of its exceptional components intersect each other.

(vi) The *stable model* of a quasistable curve is the (unique) stable curve $C$ obtained by contracting every exceptional component of $X$ to a point. The blow down map $\beta : X \to C$ is also called the stable model of $X$.

**Definition 2** (i) A *spin curve* resp. *Prym curve* of genus $g$ is a triple $(X; \mathcal{L}; b)$, where $X$ is a quasistable curve with stable model $\beta : X \to C$, $\mathcal{L}$ is a line bundle on $X$. For a spin curve, $b$ is a homomorphism $b : \mathcal{L}^{\otimes 2} \to \omega_X$ such that the restriction of $\mathcal{L}$ to any exceptional component $E$ is isomorphic to $\mathcal{O}_E(1)$ and the restriction of $b$ to the non-exceptional subcurve $\tilde{X}$ induces an isomorphism $\mathcal{L}^{\otimes 2}_{|\tilde{X}} \to \omega_{\tilde{X}}$. For a Prym curve replace $\omega_X$ by $\mathcal{O}_X$ and $\omega_{\tilde{X}}$ by $\mathcal{O}_{\tilde{X}}$ in the above definition, and additionally forbid the case $\mathcal{L} \cong \mathcal{O}_X$. The curve $X$ is called the *support* of the spin- resp. Prym curve, the pair $(\mathcal{L}; b)$ a spin- resp. Prym structure on $X$. A spin- resp. Prym curve is called smooth if $X$ is smooth.

(ii) We use the definition of isomorphisms of spin curves resp. Prym curves as for example given in [Cor89] resp. [FL10]. Thus isomorphisms of spin- resp. Prym curves are for us isomorphisms of the underlying quasistable curve $X$ compatible with the extra structure, and do not include a morphism of the extra structure, as for example in [Cor91] and [Lud10]. I.e. a isomorphism $\varphi : (X; \mathcal{L}; b) \to (X'; \mathcal{L}'; b')$ of spin- resp. Prym curves is a isomorphism $\varphi : X \to X'$ such that there is an isomorphism $\varphi^* \mathcal{L}' \cong \mathcal{L}$ which is compatible with $b$. This choice of definition influences the number of automorphisms of the objects.

(iii) For a given quasistable curve $X$ we call every line bundle (i.e. invertible sheaf) $\mathcal{L}$ that fits into the definition of a spin curve or Prym curve with support $X$ a spin sheaf resp. a Prym sheaf of $X$. **We also call the trivial sheaf a Prym sheaf, and speak of nontrivial Prym sheaves if we want to exclude it.**

(iv) Let $(X;\mathcal{L};b)$, $(X';\mathcal{L}';b')$ be two spin- or two Prym curves, Let $C$, $C'$ be the stable models of $X$ resp. $X'$, let $N$, $N'$ be the sets of nodes of $C$ resp. $C'$, to which exceptional components are contracted ("exceptional nodes"). Then there is a surjective homomorphism of isomorphism groups

$$\psi' : Isom(X, X') \to Isom((C;N),(C';N'))$$

which can of course be restricted to a group homomorphisms

$$\psi : Isom((X;\mathcal{L};b),(X';\mathcal{L}';b')) \to Isom((C;N),(C';N'))$$

The isomorphisms lying in the kernel of $\psi$ are called *inessential isomorphisms*. In case of $(X;\mathcal{L};b) = (X';\mathcal{L}';b')$ we speak of *inessential automorphisms*.

For every $g \geq 2$ there exist coarse moduli spaces $\overline{S}_g$ and $\overline{R}_g$ for spin curves resp. Prym curves of genus $g$. They are projective algebraic varieties of dimension $3g-3$ and have only finite quotient singularities. The open subsets parametrizing smooth spin- resp. Prym curves are denoted by $S_g$ and $R_g$. $\overline{S}_g$ consists of two connected components $\overline{S}_g^+$ and $\overline{S}_g^-$ prametricing even resp. odd spin curves.

**Definition 3** We denote by $\pi_R : \overline{R}_2 \longrightarrow \overline{M}_2$, $\pi_+ : \overline{S}_2^+ \longrightarrow \overline{M}_2$ and $\pi_- : \overline{S}_2^- \longrightarrow \overline{M}_2$ the "forgetful morphisms", which corresponds to discarding the additional Prym or spin structure, and passing from $X$ to its stable model $C$.

**Notation for other moduli spaces used in this text:**

$\overline{M}_{g,n}$, $\overline{S}_{g,n}$, $\overline{S}_{g,n}^+$, and so on, denote the moduli spaces of genus $g$ stable curves, spin curves, even spin curves, and so on, together with $n$ ordered marked points on the underlying curve.

$\overline{S}_{g,n}^{(r_1,...,r_n)}$ resp. $\overline{R}_{g,n}^{(r_1,...,r_n)}$, for $r_1,...,r_n \in \mathbb{Z}$, are moduli spaces of *twisted* spin- resp. Prym curves with $n$ ordered marked points. Such twisted spin resp. Prym curves are defined varying the definition of a spin- resp. Prym curve as follows: If $(p_1,...,p_n)$ are the marked points on $X$, then the line bundle $\mathcal{L}$ on $X$ is a square root of $\omega_X(r_1p_1 + ....r_np_n)$ resp. $\mathcal{O}_X(r_1p_1 + .... + r_np_n)$, instead of $\omega_X$ resp. $\mathcal{O}_X$.

## 1.2 Cohomology and rational Chow ring for moduli spaces.

We will work with the rational Chow ring as well as with the rational cohomology of moduli spaces. We denote them by $A^*_{\mathbb{Q}}(...)$ resp. $H^*_{\mathbb{Q}}(...)$.

We compile some results by J.H.M Steenbrink from [Ste77] about the cohomology of what he calls $V$-manifolds, which are what we would nowadays call the underlying spaces of orbifolds. All moduli spaces we are concerned with in this text are $V$-manifolds.

**Summary 4** *Let $X$ be a projective $V$-manifold. Then*

*(i) The hard Lefschetz theorem holds, i.e.: Let $L \in H^2(X, \mathbb{Z})$ be the cohomology class of an ample divisor on $X$. Then for all $q \in \mathbb{N}$ the map $\omega \mapsto L^q \wedge \omega$ induces an isomorphism between $H^{n-q}(X, \mathbb{C})$ and $H^{n+q}(X, \mathbb{C})$. ([Ste77] Thm. 1.13)*

*(ii) The canonical Hodge structure of $H^k(X)$, that would be mixed for an arbitrary singular variety, is pure of weight $k$ for all $k \geq 0$. ([Ste77] Cor. 1.5)*

Part (ii) allows us to speak of the pure Hodge structure on our moduli spaces, and especially to define Hodge numbers.

In [Mum83] D. Mumford introduced the rational Chow ring of $Q$-varieties and $Q$-stacks. Our moduli spaces are $Q$-stacks (with smooth global covers) so we can use Mumfords results. We summarize the ones we will use:

**Summary 5** *Let $X$ be an algebraic variety that is a $Q$-variety or a $Q$-stack, with global Cohen-Macaulay cover. then:*

*(i) There is a "natural" way to define an intersection product $\bullet . \bullet$ on the rational Chow group of $X$, making it into the Chow ring $A^*_{\mathbb{Q}}(X)$ we are going to use in our computations. (C.f. [Mum83] section § 3.)*

*(ii) To a closed codimension $n$ subvariety $Y$ of one of our moduli spaces, one can assign classes in the rational Chow ring in two ways. One is the usual of just taking the corresponding cycle class $[Y]$ in the Chow group $A^n_Q(X)$. The other is to take the $Q$-class $[Y]_Q$ of $Y$ as defined in [Mum83] § 3. This corresponds to considering the cycle of $Y$ on the moduli stack.*

*(iii) For our moduli spaces, between these two classes the relation $[Y] = n[Y]_Q$ holds, where $n$ is the number of automorphisms of an object parametrized by a general point of $Y$.*

*(iv) Intersections of $Q$-classes in the rational Chow ring, can be computed on smooth sheets $X_\alpha$ mapping to dense open parts of $X$ (c.f. [Mum83] §3). For our moduli spaces, like for $\overline{M}_g$, these sheets can be taken to be certain moduli spaces paremetrizing spin- rep. Prym curves together with a kind of level structure (c.f. [Mum83] § 3). Locally at any point, the $X_\alpha$ are isomorphic to the deformation space of the object parametrized by this point. On the deformation space each two subspaces parametrizing curves of two given topological types, meet like subvectorspaces of a vectorspace, since local coordinates can be choosen such that one coordinate each coresponds to smoothing of one of the nodes of the curve (C.f. [Cor89] § 5). Therefore if one has two cycles $Y$, $Z$ parametrizing generically curves of a given topological type (as will usually be the case for the cycles appearing in this text), and their intersection $Y \cap Z = S$ is proper, then one can treat the intersection also as transversally in computing the intersection of the $Q$-classes. Thus in this case $[Y]_Q.[Z]_Q = [S]_Q$.*

*(v) A morphism $f : X \to Y$ of $Q$-stacks (with global Cohen-Macauley cover), induces a pullback $f^* : A^*_{\mathbb{Q}}(Y) \to A^*_{\mathbb{Q}}(X)$ that is a ring homomorphism. If $W$ is a closed subvariety of $Y$ such that codim $f^{-1}(W) =$ codim $W$, and if we denote by $S$ the set of components of $f^{-1}(W)$ then:*

$$f^*([W]_Q) = \sum_{V_k \in S} i_k \cdot [V_k]_Q$$

*where $i_k$ can be calculated as the the ramification index of the map $f_\alpha : X_\alpha \to Y_\alpha$ belonging to $f$, in the locus corresponding to $V_k$ on one of the smooth sheets $X_\alpha$. As mentioned above, in our cases these sheets locally are local universal deformation spaces. (C.f. [Mum83] Section §3., especially Prop. 3.8.)*

*(vi) For the pullback $f^*$ just introduced and the usual pushforward $f_*$ the projection formula (also called push-pull formula) holds:*

$$f_*(a.f^*b) = f_*a.b$$

*For every $a \in A^*_{\mathbb{Q}}(X)$ and $b \in A^*_{\mathbb{Q}}(Y)$.*

In section 2 we will show that our moduli spaces are even global quotients of a manifold by a finite group $G$, so in our special case Steenbrink's and Mumford's results could be shown more easily:

**Lemma 6** *Let $X$ be a smooth algebraic variety, let $G$ be a finite group acting algebraically on $X$ and let $Y = X/G$ be the quotient. Then*
*(i) $H^*_{\mathbb{Q}}(Y) = \left(H^*_{\mathbb{Q}}(X)\right)^G$ (C.f. [Bre72] Page 120.)*
*(ii) $A^*_{\mathbb{Q}}(Y) = \left(A^*_{\mathbb{Q}}(X)\right)^G$ (C.f. [Ful98], Example 1.7.6.)*

### 1.3 Further notation and conventions for this article

1. If we denote a cycle class of a moduli space by 1 we mean by this the $Q$-Class of the whole space.

2. We sometimes speak of "the stratification according to topological type (of the underlying stable curves)" of the spaces $\overline{R}_2$, $\overline{S}_2^+$ or $\overline{S}_2^-$. What is meant by this is explained in the appendix.

3. We call "closed strata" of these stratifications, the closures of all their strata, not only the strata that are already closed (i.e. points).

4. If there appears a cycle class in our computations that is not written as a product of boundary classes, it is usually the class of one of the closed strata just mentioned. (For example $[C^+]_Q$, $[X^-]$, $[E'{,}']$.) The (closed) strata are described in the appendix, and they will be used in the main body of the article without defining them there.

5. If $O$ is an object of the kind parametrized by a moduli space $M$, then we denote the point in $M$ prametrizing $O$ as $[O]$. For example if $(X;\mathcal{L};b)$ is a Prym curve of genus $g$, then $[(X;\mathcal{L};b)]$ is the corresponding point in $\overline{R}_g$.

6. Usually instead of $a.b$ we write $ab$ for the intersection of cycle classes $a, b$ in the Chow ring.

## 1.4 Some lemmata for extending morphisms

We call a morphism of complex analytic spaces *finite* if it is proper and has finite fibers. The following lemmata can be proven quite easily using basic theorems form complex analysis and commutative algebra.

**Lemma 7** *Let $X$, $Y$ be complex analytic spaces, $X$ normal, and $U$ a dense open subset of $X$. If $f : U \to Y$ is a holomorphic map, and $\tilde{f} : X \to Y$ is a continous map extending $f$, then $\tilde{f}$ is holomorphic.*

**Lemma 8** *(i) Let $X$, $S$ and $M$ be complex analytic spaces, $X$ normal, $U \subset X$ an open subset. Let $\pi : S \longrightarrow M$ be a finite holomorphic map, and let $g : X \longrightarrow M$ and $f : U \longrightarrow S$ be holomorphic maps, such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[rrd, "g"'] & U \arrow[l, hook'] \arrow[r, "f"] & S \arrow[d, "\pi"] \\ & & M \end{tikzcd}$$

*Then $f$ extends to a holomorphic map $\tilde{f} : X \longrightarrow S$, compatible with the diagram.*
*(ii) If furthermore $g$ is finite, then $\tilde{f}$ is finite too.*

**Lemma 9** *Let $X$, $Y$ be algebraic varieties, $Y$ normal. Let $f : X \to Y$ be a finite morphism of degree 1, then $f$ is an isomorphism.*

## 1.5 The boundary components of $\overline{R}_2$

In the following section we quote results from [FL10] we are going to use.
We call the irreducible components of $\overline{R}_2 \smallsetminus R_2$ the *boundary components* of $\overline{R}_2$. There are exactly 5 such components. They have codimension 1, so they are divisors of $\overline{R}_2$. The boundary divisors of $\overline{R}_2$ lie above the two boundary divisors $\Delta_0$ and $\Delta_1$ of $\overline{M}_2$, with respect to the forgetful map $\pi$. We describe the boundary divisors, by explaining which kind of Prym curves $(C;\mathcal{L};b)$ their general points parametrize.

1. $D_1$: Here $C$ has two irreducible components (of genus 1), meeting in one node, such that restricting the Prym sheaf $\mathcal{L}$ to one of the components yields the trivial sheaf.

2. $D_{1:1}$: Here $C$ has two irreducible components meeting in one node, and restricting $\mathcal{L}$ to either component yields a nontrivial Prym sheaf.

3. $D_0'$: Here $C$ has one node, the normalization $\tilde{C}$ of $C$ is connected and the pullback of $\mathcal{L}$ to the normalization is a nontrivial Prym sheaf of $\tilde{C}$.

4. $D_0''$: Here $C$ has one node, the normalization $\tilde{C}$ of $C$ is connected and the pullback of $\mathcal{L}$ to the normalization is the trivial sheaf $\mathcal{O}_{\tilde{C}}$

5. $D_0^r$: Here $C$ consists of two irreducible components, one is a smooth genus 1 curve $D$, the other an exceptional component $E$, i.e. a smooth genus 0 curve meeting $D$ in two points. Restricting $\mathcal{L}$ to $D$ yields a Prym sheaf on $D$. If $\tilde{D}$ and $\tilde{E}$ are the two connected components of the normalization $\tilde{C}$ of $C$, and if $p, q$ are the two points on $\tilde{D}$ lying over the points of $C$ in which $D$ and $E$ meet, then the Pullback of $\mathcal{L}$ to $\tilde{D}$ is a square root of $\mathcal{O}_{\tilde{D}}(-q-p)$.

To the boundary components we assign elements of $A_{2,\mathbb{Q}}(\overline{R}_2)$ by taking $Q$-classes:

$$d_1 := [D_1]_Q, \quad d_{1:1} := [D_{1:1}]_Q, \quad d_0' := [D_0']_Q, \quad d_0'' := [D_0'']_Q, \quad d_0^r := [D_0^r]_Q$$

we often call these the *boundary classes* of $\overline{R}_2$. Equivalently one defines the boundary classes $\delta_0$ and $\delta_1$ of $\overline{M}_2$.
The forgetful map $\pi_R : \overline{R}_2 \longrightarrow \overline{M}_2$, is ramified in codimension 1 only at $D_0^r$ (therefore the r). The boundary classes of $\overline{M}_2$ pull back to $\overline{R}_2$ as follows:

$$\pi^*(\delta_0) = d_0' + d_0'' + 2d_0^r \qquad \text{and} \qquad \pi^*(\delta_1) = d_1 + d_{1:1}$$

The boundary components $A_0^+, B_0^+, A_1^+, B_1^+$ of $\overline{S}_2^+$ and $A_0^-, B_0^-, A_1^-$ of $\overline{S}_2^-$ are described in [BF09a]. Again we define corresponding classes:

$$\alpha_0^+ := [A_0^+]_Q, \quad \beta_0^+ := [B_0^+]_Q, \quad \alpha_1^+ := [A_1^+]_Q, \quad \beta_1^+ := [B_1^+]_Q,$$
$$\alpha_0^- := [A_0^-]_Q, \quad \beta_0^- := [B_0^-]_Q, \quad \alpha_1^- := [A_1^-]_Q$$

The pullbacks of $\delta_0$ and $\delta_1$ to these spaces are:

$$\pi_+^*(\delta_0) = \alpha_0^+ + 2\beta_0^+, \quad \pi_+^*(\delta_1) = 2\alpha_1^+ + 2\beta_1^+,$$
$$\pi_-^*(\delta_0) = \alpha_0^- + 2\beta_0^-, \quad \pi_-^*(\delta_1) = 2\alpha_1^-$$

# 2 Moduli spaces of stable hyperelliptic spin- and Prym curves

**Definition 10** By $HM_g$, $HS_g^+$, $HS_g^-$ and $HR_g$ we denote the loci of hyperelliptic curves in $M_g$, $S_g^+$, ... The closures of these loci in $\overline{M}_g$, $\overline{S}_g^+$, $\overline{S}_g^-$ resp. $\overline{R}_g$ we denote by $\overline{HM}_g$, $\overline{HS}_g^+$, $\overline{HS}_g^-$, resp. $\overline{HR}_g$. We call those compact spaces moduli spaces of stable hyperelliptic curves resp. stable hyperelliptic spin/Prym curves.

In this section we show that the normalizations of these compact moduli spaces, are isomorphic to disjoint unions of several of what we call moduli spaces of stable genus 0 curves with *partitioned* marked points (c.f. Definition 11). These moduli spaces can be described as quotients by finite groups acting on moduli spaces $\overline{M}_{0,2g+2}$ of stable genus 0 curves with $2g+2$ *ordered* marked points. The cohomology rings of the latter moduli spaces are known by work of S. Keel ([Kee92]).

To construct the isomorphisms we will use the fact, that for every set of $2g+2$ distinct points in $\mathbb{P}^1$ there is a (unique up to isomorphism) $2:1$ cover $h: C \longrightarrow \mathbb{P}^1$ ramified exactly over the given points, and $C$ is a genus $g$ smooth hyperelliptic curve, and the fact that every hyperelliptic curve can be obtained in this way. The spin- resp. Prym sheaves on $C$ can be recovered as the invertible sheafs corresponding to certain divisors that are linear combinations of the ramification points. Using admissible $2:1$ covers of stable genus 0 curves with $2g+2$ marked points, one can extend this correspondence to the asserted isomorphisms.

Probably everything proven in this section is somehow known.

## 2.1 Admissible (double) covers

**Definition 11** By $\overline{M}_{g,(n_1,...,n_l)}$ we denote the coarse moduli space of the following moduli problem: Stable curves of genus $g$ with $n_1+n_2+...+n_l$ unordered marked points that are divided into $l$ disjoint sets $A_1,...,A_l$ such that $\#A_i = n_i$ for all $i \in \{1,...,l\}$. Formally the objects are pairs $(X;(A_1,...,A_l))$, where $X$ is a genus $g$ curve such that $X$ becomes stable if one marks the points on it contained in the $A_i$. The moduli space one gets by changing the objects of the moduli problem to $(X;\{A_1,...,A_l\})$, i.e. by having a set of sets instead of a tuple of sets in the defining data, we denote by $\overline{M}_{g,[n_1,...,n_l]}$.

We often call moduli spaces of the latter type, moduli spaces of curves of genus $g$ with *partitioned* marked points.

By $M_{g,(n_1,...,n_l)}$ resp. $M_{g,[n_1,...,n_l]}$ we denote the corresponding moduli spaces of smooth curves.

**Remark & Definition 12** (i) For $n := n_1+n_2+...+n_l$ one can construct the moduli space $\overline{M}_{g,(n_1,...,n_l)}$ as the quotient of $\overline{M}_{g,n}$. Divide the set of marked points $\{1,...,n\}$ into disjoint subsets $A'_1,...,A'_l$ of the appropriate size $\#A'_i = n_i$. Then take the quotient of $\overline{M}_{g,n}$ induced by the action of $S_{n_1}\times ... \times S_{n_l}$ permuting the indices inside the sets $A'_1,...,A'_n$. $\overline{M}_{g,[n_1,...,n_l]}$ can be constructed as the quotient of $\overline{M}_{g,(n_1,...,n_l)}$ by the action permuting the indices of those of the sets $A_1,...,A_n$ having the same cardinality.

(ii) For genus 0 we fix some of the quotient morphisms, we are going to use later. Let

$$\pi_{(n_1,...,n_l)} : \overline{M}_{0,n} \to \overline{M}_{0,(n_1,...,n_l)}$$

be the quotient morphism corresponding to the choice $A'_1 := \{1,...,n_1\}$, $A'_2 = \{n_1+1, n_1+2,...,n_1+n_2\}$, and so on.

Let

$$\pi_{[n_1,...,n_l]} : \overline{M}_{0,n} \to \overline{M}_{0,[n_1,...,n_l]}$$

be the composition of $\pi_{(n_1,...,n_l)}$ with the quotient morphism $\overline{M}_{0,(n_1,...,n_l)} \to \overline{M}_{0,[n_1,...,n_l]}$.

**Definition 13** (i) Let $(D; \{p_1, ..., p_n\})$ be a stable genus 0 curve with $n$ unordered marked points. For us an admissible $d : 1$ cover of $(D; \{p_1, ..., p_n\})$ is a regular morphism $f : Y \to D$ such that $Y$ is a connected nodal curve, and:

1. For $D_{ns}$ the nonsingular locus of $D$, $f^{-1}(D_{ns}) = Y_{ns}$ and the restriction of $f$ to $f^{-1}(D_{ns})$ is a $d : 1$ cover simply ramified over the marked points and unramified everywhere else.
2. For every node $q$ of $D$, every point in $f^{-1}(q)$ is a node of $Y$ and for every such node $r$ the two branches of $Y$ in $r$ are mapped to the two branches of $D$ near $q$, both with the same ramification index in $r$.

(ii) An isomorphism between two admissible covers $f : Y \to D$ and $f' : Y' \to D'$ is an isomorphism $\varphi : Y \to Y'$ such that there is an isomorphism $\psi : D \to D'$ for which $\psi \circ f = f' \circ \varphi$.

We compile some facts about admissible covers, especially $2 : 1$ covers, which we mostly take from [AL02].

**Summary 14** *(i) There is a coarse moduli space $\overline{H}_{d,g}$ of admissible $d : 1$-covers of stable genus 0 curves with $2(g + d) - 2$ marked points.*

*(ii) The covering space $Y$ of an admissible $2 : 1$ cover $f : Y \to D$ is a semistable curve, all whose irreducible components are smooth.*

*(iii) There are isomorphisms $\overline{M}_{0,[2g+2]} \cong \overline{H}_{2,g} \cong \overline{HM}_g$. They are explicitly constructed in [AL02] by describing how to associate to a family of stable genus 0 curves with $2g+2$ unordered marked points a unique family of admissible $2 : 1$ covers, and how to associate to a family of such admissible covers a unique family of stable hyperelliptic curves. (The latter is just by contracting all exceptional components of $Y$, i.e. stable reduction.)*

*(iv) Thus, in particular, the $2 : 1$ admissible cover $f : Y \to D$ of a stable genus 0 curve with an even number of unordered marked points is defined uniquely up to isomorphism.*

**Caution:** Sometimes we will just mean the $Y$ of $f : Y \to D$ when we talk about the admissible $2 : 1$ cover of a stable genus 0 curve with marked points.

## 2.2 Relation to moduli spaces of stable genus 0 curves with partitioned marked points.

**Lemma 15** *For $g \geq 2$, let $p_1, ..., p_{2g+2}$ be distinct points in $\mathbb{P}^1$, and $h : Y \longrightarrow \mathbb{P}^1$ the (unique) $2 : 1$ cover of $\mathbb{P}^1$ ramified exactly over these points. Then $Y$ is a genus $g$ hyperelliptic curve. For $i = 1, ..., 2g + 2$, define $q_i := h^{-1}(p_i)$. Let $Q$ be the set of all $q_i$ and denote by $P_n$ the set of possible partitions of $Q$ into a set of $n$ elements and a set of $2g + 2 - n$ elements. I.e.:*

$$P_n := \{\{A, B\} \mid A, B \subseteq Q,\ A \uplus B = Q,\ \#A = n,\ \#B = 2g + 2 - n\}$$

*Let $J_R(Y)$, $J_S(Y)$, $J_+(Y)$, $J_-(Y)$ be the sets of isomorphism classes of nontrivial Prym sheaves, resp. spin sheaves, resp. even spin sheaves, resp. odd spin sheaves on $Y$. (Of course $J_S(Y) = J_+(Y) \uplus J_-(Y)$.) Then we have:*

*For any $\{A, B\} \in P_n$ and $r_1, ..., r_n$ the points in $A$.*

*(i) For all even $2 \leq n \leq g+1$:*

1. *$\phi_{R,n}(\{A, B\}) := \mathcal{O}_Y(r_1 + ... + r_{\frac{n}{2}} - r_{\frac{n}{2}+1} - ... - r_n)$ is a nontrivial Prym sheaf of $Y$. Its isomorphism class is independent of the ordering of the points $r_i$, as well as of the choice of $A$, necessary in the case $n = g+1$. Thus the following map is well defined.*
2. *The map $\phi_{R,n} : P_n \to J_R(Y)$, $\{A, B\} \mapsto \phi_{R,n}(\{A, B\})$ is injective.*
3. *The map $\phi_R : \biguplus_{\substack{2\leq n\leq g+1,\\ n \text{ even}}} P_n \to J_R(Y)$, obtained as union of the maps $\phi_{R,n}$ is a bijection.*

*(ii) Analogously for spin structures:*

1. *If $g$ is even, then for all $0 \leq n \leq g+1$, with $n$ odd:*
   *$\phi_{S,n}(\{A, B\}) := \mathcal{O}_Y((g-2) \cdot q_1 + r_1 + r_2 + ... + r_{\frac{n+1}{2}} - r_{\frac{n+1}{2}+1} - ... - r_n)$ is a spin sheaf of $Y$.*
2. *If $g$ is odd, then for all $0 \leq n \leq g+1$, with $n$ even:*
   *$\phi_{S,n}(\{A, B\}) := \mathcal{O}_Y(g \cdot q_1 + r_1 + r_2 + ... + r_{\frac{n}{2}} - r_{\frac{n}{2}+1} - ... - r_n)$ is a spin sheaf of $Y$.*
3. *In both cases the isomorphism class of $\phi_{S,n}(\{A, B\})$ is independent of the ordering of the points $r_i$ and $q_i$, as well as of the choice of $A$, necessary in the case $n = g+1$. Thus the map $\phi_{S,n} : P_n \to J_S(Y)$, $\{A, B\} \mapsto \phi_{R,n}(\{A, B\})$ is well defined. It is injective, and the map $\phi_S : \biguplus_{\substack{1\leq n\leq g+1,\\ n \text{ odd}}} P_n \to J_S(Y)$, obtained as union of the maps $\phi_{S,n}$, is a bijection.*

*(iii) For every $g \geq 2$ the bijection $\phi_S$ of course splits into two bijections $\phi_+ : (\phi_S)^{-1} J_+(Y) \to J_+(Y)$ and $\phi_- : (\phi_S)^{-1} J_-(Y) \to J_-(Y)$. They can also be written (by describing $(\phi_S)^{-1} J_+(Y)$ and $(\phi_S)^{-1} J_-(Y)$ explicitly) as:*

$$\phi_+ : \biguplus_{\substack{1\leq n\leq g+1,\\ n\equiv g+1 \bmod 4}} P_n \to J_+(Y)$$

*and*

$$\phi_- : \biguplus_{\substack{1\leq n\leq g+1,\\ n\equiv g-1 \bmod 4}} P_n \to J_-(Y)$$

**Proof:** It is easy to show that, for all $i, j \in \{1, ..., 2g+2\}$, $2q_i - 2q_j \sim 0$. I.e. all $2q_i$ are equivalent.

Using this, all claims of part (i) follow form what is shown in § 5.2.3. in [Dol10].

All assertions of (ii) follow from the fact that the canonical sheaf of $Y$ is equivalent to $(2g-2)q_i$ for any $i \in \{1, ..., 2g+2\}$ and the corresponding assertions of part (i) of the Lemma.

For (iv): From Lemma 5.2.1. in [Dol10] it follows that $h^0(\phi_{S,n}(\{A, B\}))$ is even if $g - n + 1 \equiv 0 \mod 4$ and odd if $g - n + 1 \equiv 2 \mod 4$. This proves part (iv) of the Lemma. □

**Lemma 16** *If by $X^{\sim}$ we denote the normalization of a variety $X$ then:*
*(i) For all $g \geq 2$ there is an isomorphism:*

$$b : \overline{M}_{0,[2g+2]} \xrightarrow{\cong} \overline{HM}_g$$

*(ii) For all $g \geq 2$ there is an isomorphism:*

$$a_R : \biguplus_{\substack{2 \leq n \leq g+1, \\ n \text{ even}}} \overline{M}_{0,[n,2g+2-n]} \xrightarrow{\cong} (\overline{HR}_g)^{\sim}$$

*(iii) For all $g \geq 2$ there are isomorphisms:*

$$a_+ : \biguplus_{\substack{0 \leq n \leq g+1, \\ n \equiv g+1 \text{ mod } 4}} \overline{M}_{0,[n,2g+2-n]} \xrightarrow{\cong} (\overline{HS}_g^+)^{\sim}$$

*and*

$$a_- : \biguplus_{\substack{0 \leq n \leq g+1, \\ n \equiv g-1 \text{ mod } 4}} \overline{M}_{0,[n,2g+2-n]} \xrightarrow{\cong} (\overline{HS}_g^-)^{\sim}$$

*All the isomorphism above map boundary points to boundary points (after composing the isomorphisms with the normalization map).*

**Proof:** (i) The isomorphism of (i) is constructed in [AL02] (also c.f. Summary 14), it maps boundary points to boundary points, as can easily be checked by looking at the construction there.

On the interior of the moduli spaces the restricted morphism $b' : M_{0,[2g+2]} \longrightarrow HM_g$ acts in the following way: Let $(D; \{p_1, ..., p_{2g+2}\})$ be a smooth rational curve with $2g+2$ unordered marked points. Let $Y$ be the unique $2:1$ cover of $D$ ramified over exactly the points $p_i$. The morphism $b'$ assigns to $[(D; \{p_1, ..., p_{2g+2}\}] \in M_{0,2g+2}$ the point $[Y] \in HM_g$. Every smooth hyperelliptic curve $Y$ of genus $g$ is a $2:1$ cover of $\mathbb{P}^1$ ramified in $2g+2$ Points, thus $b'$ is surjective. Since two smooth pointed curves $(D; \{p_1, ..., p_{2g+2}\})$ and $(D'; \{p'_1, ..., p'_{2g+2}\})$ are isomorphic if and only if the covers $Y$ and $Y'$ are isomorphic, $b'$ is of degree 1. Both $M_{0,[2g+2]}$ and $M_g$ are normal varieties, thus this implies that $b'$ is an isomorphism. For the description of the isomorphism $b$, extending $b'$ to the compactified moduli spaces, c.f. [AL02].

Now we prove (ii): A morphism

$$a'_R : \biguplus_{\substack{2 \le n \le g+1, \\ n \text{ even}}} M_{0,[n,2g+2-n]} \longrightarrow HR_g$$

can be defined in the following way: For $2 \le n \le g+1$, $n$ even, $a'_R$ assigns to $[(D;\{A,B\})] \in M_{0,[n,2g+2-n]}$ the point $[(Y;\phi_{R,n}(\{A,B\}))] \in HR_g$, where $Y$ is defined as above, and $\phi_{R,n}(\{A,B\})$ is defined as in Lemma 15 (i). The morphism $a'_R$ is surjective and $1:1$ by Lemma 15 (i). Let

$$\psi : \biguplus_{\substack{2 \le n \le g+1, \\ n \text{ even}}} \overline{M}_{0,[n,2g+2-n]} \longrightarrow \overline{M}_{0,[2g+2]}$$

and

$$\pi : \overline{HR}_g \longrightarrow \overline{HM}_g$$

be the forgetful morphisms. Using the abbreviations $N := \biguplus_{\substack{2 \le n \le g+1, \\ n \text{ even}}} M_{0,[n,2g+2-n]}$ and $\overline{N} := \biguplus_{\substack{2 \le n \le g+1, \\ n \text{ even}}} \overline{M}_{0,[n,2g+2-n]}$, we get the commutative diagram

$$\begin{tikzcd}
\overline{N} \arrow[drr, "b\circ\phi"'] & N \arrow[l, hook'] \arrow[r, "a'_R"] & \overline{HR}_g \arrow[d, "\pi"] \\
 & & \overline{HM}_g
\end{tikzcd}$$

Thus by Lemma 8 $a'_R$ extends to a finite surjective Morphism

$$a_R : \biguplus_{\substack{2 \le n \le g+1, \\ n \text{ even}}} \overline{M}_{0,[n,2g+2-n]} \longrightarrow \overline{HR}_g.$$

It has degree 1 and must be an isomorphism since both varieties are normal (c.f. Lemma 9).

Part (iii) of our Lemma is proven analogously to part (ii), by using part (ii) and (iii) of Lemma 15 instead of part (i). The isomorphisms of part (ii) and (iii) of our Lemma map boundary points to boundary points because they are compatible with the isomorphism $b$ of Part (i), and this one does. □

**Remark:** While $\overline{HM}_g$ is a normal variety for all $g \ge 2$ (since it is isomorphic to $\overline{M}_{0,[2g+2]}$), the spaces $\overline{HS}_g^+$, $\overline{HS}_g^-$ and $\overline{HR}_g$ in gereral are not. Take for example in $\overline{S}_3^-$ a point coresponding to a spin curve $(X;\mathcal{L};b)$ with $X$ consisting of two disjoint smooth genus 1 curves and two exceptional components, such that each exceptional component meets each genus 1 component in exactly one point. Now let $D'$ be the local universal deformation space of $(X;\mathcal{L};b)$, and let $D$ be the local universal deformation space of $C$, the stable model of $X$. Then the forgetful map $\varphi : D' \to D$ is $4:1$ and simply ramified over each of the two subspaces of $D$ coresponding to the two nodes of $C$, which are blown up in $X$. One can define local coodinates $x, y, z_1, ..., z_4$ around the special point of $D$, such that the two subspaces just mentioned are the spaces $x = 0$ and $y = 0$. Then, for suitably choosen local coordinates on $D$, $\varphi$ is described by $x' \mapsto (x')^2$, $y' \mapsto (y')^2$ and $z'_i \mapsto z'_i$ for $i = 1, ..., n$. The hyperelliptic involution on $C$ swaps the two nodes, thus the hyperelliptic locus in $D$ is invariant under swaping the coordinates $x$ and $y$. Since the hyperelliptic locus is also normal, it follows that it can localy be described by $x = y$ (for a possible choice of coordinates). Thus the hyperelliptic locus in $D'$ is described by $(x')^2 = (y')^2$, thereby having a singularity of codimension 1. As one can check, this singularity is retained when quotienting $D'$ by the action of the automorphism group of $(X; \mathcal{L}; b)$, hence the hyperelliptic locus $\overline{HS}_3^-$ in $\overline{S}_3^-$ is not normal.

## 2.3 Some properties of $\overline{M}_{0,n}$

The moduli spaces $\overline{M}_{0,n}$ $(n \geq 3)$ of stable genus 0 curves with ordered marked points where examined by S. Keel in [Kee92]. Among other things he computed their cohomology ring (and, what is the same for these spaces, the Chow ring) for all $n \geq 3$. We summarize some facts about these spaces we are going to use from [Kee92].

**Summary 17 (S. Keel)**

*For all $n \geq 3$:*

*(i) $\overline{M}_{0,n}$ is a smooth rational projective variety of dimension $n-3$.*

*(ii) For every $S \subsetneq \{1, ..., n\}$ such that $\#S \geq 2$ and $\#(\{1, ..., n\} \smallsetminus S) \geq 2$, there is a boundary divisor $D^S$ of $\overline{M}_{0,n}$, a general point of which corresponds to a rational curve with two smooth irreducible components meeting in one node, such that the marked point with indices in $S$ lie on one of the components, and the marked points with indices in $S^c := \{1, ..., n\} \smallsetminus S$ lie on the other component. (Of course $D^S$ and $D^{S^c}$ are the same divisor.)*

*The boundary $\overline{M}_{0,n} \smallsetminus M_{0,n}$ of $\overline{M}_{0,n}$ is exactly the union of the divisors just described.*

*(iii) The cohomology ring of $\overline{M}_{0,n}$ is generated by the boundary components, and is isomorphic to the chow ring by the cycle map.*

*(iv) More specific:*

$$H^*(\overline{M}_{0,n}) \cong A^*(\overline{M}_{0,n}) = \frac{\mathbb{Z}[\{D^S | S \subsetneq \{1, ..., n\},\ \#S \geq 2,\ \#S^c \geq 2\}]}{\{\textit{the following relations}\}}$$

*The relations in the Chow ring are:*

1. *For all $S \subsetneq \{1, ..., n\}$ such that $\#S \geq 2$ and $\#S^c \geq 2$: $D^S = D^{S^c}$*

2. *For every for $i, j, k, l \in \{1, ..., n\}$:*

$$\sum_{\substack{S \subsetneq \{1,...,n\}, \\ i,j \in S, \\ k,l \notin S}} D^S = \sum_{\substack{S \subsetneq \{1,...,n\}, \\ i,k \in S, \\ j,l \notin S}} D^S = \sum_{\substack{S \subsetneq \{1,...,n\}, \\ i,l \in S, \\ j,k \notin S}} D^S \tag{1}$$

*3. For all $S, T \subsetneq \{1, ..., n\}$ such that $\#S$, $\#T$, $\#S^c$, $\#T^c \geq 2$: $D^S D^T = 0$ if not one of the following conditions holds:*

$$S \subseteq T, \quad T \subseteq S, \quad S \subseteq T^c, \quad S^c \subseteq T$$

**Definition 18** We use the following short notation for the boundary components of $\overline{M}_{0,n}$: If $\{a_1, ..., a_l\}$, $2 \leq l \leq n-2$, is a subset of $\{1, ..., n\}$ we will denote the corresponding boundary divisor $D^{\{a_1,...,a_l\}}$ by $[a_1, ..., a_l]$.

## 2.4 Conclusions

**Corollary 19** *For all $g \geq 2$ and every $\overline{X} \in \{\overline{HM}_g, (\overline{HS}_g^+)^\sim, (\overline{HS}_g^-)^\sim, (\overline{HR}_g)^\sim\}$ we have:*

*(i) Every connected component of $\overline{X}$ is unirational.*

*(ii) $A^*_{\mathbb{Q}}(\overline{X}) \cong H^*_{\mathbb{Q}}(\overline{X})$, as graded $\mathbb{Q}$-algebras, after adjusting the grading of $A^*_{\mathbb{Q}}(\overline{X})$ by a factor 2. In particular $H^n_{\mathbb{Q}}(\overline{X}) = 0$ for all odd $n$.*

*(iii) $Pic_{\mathbb{Q}}(\overline{X}) \cong A^1_{\mathbb{Q}}(\overline{X})$*

*(iv) $A^1_{\mathbb{Q}}(\overline{X})$ is generated by the boundary divisors of $\overline{X}$. (Meaning the preimages of the boundary components of the moduli space on its normalization.)*

*(v) $h^{p,0}(\overline{X}) = 0$ for $p > 0$.*

**Proof:** For all claims it suffices to show them for every connected component of $\overline{X}$. Let $\overline{Y}$ be such a component, $Y$ its Interior. Then, by Lemma 16 and the Remark 12, $\overline{Y} \cong \overline{M}_{0,2g+2}/G$ for some subgroup G of $S_{2g+2} \times S_2$.

(i): $\overline{Y} \cong \overline{M}_{0,2g+2}/G$ is of course covered by $\overline{M}_{0,2g+2}$, and all spaces $\overline{M}_{0,n}$ are rational (Summary 17 (i)).

(ii): By Summary 17 (iii), $A^*_{\mathbb{Q}}(\overline{M}_{0,2g+2}) \cong H^*_{\mathbb{Q}}(\overline{M}_{0,2g+2})$. Using Lemma 6 we get:

$$A^*_{\mathbb{Q}}(\overline{Y}) \cong A^*_{\mathbb{Q}}(\overline{M}_{0,2g+2}/G) \cong (A^*_{\mathbb{Q}}(\overline{M}_{0,2g+2}))^G$$

$$\cong (H^*_{\mathbb{Q}}(\overline{M}_{0,2g+2}))^G \cong H^*_{\mathbb{Q}}(\overline{M}_{0,2g+2}/G) \cong H^*_{\mathbb{Q}}(\overline{Y})$$

(iii): $\overline{Y}$ is normal, so the Picard group is in a natural way a subgroup of the divisor class group, c.f. [Har77] Remark 6.11.2. and Prop. 6.15. Thus there is an injection

$$Pic_{\mathbb{Q}}(\overline{Y}) \longrightarrow A^1_{\mathbb{Q}}(\overline{Y})$$

Since $\overline{Y} \cong \overline{M}_{0,2g+2}/G$ has only finite quotient singularities, it is $\mathbb{Q}$-factorial, i.e. every Weil-divisor is $\mathbb{Q}$-Cartier. Thus the map is also surjective.

(iv): By Summary 17, $A^1(\overline{M}_{0,2g+2}) = A_{(2g-1)-1}(\overline{M}_{0,2g+2})$ is generated by the boundary classes, i.e. the map $A_{(2g-1)-1}(\overline{M}_{0,2g+2} \setminus M_{0,2g+2}) \longrightarrow A_{(2g-1)-1}(\overline{M}_{0,2g+2})$ is surjective. The exact sequence

$$A_{(2g-1)-1}(\overline{M}_{0,2g+2} \setminus M_{0,2g+2}) \longrightarrow A_{(2g-1)-1}(\overline{M}_{0,2g+2}) \longrightarrow A_{(2g-1)-1}(M_{0,2g+2}) \longrightarrow 0$$

then yields $A_{\mathbb{Q},(2g-1)-1}(M_{0,2g+2}) = A_{(2g-1)-1}(M_{0,2g+2}) = 0$. By Lemma 6, then

$$A_{\mathbb{Q},(2g-1)-1}(Y) \cong A_{\mathbb{Q},(2g-1)-1}(M_{0,2g+2}/G) \cong (A_{\mathbb{Q},(2g-1)-1,}(M_{0,2g+2}))^G = 0$$

Again using an exact sequence like the one above we conclude that $A_{\mathbb{Q},(2g-1)-1}(\overline{Y} \setminus Y) \longrightarrow A_{\mathbb{Q},(2g-1)-1}(\overline{Y})$ is surjective, i.e. that $A_{\mathbb{Q},(2g-1)-1}(\overline{Y}) \cong A^1_{\mathbb{Q}}(\overline{Y})$ is generated by the boundary classes.

(v): According to [Kee92], every $\overline{M}_{0,2g+2}$, is rational. Thus $H^{p,0}(\overline{M}_{0,2g+2}) \cong H^{p,0}(\mathbb{P}^{n-3}) = 0$ for all $p > 0$, since all $h^{p,0}$ are birational invariants (c.f. [GH94] p. 494). This implies $H^{p,0}(\overline{Y}) = (H^{p,0}(\overline{M}_{0,2g+2}))^G = 0$. □

# 3 Morphisms to $\overline{S}_2$ and $\overline{R}_2$.

In this section we introduce several finite morphisms from other moduli spaces to $\overline{R}_2$, $\overline{S}_2^+$ and $\overline{S}_2^-$. They will later be used to determine relations between cycle classes on our moduli spaces, by pushing forward known relations, or by using push-pull formula.

## 3.1 The morphisms $a_R$, $a_+$ and $a_-$ in the case of genus 2

In the case of genus 2, all smooth curves are hyperelliptic, hence $\overline{HM}_2 = \overline{M}_2$, $\overline{HR}_2 = \overline{R}_2$, $\overline{HS}_2^+ = \overline{S}_2^+$ and $\overline{HS}_2^- = \overline{S}_2^-$. Thus the conclusions listed in Corollary 19, apply to the moduli spaces we are interested in. Lemma 16 in this special case reads

**Lemma 20** ***(& Definition)***
*There are Isomorphisms*

$$b : \overline{M}_{0,[6]} \xrightarrow{\cong} \overline{M}_2 \quad \textit{resp.}$$

$$a_R : \overline{M}_{0,[2,4]} \xrightarrow{\cong} \overline{R}_2 \quad \textit{resp.} \quad a_+ : \overline{M}_{0,[3,3]} \xrightarrow{\cong} \overline{S}_2^+ \quad \textit{resp.} \quad a_- : \overline{M}_{0,[1,5]} \xrightarrow{\cong} \overline{S}_2^-$$

*These isomorphisms map the boundary of $\overline{M}_{0,6}$ onto the boundary of the images.*

*We define surjective finite morphisms, by composing every one of isomorphisms above with the appropriate quotient morphism out of, $\pi_{0,[6]})$, $\pi_{0,[2,4]}$, $\pi_{0,[3,3]}$, and $\pi_{0,[1,5]}$ from Definition 12:*

$$g : \overline{M}_{0,6} \xrightarrow{720:1} \overline{M}_2,$$

$$f_R : \overline{M}_{0,6} \xrightarrow{48:1} \overline{R}_2, \quad f_+ : \overline{M}_{0,6} \xrightarrow{72:1} \overline{S}_2^+ \quad \textit{and} \quad f_- : \overline{M}_{0,6} \xrightarrow{120:1} \overline{S}_2^-.$$

**Proof:** Everything except the degrees of the finite surjective morphisms is just a special case of Lemma 16, since all genus 2 curves are hyperelliptic, and since our moduli spaces are normal. The degrees equal those of the forgetful morphisms $\pi_{0,[6]})$, $\pi_{0,[2,4]}$, $\pi_{0,[3,3]}$, $\pi_{0,[1,5]}$, which can easily be counted. □

By the Lemma we even know:

$$H^*_{\mathbb{Q}}(\overline{R}_2) \cong (H^*_{\mathbb{Q}}(\overline{M}_{0,6}))^{S_2 \times S_4}, \quad H^*_{\mathbb{Q}}(\overline{S}_2^+) \cong (H^*_{\mathbb{Q}}(\overline{M}_{0,6}))^{S_3 \times S_3 \times S_2},$$

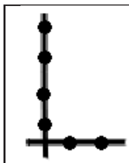
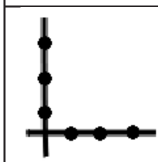
$$H^*_{\mathbb{Q}}(\overline{S}_2^-) \cong (H^*_{\mathbb{Q}}(\overline{M}_{0,6}))^{S_1 \times S_5}$$

where the group actions are those of Remark 12. As the cohomology of $\overline{M}_{0,6}$ is known (c.f. Summary 17), probably a computer algebra program could compute these invariant subrings. In this article we instead compute the rational cohomology of $\overline{R}_2$ and $\overline{S}_2$ by hand. For our computation we need some more information about the isomorphism $a_R$, $a_+$ and $a_-$, and the finite surjective maps $f_R$, $f_+$, and $f_-$ defined from them.

First we examine which boundary components are identified by the isomorphisms $b$, $a_R$, $a_+$ and $a_-$. On $\overline{M}_{0,[6]}$ there are two boundary divisors corresponding to the two types of smooth curves genus 0 curves with 6 unordered marked points and one node, which are:

1. Two smooth genus 0 curves meeting in one node, with 4 marked points on one curve, 2 marked points on the other.
2. Two smooth genus 0 curves meeting in one node, with 3 marked points on each curve.

From the theory of admissible covers it is clear that the first boundary divisor is mapped to $\Delta_0$ and the second to $\Delta_1$ by $b$. If we describe boundary divisors of $\overline{M}_{0,[6]}$ by diagrams of the objects corresponding to general points, we get the table

| Bound. Div. of $\overline{M}_{0,[6]}$ | is mapped to |
|---|---|
|  | $\Delta_0$ |
|  | $\Delta_1$ |

For the isomorphisms $a_R$, $a_+$ and $a_-$ we use that there is the following commutative diagram for $a_R$, and analogous diagrams for $a_+$ and $a_-$.

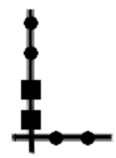
$$\begin{tikzcd}
\overline{M}_{0,[2,4]} \arrow[r, "a_R"] \arrow[d, "\psi"] & \overline{R}_2 \arrow[d, "\pi_R"] \\
\overline{M}_{0,[6]} \arrow[r, "b"] & \overline{M}_2
\end{tikzcd}$$

Here $\psi : \overline{M}_{0,[2,4]} \to \overline{M}_{0,[6]}$ is the forgetful morphism.

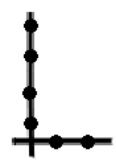
Let $C$ be a boundary component of $\overline{M}_{0,[2,4]}$ and $D$ the boundary component of $\overline{R}_2$ it is mapped to by $a_R$. Take a general point $y$ in the boundary component of $\overline{M}_2$ underlying

$D$, and let $x := b^{-1}(x)$ be its preimage in $\overline{M}_{0,[6]}$. Then the number $M$ of elements in the fiber $\psi^{-1}(x)$ lying in $C$ must be equal to the number $N$ of elements of the fiber $\pi_R^{-1}(y)$ lying in $D$. Knowing the numbers $N$ and $M$ for all boundary components, and the behavior of $b$ suffices to see which components get identified. The number $M$ can be counted if we draw for every boundary component of $\overline{M}_{0,[2,4]}$ the diagram of the objects corresponding to a general point. In those diagrams we will always denote a elements of $A$ by squares and elements of $B$ by dots. (In the case of $\overline{M}_{0,[3,3]}$ we do not distinguish between $A$ an $B$, and only say that squares and dots belong to different sets.)

For example

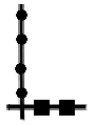
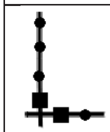


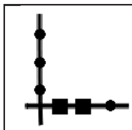
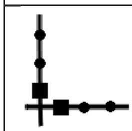
describes an object $(X; \{A, B\})$ where $X$ consists of two smooth genus 0 curves meeting in one node, $A$ is a set of two points on $X$, $B$ a set of 4 points on $X$, $A$ and $B$ disjoint, such that one of the two smooth curves contains two elements of $B$ and all two elements of $A$, while the other curve contains the remaining two elements of $B$. One can count that there are $M = \binom{4}{2} = 6$ objects of this type lying over the corresponding object in $\overline{M}_{0,[6]}$ described by the diagram

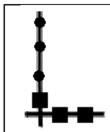
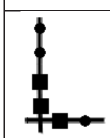


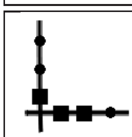
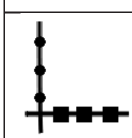
The numbers $N$ can be determined, using the descriptions of the boundary divisors in section 1.5. For example for $D_0'$ the number is $N = 6$. Indeed, if $(X; \mathcal{L}; b)$ is a general object parametrized by $D_0'$, then $\mathcal{L}$ can be obtained by taking a nontrivial Prym sheaf on the normalization of $X$, and gluing the sheaves fibers over the points identified by the normalization map, in one of two possible ways. The Normalization is an elliptic curve, so there are are 3 nontrivial Prym sheaves on it, and gluing them in the two possible ways yields 6 nonisomorphic Prym sheaves on $X$.

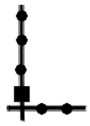
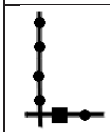
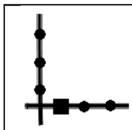
After computing $M$ and $N$ for all boundary components one can conclude that the two boundary components from our examples get identified. The following tables list the identifications for all boundary components, together with the numbers $N$ and $M$

| Bound. Div. of $\overline{M}_{0,[2,4]}$ | is mapped to | $M = N$ |
|---|---|---|
|  | $D_0'$ | 6 |
|  | $D_0''$ | 1 |
|  | $D_0^r$ | 4 |

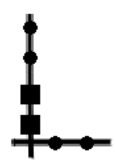
| Bound. Div. of $\overline{M}_{0,[2,4]}$ | is mapped to | $M = N$ |
|---|---|---|
|  | $D_1$ | 6 |
|  | $D_{1:1}$ | 9 |

| Bound. Div. of $\overline{M}_{0,[3,3]}$ | is mapped to | $M = N$ |
|---|---|---|
|  | $A_0^+$ | 4 |
|  | $B_0^+$ | 3 |

| Bound. Div. of $\overline{M}_{0,[3,3]}$ | is mapped to | $M = N$ |
|---|---|---|
|  | $A_1^+$ | 9 |
|  | $B_1^+$ | 1 |

| Bound. Div. of $\overline{M}_{0,[1,5]}$ | is mapped to | $M = N$ |
|---|---|---|
|  | $A_0^-$ | 4 |
|  | $B_0^-$ | 1 |
|  | $A_1^+$ | 6 |

Now we can determine how $f_R$, $f_+$ and $f_-$ behave on the boundary components of $\overline{M}_{0,6}$. Using the notation introduced in Definition 18, all these boundary components are of the form $[i_1, i_2]$ or $[j_1, j_2, j_3]$ ($i_1, i_2, j_1, j_2, j_3 \in \{1, 2, 3, 4, 5, 6\}$). To which component a boundary component of $\overline{M}_{0,6}$ is mapped, can be seen using the tables above. The degree of the map on a given boundary component one gets as in the following example: The boundary component $[3, 4]$ is mapped to $D_0'$. A general point of $[3, 4]$ is thus mapped by $f_R$ to a point of $D_0' \subset \overline{R}_2$ corresponding in $\overline{M}_{0,[2,4]}$ to a diagram of the form



One gets that the degree of $f_R$ on $[3, 4]$ is 4 by counting how many nonisomorphic possibilities there are to assign indices $1, ..., 6$ to the marked points of the diagram, such that the dots get $3, 4, 5, 6$, the squares get $1, 2$ and such that 3 and 4 go to the component with only two marked points. There are 8 possibilities, but swapping 3 and 4 yields isomorphic objects.

Behavior of $f_R : \overline{M}_{0,6} \xrightarrow{48:1} \overline{R}_2$. For arbitrary $b_1, b_2 \in \{3, 4, 5, 6\}$ we have:

- Boundary components of the form $[b_1, b_2]$ are mapped $4 : 1$ (each) onto $D_0'$.
- The boundary component $[1, 2]$ is mapped $24 : 1$ onto $D_0''$.
- Boundary components of the form $[1, b_1]$ or $[2, b_1]$ are mapped $6 : 1$ (each) onto $D_0^r$.
- Boundary components of the form $[1, 2, b_1]$ are mapped $12 : 1$ (each) onto $D_1$.
- Boundary components of the form $[1, b_1, b_2]$ (or equivalently $[2, b_1, b_2]$) are mapped $8 : 1$ (each) onto $D_{1:1}$.

Behavior of $f_+ : \overline{M}_{0,6} \xrightarrow{72:1} \overline{S}_2^+$. For arbitrary $a_1, a_2 \in \{1,2,3\}$ and $b_1, b_2 \in \{4,5,6\}$ we have:

- Boundary components of the form $[a_1, a_2]$ or $[b_1, b_2]$ are mapped $6:1$ (each) onto $A_0^+$.
- Boundary components of the form $[a_1, b_1]$ are mapped $8:1$ (each) onto $B_0^+$.
- Boundary components of the form $[a_1, a_2, b_1]$ (or equivalently $[a_1, b_1, b_2]$) are mapped $8:1$ (each) onto $A_1^+$.
- The boundary component $[1,2,3]$ is mapped $72:1$ onto $B_1^+$.

Behavior of $f_- : \overline{M}_{0,6} \xrightarrow{120:1} \overline{S}_2^-$. For arbitrary $b_1, b_2 \in \{2,3,4,5,6\}$:

- Boundary components of the form $[1, b_1]$ are mapped $24:1$ (each) onto $B_0^-$.
- Boundary components of the form $[b_1, b_2]$ are mapped $6:1$ (each) onto $A_0^-$.
- Boundary components of the form $[1, b_1, b_2]$ are mapped $12:1$ (each) onto $A_1^-$.

We now use this to compute:

**Lemma 21** *There are the following relations between cycle classes on our moduli spaces:*
*(i) In* $A_{2,\mathbb{Q}}(\overline{R}_2)$*:* $d_0' + 6d_0'' - 3d_0^r + 12d_1 - 8d_{1:1} = 0$
*(ii) In* $A_{2,\mathbb{Q}}(\overline{S}_2^+)$*:* $3\alpha_0^+ - 4\beta_0^+ - 8\alpha_1^+ + 72\beta_1^+ = 0$

**Proof:** (i): Using equation (1) from Summary 17 with $i,j,k,l := 1,2,3,4$ we get

$$[1,2] + [1,2,5] + [1,2,6] + [1,2,5,6] = [1,3] + [1,3,5] + [1,3,6] + [1,3,5,6]$$

which is the same as

$$0 = [1,2] + [1,2,5] + [1,2,6] + [3,4] - [1,3] - [1,3,5] - [1,3,6] - [2,4]$$

Pushing this relation forward by $f_R$ we get:

$$0 = 24[D_0''] + 12[D_1] + 12[D_1] + 4[D_0'] - 6[D_0^r] - 8[D_{1:1}] - 8[D_{1:1}] - 6[D_0^r]$$
$$= 4[D_0'] + 24[D_0''] - 12[D_0^r] + 24[D_1] - 16[D_{1:1}]$$

Using the automorphism numbers from the tables in the appendix, this can be written as

$$0 = 8d_0' + 48d_0'' - 24d_0^r + 96d_1 - 64d_{1:1}$$
$$\Leftrightarrow 0 = d_0' + 6d_0'' - 3d_0^r + 12d_1 - 8d_{1:1}$$

(ii): Using equation (1), this time with $i,j,k,l := 1,2,4,5$, we get

$$[1,2] + [1,2,3] + [1,2,6] + [1,2,3,6] = [1,4] + [1,3,4] + [1,4,6] + [1,3,4,6]$$

Pushing this relation forward by $f_+$, and proceeding like in part (i) we get:

$$0 = 24\alpha_0^+ - 32\beta_0^+ - 64\alpha_1^+ + 576\beta_1^+$$

$$\Leftrightarrow 0 = 3\alpha_0^+ - 4\beta_0^+ - 8\alpha_1^+ + 72\beta_1^+$$

□

## 3.2 Morphisms to the boundary components of $\overline{R}_2$ and $\overline{S}_2$

Now we come to several finite surjective morphisms from other moduli spaces to different boundary components of $\overline{R}_2$, $\overline{S}_2^+$ and $\overline{S}_2^-$. Later they will be used to determine relations between intersection products of boundary components via push-pull formula.

### 3.2.1 Morphisms from $\overline{M}_{0,5}$

Fist we define a Morphism $c : \overline{M}_{0,5} \times \overline{M}_{0,3} \to [5,6] \subseteq \overline{M}_{0,6}$. ([5,6] is one of the boundary divisors of $\overline{M}_{0,6}$, c.f. Definition 18.) To the pair of $[(C;(q_0,...,q_4))] \in \overline{M}_{0,5}$ and $[(C';(q_0',...,q_2'))] \in \overline{M}_{0,3}$ the morphism $c$ assigns $[D;(p_1,...,p_6)] \in [5,6] \subset \overline{M}_{0,6}$, where $D$ is the curve obtained from $C$ and $C'$ by gluing the points $q_0$ and $q_0'$, and where $p_1,...,p_4$ are defined as the images of $q_1,...,q_4$ at $D$, and $p_5$ resp. $p_6$ are defined as the images of $q_1'$ resp. $q_2'$. $\overline{M}_{0,3}$ is just a point, so there is an isomorphism $i : \overline{M}_{0,5} \to \overline{M}_{0,5} \times \overline{M}_{0,3}$. The composed map $c \circ i$ is a finite degree 1 morphism onto $[5,6]$. We compose this morphism with $f_R$ and get a finite Morphism:

$$h_0' : \overline{M}_{0,5} \xrightarrow{4:1} D_0'$$

$h_0'$ is $4:1$ because that is the degree of $f_R$ on $[5,6]$ (c.f. section 3.1).
By composing $c \circ i$ with $f_-$ one gets a morphism

$$h_0^\alpha : \overline{M}_{0,5} \xrightarrow{6:1} A_0^-$$

Similar to what was done in section 2.2 for $f_R$, $f_+$ and $f_-$, one can determine the behavior of these two morphisms on the boundary of $\overline{M}_{0,5}$. We describe the images of the boundary components in terms of the classes of closed strata of the stratification by topological type of $\overline{R}_2$ resp. $\overline{S}_2^-$. These strata are described in the appendix. The boundary divisors of $\overline{M}_{0,5}$ are (for our choice of the indices of the marked points) all of the form $[i_1,i_2]$ ($i_1,i_2 \in \{0,1,2,3,4\}$).

Behavior of $h_0' : \overline{M}_{0,5} \xrightarrow{4:1} D_0' \subset \overline{R}_2$. For arbitrary $a \in \{1,2\}$ and $b \in \{3,4\}$:

- The boundary component $[1,2]$ is mapped $2:1$ onto $E^{','' } = D_0' \cap D_0''$.
- Boundary components of the form $[a,b]$ are mapped $1:1$ (each) onto $E^{',r} = D_0' \cap D_0^r$.
- The boundary component $[3,4]$ is mapped $2:1$ onto $E^{','}$.

- Boundary components of the form $[0,a]$ are mapped $2:1$ (each) onto $F'_{1:1} = D'_0 \cap D_{1:1}$.
- Boundary components of the form $[0,b]$ are mapped $2:1$ (each) onto $F'_1 = D'_0 \cap D_1$.

Behavior of $h_0^\alpha : \overline{M}_{0,5} \xrightarrow{6:1} A_0^- \subset \overline{S}_2^-$. For arbitrary $b_1, b_2 \in \{2,3,4\}$:

- Boundary components of the form $[b_1, b_2]$ are mapped $2:1$ (each) onto $C^-$. ($2:1$ because two nonisomorphic diagrams of $\overline{M}_{0,5}$ are assigned two different but isomorphic diagrams of $\overline{M}_{0,[1,5]} \cong \overline{S}_2^-$.)
- Boundary components of the form $[1, b_1]$ are mapped $2:1$ (each) onto $D^- = A_0^- \cap B_0^-$.
- The boundary component $[0,1]$ is mapped $6:1$ onto $X^-$.
- Boundary components of the form $[0, b_1]$ are mapped $2:1$ (each) onto $Y^-$.

we use this to compute:

**Lemma 22** *There are the following relations between classes in the Chow ring of our moduli spaces:*
*(i) In* $A_{1,\mathbb{Q}}(\overline{R}_2)$*:* $2d'_0 d''_0 + 4d'_0 d_1 - 4d'_0 d_{1:1} - d'_0 d^r_0 = 0$
*(ii) In* $A_{1,\mathbb{Q}}(\overline{S}_2^-)$*:* $16[X^-]_Q + [C^-]_Q - 4\alpha_0^- \alpha_1^- - \alpha_0^- \beta_0^- = 0$
*(iii) In* $A_{1,\mathbb{Q}}(\overline{R}_2)$*:* $[E'^{,r}]_Q = 2[E'^{,\prime}]_Q + [E'^{,\prime\prime}]_Q$

**Proof:** (i): Using equation 1 with $i,j,k,l := 0,1,2,3$ we get

$$[0,1] + [2,3] = [0,3] + [1,2]$$

Pushing this relation forward by $h'_0$ we get:

$$0 = 2[D'_0 \cap D_1] + 2[D'_0 \cap D''_0] - 2[D'_0 \cap D_{1:1}] - [D'_0 \cap D^r_0]$$

Using the automorphism numbers from the appendix this can be written as

$$0 = 8d'_0 d_1 + 4d'_0 d''_0 - 8d'_0 d_{1:1} - 2d'_0 d^r_0$$

$$\Leftrightarrow 2d'_0 d''_0 + 4d'_0 d_1 - 4d'_0 d_{1:1} - d'_0 d^r_0 = 0$$

(ii): We again use the equation

$$0 = [0,3] + [1,2] - [0,1] - [2,3]$$

and now push it forward by $h_0^\alpha$. Then proceeding as above, we arive at

$$0 = 12[X^-]_Q + [C^-]_Q - 4[Y^-]_Q - \alpha_0^- \beta_0^-$$

Now we use that $A_0^- \cap A_1^- = X^- \cup Y^-$ is a proper intersection. We can treat all proper intersections of $Q$-classes of closed strata of the stratifications by topological type, as transversal intersections, as those closed strata meet transversally on the deformation space (c.f. Summary 5 (iv)). Thus $\alpha_0^- \alpha_1^- = [X^-]_Q + [Y^-]_Q$. Using this one can rewrite the equation as

$$0 = 16[X^-]_Q + [C^-]_Q - 4\alpha_0^- \alpha_1^- - \alpha_0^- \beta_0^-$$

(iii) Using equation 1 with $i, j, k, l := 1, 2, 3, 4$ we get

$$[1,2] + [3,4] = [1,3] + [2,4]$$

Pushing this relation forward by $h_0'$ and using the automorphism numbers from the appendix we get:

$$4[E^{\prime,\prime\prime}]_Q + 8[E^{\prime,\prime}]_Q = 2[E^{\prime,r}]_Q$$
$$\Leftrightarrow [E^{\prime,\prime\prime}]_Q + 2[E^{\prime,\prime}]_Q = [E^{\prime,r}]_Q$$

□

### 3.2.2 Other morphisms to the boundary components

For $\overline{R}_2$ we will use the following morphisms. We describe how they behave on general points.

$$\tau_1 : \overline{M}_{1,1} \times \overline{R}_{1,1} \xrightarrow{1:1} D_1$$

For $[(X;p)] \in \overline{M}_{1,1}$ and $[(Y;\mathcal{L};b;q)] \in \overline{R}_{1,1}$ the image of the pair is the point in $D_1$ parametrizing the following Prym curve $(X';\mathcal{L}')$: The quasistable curve $X'$ is generated by gluing the points $p$ and $q$ on the curves $X$ and $Y$. The Prym sheaf $\mathcal{L}'$ is obtained from the trivial sheaf on $X$ and the Prym sheaf $\mathcal{L}$ on $Y$, by identifying the fibers over $p$ resp. $q$. All possible choices of identification yield isomorphic Prym sheaves.

$$\tau_{1:1} : \overline{R}_{1,1} \times \overline{R}_{1,1} \xrightarrow{2:1} D_{1:1}$$

This morphism is defined analogously to $\tau_1$ . It is of degree 2 since a pair $([(X;\mathcal{L};b;p)], [(X';\mathcal{L}';b';p')]) \in \overline{R}_{1,1} \times \overline{R}_{1,1}$ and the transposed pair are mapped to the same point in $D_{1:1}$.

$$\tau_0'' : \overline{M}_{1,2} \xrightarrow{1:1} D_0''$$

A point $[(X;p,q)] \in \overline{M}_{1,2}$ is mapped to the point parameterizing the following Prym curve $(X';\mathcal{L})$: The underlying quasistable curve $X'$ is obtained by gluing the points $p$ and $q$. There are two ways to glue the fibers of the trivial bundle of $X$ over the points $p$ and $q$ such that a Prym bundle on $X'$ is obtained. One way yields the trivial bundle on $X'$, the other one yields the nontrivial Prym bundle $\mathcal{L}$.

$$\tau_0^r : \overline{R}_{1,2}^{(-1,-1)} \xrightarrow{1:1} D_0^r$$

A point $[(X;\mathcal{L};p,q)] \in \overline{R}_{1,2}^{(-1,-1)}$ is mapped to the point parametrizing the following Prym curve $(X';\mathcal{L}')$: The underlying quasistable curve $X'$ is obtained by gluing the points $p$ and $q$, and then blowing up the node. $\mathcal{L}'$ is the Prym bundle on $X$, such that if $\tilde{X}$ is stable subcurve of $X$ and $E$ the exceptional component, $\mathcal{L}'_{|\tilde{X}} \cong \mathcal{L}$ and $\mathcal{L}'_{|E} \cong \mathcal{O}_E(1)$.

$$\tau_0' : \overline{M}_{0,(2,2,1)} \xrightarrow{1:1} D_0'$$

The morphism $h_0'$ factors through $\overline{M}_{0,(2,2,1)}$, and we use this to define $\tau_0'$.

For $\overline{S}_2^+$ we will use the following morphisms.

$$\rho_0^\alpha : \overline{S}_{1,2}^{(1,1)} \xrightarrow{1:1} A_0^+$$

A point $[(X;\mathcal{L};b;p,q)] \in \overline{S}_{1,2}^{1,1}$ is mapped to the point parametrizing the following spin curve $(X';\mathcal{L}')$: The underlying quasistable curve $X'$ is obtained by gluing the points $p$ and $q$. There are two ways to glue the fibers of the the bundle $\mathcal{L}$ of $X$ over the points $p$ and $q$ such that a spin bundle on $X'$ is obtained. One way yields an odd bundle, the other one the even bundle $\mathcal{L}'$. (This is implicit in [Cor89], Example 3.2)

$$\rho_0^\beta : \overline{S}_{1,2}^+ \xrightarrow{1:1} B_0^+$$

Defined analogously to $\tau_0^r$.

$$\rho_1^\alpha : \overline{S}_{1,1}^+ \times \overline{S}_{1,1}^+ \xrightarrow{2:1} A_1^+$$

Defined analogously to $\tau_1$, but the node is blown up.

$$\rho_1^\beta : \overline{S}_{1,1}^- \times \overline{S}_{1,1}^- \xrightarrow{2:1} B_1^+$$

Defined analogously to $\rho_1^\alpha$

For $\overline{S}_2^-$ there are the following morphisms.

$$\eta_0^\alpha : \overline{S}_{1,2}^{(1,1)} \xrightarrow{1:1} A_0^-$$

Defined analogously to $\rho_0^\alpha$.

$$\eta_0^\beta : \overline{S}_{1,2}^- \xrightarrow{1:1} B_0^-$$

Defined analogously to $\rho_0^\beta$.

$$\eta_1^\alpha : \overline{S}_{1,1}^+ \times \overline{S}_{1,1}^- \xrightarrow{1:1} A_1^-$$

Defined analogously to $\rho_1^\alpha$.

Now we gather facts about some of the moduli spaces of pointed curves that the domains of the morphisms just defined consist of. Especially this will be facts about the rational Chow groups of these spaces.

1. $\overline{M}_{1,1}$ has only one boundary divisor: $\tilde{\Delta}_0$. It parametrizes curves with one node. The corresponding $Q$-class we call $\tilde{\delta_0} := [\tilde{\Delta}_0]_Q$.

2. $\overline{R}_{1,1}$ has boundary divisors $\tilde{D_0''}$ and $\tilde{D_0^r}$, defined analogously to $D_0''$ and $D_0^r$. The corresponding $Q$-classes we call $\tilde{d_0''}$ and $\tilde{d_0^r}$. $\overline{R}_{1,1}$ is isomorphic to $\mathbb{P}^1$, thus $\tilde{d_0''} = \tilde{d_0^r}$ in the Chow group.

3. $\overline{M}_{1,2}$ has boundary divisors $\hat{\Delta}_0$ and $\hat{\Delta}_1$. A curve parametrized by a general point of $\hat{\Delta}_0$ is irreducible with one node. A general curve parametrized by $\hat{\Delta}_1$ consists of two irreducible components, one smooth elliptic curve and one smooth rational curve with two marked points. The corresponding $Q$-classes we call $\hat{\delta_0}$ and $\hat{\delta_1}$.

4. $\overline{R}_{1,2}$ has boundary divisors $\hat{D_0''}$, $\hat{D_0^r}$ and $\hat{D}_1$. Where $\hat{D_0''}$ and $\hat{D_0^r}$ are defined analogously to $D_0''$ and $D_0^r$. For a Prym curve $(X;\mathcal{L};b;p,q)$ parametrized by a general point of $\hat{D}_1$, $X$ consists of two irreducible components, one smooth elliptic curve and one smooth rational curve with two marked points. The Prym sheaf $\mathcal{L}$ is nontrivial restricted to the elliptic curve and (necessarily) trivial restricted to the rational curve. The $Q$-classes $\hat{d_0''}$ and $\hat{d_0^r}$ are equivalent in the Chow group, because they are the pullbacks of the corresponding classes on $\overline{R}_{1,1}$.

5. $\overline{S}^-_{1,1}$ and $\overline{S}^-_{1,2}$ are just $\overline{M}_{1,1}$ respectively $\overline{M}_{1,2}$ because a odd Prym sheaf on a genus 1 curve is trivial. **In later computations, we will usually replace $\overline{S}^-_{1,1}$ and $\overline{S}^-_{1,2}$ by $\overline{M}_{1,1}$ respectively $\overline{M}_{1,2}$ without further mentioning**.

6. $\overline{S}^+_{1,1}$: The boundary divisors are $\tilde{A}_0^+$ and $\tilde{B}_0^+$. Defined analogously to $A_0^+$ and $B_0^+$. The corresponding $Q$-classes $\tilde{\alpha}_0^+$ and $\tilde{\beta}_0^+$ are equivalent in the Chow group, since $\overline{S}^+_{1,1} \cong \mathbb{P}^1$.

7. $\overline{S}^+_{1,2}$: The boundary divisors are $\hat{A}_0^+$, $\hat{B}_0^+$ and $\hat{A}_1^+$. The $Q$-classes $\hat{\alpha}_0^+$ and $\hat{\beta}_0^+$ are equivalent in the Chow group, since they are the pullbacks of the corresponding classes on $\overline{S}_{1,1}$.

8. $\overline{S}^{(1,1)}_{1,2}$: There are, among others, the boundary divisors $\check{A}_0$ and $\check{B}_0$ whose general points parametrize irreducible curves with one node that is blown up in the case of $\check{B}_0$. The $Q$ classes $\check{\alpha}_0$ and $\check{\beta}_0$ are not equivalent.

The facts just listed are probably all known (for some of them c.f. [BF09a], Page 8, and [BF09b]). One way of proving them, is to use that the moduli spaces of curves with one marked points appearing, are all isomorphic to certain quotients of $\overline{M}_{0,4}$. The moduli spaces of curves with two marked points appearing, are, after forgetting the order of the two marked points, isomorphic to certain quotients of $\overline{M}_{0,5}$. For an example look at Part (ii) of the following Lemma. Forgetting the order of the two marked points on the genus 1 curves does not change the coarse moduli spaces.

**Lemma 23** *(i) Let*

$$\pi_{(2,2,1)} : \overline{M}_{0,5} \to \overline{M}_{0,(2,2,1)}, \quad [(X;(p_1,....,p_4,p_0))] \mapsto [(X;(\{p_1,p_2\},\{p_3,p_4\},\{p_0\}))]$$

*be the quotient morphisms of Definition 12, and let $a \in \{1,2\}$ and $b \in \{3,4\}$ be arbitrary. We define*

$$C'' := \pi_{(2,2,1)}([1,2]), \quad C' := \pi_{(2,2,1)}([3,4]), \quad C^r := \pi_{(2,2,1)}([a,b]),$$

$$C_{1:1} := \pi_{(2,2,1)}([a,0]), \quad C_1 := \pi_{(2,2,1)}([b,0])$$

*These images are independent of the choice of $a$ and $b$, which implies that the moduli space $\overline{M}_{0,(2,2,1)}$ has exactly the five boundary components $C'$, $C''$, $C^r$, $C_1$ and $C_{1:1}$.*

*(ii) There is an isomorphism $\overline{M}_{0,[4,1]} \to \overline{M}_{1,[2]} \cong \overline{M}_{1,2}$. By combining this with the forgetful morphism $\overline{M}_{0,(2,2,1)} \to \overline{M}_{0,[4,1]}$ we define a finite surjective morphism $\theta : \overline{M}_{0,(2,2,1)} \to \overline{M}_{1,2}$*

**Proof:** (i): Easy to check. (For the notation used, c.f. Definition 18.)

(ii): To a point $[(D;A,p)] \in \overline{M}_{0,[4,1]}$, let $f : Y \to D$ be the admissible $2:1$ cover of $(D;A)$, and let $Q$ be the set $f^{-1}(p)$. Then $[(D;A,p)] \mapsto [(Y;Q)]$ defines a morphism $\theta' : \overline{M}_{0,[4,1]} \to \overline{M}_{1,[2]} \cong \overline{M}_{1,2}$. It is easy to check that it is 1:1 on the locus of smooth curves. Since both moduli spaces are normal projective varieties this suffices to prove that $\theta'$ is an isomorphism. $\square$

**Lemma 24** *The following table shows the pushforwards of several classes by the morphisms defined in this section.*

| ***Morphism*** | ***class*** | ***Pushforward*** |
|---|---|---|
| $\tau'_0$ | $1$ | $2d'_0$ |
| $\tau'_0$ | $c'$ | $2[E',']_Q$ |
| $\tau'_0$ | $c''$ | $d'_0 d''_0$ |
| $\tau'_0$ | $c^r$ | $2d'_0 d^r_0$ |
| $\tau'_0$ | $c_1$ | $4d'_0 d_1$ |
| $\tau'_0$ | $c_{1:1}$ | $4d'_0 d_{1:1}$ |
| $\tau''_0$ | $1$ | $2d''_0$ |
| $\tau''_0$ | $\hat{\delta}_0$ | $2d'_0 d''_0$ |
| $\tau''_0$ | $\hat{\delta}_1$ | $2d'_0 d_1$ |
| $\tau_1$ | $\tilde{d}''_0 \otimes 1$ | $d''_0 d_1$ |
| $\tau_1$ | $\tilde{d}''_0 \otimes 1$ | $d''_0 d_1$ |
| $\tau_1$ | $\tilde{d}^r_0 \otimes 1$ | $d^r_0 d_1$ |
| $\tau_1$ | $1 \otimes \tilde{\delta}_0$ | $d'_0 d_1$ |
| $\tau_{1:1}$ | $\tilde{d}''_0 \otimes 1$ | $d'_0 d_{1:1}$ |
| $\tau_{1:1}$ | $1 \otimes \tilde{d}''_0$ | $d'_0 d_{1:1}$ |
| $\tau_{1:1}$ | $\tilde{d}^r_0 \otimes 1$ | $d^r_0 d_{1:1}$ |
| $\tau_{1:1}$ | $1 \otimes \tilde{d}^r_0$ | $d^r_0 d_{1:1}$ |

| ***Morphism*** | ***class*** | ***Pushforward*** |
|---|---|---|
| $\rho_0^\alpha$ | $1$ | $2\alpha_0^+$ |
| $\rho_0^\alpha$ | $\check{\alpha}_0$ | $4[C^+]_Q$ |
| $\rho_0^\alpha$ | $\check{\beta}_0$ | $2\alpha_0^+\beta_0^+$ |
| $\rho_0^\beta$ | $1$ | $2\beta_0^+$ |
| $\rho_0^\beta$ | $\hat{\alpha}_0^+$ | $2\alpha_0^+\beta_0^+$ |
| $\rho_0^\beta$ | $\hat{\beta}_0^+$ | $4[E]_Q$ |
| $\rho_1^\alpha$ | $\tilde{\alpha}_0^+\otimes 1$ | $2\alpha_0^+\alpha_1^+$ |
| $\rho_1^\alpha$ | $1\otimes\tilde{\alpha}_0^+$ | $2\alpha_0^+\alpha_1^+$ |
| $\rho_1^\alpha$ | $\tilde{\beta}_0^+\otimes 1$ | $2\beta_0^+\alpha_1^+$ |
| $\rho_1^\alpha$ | $1\otimes\tilde{\beta}_0^+$ | $2\beta_0^+\alpha_1^+$ |
| $\rho_1^\beta$ | $\tilde{\delta}_0\otimes 1$ | $2\alpha_0^+\beta_1^+$ |
| $\rho_1^\beta$ | $1\otimes\tilde{\delta}_0$ | $2\alpha_0^+\beta_1^+$ |

| ***Morphism*** | ***class*** | ***Pushforward*** |
|---|---|---|
| $\eta_0^\alpha$ | $1$ | $2\alpha_0^-$ |
| $\eta_0^\alpha$ | $\check{\alpha}_0$ | $4[C^-]_Q$ |
| $\eta_0^\alpha$ | $\check{\beta}_0$ | $2\alpha_0^-\beta_0^-$ |
| $\eta_0^\beta$ | $1$ | $2\beta_0^-$ |
| $\eta_0^\beta$ | $\hat{\delta}_0$ | $2\alpha_0^+\beta_0^+$ |
| $\eta_1^\alpha$ | $\tilde{\alpha}_0^+\otimes 1$ | $2[X^-]_Q$ |
| $\eta_1^\alpha$ | $\tilde{\beta}_0^+\otimes 1$ | $2\beta_0^-\alpha_1^-$ |
| $\eta_1^\alpha$ | $1\otimes\tilde{\delta}_0$ | $2[Y^-]_Q$ |

**Proof:** By counting the degree of the given morphism on the given cycle, and comparing the automorphism number of an object parametrized by a general point of the cycle, with the automorphism number of the object parametrized by the image of such a point, under the given morphism. $\square$

## 3.3 Hodge classes

Another type of cycle classes used in our computation, beside classes of closed strata of the stratifications according to topological type, are first Chern classes of the Hodge bundles on moduli spaces, and their pullbacks.

**Definition 25** Let $\tilde{\pi}_R : \overline{R}_{1,1} \longrightarrow \overline{M}_{1,1}$, $\tilde{\pi}^+ : \overline{S}_{1,1}^+ \longrightarrow \overline{M}_{1,1}$, $\hat{\pi}^+ : \overline{S}_{1,2}^+ \longrightarrow \overline{M}_{1,2}$, and $\check{\pi} : \overline{S}_{1,2}^{(1,1)} \longrightarrow \overline{M}_{1,2}$ be the usual forgetful morphisms, and let $\theta : \overline{M}_{0,(2,2,1)} \to \overline{M}_{1,2}$ be the morphism of Lemma 23 (ii). Let $\lambda$, $\tilde{\lambda}$ resp. $\hat{\lambda}$ be the first Chern class of the Hodge bundle on $\overline{M}_2$, $\overline{M}_{1,1}$ resp. $\overline{M}_{1,2}$.

We define classes :

$$l := (\pi_R)^*\lambda, \quad l^+ := (\pi_+)^*\lambda, \quad l^- := (\pi_-)^*\lambda, \quad \tilde{l} := (\tilde{\pi}_R)^*\tilde{\lambda},$$

$$\tilde{l}^+ := (\tilde{\pi}^+)^*\tilde{\lambda}, \quad \hat{l}^+ := (\hat{\pi}^+)^*\hat{\lambda}, \quad \check{l} := (\check{\pi})^*\hat{\lambda}, \quad \bar{l} := \theta^*\hat{\lambda}$$

**Lemma 26** *We can describe the pullbacks of $l$, $l^+$ and $l^-$ by the boundary morphisms in the following way*
*(i)* $(\tau_1)^* l = \tilde{\lambda} \otimes 1 + 1 \otimes \tilde{l}$
*(ii)* $(\tau_{1:1})^* l = \tilde{l} \otimes 1 + 1 \otimes \tilde{l}$
*(iii)* $(\tau_0')^* l = \bar{l}$
*(iv)* $(\tau_0'')^* l = \hat{\lambda}$
*(v)* $(\rho_0^\alpha)^* l^+ = \check{l}$
*(vi)* $(\rho_0^\beta)^* l^+ = \hat{l}^+$
*(vii)* $(\rho_1^\alpha)^* l^+ = \tilde{l}^+ \otimes 1 + 1 \otimes \tilde{l}^+$
*(viii)* $(\rho_1^\beta)^* l^+ = \tilde{\lambda} \otimes 1 + 1 \otimes \tilde{\lambda}$
*(ix)* $(\eta_0^\alpha)^* l^- = \check{l}$
*(x)* $(\eta_0^\beta)^* l^- = \hat{\lambda}$
*(xi)* $(\eta_1^\alpha)^* l^- = \tilde{\lambda} \otimes 1 + 1 \otimes \tilde{l}^+$

**Proof:** First consider the commutative diagram

$$\begin{tikzcd}
\overline{S}_{1,2}^{(1,1)} \arrow[r, "\rho_0^\alpha"] \arrow[d, "\check{\pi}"'] & \overline{S}_2^+ \arrow[d, "\pi_+"] \\
\overline{M}_{1,2} \arrow[r, "f"] & \overline{M}_2
\end{tikzcd}$$

where $f$ is the morphism corresponding to gluing the two marked points on a curve. Because of the way $l^+$ and $\check{l}$ are defined, it suffices to show $\hat{\lambda} = f^*\lambda$ in order to prove (v). That this equation indeed is true, is shown in [Mum83], § 10. The assertions (iii), (iv), (vi), (ix) and (x) can be proved in the same way.

Now we consider the commutative diagram:

$$\begin{tikzcd}
\overline{R}_{1,1} \times \overline{R}_{1,1} \arrow[r, "\tau_{1:1}"] \arrow[d, "\tilde{\pi}_R \times \tilde{\pi}_R"'] & \overline{R}_2 \arrow[d, "\pi_R"] \\
\overline{M}_{1,1} \times \overline{M}_{1,1} \arrow[r, "g"] & \overline{M}_2
\end{tikzcd}$$

Where $g$ is the morphism corresponding to gluing two genus 1 curves, each with one marked point, together at those marked points. In [[Mum83], § 10. $g^*\lambda = \tilde{\lambda} \otimes 1 + 1 \otimes \tilde{\lambda}$ is proven (there the notation is slightly different). From this (i) follows. (ii), (vii), (viii) and (xi) can be proved analogously.

□

If $\lambda$ is the fist Chern class of the Hodge bundle on a $\overline{M}_{1,n}$, $n \geq 1$ arbitrary, then for $\delta_0$ the $Q$ class of the divisor of $\overline{M}_{1,n}$ parametrizing irreducible curves with one node, $\lambda = \frac{1}{12}\delta_0$ (c.f. [BF09a] Page 8). By pulling these relations back one obtains the following equation:

**Lemma 27**
*(i)* $\tilde{\lambda} = \frac{1}{12}\tilde{\delta}_0$
*(ii)* $\hat{\lambda} = \frac{1}{12}\hat{\delta}_0$
*(iii)* $\bar{l} = \frac{1}{12}(2c' + 2c'' + 2c^r)$
*(v)* $\tilde{l} = \frac{1}{12}\tilde{\theta}^*\tilde{\delta}_0 = \frac{1}{12}(\tilde{d}_0'' + 2\tilde{d}_0^r) = \frac{1}{4}\tilde{d}_0^r$
*(vi)* $\check{l} = \frac{1}{12}(\check{\theta})^*\hat{\delta}_0 = \frac{1}{12}(\check{\alpha}_0 + 2\check{\beta}_0)$
*(v)* $\tilde{l}^+ = \frac{1}{12}\tilde{\theta}^*\tilde{\delta}_0 = \frac{1}{12}(\tilde{\alpha}_0^+ + 2\tilde{\alpha}_0^+) = \frac{1}{4}\tilde{\alpha}_0^+$
*(v)* $\hat{l}^+ = \frac{1}{12}(\hat{\alpha}_0^+ + 2\hat{\alpha}_0^+) = \frac{1}{4}\hat{\alpha}_0^+$

**Lemma 28** *All the following products are equal to* 0 *in the rational Chow rings they are contained in.*

$$l^2 d_0', \quad l^2 d_0'', \quad l^2 d_0^r, \quad (l^+)^2\alpha_0^+, \quad (l^+)^2\beta_0^+, \quad (l^-)^2\alpha_0^-, \quad (l^-)^2\beta_0^-$$

**Proof:** Take for example $(l^+)^2\alpha_0^+$. Using the boundary morphism $\rho_0^\alpha : \overline{S}_{1,2}^{(1,1)} \xrightarrow{1:1} A_0^+$ and the fact that $\alpha_0^+ = \frac{1}{2}(\rho_0^\alpha)_*(1)$ we can write $(l^+)^2\alpha_0^+$ by the projection formula as $\frac{1}{2}(\rho_0^\alpha)_*(\rho_0^\alpha)^*(l^+)^2$. According to Lemma 26 $(\rho_0^\alpha)^*(l^+) = \check{l}$, thus $(l^+)^2\alpha_0^+ = \frac{1}{2}(\rho_0^\alpha)_*(\check{l})^2$. By definition $\check{l} = (\check{\pi})^*\hat{\lambda}$. But $\hat{\lambda}$ is, as shown in [Mum83] § 10., equal to the pullback of $\tilde{\lambda}$ from $\overline{M}_{1,1}$ to $\overline{M}_{1,2}$. $\overline{M}_{1,1}$ is one dimensional, thus $(\tilde{\lambda})^2 = 0$. This implies $(\check{l})^2 = 0$, which pushed forward by $\rho_0^\alpha$ yields $(l^+)^2\alpha_0^+ = 0$. That the other products listed in the Lemma are equal to 0 can be proved analogously. □

# 4 Computation of the rational cohomology

## 4.1 The rational Picard group

**Lemma 29** *The Chow groups $A_{2,\mathbb{Q}}(\overline{R}_2)$, $A_{2,\mathbb{Q}}(\overline{S}_2^+)$ and $A_{2,\mathbb{Q}}(\overline{S}_2^-)$, are isomorphic to the rational Picard groups $Pic_{\mathbb{Q}}(\overline{R}_2)$, $Pic_{\mathbb{Q}}(\overline{S}_2^+)$ respectively $Pic_{\mathbb{Q}}(\overline{S}_2^-)$, an they are generated by the boundary divisors of the moduli spaces. Furthermore the linear relations of Lemma 21 are the only ones. Thus:*
*(i)* $A_{2,\mathbb{Q}}(\overline{R}_2) = (d_0'\mathbb{Q} \oplus d_0''\mathbb{Q} \oplus d_0^r\mathbb{Q} \oplus d_1\mathbb{Q} \oplus d_{1:1}\mathbb{Q})/(d_0' + 6d_0'' - 3d_0^r + 12d_1 - 8d_{1:1})\mathbb{Q}$
*(ii)* $A_{2,\mathbb{Q}}(\overline{S}_2^+) = (\alpha_0^+\mathbb{Q} \oplus \beta_0^+\mathbb{Q} \oplus \alpha_1^+\mathbb{Q} \oplus \beta_1^+\mathbb{Q})/(3\alpha_0^+ - 4\beta_0^+ - 8\alpha_1^+ + 72\beta_1^+)\mathbb{Q}$
*(iii)* $A_{2,\mathbb{Q}}(\overline{S}_2^-) = \alpha_0^-\mathbb{Q} \oplus \beta_0^-\mathbb{Q} \oplus \alpha_1^-\mathbb{Q}$

**Proof:** That the second Chow groups are generated by boundary divisors and are isomorphic to the rational Picard groups is a special case of Corollary 19 (iv) resp. (iii).

It remains to show that there are no linear relations between the boundary classes other than those of lemma 21.

To do this we compute the intersection numbers of all boundary classes with the $Q$-classes of all the 2-dimensional closed strata of the stratifications of our moduli spaces according to topological type. These are the cycles lying above the cycles $\Delta_{00}$ and $\Delta_{01}$ of $\overline{M}_2$ with respect to the forgetful morphisms. They are described in the appendix. For a codimension 1 cycle $d$ and a codimension 2 cycle $e$ we take the intersection number to be the number $n$ such that $de = n[x]$ where $x$ is a general point of the moduli space. Note that in the definition we use the class $[x]$, not $[x]_Q$, to be consistent with [Mum83]. For $\overline{R}_2$ we get the intersection numbers:

| Underlying stratum of $\overline{M}_2$ | stratum class | $d_0'$ | $d_0''$ | $d_0^r$ | $d_1$ | $d_{1:1}$ |
|---|---|---|---|---|---|---|
| $\Delta_{00}$ | $[E^{','}]_Q$ | $-\frac{1}{2}$ | $\frac{1}{4}$ | $0$ | $0$ | $\frac{1}{8}$ |
| $\Delta_{00}$ | $[E^{','' }]_Q$ | $0$ | $-\frac{1}{2}$ | $0$ | $\frac{1}{4}$ | $0$ |
| $\Delta_{00}$ | $[E^{',r}]_Q$ | $-1$ | $0$ | $0$ | $\frac{1}{4}$ | $\frac{1}{4}$ |
| $\Delta_{00}$ | $[E^{r,r}]_Q$ | $\frac{1}{4}$ | $0$ | $-\frac{1}{4}$ | $0$ | $\frac{1}{8}$ |
| $\Delta_{01}$ | $[F_1']_Q$ | $0$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $-\frac{3}{48}$ | $0$ |
| $\Delta_{01}$ | $[F_1'']_Q$ | $\frac{1}{4}$ | $0$ | $0$ | $-\frac{1}{48}$ | $0$ |
| $\Delta_{01}$ | $[F_1^r]_Q$ | $\frac{1}{4}$ | $0$ | $0$ | $-\frac{1}{48}$ | $0$ |
| $\Delta_{01}$ | $[F_{1:1}']_Q$ | $\frac{1}{4}$ | $0$ | $\frac{1}{4}$ | $0$ | $-\frac{3}{48}$ |
| $\Delta_{01}$ | $[F_{1:1}^r]_Q$ | $\frac{1}{4}$ | $0$ | $\frac{1}{4}$ | $0$ | $-\frac{3}{48}$ |

If we have a linear relation $\alpha_1 d_0' + \alpha_2 d_0'' + \alpha_3 d_0^r + \alpha_4 d_1 + \alpha_5 d_{1:1} = 0$ between the boundary components, the vector $\alpha = (\alpha_1, ..., \alpha_5)$ has to lie in the kernel of the $9 \times 5$ matrix formed by the intersection numbers in the table above. One can check, that this matrix has rank 4 and thus has 1-dimensional kernel, and that the relation $d_0' + 6d_0'' - 3d_0^r + 12d_1 - 8d_{1:1}$ indeed lies in its kernel.

For $\overline{S}^+$ the intersection numbers are:

| Underlying stratum of $\overline{M}_2$ | stratum class | $\alpha_0^+$ | $\beta_0^+$ | $\alpha_1^+$ | $\beta_1^+$ |
|---|---|---|---|---|---|
| $\Delta_{00}$ | $[C^+]_Q$ | $-1$ | $\frac{1}{4}$ | $\frac{1}{16}$ | $\frac{1}{16}$ |
| $\Delta_{00}$ | $[D^+]_Q$ | $0$ | $-\frac{1}{4}$ | $\frac{1}{8}$ | $0$ |
| $\Delta_{00}$ | $[E]_Q$ | $0$ | $-\frac{1}{8}$ | $\frac{1}{16}$ | $0$ |
| $\Delta_{01}$ | $[X^+]_Q$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $-\frac{3}{192}$ | $0$ |
| $\Delta_{01}$ | $[Y^+]_Q$ | $\frac{1}{8}$ | $0$ | $0$ | $-\frac{1}{192}$ |
| $\Delta_{01}$ | $[Z^+]_Q$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $-\frac{3}{192}$ | $0$ |

One can check that the $6 \times 4$ matrix formed by the intersection numbers, has rank 3, and that $3\alpha_0^+ - 4\beta_0^+ - 8\alpha_1^+ + 72\beta_1^+$ lies inside the kernel.

For $\overline{S}^-$ the intersection numbers are:

| Underlying stratum of $\overline{M}_2$ | stratum class | $\alpha_0^-$ | $\beta_0^-$ | $\alpha_1^-$ |
|---|---|---|---|---|
| $\Delta_{00}$ | $[C^-]_Q$ | $-1$ | $\frac{1}{4}$ | $\frac{1}{8}$ |
| $\Delta_{00}$ | $[D^-]_Q$ | $0$ | $-\frac{1}{4}$ | $\frac{1}{8}$ |
| $\Delta_{01}$ | $[X^-]_Q$ | $\frac{1}{8}$ | $0$ | $-\frac{1}{192}$ |
| $\Delta_{01}$ | $[Y^-]_Q$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $-\frac{3}{192}$ |
| $\Delta_{01}$ | $[Z^-]_Q$ | $\frac{1}{8}$ | $0$ | $-\frac{1}{192}$ |

The $5 \times 3$ matrix formed by the intersection numbers has rank 3.

As examples we will compute some intersection numbers from the tables above. The other numbers can be computed analogously. From [Mum83], Theorem 10.1, we know that $\delta_0[\Delta_{00}]_Q = -\frac{1}{4}p$, $\delta_1[\Delta_{00}]_Q = \frac{1}{8}p$, $\delta_1[\Delta_{01}]_Q = -\frac{1}{48}p$ and $\delta_0[\Delta_{01}]_Q = \frac{1}{4}p$, where $p$ is the class $[y]$ of a general point of $\overline{M}_2$.

For $\overline{X} \in \{\overline{R}_2, \overline{S}_2^+, \overline{S}_2^-\}$ let $S$ be one of the codimension 2 cycles on $\overline{X}$ listed in the tables above. If $\pi : \overline{X} \to \overline{M}_2$ is the forgetful morphism, then $\pi_* S = mD$ for some $m \in \mathbb{Q}$, and for $D$ the $Q$-class of the reduced image of $S$ under $\pi$, thus $D = [\Delta_{00}]_Q$ or $D = [\Delta_{01}]_Q$. The number $m$ is listed for all cycles $S$ in the appendix. Thus one can compute the intersection number $n$ of $S$ with the pullback of $\delta_i$ $(i = 0, 1)$ by using the forgetful map $\pi$ and the projection formula:

$$\pi^*\delta_i S = n[x] \quad \Leftrightarrow \quad \delta_i \pi_* S = n[y] = np$$

$$\Leftrightarrow \quad mD\delta_i = np$$

Where $D\delta_i$ is one of the four known intersections on $\overline{M}_2$ mentioned above.

For the example $E^{\prime,\prime}$ we have $(\pi_R)_*[E^{\prime,\prime}]_Q = [\Delta_{00}]_Q$, thus $\pi^*\delta_0[E^{\prime,\prime}]_Q = -\frac{1}{4}[x]$ and $\pi^*\delta_1[E^{\prime,\prime}]_Q = \frac{1}{8}[x]$.

We also have $D_0^r \cap E^{\prime,\prime} = D_1 \cap E^{\prime,\prime} = \emptyset$ (as one can show using the description of these strata in the appendix), so the corresponding intersection numbers are 0. Using $(\pi_R)^*\delta_0 = d_0' + d_0'' + 2d_0^r$ and $(\pi_R)^*\delta_1 = d_1 + d_{1:1}$, we get $d_1[E^{\prime,\prime}]_Q = \frac{1}{8}[x]$ and

$$(d_0' + d_0'')[E^{\prime,\prime}]_Q = -\frac{1}{4}[x] \tag{2}$$

The intersection $D_0'' \cap E^{\prime,\prime} = G'$ (use description in the appendix) is proper, so by Summary 5 (iv) we can treat the intersection as transversal and we get $d_0''[E^{\prime,\prime}]_Q = [G']_Q$. $G'$ consist of one point, and the corresponding Prym curve has 4 automorphisms (c.f. appendix), thus $d_0''[E^{\prime,\prime}]_Q = \frac{1}{4}[x]$. By plugging this into equation (2) we obtain the last intersection number $d_0'[E^{\prime,\prime}]_Q = -\frac{1}{2}[x]$.

All rows in the above tables can be computed in this way, except for the ones containing the intersection numbers of $E^{\prime,\prime\prime}$, $E^{\prime,r}$ and $D^-$. In computing the first two one has to use additionally the relation $[E^{\prime,r}]_Q = 2[E^{\prime,\prime}]_Q + [E^{\prime,\prime\prime}]_Q$. For the intersections with $[D^-]_Q$ one uses the relation $12[X^-]_Q + [C^-]_Q - 4[Y^-]_Q = [D^-]_Q$. Both relations are proven in Lemma 22. $\square$

**Remark:** In [BF09a], Page 5-6, it is claimed that the boundary components of $S_2^+$ (and $S_2^-$) are independent, which results in wrong Betti (and Hodge) numbers computed for $S_2^+$. It is claimed that Cornalba's proof of independence of the boundary classes for genus $g \geq 3$ in [Cor89], can also be applied to $g = 2$. Cornalba's proof works similar to the proof of the lemma above by computing intersections of the boundary classes with various test curves. The proof does not extend to genus 2, because some of the families used do not yield test curves in the genus 2 case but only points. (For example one family is constructed by attaching a fixed elliptic curve to a moving point on a fixed $g - 1$ curve. For genus $g = 2$ all the curves in the family are isomorphic.).

## 4.2 Hodge numbers

**Theorem 30** *For every $\overline{X} \in \{\overline{R}_2, \overline{S}_2^+, \overline{S}_2^-\}$, the rational cohomology of $\overline{R}_2$ is algebraic, i.e. all odd cohomology groups vanish, and for all $n \in \mathbb{N}$ we have $H^{2n}_{\mathbb{Q}}(\overline{X}) = A^n_{\mathbb{Q}}(\overline{X})$. Furthermore:*

*(i) The boundary classes generate the $\mathbb{Q}$-vectorspace $H^2_{\mathbb{Q}}(\overline{X})$.*

*(ii) There is an ample divisor $L$ which is a linear combination of the boundary classes of $\overline{X}$, such that $LH^2_{\mathbb{Q}}(\overline{X}) = H^4_{\mathbb{Q}}(\overline{X})$. Thus the products of $L$ with the boundary classes generate the $\mathbb{Q}$-vectorspace $H^4_{\mathbb{Q}}(\overline{X})$.*

*Hence the boundary classes generate the $\mathbb{Q}$-algebras $H^*_{\mathbb{Q}}(\overline{X})$ and $A^*_{\mathbb{Q}}(\overline{X})$.*

**Proof:** All except part (ii) follows as a special case from Corollary 19 (ii) and (iv).

Proof of (ii): $\overline{S}_2$ being projective, there is an ample divisor on this space. Like every divisor, according to lemma 29, it is equivalent to a linear combination $L$ of boundary classes. Of course $L$ is also ample. According to the Hard Lefshetz Theorem, multiplication with $L$ induces an isomorphism from $H^2_{\mathbb{Q}}(\overline{X})$ to $H^4_{\mathbb{Q}}(\overline{X})$. The Hard Lefshetz Theorem holds for our moduli spaces according to Summary 4 (i) □

**Theorem 31** *$\overline{R}_2$, $\overline{S}_2^+$ and $\overline{S}_2^-$ all have Hodge diamonds of the following form*

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & n & & 0 & \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & n & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

*with $n = 4$ for $\overline{R}_2$ and $n = 3$ for $\overline{S}_2^+$ as well as $\overline{S}_2^-$.*

**Proof:** For every $\overline{X} \in \{\overline{R}_2, \overline{S}_2^+, \overline{S}_2^-\}$ $h^{2,0}(\overline{X}) = 0$ by Corollary 19 (v) , thus, due to the symmetries of the Hodge diamond, also $h^{0,2}(\overline{X}) = 0$, $h^{1,3}(\overline{X}) = 0$ and $h^{3,1}(\overline{X}) = 0$. Theorem 30 then yields $h^{1,1}(\overline{X}) = h^{2,2}(\overline{X})$, and the value for $n = h^{1,1}(\overline{X})$ is given by Lemma 29. □

## 4.3 The cohomology rings in terms of generators and relations.

By Theorem 30 we know that for our moduli spaces the Chow ring and the rational cohomology ring coincide, and that they are generated by the boundary classes. Now we determine the graded ring structures:

**Theorem 32** *(i) The rational Chow ring $A^*_{\mathbb{Q}}(\overline{R}_2)$ is as a graded $\mathbb{Q}$-Algebra isomorphic to the quotient $\mathbb{Q}[d_0', d_0'', d_0^r, d_1, d_{1:1}]/I$, where $I$ is the homogeneous ideal generated by the following (independent) elements:*

$$d_0' + 6d_0'' - 3d_0^r + 12d_1 - 8d_{1:1},$$

$$d_0''d_{1:1}, \qquad d_0''d_0^r, \qquad d_1d_{1:1},$$

$$d_1(d_0'' - d_0^r), \qquad d_{1:1}(d_0' - d_0^r), \qquad 4(d_{1:1})^2 + d_0^r d_{1:1}, \qquad 2d_0'd_0'' + 4d_0'd_1 - 4d_0'd_{1:1} - d_0'd_0^r,$$

$$d_0'(d_0^r)^2, \qquad (d_0')^2 d_0''$$

*(ii)* $A^*_{\mathbb{Q}}(\overline{S}_2^+) \cong \mathbb{Q}[\alpha_0^+, \beta_0^+, \alpha_1^+, \beta_1^+]/J$*, where* $J$ *is the homogeneous ideal generated by the following (independent) elements:*

$$3\alpha_0^+ - 4\beta_0^+ - 8\alpha_1^+ + 72\beta_1^+,$$

$$\alpha_1^+\beta_1^+, \qquad \beta_0^+\beta_1^+, \qquad \alpha_0^+\alpha_1^+ - \beta_0^+\alpha_1^+,$$

$$(\alpha_0^+)^2\beta_0^+, \qquad (\alpha_0^+)^2(\alpha_1^+ - \beta_1^+)$$

*(iii)* $A^*_{\mathbb{Q}}(\overline{S}_2^-) \cong \mathbb{Q}[\alpha_0^-, \beta_0^-, \alpha_1^-]/K$*, where* $K$ *is the homogeneous ideal generated by the following (independent) elements:*

$$24(\alpha_1^-)^2 + \alpha_0^-\alpha_1^- + 2\beta_0^-\alpha_1^-, \qquad 12(\beta_0^-)^2 + 24\beta_0^-\alpha_1^- + \alpha_0^-\beta_0^-,$$

$$3(\alpha_0^-)^2 - 4\alpha_0^-\beta_0^- - 8\alpha_0^-\alpha_1^- + 80\beta_0^-\alpha_1^-$$

**Proof:** The general idea of the proof and many of its steps are adopted from [BF09a].

The rational Chow rings of our Moduli spaces are generated by the boundary components according to Theorem 30. Thus there is a surjective morphism from the quotient algebras of our Theorem to these Chow rings, if only the elements listed above as generators of the ideals of relations $I$, $J$ and $K$, indeed equal zero in the rational Chow ring.

If this is shown, the following fact implies, that the morphisms are even isomorphisms: The homogeneous components of the algebra $\mathbb{Q}[d_0', d_0'', d_0^r, d_1, d_{1:1}]/I$ have $\mathbb{Q}$-vectorspace dimensions $1, 4, 4, 1, 0, 0, ...$, whereas the homogeneous components of $\mathbb{Q}[\alpha_0^+, \beta_0^+, \alpha_1^+, \beta_1^+]/J$ and $\mathbb{Q}[\alpha_0^-, \beta_0^-, \alpha_1^-]/K$ have dimensions $1, 3, 3, 1, 0, 0, ...$, as one can check using a coputer algebra system like Macaulay 2. These are exactly the vectorspace dimensions of the homogeneous components of the rational Chow rings (according to theorem 31).

To prove most of the relations, we will use the finite morphisms onto boundary components described in section 3.2.2. By these morphisms we will push forward classes and relations. Many of the relations we will push forward are already described in section 3.2.2. Pushforwards of boundary cycles are listed in the tables of Lemma 24. In the computations we will use these facts without mentioning that we take them from section 3.2.2.

First we prove the relations for $\overline{R}_2$.

The linear relation

$$d_0' + 6d_0'' - 3d_0^r + 12d_1 - 8d_{1:1} = 0 \tag{3}$$

holds by Lemma 21.

A Prym curve corresponding to a point in $D_0''$ can not correspond to a point in $D_{1:1}$. The preimage of such a point under $\tau_0'' : \overline{M}_{1,2} \longrightarrow D_0''$, would have to correspond to a reducible curve. Such a curve is of the following form: It consist of a component $D$ of genus 1, and a component $E \cong \mathbb{P}^1$ with two marked points on it. $D$ and $E$ meet in one node. The Prym curve generated by gluing the marked points has a genus 1 component corresponding to $D$. Restricted to this component its Prym sheaf is trivial. The Prym curve can thus not correspond to a point in $D_{1:1}$. So $D_0'' \cap D_{1:1} = 0$, and:

$$d_0'' d_{1:1} = 0 \tag{4}$$

Similarly one can prove

$$d_0'' d_0^r = 0 \tag{5}$$

and

$$d_1 d_{1:1} = 0 \tag{6}$$

Now we use the morphism $\tau_1 : \overline{M}_{1,1} \times \overline{R}_{1,1} \longrightarrow D_1$. In $A^1_{\mathbb{Q}}(\overline{R}_{1,1})$ the relation $\tilde{d}_0'' = \tilde{d}_0^r$ holds . Thus we also have $1 \otimes \tilde{d}_0'' = 1 \otimes \tilde{d}_0^r$ in $A^1_{\mathbb{Q}}(\overline{M}_{1,1} \times \overline{R}_{1,1})$. Pushing this forward by $\tau_1$ one gets:

$$\begin{aligned} (\tau_1)_*(1 \otimes \tilde{d}_0'') &= (\tau_1)_*(1 \otimes \tilde{d}_0^r) \\ \Leftrightarrow \quad d_1 d_0'' &= d_1 d_0^r \end{aligned}$$

$$\Leftrightarrow \quad d_1(d_0'' - d_0^r) = 0 \tag{7}$$

Similarly, but using the $\tau_{1:1} : \overline{R}_{1,1} \times \overline{R}_{1,1} \longrightarrow D_{1:1}$, we get:

$$d_{1:1}(d_0' - d_0^r) = 0 \tag{8}$$

According to [Mum83], page 321, in $A^*_{\mathbb{Q}}(\overline{M}_2)$ the relation $10\lambda = \delta_0 + 2\delta_1$ holds. Pulling this back by $\pi_R$ to $\overline{R}_2$ one gets:

$$l = \frac{1}{10}(d_0' + d_0'' + 2d_0^r + 2d_1 + 2d_{1:1}) \tag{9}$$

Multiplying equation (9) with $d_{1:1}$ and using equations (6), (4) and (8) yields:

$$d_{1:1} l = \frac{1}{10}(3d_{1:1}d_0^r + 2(d_{1:1})^2) \tag{10}$$

On the other hand, because of $d_{1:1} = \frac{1}{2}(\tau_{1:1})_*(1))$ we can write $d_{1:1}l = \frac{1}{2}(\tau_{1:1})_*((\tau_{1:1})^*l)$ by the projection formula. According to the Lemmata 26 and 27

$$(\tau_{1:1})^*l = \tilde{l} \otimes 1 + 1 \otimes \tilde{l} = \frac{1}{4}(\tilde{d_0^r} \otimes 1) + \frac{1}{4}(1 \otimes \tilde{d_0^r})$$

We use $d_{1:1}d_0^r = (\tau_{1:1})_*(\tilde{d_0^r} \otimes 1) = (\tau_{1:1})_*(1 \otimes \tilde{d_0^r})$ and get:

$$\begin{aligned} d_{1:1}l = \frac{1}{2}(\tau_{1:1})_*((\tau_{1:1})^*l) &= \frac{1}{2}(\tau_{1:1})_*(\frac{1}{4}(\tilde{d_0^r} \otimes 1) + \frac{1}{4}(1 \otimes \tilde{d_0^r})) \\ &= \frac{1}{2}\frac{1}{4}(d_{1:1}d_0^r + d_{1:1}d_0^r) = \frac{1}{4}d_{1:1}d_0^r \end{aligned}$$

By subtracting the equation $d_{1:1}l = \frac{1}{4}d_{1:1}d_0^r$ from equation (10), and multiplying by 20, one gets:

$$4(d_{1:1})^2 + d_0^r d_{1:1} = 0 \tag{11}$$

The last codimension 2 relation

$$2d_0'd_0'' + 4d_0'd_1 - 4d_0'd_{1:1} - d_0'd_0^r \tag{12}$$

we have proven earlier (Lemma 22).

To obtain the codimension 3 relations we use that $l^2d_0' = l^2d_0'' = l^2d_0^r = 0$ (cf. lemma 28).

Because of $d_0'' = \frac{1}{2}(\tau_0'')_*1$ we can write $d_0''l = \frac{1}{2}(\tau_0'')_*((\tau_0'')^*l)$. According to Lemma 26 and 27 one has

$$(\tau_0'')^*l = \hat{\lambda} = \frac{1}{12}\hat{\delta}_0$$

By using $d_0'd_0'' = \frac{1}{2}(\tau_0'')_*\hat{\delta}_0$ we get

$$d_0''l = \frac{1}{2}(\tau_0'')_*(\frac{1}{12}\hat{\delta}_0) = \frac{1}{12}d_0'd_0''$$

Thus $0 = l^2d_0'' = \frac{1}{12}ld_0'd_0'' = \frac{1}{144}(d_0')^2d_0''$, and so

$$(d_0')^2d_0'' = 0 \tag{13}$$

Using $d_0' = \frac{1}{2}(\tau_0')_*1$ we can write $d_0'l = \frac{1}{2}(\tau_0')_*((\tau_0')^*l)$. According to Lemma 26 and 27 one has

$$(\tau_0')^*l = \bar{l} = \frac{1}{6}(c' + c'' + c^r)$$

By using the pushforwards of Lemma 24 we get

$$d_0''l = \frac{1}{2}(\tau_0')_*(\frac{1}{6}(c' + c'' + c^r)) = \frac{1}{12}(2[E'^{,\prime}]_Q + d_0'd_0'' + 2d_0'd_0^r)$$

Together with the relation $2[E'^{,\prime}]_Q + d_0'd_0'' = d_0'd_0^r$ of Lemma 22 (iii), this yields

$$d_0'l = \frac{1}{4}d_0'd_0^r$$

Thus $0 = l^2d_0' = \frac{1}{4}ld_0'd_0^r = \frac{1}{16}d_0'(d_0^r)^2$, and so

$$d_0'(d_0^r)^2 = 0 \tag{14}$$

We have proven that the generators of the ideal $I$ are indeed equal to 0 in the rational Chow ring of $\overline{R}_2$.

Now we prove the relations on $\overline{S}_2^+$
The linear relation

$$3\alpha_0^+ - 4\beta_0^+ - 8\alpha_1^+ + 72\beta_1^+ = 0 \tag{15}$$

holds by Lemma 21.
Similar to what was done for $\overline{R}_2$ above, one can show that $A_1^+ \cap B_1^+ = \emptyset$ and $B_0^+ \cap B_1^+ = \emptyset$, so we have the relations

$$\alpha_1^+\beta_1^+ = 0 \tag{16}$$

and

$$\beta_0^+\beta_1^+ = 0 \tag{17}$$

Proceeding similarly as in the proof of equation 7 and using the morphism $\rho_1^\alpha : \overline{S}_{1,1}^+ \times \overline{S}_{1,1}^+ \longrightarrow A_1^+$ we get:

$$\alpha_1^+(\alpha_0^+ - \beta_0^+) = 0 \tag{18}$$

To obtain the codimension 3 relations, similar to the case of $\overline{R}_2$ we use that $\alpha_0^+(l^+)^2 = \beta_0^+(l^+)^2 = 0$ (c.f. Lemma 28).
Because of $\beta_0^+ = \frac{1}{2}(\rho_0^\beta)_*1$ we can write $\beta_0^+l^+ = \frac{1}{2}(\rho_0^\beta)_*((\rho_0^\beta)^*l^+)$. According to Lemma 26 and 27 one has

$$(\rho_0^\beta)^*l^+ = \hat{l}^+ = \frac{1}{4}\hat{\alpha}_0^+$$

By using $\alpha_0^+\beta_0^+ = \frac{1}{2}(\rho_0^\beta)_*\hat{\alpha}_0^+$ we get

$$\beta_0^+l^+ = \frac{1}{2}(\tau_0')_*(\frac{1}{4}\hat{\alpha}_0^+) = \frac{1}{4}\alpha_0^+\beta_0^+$$

Thus $0 = \beta_0^+(l^+)^2 = \frac{1}{4}\alpha_0^+\beta_0^+l^+ = \frac{1}{16}(\alpha_0^+)^2\beta_0^+$, and so

$$(\alpha_0^+)^2\beta_0^+ = 0 \tag{19}$$

We would also like to make use of $\alpha_0^+(l^+)^2 = 0$, by expressing $\alpha_0^+(l^+)^2$ in a nontrivial way as a product of boundary classes, but the morphism $\rho_0^\alpha$ does not help. We instead use equation (15) to write $3\alpha_0^+$ as $4\beta_0^+ + 8\alpha_1^+ - 72\beta_1^+$ and to get $0 = (4\beta_0^+ + 8\alpha_1^+ - 72\beta_1^+)(l^+)^2$. Because of $\beta_0^+(l^+)^2 = 0$ this simplifies to

$$(\alpha_1^+ - 9\beta_1^+)(l^+)^2 = 0 \tag{20}$$

We can write $\alpha_1^+l^+ = \frac{1}{4}(\rho_1^\alpha)_*((\rho_1^\alpha)^*l^+)$, and here the Lemmata 26 and 27 yield

$$(\rho_1^\alpha)^*l^+ = \tilde{l}^+ \otimes 1 + 1 \otimes \tilde{l}^+ = \frac{1}{4}(\tilde{\alpha}_0^+ \otimes 1 + 1 \otimes \tilde{\alpha}_0^+)$$

By using $\alpha_0^+\alpha_1^+ = \frac{1}{2}(\rho_1^\alpha)_*(\tilde{\alpha}_0^+ \otimes 1) = \frac{1}{2}(\rho_1^\alpha)_*(1 \otimes \tilde{\alpha}_0^+)$ we get

$$\alpha_1^+l^+ = \frac{1}{4}(\rho_1^\alpha)_*(\frac{1}{4}(\tilde{\alpha}_0^+ \otimes 1 + 1 \otimes \tilde{\alpha}_0^+)) = \frac{1}{4}\alpha_0^+\alpha_1^+$$

Analogously, from $\beta_1^+l^+ = \frac{1}{4}(\rho_1^\beta)_*((\rho_1^\beta)^*l^+)$ we get to

$$\beta_1^+l^+ = \frac{1}{4}(\rho_1^\beta)_*(\frac{1}{12}(\tilde{\alpha}_0^+ \otimes 1 + 1 \otimes \tilde{\alpha}_0^+)) = \frac{1}{12}\alpha_0^+\beta_1^+$$

By using $\alpha_1^+l^+ = \frac{1}{4}\alpha_0^+\alpha_1^+$ and $\beta_1^+l^+ = \frac{1}{12}\alpha_0^+\alpha_1^+$ one can now rewrite equation (20)

$$0 = (\alpha_1^+ - 9\beta_1^+)(l^+)^2 = \alpha_0^+(\frac{1}{4}\alpha_1^+ - 9\frac{1}{12}\beta_1^+)l^+ = (\alpha_0^+)^2(\frac{1}{16}\alpha_1^+ - 9\frac{1}{144}\beta_1^+)$$

Thus

$$(\alpha_0^+)^2(\alpha_1^+ - \beta_1^+) = 0 \tag{21}$$

(The codimension 3 relations computed in [BF09a], except of $(\alpha_0^+)^2\beta_0^+ = 0$, are incompatible with our results.)

Now we come to the relations on $\overline{S}_2^-$.
The relation $12(\delta_1)^2 + \delta_0\delta_1 = 0$ holds on $\overline{M}_2$ as follows directly from Theorem 10.1. of [Mum83]. Pulling this relation back by $\pi_-$ yields the first relation

$$24(\alpha_1^-)^2 + \alpha_0^-\alpha_1^- + 2\beta_0^-\alpha_1^- = 0 \tag{22}$$

Pulling back the relation $10\lambda = \delta_0 + 2\delta_1$ by $\pi_-$ one gets:

$$l^- = \frac{1}{10}(\alpha_0^- + 2\beta_0^- + 4\alpha_1^-) \tag{23}$$

Multiplication by $\beta_0^-$ yields:

$$l^-\beta_0^- = \frac{1}{10}(\alpha_0^-\beta_0^- + 2(\beta_0^-)^2 + 4\beta_0^-\alpha_1^-) \tag{24}$$

On the other hand, because of $\beta_0^- = \frac{1}{2}(\eta_0^\beta)_*(1))$, we can write $\beta_0^- l^- = \frac{1}{2}(\eta_0^\beta)_*((\eta_0^\beta)^* l)$. According to the Lemmata 26 and 27

$$(\eta_0^\beta)^* l^- = \hat{\lambda} = \frac{1}{12}\hat{\delta_0}$$

We use $\alpha_0^-\beta_0^- = \frac{1}{2}(\eta_0^\beta)_*\hat{\delta_0}$ and get:

$$l^-\beta_0^- = \frac{1}{2}(\eta_0^\beta)_*(\frac{1}{12}\hat{\delta_0}) = \frac{1}{12}\alpha_0^-\beta_0^- \quad (*)$$

By subtracting the equation $\beta_0^- l^- = \frac{1}{12}\alpha_0^-\beta_0^-$ from equation (24), and multiplying by 60, one gets:

$$12(\beta_0^-)^2 + 24\beta_0^-\alpha_1^- + \alpha_0^-\beta_0^- \tag{25}$$

(In [BF09a] it is claimed that $l^-\beta_0^- = \frac{1}{6}\alpha_0^-\beta_0^-$ instead of $(*)$, from this then follows $3(\beta_0^-)^2 + 6\beta_0^-\alpha_1^- - \alpha_0^-\beta_0^-$ instead of equation (25).)

To get the last relation we first compute three relations containing classes that can not immediately be written as products of boundary classes (for the description of the closed strata defining these classes, c.f. the appendix). The fist of these relations we take from Lemma 22:

$$16[X^-]_Q + [C^-]_Q - 4\alpha_0^-\alpha_1^- - \alpha_0^-\beta_0^- = 0 \tag{26}$$

In $A^1_{\mathbb{Q}}(\overline{S}^+_{1,1})$ the relation $\tilde{\alpha}_0^+ = \tilde{\beta}_0^+$ holds, which implies for $A^1_{\mathbb{Q}}(\overline{S}^+_{1,1} \times \overline{M}_{1,1})$ the relation $\tilde{\alpha}_0^+ \otimes 1 = \tilde{\beta}_0^+ \otimes 1$. Pushing this forward by the morphism $\eta_1^\alpha : \overline{S}^+_{1,1} \times \overline{M}_{1,1} \longrightarrow A_0^- \subset \overline{S}_2^-$ yields:

$$[X^-]_Q = \beta_0^-\alpha_1^- \tag{27}$$

(In [BF09a] the authors claim, that one can get the equation $\alpha_0^-\alpha_1^- = \beta_0^-\alpha_1^-$ instead of equation (27). Using the projection formula and the morphism $\eta_1^\alpha$ they obtain the equation $\alpha_0^-\alpha_1^- - (\eta_0^\alpha)_*(1 \otimes \delta_0) = \beta_0^-\alpha_1^-$. Then they claim that $(\eta_1^\alpha)_*(1 \otimes \delta_0) = \frac{1}{2}\alpha_0^-\alpha_1^-$, from which their equation would follow. If I understand them correctly, they assume that $\overline{S}^+_{1,1} \times \Delta_0$ is mapped $1:1$ onto $A_0^- \cap A_1^-$ by $\eta_1^\alpha$. This would be wrong. $\overline{S}^+_{1,1} \times \Delta_0$ is only mapped onto $Y^-$, which is one of the two irreducible components of $A_0^- \cap A_1^-$, the other being $X^-$. There is no a priori reason for $[Y^-]_Q$ and $[X^-]_Q$ to be equivalent, so their equation does not follow. As one can check after computing all relations, the equation does not hold.)

By multiplying equation (23) with $\alpha_0^-$ one gets

$$l^-\alpha_0^- = \frac{1}{10}((\alpha_0^-)^2 + 2\alpha_0^-\beta_0^- + 4\alpha_0^-\alpha_1^-) \tag{28}$$

On the other hand, because of $\alpha_0^- = \frac{1}{2}(\eta_0^\alpha)_*(1))$, we can write $\alpha_0^- l^- = \frac{1}{2}(\eta_0^\alpha)_*((\eta_0^\alpha)^* l)$. According to the Lemmata 26 and 27

$$(\eta_0^\alpha)^* l^- = \check{l} = \frac{1}{12}(\check{\alpha}_0 + 2\check{\beta}_0)$$

We use $[C^-]_Q = \frac{1}{4}(\eta_0^\alpha)_*\check{\alpha}_0$ and $\alpha_0^-\beta_0^- = \frac{1}{2}(\eta_0^\alpha)_*\check{\beta}_0$ to get :

$$l^-\alpha_0^- = \frac{1}{2}(\eta_0^\alpha)_*(\frac{1}{12}(\check{\alpha}_0 + 2\check{\beta}_0)) = \frac{1}{6}([C^-]_Q + \alpha_0^-\beta_0^-)$$

By subtracting the equation $l^-\alpha_0^- = \frac{1}{6}([C^-]_Q + \alpha_0^-\beta_0^-)$ from equation (28), and multiplying by 30, one gets:

$$5[C^-]_Q = 3(\alpha_0^-)^2 + \alpha_0^-\beta_0^- + 12\alpha_0^-\alpha_1^- \tag{29}$$

Plugging equation (27) into equation (26) yields:

$$16\beta_0^-\alpha_1^- + [C^-]_Q - 4\alpha_0^-\alpha_1^- - \alpha_0^-\beta_0^- = 0$$

By multiplying this with 5 and plunging in equation (29) we get

$$3(\alpha_0^-)^2 - 4\alpha_0^-\beta_0^- - 8\alpha_0^-\alpha_1^- + 80\beta_0^-\alpha_1^- \tag{30}$$

This is the last relation we had to check. □

**Remarks:** (i) One can test these relations by pulling the known relations $\delta_0\delta_1 + 12(\delta_0)^2 = 0$ and $528(\delta_1)^3 + (\delta_0)^3 = 0$ (known from [Mum83]) back from $\overline{M}_2$ to our moduli spaces and check whether they are fulfilled in the rings that Theorem 32 claims to be to the rational Chow rings.
(ii) While the cohomology rings of $\overline{S}_2^+$ and $\overline{S}_2^-$ have, according to our computation, the same Betti numbers, they are still nonisomorphic as is clear by the fact that the relations in codimension 1 and 2 determine the cohomology ring of $\overline{S}_2^-$ completely, whereas for $\overline{S}_2^+$, codimension 3 relations are needed.

# 5 Appendix

## 5.1 Stratifications "by topological type"

We now describe the strata of the stratifications of $\overline{M}_2$, $\overline{S}_2^+$, $\overline{S}_2^-$ and $\overline{R}_2$ according to the topological type of the curves. For one of the moduli spaces beside $\overline{M}_2$ we mean by this the irreducible components of the preimages of the strata of $\overline{M}_2$ under the forgetful morphism. In what follows we do rather describe the closures of the strata than the strata themselves. We call these closures the closed strata of the stratifications according to topological type.

The description of the stratifications of $\overline{S}_2^+$ and $\overline{S}_2^-$ can be found in the appendix of [BF09a]. We use the symbols introduced there for the strata, instead to denote the closures of the strata, because some of the associated cycle classes appear in our computations. The stratification of $\overline{M}_2$ is described in [Mum83] §9., we will use the notation introduced there for the closed strata.

We now describe the closed strata of $\overline{R}_2$. We group the strata according to the strata of $\overline{M}_2$ they are lying over. For every stratum we will explain how a Prym curve $(X; L; b)$ parametrized by a general point looks like. We call such a Prym curve a general Prym curve of the stratum.

Strata over $\Delta_0$: For these cycles the underlying stable model $C$ of a generic Prym curve is irreducible with one node. The cycles are the divisors $D_0'$, $D_0''$ and $D_0^r$ described in section 1.5

Strata over $\Delta_1$: For these cycles the underlying stable model $C$ of a generic Prym curve consist of two smooth irreducible components meeting in one node. Again these divisors ($D_1$ and $D_{1:1}$) are described in section 1.5

Strata over $\Delta_{00}$: For these codimension 2 strata the underlying stable model $C$ of a general Prym curve is an irreducible curve with two nodes.

1. $E^{\prime,\prime}$. General Prym curve: $X = C$, normalizing either of the two nodes and pulling back $\mathcal{L}$ to this partial normalization yields a nontrivial Prym sheaf.
2. $E^{\prime,\prime\prime}$. General Prym curve: $X = C$, normalizing one of the two nodes and pulling back $\mathcal{L}$ to this partial normalization yields in one case a nontrivial Prym sheaf, in the other case the trivial sheaf, depending on which node was normalized .
3. $E^{\prime,r}$. General Prym curve: $X$ is obtained from $C$ by blowing up one of the two nodes.
4. $E^{r,r}$. General Prym curve: $X$ is obtained from $C$ by blowing up both nodes.

Strata over $\Delta_{01}$: For these codimension 2 strata the underlying stable model $C$ of a general Prym curve consists of two irreducible components, one of them, called $C_1$, is smooth, the other one, $C_2$, has a node.

1. $F_1'$. General Prym curve: $X = C$, $\mathcal{L}_{|C_1}$ is nontrivial, $\mathcal{L}_{|C_2}$ is trivial.
2. $F_1''$. General Prym curve: $X = C$, $\mathcal{L}_{|C_1}$ is trivial, $\mathcal{L}_{|C_2}$ is nontrivial.
3. $F_1^r$. General Prym curve: $X$ is obtained from $C$ by blowing up the node on $C_2$, $\mathcal{L}_{|C_1}$ is trivial.
4. $F_{1:1}'$. General Prym curve: $X = C$, both $\mathcal{L}_{|C_1}$ and $\mathcal{L}_{|C_2}$ are nontrivial.

5. $F^r_{1:1}$. General Prym curve: $X$ is obtained from $C$ by blowing up the node on $C_2$, $\mathcal{L}_{|C_1}$ is nontrivial.

Strata over $C_{000}$: For these codimension 3 strata the underlying stable model $C$ of a general Prym curve consists of two irreducible smooth rational components meeting in three nodes.

1. $G'$. General Prym curve: $X = C$
2. $G^r$. General Prym curve: $X$ is obtained from $C$ by blowing up one of the nodes.

Strata over $C_{001}$: For these codimension 3 strata the underlying stable model $C$ of a general Prym curve consists of two irreducible components $C_1$ and $C_2$ meeting in one node, each irreducible component having one node.

1. $H'_1$. General Prym curve: $X = C$, restricting $\mathcal{L}$ to one of the components yields a nontrivial Prym sheaf, restricting to the other yields the trivial sheaf.
2. $H^r_1$. General Prym curve: $X$ is obtained from $C$ by blowing up the node on one of the components, $\mathcal{L}$ is trivial restricted to the component not blown up.
3. $H'_{1:1}$. General Prym curve: $X = C$, $\mathcal{L}$ is nontrivial on both components.
4. $H^r_{1:1}$. General Prym curve: $X$ is obtained from $C$ by blowing up the node on one of the components, $\mathcal{L}$ is nontrivial on both components.
5. $H^{r,r}_{1:1}$. General Prym curve: $X$ is obtained from $C$ by blowing up the nodes on both components.

### 5.2 Comparison of automorphisms

As we have shown, there is an isomorphism of coarse moduli spaces $a_R : \overline{M}_{0,[2,4]} \stackrel{\cong}{\to} \overline{R}_2$, and also $\overline{S}_2^+$ and $\overline{S}_2^-$ are isomorphic to moduli spaces of stable genus 0 curves with partitioned marked Points. Now, lets say for $x \in \overline{M}_{0,[2,4]}$, one can ask how the Automorphisms of objects in the class $x$ and objects in the class $a_R(x)$ fit together.

The fact that we know $a_R$ explicitly only on an open subset of $\overline{M}_{0,[2,4]}$ makes it difficult to compare the automorphisms on both sides directly. But one can overcome this difficulty by extending the automorphisms to the local universal deformation spaces of the Prym curve belonging to $a_R(x)$, respectively to the local universal deformation space of the stable genus 0 curve with marked points belonging to $x$. This is helpful because on the loci of smooth curves of these deformation spaces, i.e. almost everywhere, one knows explicitly how Prym structures and marked points correspond to each other.

(We only spoke about $a_R$ in this introduction, but everything also applies to $a_+$ and $a_-$.)

**General notation (part 1):** For the rest of the section let $\overline{M}_{0,\bullet}$ denote one of the moduli spaces $\overline{M}_{0,[1,5]}$, $\overline{M}_{0,[2,4]}$ and $\overline{M}_{0,[3,3]}$. We denote by $\overline{Q}_2$ the one of the moduli spaces $\overline{R}_2$, $\overline{S}_2^+$ and $\overline{S}_2^-$ the space $\overline{M}_{0,\bullet}$ is isomorphic to. We call $a_\bullet : \overline{M}_{0,\bullet} \to \overline{Q}_2$ the corresponding isomorphism (one out of $a_R$, $a_+$ and $a_-$). (C.f. Lemma 20)

If not specified otherwise $(D; \{A, B\})$ is always a genus 0 curve $D$ together with two disjoint sets $A, B$ of marked points, such that $(D; \{A, B\})$ is parametrized by a point $x \in \overline{M}_{0,\bullet}$ . We define $y := a_\bullet(x)$.

$f : Y \to D$ is always the admissible $2:1$ cover of $(D; A \cup B)$.

**Definition 33** If $(D; M)$ is a stable genus 0 curve with a set $M$ of marked points, then we call those irreducible components of $D$ the *extremities of* $(D; M)$ which meet the rest of $D$ only in one point and which carry only two of the marked points.

**Lemma 34** *(i) For $f : Y \to D$ as in the general assumption, $E$ an extremity of $(D; A \cup B)$, the preimage $f^{-1}(E)$ is an exceptional component of the quasistable curve $Y$.*

*(ii) Let $Cont_1 : Y \to C$ be the contraction of all extremal components of $Y$, $C$ the stable model of $Y$. Let $cont_1 : D \to \hat{D}$ be the contraction of all extremities of $(D; A \cup B)$. Then there is a unique finite $2:1$ morphisms $C \to \hat{D}$ fitting into the following commutative diagram.*

$$\begin{tikzcd} Y \arrow[r, "Cont_1"] \arrow[d, "f"'] & C \arrow[d, "\hat{f}"'] \\ D \arrow[r, "cont_1"'] & \hat{D} \end{tikzcd}$$

*(iii) There is a (not necessarily unique) way to blow up nodes of the curve $C$ such that the variety $X$ obtained by this can be equipped with a structure $(\mathcal{L}; b)$ such that $[(X; \mathcal{L}; b)] = y$. We pick such a blowup morphism $Cont_2 : X \to C$.*

**Proof:** (i): Look at the explicit description of an admissible $2:1$ cover in the proof of Cor. 2.5 in [AL02].

(ii): For every exceptional component $E$ contracted to a point by $Cont_1$, $cont_1$ contracts the extremity $f(E)$ to a point. Considering this, it is obvious how to define $\hat{f} : Y \to \hat{D}$ to fit the diagram.

(iii): By the construction of $a_\bullet$ in the proof of Lemma 16, it is clear that

$$\begin{tikzcd} \overline{M}_{0,\bullet} \arrow[r, "a_\bullet"] \arrow[d, "\pi"'] & \overline{Q}_2 \arrow[d, "\pi'"'] \\ \overline{M}_{0,[6]} \arrow[r, "b"'] & \overline{M}_2 \end{tikzcd}$$

commutes, and thus for any $(X; \mathcal{L}; b)$ with $[(X; \mathcal{L}; b)] = y$, $X$ has the same stable model $C$ as $Y$. $\square$

**General notation (part 2):** We will keep the notation of Lemma 34 for the rest of the section. We fix one $Cont_2 : X \to C$ as in part (iii).

We will denote by $\hat{A}$ resp. $\hat{B}$ the set of all points of $\hat{D}$ that come from those marked points in $A$ resp. $B$ that lie on components of $D$ not contracted by $cont_1$. By $H$ we will denote the set of points of $\hat{D}$ to which extremities of $(D; A \cup B)$ are contracted by $cont_1$. We set $G := \hat{A} \cup \hat{B}$. The object $(\hat{D}; (G, H))$ is then stable and parametrized by a point of some $\overline{M}_{0,(i_1,i_2)}$ with $3 \leq i_1 + i_2 \leq 6$. If we want to retain more information about the extremities contracted to points of $H$, we decompose this set into $H_A$, $H_B$, $H_{AB}$, where $H_A$ contains the points to which extremities carrying only marked points of $A$ are contracted, $H_B$ contains those coming from extremities with marked point only from $B$, while to the points of $H_{A,B}$ extremities that carry one point of $A$ and one point of $B$ are contracted. Note that then $(\hat{D}; \{(\hat{A}, H_A), (\hat{B}, H_B)\}, H_{AB})$ contains the full information about the isomorphism class of $(D; \{A, B\})$.

**Lemma 35** *Using the general notation for this section:*
*(i) For every $\varphi \in Aut(Y)$ and every $a, b \in \tilde{Y}$ ($\tilde{Y}$ the non-exceptional subcurve of $Y$):*

$$f(a) = f(b) \Leftrightarrow f(\varphi(a)) = f(\varphi(b))$$

*Thus for every $\varphi \in Aut(C)$ and every $a, b \in C$:*

$$\hat{f}(a) = \hat{f}(b) \Leftrightarrow \hat{f}(\varphi(a)) = \hat{f}(\varphi(b))$$

*(ii) There are natural surjective group homomorphisms*

$$Aut(Y) \xrightarrow{\chi_1} Aut(C) \xrightarrow{\chi_2} Aut((\hat{D}; (G, H))$$

*and*

$$Aut(X) \xrightarrow{\chi_1'} Aut(C)$$

*There is also a natural surjective group homomorphism*

$$\psi_1' : Aut((D; A \cup B)) \to Aut((\hat{D}; (G, H)))$$

*(The homomorphisms are defined explicitly in the proof.)*
*(iii) The kernels $Ker\,\chi_1$ resp. $Ker\,\chi_1'$ consist of those automorphisms that are nontrivial only on the exceptional components of $Y$ resp. $X$. The kernel $Ker\,\psi_1'$ consists of those $\varphi \in Aut((D; \{A, B\})$ that are nontrivial only on the extremities of $(D; A \cup B)$.*
*(iv) Let $\hat{D}_i$ be an irreducible component of $\hat{D}$, $C_i := \hat{f}^{-1}(\hat{D}_i)$, and let $\hat{f}_i : C_i \to \hat{D}_i$ be the restriction of $\hat{f}$. There is an automorphism $h_i$ on $C_i$ interchanging the two sheets of $\hat{f}_i$. We call $h_i$ the hyperelliptic involution on $C_i$ (although $C_i$ may be reducible and not a hyperelliptic curve in the usual sense). One can extend $h_i$ to an Automorphism of $C$, such that it is the identity on all components of $C$ except $C_i$. We again denote this extension by $h_i$.*
*(v) The $h_i \in Aut(C)$ belonging to the different irreducible components of $\hat{D}$ generate the kernel $Ker\,\chi_2$. We call the unique automorphism of $Ker\,\chi_2$ whose restriction to no component of $C$ is trivial the* full *hyperelliptic involution of $C$*

**Proof:** (i): As shown in [AL02], the $2:1$ admissible cover of a stable genus 0 curve $D$ with $2g+2$ marked points for $g \geq 2$ is unique up to isomorphism. There also an explicit method is given to associate to such a $D$ an admissible $2:1$ cover. We use this explicit description in our proof, and one might need to know it in order to understand the arguments.

Let $\tilde{D}$ be the subcurve of $D$ consisting of all components of $D$ that are no extremities of $(D; A \cup B)$, and let $\tilde{D}_1, ..., \tilde{D}_m$ be the irreducible components of $\tilde{D}$. For $i = 1, ...m$, let $D_i$ be the subcurve of $D$ consisting of $\tilde{D}_i$ and the extremities of $D$ attached to $\tilde{D}$, and let $Y_i$ be the part of $Y$ lying over $D_i$. Define $\tilde{Y}_i := \tilde{Y} \cap Y_i$ , and denote by $\tilde{f}_i : \tilde{Y}_i \to D_i$ the restriction of $f$ to $\tilde{Y}_i$.

Let each of $q_{i,1}, ..., q_{i,l_i}$ be a point in which $D_i$ meets one other component $D_j$ of $D$, let $Q_{i,1}, ..., Q_{i,l_i}$ be the sets of points in $Y$ lying over $q_{i,1}, ..., q_{i,l_i}$. Each $Q_{i,j}$ contains one or two points, an is contained in $\tilde{Y}_i$. Let $p_{i,1}, ..., p_{i,k_i}$ be the points on $D$ in which $\tilde{D}_i$ meets extremities, and let $P_{i,1}, ..., P_{i,k_i}$ be the sets of points in $Y$ lying over $p_{i,1}, ..., p_{i,k_i}$. For any $\varphi \in Aut(Y)$, $f \circ \varphi : Y \to D$ is again an admissible $2:1$ cover of $(D; \{A, B\})$. For any $\tilde{Y}_i$, $\varphi_{|\tilde{Y}_i}$ is an isomorphism of $(\tilde{Y}_i; \{Q_{i,1}, ..., Q_{i,l_i}\}, \{P_{i,1}, ..., P_{i,k_i}\})$ to some $(\tilde{Y}_j; \{Q_{j,1}, ..., Q_{j,l_j}\}, \{P_{j,1}, ..., P_{j,k_j}\})$, where $j = i$ is possible.

Now we prove (i) by checking several cases separately.

1. If $\tilde{Y}_i$ is a smooth connected curve of genus $\geq 2$, the assertion of (i) (and (ii)) holds, since then $\tilde{Y}_i$ has to be hyperelliptic and every hyperelliptic curve has a *unique* $g_2^1$ (C.f. [Har77], Chapt. IV, Prop. 5.3.)

2. If $\tilde{Y}_i$ is a smooth connected curve of genus 1, then, for stability reasons, either $l_i > 0$ or $k_i > 0$. But knowing one fiber of the induced $2:1$ cover, determines one $g_2^1$ on an elliptic curve uniquely. (C.f. [Har77], Chapt. IV, § 4.)

3. If $\tilde{Y}_i$ is a smooth connected curve of genus 0, then $\tilde{f}_i : \tilde{Y}_i \to \tilde{D}_i$ is ramified in exactly two points. Thus $\tilde{D}_i$ carries at most 2 points of $A \cup B$. Thus, for reasons of stability and because $\tilde{D}_i$ is not an extremity of $D$, $l_i + k_i \geq 2$ has to hold, and one of the $P$'s and $Q$'s on $\tilde{Y}_i$ has to contain two elements. So we know that (i) holds for two fibers of $\tilde{f}_i : \tilde{Y}_i \to \tilde{D}_i$. Because knowing the behavior of an isomorphism of $\mathbb{P}^1$'s in 3 points determines the isomorphism, one can quite easily conclude from this that (i) holds for all of $\tilde{Y}_i$.

4. Otherwise $\tilde{Y}_i$ consists of two connected components, which both are smooth genus 0 curves. In this case there are no points of $A \cup B$ lying on $\tilde{D}_i$, for otherwise $\tilde{f}_i : \tilde{Y}_i \to \tilde{D}_i$ was ramified there. Thus, for $\tilde{D}_i$ to be stable, we must have $l_i + k_i \geq 3$. So we know three fibers of $\tilde{f}_i$ for which (i) holds. Again it easily follows that (i) holds for all of $\tilde{Y}_i$.

(ii): $\chi_1$ is defined by $\varphi \mapsto \varphi^*$ for every $\varphi \in Aut(Y)$ where $\varphi^* \in Aut(X)$ is the automorphism defined by $\varphi^*(x) := Cont_1(\varphi((Cont_1^{-1}(x))))$ for all $x \in X$. $\chi_2$ is defined analogously. They are surjective because every automorphism of a curve can obviously be extended to any curve obtained from it by blowing up nodes. We define $\chi_3$ by $\varphi \mapsto \varphi^*$ for every $\varphi \in Aut(C)$ where $\varphi^* \in Aut(\hat{D})$ is the automorphism defined by $\varphi^*(x) := \hat{f}(\varphi((\hat{f}^{-1}(x))))$. The definition of $\varphi^*(x)$ indeed gives a point by (i). We have to check $\varphi^* \in Aut((\hat{D}; (G, H))$: $\varphi^*$ maps points in $H$ to points in $H$, because they correspond to the $p$'s introduced in the proof of (i), and we saw there that $\varphi$ maps the points lying over them again to such points. The points of $C$ lying over $G$ are exactly the smooth ramification points of $\hat{f}$, and by (i) $\varphi$ has to map such points to such.

That $\chi_2$ is surjective can be proven by again using the decomposition of $Y$ used in the proof of (i) and again checking it in the four possible cases distinguished there.

The morphism $\psi_1$ obviously exists and is surjective.

(iii): Clear from the definition of $\chi_1$ and $\chi_2$.

(iv): Obviously $h_i$ exists (uniquely). $C$ is of genus 2 and the components $C_i$ are of genus 1 at least, so there can be only two of them, and they can meet only in one point. Thus one can extend each $h_i$ to the other component by the identity.

(v): The Kernel of $\chi_3$ consists of all $\varphi \in Aut(C)$ such that $\hat{f}(\varphi(a)) = \hat{f}(a)$ for all $a \in C$. Quite obviously the $h_i$ generate this group. $\square$

**Lemma 36** *Let $x \in \overline{M}_{0,\bullet}$ be a point parametrizing $(D; \{A, B\})$ and $y := a_\bullet(y)$ its image in $\overline{Q}_2$. Let $(X; \mathcal{L}; b)$ be a object parametrized by $y$. Then:*

*(i) $Aut((D; \{A, B\}))$ is a subgroup of $Aut((D; A \cup B))$ and we call the restriction of the Morphism $\psi_1'$ of Lemma 35*

$$\psi_1 : Aut((D; \{A, B\})) \to Aut((\hat{D}; (G, H)))$$

*$Aut((X; \mathcal{L}; b))$ is a subgroup of $Aut(X)$. We call the restriction of $\chi_2 \circ \chi_1'$ to this subgroup*

$$\psi_2 : Aut((X; \mathcal{L}; b)) \to Aut((\hat{D}; (G, H)))$$

*From now on we use the abbreviations $M := Aut((D; \{A, B\}))$ and $N := Aut((X; \mathcal{L}; b))$.*

*(ii) The group $Aut((\hat{D}; \{(\hat{A}, H_A), (\hat{B}, H_B)\}, H_{AB}))$, (for the definition of this, c.f. the general notation (part 2) for this section), is a subgroup of $Aut((\hat{D}; G, H))$ and:*

$$\psi_2(N) = \psi_1(M) = Aut((\hat{D}; \{(\hat{A}, H_A), (\hat{B}, H_B)\}, H_{AB}))$$

*(iii) Let $r$ be the number of extremities of $(D; A \cup B)$, let $r'$ be the number of those extremities whose two marked points either lie both in $A$ or lie both in $B$. Let $s$ be the number of irreducible components of $D$. Let $I$ be the group of irrelevant automorphisms of $(X; \mathcal{L}; b)$. We define $h := \#\psi_1(M) = \#\psi_2(N)$, $i := \#I$, $m := \#M$ and $n := \#N$, then:*

$$m = 2^{r'} \cdot h, \qquad n = 2^{(s-r)} \cdot i \cdot h$$

*and thus*

$$n = 2^{(s-r-r')} \cdot i \cdot m$$

*One can also write $i$ as $2^{u-1}$ where $u$ is the number of connected components of $\tilde{X}$ the non-exceptional subcurve of $X$.*

**Proof:** The different assertions that one automorphism group is a subgroup of another one, made in parts (i) and (ii), are all quite obvious.

The first thing we prove is the first equation of part (ii).

We are in the situation described by the following commutative diagram.

$$\begin{tikzcd}
 & Y \arrow[d, "f"] \arrow[dr, "Cont_1"] & & X \arrow[dl, "Cont_2"] & \mathcal{L}, b \arrow[l, squiggly] \\
\{A,B\} \arrow[r, squiggly] & D \arrow[dr, "cont_1"'] & C \arrow[d, "\hat{f}"] & & \\
 & (G,H) \arrow[r, squiggly] & \hat{D} & &
\end{tikzcd}$$

Where the curly arrows are meant to symbolize that additional structures are attached to $D$, $\hat{D}$ and $X$

From now on, we will work in the category of complex analytic spaces.

Let $(\mathcal{D} \to S; \{\mathcal{A}, \mathcal{B}\})$ be the local universal deformation of $(D; \{A, B\})$. Part of the deformation is a identification $\psi$ of the "central fiber" lying over a special point $s_0 \in S$ with $(D; \{A, B\})$. $\mathcal{A}$ and $\mathcal{B}$ are then sets of sections on $\mathcal{D}$ meeting the central fibers in the points of $A$ resp. $B$. Possibly after making a base change on $S$ we can extend the $2:1$ admissible cover $f : Y \to D$ to a local universal deformation $\mathbf{f} : \mathcal{Y} \to \mathcal{D}$ over $S$. I.e. $\mathbf{f}$ restricted to the fiber over $s_0$ can be identified with $f$, in a way compatible with $\psi$. To see that this is possible, c.f. [HM82], Page 61-62. There the same thing is done for the local universal deformation of a stable genus 0 curve with *ordered* marked points, but, since the elements of $\mathcal{A}$ and $\mathcal{B}$ are sections (not multi-sections) on our local deformation space, we can just put an arbitrary ordering on them and by this make $(\mathcal{D} \to S; \{\mathcal{A}, \mathcal{B}\})$ into a local universal deformation of a curve with ordered marked points.

If we denote the morphism contracting the exceptional components of $\mathcal{Y}$ by $\mathbf{Cont_1} : \mathcal{X} \to \mathcal{C}$ and the one contracting the extremities of $(\mathcal{D}, \mathcal{A} \cup \mathcal{B})$ by $\mathbf{cont_1} : \mathcal{D} \to \hat{\mathcal{D}}$, there is, analogously to Lemma 34 (ii), a morphism of families $\hat{\mathbf{f}} : \mathcal{C} \to \tilde{\mathcal{D}}$ forming a commutative diagram with $\mathbf{Cont_1}$, $\mathbf{cont_1}$ and $\mathbf{f}$.

Blowing up the appropriate nodes of $C$ (c.f. Lemma 34 (iii))) and the loci in the deformation $\mathcal{C}$ to which these nodes extend, we arrive at an isomorphism $\mathbf{Cont_2} : \mathcal{X} \to \mathcal{C}$, such that $\mathcal{X} \to S$ is a deformation of $X$. After making a base change we can extend $(\mathcal{L}; b)$ to a Prym- resp. spin structure $(\mathbf{L}; \mathbf{b})$ on $\mathcal{X}$. (C.f. [Cor89], Page 570.)

Now we have deformations over $(S, s_0)$, forming the diagram

$$\begin{tikzcd}
 & \mathcal{Y} \arrow[d, "\mathbf{f}"] \arrow[dr, "\mathbf{Cont_1}"] & & \mathcal{X} \arrow[dl, "\mathbf{Cont_2}"] & \mathbf{L}, \mathbf{b} \arrow[l, squiggly] \\
\{\mathcal{A}, \mathcal{B}\} \arrow[r, squiggly] & \mathcal{D} \arrow[dr, "\mathbf{cont_1}"'] & \mathcal{C} \arrow[d, "\hat{\mathbf{f}}"] & & \\
 & & \hat{\mathcal{D}} & &
\end{tikzcd}$$

And by restricting these families to the fibers over $s_0$ we get back to the diagram above.

We now prove the fist equation of part (ii) of our Lemma. We have to show that for $\varphi \in Aut((\hat{D}; (G, H)))$,

$$\varphi \in Im(\psi_1) \Leftrightarrow \varphi \in Im(\psi_2) \qquad (*)$$

In what follows, we use Lemma 35 (ii). First lift an automorphism $\varphi \in Aut((\tilde{D};(G,H)))$ to an element of $Aut(f : Y \to D)$, where we denote by $Aut(f : Y \to D)$ the automorphisms of the admissible $2:1$ cover, i.e. automorphisms $\psi$ of $Y$ for which there exists an automorphism $\psi$ of $(D;A\cup B)$ such that $\psi' \circ f = f \circ \psi$. We extend this automorphism to an automorphism $\breve{\varphi} \in Aut(\mathbf{f} : \mathcal{Y} \to \mathcal{D})$, which is possible because $\mathbf{f} : \mathcal{Y} \to \mathcal{D}$ is a local universal deformation of the $2:1$ admissible cover $f : Y \to D$. By the definition of $Aut(\mathbf{f} : \mathcal{Y} \to \mathcal{D})$, $\breve{\varphi}$ induces an automorphism of $(\mathcal{D};\mathcal{A}\cup\mathcal{B})$ we call $\breve{\varphi}_1$. We restrict $\breve{\varphi}$ to $\mathcal{C}$ and lift it to an automorphism $\breve{\varphi}_2 \in Aut(\mathcal{X})$. Restricting $\breve{\varphi}_1$ and $\breve{\varphi}_2$ to the central fibers, yields automorphism $\varphi_1 \in Aut((D;A\cup B))$ and $\varphi_2 \in Aut(X)$ which are liftings of $\varphi \in Aut((D;(G,H)))$ via the homomorphisms of Lemma 35 (ii).

We have $\varphi \in Im(\psi_1)$ iff $\varphi_1$ respects the structure $\{A,B\}$, and $\varphi \in Im(\psi_2)$ iff $\varphi_2$ respects the structure $(\mathcal{L};b)$. Respecting the structure here means that $\{\varphi_1(A),\varphi_1(B)\} = \{A,B\}$, respectively $\varphi_2^*\mathcal{L} \cong \mathcal{L}$, compatible with $b$. This in turn is equivalent to $\breve{\varphi}_1$ respecting $\{\mathcal{A},\mathcal{B}\}$, respectively $\breve{\varphi}_2$ respecting $(\mathbf{L};\mathbf{b})$.

Let $S'$ be the open dense subset of the base space $S$ over which all fibers of $\mathcal{D} \to S$ are smooth. Then over $S'$ the contraction morphisms $\mathbf{Cont_1}$, $\mathbf{Cont_2}$ and $\mathbf{cont_1}$ are all isomorphisms, and the diagram of deformations above collapses to :

$$\begin{tikzcd}
 & \mathcal{C}' \arrow[d, "\mathbf{f}'=\hat{\mathbf{f}}'"] & \mathbf{L}', \mathbf{b}' \arrow[l, squiggly] \\
\{\mathcal{A}',\mathcal{B}'\} \arrow[r, squiggly] & \mathcal{D}' &
\end{tikzcd}$$

(the $'$ indicating restriction to the preimages of $S'$)

On $\mathcal{C}'$ we can also define a spin- resp. Prym structure in the following way: Define a divisor $E$ as such a linear combination of the preimages under $\mathbf{f}'$ of the sections in $\mathcal{A}'$ as is described (for points) in Lemma 15 (i)-(iii), and then let $\mathbf{L}''$ be the line bundle on $\mathcal{C}'$ coresponding to $E$. One can show that $\mathbf{L}' \cong \mathbf{L}''$ using that restricted to the fiber over any point of $S'$, $\mathbf{L}''$ and $\mathbf{L}'$ are isomorphic, and using that $(\mathcal{X};\mathbf{L};\mathbf{b})$ is the local universal deformation of every one of its fibers (c.f. [Cor89] Page 574).

$\breve{\varphi}_1$ and $\breve{\varphi}_2$ can also be restricted to the preimages of $S'$, where we denote them by $\breve{\varphi}_1'$ and $\breve{\varphi}_2'$. Now $\breve{\varphi}_1$ respects $\{\mathcal{A},\mathcal{B}\}$ iff $\breve{\varphi}_1'$ respects $\{\mathcal{A}',\mathcal{B}'\}$, while $\breve{\varphi}_2$ respects $(\mathbf{L};\mathbf{b})$ iff $\breve{\varphi}_2'$ respects $(\mathbf{L}';\mathbf{b}')$.

But $\mathbf{L}'$ as shown above is just the line bundle coresponding to $E$, and looking at the definition of the divisor $E$ and at Lemma 15 one sees that its class does not change under permutations of the sections in $\mathcal{A}'\cup\mathcal{B}'$ wich respect the partition into $\mathcal{A}'$ and $\mathcal{B}'$. So $\breve{\varphi}_2'$ respects $(\mathbf{L}';\mathbf{b}')$ iff $\breve{\varphi}_1'$ respects $\{\mathcal{A}',\mathcal{B}'\}$.

Going back in our chain of equivalences of "respecting" conditions, the previous sentence translates to, $\varphi_2 \in Aut((X;\mathcal{L};b) \Leftrightarrow \varphi_1 \in Aut((D;\{A,B\}$, which implies the equivalence $(*)$ we wanted to prove.

That also the second equation of Part (ii) holds is easy to check, considering how

$(\tilde{D};\{(\tilde{A},H_A),(\tilde{B},H_B)\},H_{AB})$ contains the information which kinds of marked points the contracted extremities of $D$ carried.

(iii): We know $Ker\,\psi_1 = Ker\,\psi_1' \cap Aut((D;\{A,B\}))$. By Lemma 35 (iii) this means that $Ker\,\psi_1$ consists of those Automorphisms of $(D;\{A,B\})$ that are nontrivial only on Extremities of $(D; A \cup B)$. For every such extremity carrying marked points only from the set $A$ or only from the set $B$, there is an Automorphism of $(D;\{A,B\})$ that swaps the two marked points and is trivial away form the extremity. These automorphisms generate $Ker\,\psi_1$ which consist thus of $2^{r'}$ elements. This together with (ii) implies:

$$h = m/2^{r'} \quad \Leftrightarrow \quad m = 2^{r'} \cdot h$$

To get the next equation we use

$$\#(Ker\,\psi_2) = \#\Big(Ker\,\chi_1' \cap Aut((X;\mathcal{L};b))\Big) \cdot \#\Big(Ker\,\chi_2 \cap \chi_1'\big(Aut((X;\mathcal{L};b))\big)\Big)$$

Considering Lemma 35 (iii) and the definition of the irrelevant automorphisms of $(X;\mathcal{L};b)$ (c.f. preliminaries), we see that $Ker\,\chi_1' \cap Aut((X;\mathcal{L};b))$ is just the group of irrelevant automorphisms. Since the "hyperelliptic involutions" generating $Ker\,\chi_2$ (c.f. Lemma 35 (v)) act trivially on all Prym- or spin sheaves, $Ker\,\chi_2$ is contained in $\chi_1'\big(Aut((X;\mathcal{L};b))\big)$. By Lemma 35 (iv), $\#(Ker\,\chi_2) = 2^{s-r}$. This implies:

$$h = n/2^{s-r} \quad \Leftrightarrow \quad n = 2^{s-r} \cdot i \cdot h$$

For the last assertion of (iii), c.f. [Lud10] Prop. 2.7., in the case of spin curves. (There the number of irrelevant automorphisms is $2^u$ instead of $2^{u-1}$ due to the different definition of automorphisms). For Prym curves c.f. [FL10] Remark 6.3. □

### 5.3 Automorphism numbers

**Lemma 37** *Let $p_1, ..., p_n$ be $n$ distinct points of $\mathbb{P}^1$ in general position. We describe, for different $n \in \mathbb{N}$, the group $A := Aut(\mathbb{P}^1;\{p_1,...,p_n\})$ of automorphisms of $\mathbb{P}^1$ that map points of the set $\{p_1,...,p_n\}$ again to points of this set.*

*(i) For $n \leq 2$, $A$ is an infinite group.*

*(ii) For $n = 3$, $A$ has 6 elements corresponding to the permutations of the 3 points.*

*(iii) For $n = 4$, $A$ has 4 elements, one is the identity, the others correspond to choosing two disjoint pairs of the points, and interchanging the points in each pair.*

*(iv) For $n \geq 5$, $A$ consists only of the identity.*

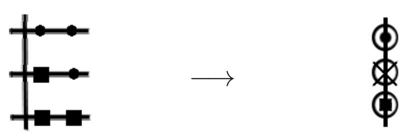
**Proof:** The automorphisms of $\mathbb{P}^1$ are the transformations $x \mapsto \frac{Ax+B}{Cx+D}$ for $A,B,C,D \in \mathbb{C}$. With this information one can check that the assertions of the Lemma are true. □.

We can use Lemma 37 together with Lemma 36 (iii) to compute the number of automorphisms of a general Prym- or spin curve of one of the strata of the stratifications by topological type. Lemma 36 allows to reduce to computing the automorphism number of the corresponding genus 0 curve with 6 sorted marked points. The diagrams of these corresponding objects are listed in the table below, and using Lemma 37 their automorphisms can quite easily be counted.

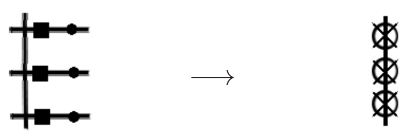
It may be even easier to draw a diagram for the object $(\tilde{D}; \{(\tilde{A}, H_A), (\tilde{B}, H_B)\}, H_{AB})$, count the number $h$ of automorphisms it allows and compute $n$ using the formula $n = 2^{(s-r)} \cdot i \cdot h$. Note that $s - r$ is just the number of irreducible components of $\tilde{D}$.

**Example:** We take the diagram of the object $(D; \{A, B\})$ corresponding to a general object of a given stratum, and reduce it to a diagram of $(\tilde{D}; \{(\tilde{A}, H_A), (\tilde{B}, H_B)\}, H_{AB})$ in the following way: We keep the markings that do not lie on extremities, and we introduce for every point to which an extremity is contracted a circle, in the center of which we insert a dot if the extremity carried two dots, a square if the extremity carried two squares, and a cross if the extremity carried one square and one dot. A automorphism must either take all symbols to symbols of the same kind (i.e. dots to dots, squares to squares, circled dots to circled dots,...) or it it must take all dots to squares and vice versa, all circled dots to circled squares and vice versa, and take circled crosses to circled crosses.

For example, in the case of the stratum $L^+$ we get

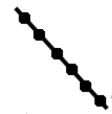
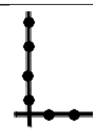
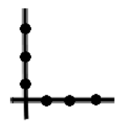
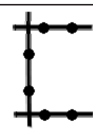
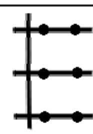
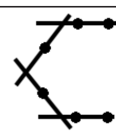


For $M^+$ we get

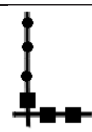
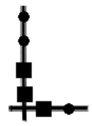
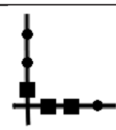
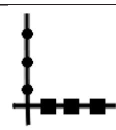
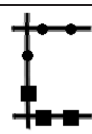
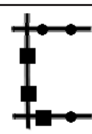
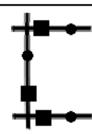


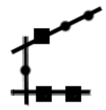
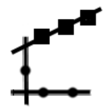
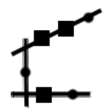
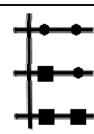
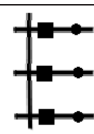
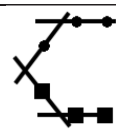
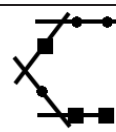
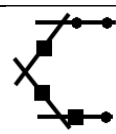
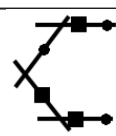
Now, using Lemma 37 (ii), it is clear that $h = 2$ for $L^+$ (it is possible to swap the square and the dot), and $h = 6$ for $M^+$. Since both diagrams have only one irreducible component, $s - r = 1$ in both cases. The nonexceptional subcurve of a general object of $L^+$ has one connected components, so here $i = 2^{1-1} = 1$, while for $M^+$ the nonexceptional subcurve has two connected components, so there $i = 2^{2-1} = 2$. Putting all this together we get that the automorphism number $n$ is 4 for $L^+$ and 24 for $M^+$.

The following table contains for each closed stratum of $\overline{M}_2$ the automorphism numbers of objects corresponding to general points of the different closed strata of $\overline{M}_2$, and the diagram of the object corresponding to the preimage of such a general point under the isomorphism $b : \overline{M}_{0,[6]} \to \overline{M}_2$, which can be determined easily using the explicit description of $b$ in [AL02].

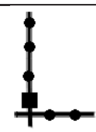
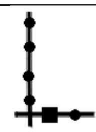
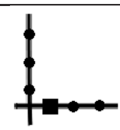
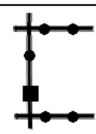
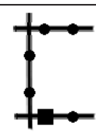
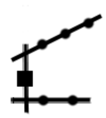
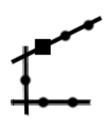
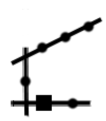
| Codim. | **Stratum** | **Diagram** | **Auto. number** |
|---|---|---|---|
| 0 | $S_2$ |  | 2 |
| 1 | $\Delta_0$ |  | 2 |
| 1 | $\Delta_1$ |  | 4 |
| 2 | $\Delta_{00}$ |  | 4 |
| 2 | $\Delta_{01}$ |  | 4 |
| 2 | $C_{000}$ |  | 12 |
| 2 | $C_{001}$ |  | 8 |

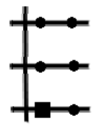
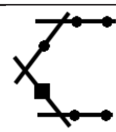
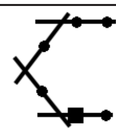
Next, the same for $S_2^+$, but with an additional column, showing to which class in $A^*_{\mathbb{Q}}(\overline{M}_2)$ the $Q$-class of each closed stratum is pushed forward by $\pi_+$. Concerning the diagram: Here we of course list the diagram belonging of the preimage of a general point under $a_+ : \overline{M}_{0,[3,3]} \to \overline{S}_2^+$. Which kind of diagram coresponds to the general spin curve of a stratum can be determined by using that we know such a corespondence already for $\overline{M}_2$, that we know the corespondence for all codimension 1 Strata from Section 3.1, and by considdering how their general spin curves can degenerate, and, when in doubt, by counting the degree of the strata over $\overline{M}_2$ and $\overline{M}_{0,[6]}$, like in section 3.1.

| Codim. | **Stratum** | **Diagram** | **Auto's** | $(\pi_+)_*([...]_Q)$ |
|---|---|---|---|---|
| 0 | $\overline{S}_2^+$ |  | 2 | $10[\overline{M}_2]_Q$ |
| 1 | $A_0^+$ |  | 2 | $4\delta_0$ |
| 1 | $B_0^+$ |  | 2 | $3\delta_0$ |
| 1 | $A_1^+$ |  | 8 | $\frac{9}{2}\delta_1$ |
| 1 | $B_1^+$ |  | 8 | $\frac{1}{2}\delta_1$ |
| 2 | $C^+$ |  | 4 | $2[\Delta_{00}]_Q$ |
| 2 | $D^+$ |  | 2 | $2[\Delta_{00}]_Q$ |
| 2 | $E$ |  | 4 | $[\Delta_{00}]_Q$ |

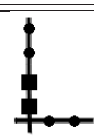
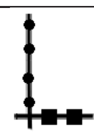
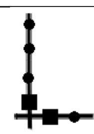
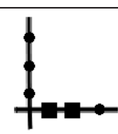
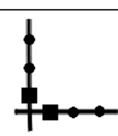
| | | | | |
|---|---|---|---|---|
| 2 | $X^+$ |  | 8 | $\frac{3}{2}[\Delta_{01}]_Q$ |
| 2 | $Y^+$ |  | 8 | $\frac{1}{2}[\Delta_{01}]_Q$ |
| 2 | $Z^+$ |  | 8 | $\frac{3}{2}[\Delta_{01}]_Q$ |
| 3 | $L^+$ |  | 4 | $3[\Delta_{000}]_Q$ |
| 3 | $M$ |  | 24 | $\frac{1}{2}[\Delta_{000}]_Q$ |
| 3 | $Q^+$ |  | 16 | $\frac{1}{2}[\Delta_{001}]_Q$ |
| 3 | $P^+$ |  | 16 | $\frac{1}{2}[\Delta_{001}]_Q$ |
| 3 | $U^+$ |  | 8 | $[\Delta_{001}]_Q$ |
| 3 | $R$ |  | 16 | $\frac{1}{2}[\Delta_{001}]_Q$ |

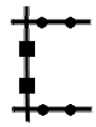
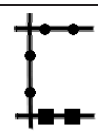
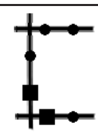
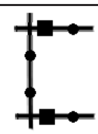
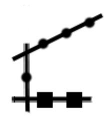
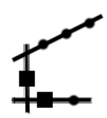
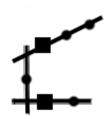
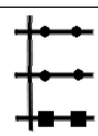
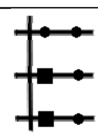
Next, the same for $\overline{S_2^-}$:

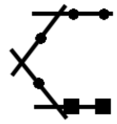
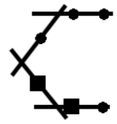
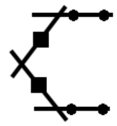
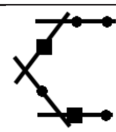
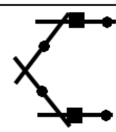
| Codim. | **Stratum** | **Diagram** | **Auto's** | $(\pi_-)_*([...]_Q)$ |
|---|---|---|---|---|
| 0 | $\overline{S}_2^-$ |  | 2 | $6[\overline{M}_2]_Q$ |
| 1 | $A_0^-$ |  | 2 | $4\delta_0$ |
| 1 | $B_0^-$ |  | 2 | $\delta_0$ |
| 1 | $A_1^-$ |  | 8 | $3\delta_1$ |
| 2 | $C^-$ |  | 2 | $2[\Delta_{00}]_Q$ |
| 2 | $D^-$ |  | 2 | $2[\Delta_{00}]_Q$ |
| 2 | $X^-$ |  | 8 | $\frac{1}{2}[\Delta_{01}]_Q$ |
| 2 | $Y^-$ |  | 8 | $\frac{3}{2}[\Delta_{01}]_Q$ |
| 2 | $Z^-$ |  | 8 | $\frac{1}{2}[\Delta_{01}]_Q$ |

| | | | | |
|---|---|---|---|---|
| 3 | $L^-$ |  | 4 | $3[\Delta_{000}]_Q$ |
| 3 | $P^-$ |  | 8 | $[\Delta_{001}]_Q$ |
| 3 | $U^-$ |  | 8 | $[\Delta_{001}]_Q$ |

Next, the same for $\overline{R}_2$:

| Codim. | **Stratum** | **Diagram** | **Auto's** | $(\pi_R)_*([...]_Q)$ |
|---|---|---|---|---|
| 0 | $\overline{R}_2$ |  | 2 | $15[\overline{M}_2]_Q$ |
| 1 | $D_0'$ |  | 2 | $6\delta_0$ |
| 1 | $D_0''$ |  | 2 | $\delta_0$ |
| 1 | $D_0^r$ |  | 2 | $4\delta_0$ |
| 1 | $D_1$ |  | 4 | $6\delta_1$ |
| 1 | $D_{1:1}$ |  | 4 | $9\delta_1$ |

| | | | | |
|---|---|---|---|---|
| 2 | $E^{\prime,\prime}$ |  | 4 | $[\Delta_{00}]_Q$ |
| 2 | $E^{\prime,\prime\prime}$ |  | 2 | $2[\Delta_{00}]_Q$ |
| 2 | $E^{\prime,r}$ |  | 2 | $4[\Delta_{00}]_Q$ |
| 2 | $E^{r,r}$ |  | 4 | $[\Delta_{00}]_Q$ |
| 2 | $F_1^{\prime}$ |  | 4 | $3[\Delta_{01}]_Q$ |
| 2 | $F_1^{\prime\prime}$ |  | 4 | $[\Delta_{01}]_Q$ |
| 2 | $F_1^{r}$ |  | 4 | $[\Delta_{01}]_Q$ |
| 2 | $F_{1:1}^{\prime}$ |  | 4 | $3[\Delta_{01}]_Q$ |
| 2 | $F_{1:1}^{r}$ |  | 4 | $[\Delta_{01}]_Q$ |
| 3 | $G^{\prime}$ |  | 4 | $3[\Delta_{000}]_Q$ |
| 3 | $G^{r}$ |  | 4 | $3[\Delta_{000}]_Q$ |

| | | | | |
|---|---|---|---|---|
| 3 | $H_1'$ |  | 4 | $2[\Delta_{001}]_Q$ |
| 3 | $H_1^r$ |  | 4 | $2[\Delta_{001}]_Q$ |
| 3 | $H_{1:1}'$ |  | 8 | $[\Delta_{001}]_Q$ |
| 3 | $H_{1:1}^r$ |  | 4 | $2[\Delta_{001}]_Q$ |
| 3 | $H_{1:1}^{r,r}$ |  | 8 | $[\Delta_{001}]_Q$ |

## References


[AL02] D. Avritzer and H. Lange. The moduli spaces of hyperelliptic curves and binary forms. *Math. Z.*, 242(4):615–632, 2002.

[BF09a] Gilberto Bini and Claudio Fontanari. On the geometry of $\overline{S}_2$. *Internat. J. Math.*, 20(5):605–621, 2009.

[BF09b] Gilberto Bini and Claudio Fontanari. A remark on the rational cohomology of $\overline{S}_{1,n}$. *Collect. Math.*, 60(3):241–247, 2009.

[Bre72] Glen E. Bredon. *Introduction to compact transformation groups.* Academic Press, New York, 1972. Pure and Applied Mathematics, Vol. 46.

[Cor89] Maurizio Cornalba. Moduli of curves and theta-characteristics. In *Lectures on Riemann surfaces (Trieste, 1987)*, pages 560–589. World Sci. Publ., Teaneck, NJ, 1989.

[Cor91] Maurizio Cornalba. A remark on the Picard group of spin moduli space. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 2(3):211–217, 1991.

[Dol10] Igor Dolgachev. *Topics in classical algebraic geometry.* 2010. Not published as hard copy yet. Available at authors homepage: `http://www.math.Isa.umich.edu/~idolga/lecturenotes.html`.

[FL10] Gavril Farkas and Katharina Ludwig. The Kodaira dimension of the moduli space of Prym varieties. *J. Eur. Math. Soc. (JEMS)*, 12(3):755–795, 2010.

[Ful98] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[GH94] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1994. Reprint of the 1978 original.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[HM82] Joe Harris and David Mumford. On the Kodaira dimension of the moduli space of curves. *Invent. Math.*, 67(1):23–88, 1982. With an appendix by William Fulton.

[Kee92] Sean Keel. Intersection theory of moduli space of stable $n$-pointed curves of genus zero. *Trans. Amer. Math. Soc.*, 330(2):545–574, 1992.

[Lud10] Katharina Ludwig. On the geometry of the moduli space of spin curves. *J. Algebraic Geom.*, 19(1):133–171, 2010.

[Mum83] David Mumford. Towards an enumerative geometry of the moduli space of curves. In *Arithmetic and geometry, Vol. II*, volume 36 of *Progr. Math.*, pages 271–328. Birkhäuser Boston, Boston, MA, 1983.

[Ste77] J. H. M. Steenbrink. Mixed Hodge structure on the vanishing cohomology. In *Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976)*, pages 525–563. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.