POLYNOMIAL INVARIANTS OF GRAPHS ON SURFACES

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ABSTRACT. For a graph embedded into a surface, we relate many combinatorial parameters of the cycle matroid of the graph and the bond matroid of the dual graph with the topological parameters of the embedding. This will give an expression of the polynomial, defined by M. Las Vergnas in a combinatorial way using matroids as a specialization of the Krushkal polynomial, defined using the symplectic structure in the first homology group of the surface.

INTRODUCTION

Since Hassler Whitney's introduction of matroids in 1935, the theory of matroids has developed mostly in two major directions. The development of matroids as a tool to be applied to graphs was mostly fostered in the area of combinatorics while matroids of collections of vectors in a vector space found a strong application to topologies of arrangements of hyperplanes in a vector space and was developed jointly in combinatorics and topology. It has been shown that the matroidal properties of such a hyperplane arrangement are very closely related to the topological properties of the arrangement. In this paper we demonstrate that the first direction of matroids associated to a graph is also tightly related to topology.

We consider graphs on surfaces. Suppose that a graph G is embedded into a surface Σ in a cellular manner, that is each connected component (face) of the complement to the graph is homeomorphic to a disc. Then we can define a dual graph G^* embedded into the same surface Σ in a natural way. With the graph G we associate its *cycle matroid* $\mathcal{C}(G)$, and with the dual graph G^* we associate its *bond matroid* $\mathcal{B}(G^*)$, the *dual matroid* of its cycle matroid $\mathcal{C}(G^*)$. In the planar case, when Σ is a sphere, $\mathcal{B}(G^*)$ is a dual matroid of the dual graph, so it is isomorphic to $\mathcal{C}(G) = \mathcal{B}(G^*)$. Thus the difference between $\mathcal{C}(G)$ and $\mathcal{B}(G^*)$ can be considered as a measure of non-planarity of the embedding and should reflect the topology of the pair (Σ, G). Various combinatorial parameters of the matroids $\mathcal{C}(G)$ and $\mathcal{B}(G^*)$ can be assembled into the Las Vergnas polynomial of a matroid perspective $\mathcal{B}(G^*) \to \mathcal{C}(G)$ introduced in [LV1, LV2, LV3].

On the other hand, the topological parameters of the embedding of G into Σ can be assembled into the Krushkal polynomial introduced in [Kru]. In this paper we show that the Las Vergnas polynomial is a specialization of the Krushkal polynomial. In the process, we relate many combinatorial parameters of the matroids $\mathcal{C}(G)$ and $\mathcal{B}(G^*)$ with the topological parameters of the embedding.

In Section 1 we briefly review matroids and their combinatorial parameters, and introduce the Las Vergnas polynomial. We introduce the Krushkal polynomial in Section 2. The main theorem is formulated in Section 3, wherein we also get the duality property of the Las Vergnas polynomial as a simple consequence of our main theorem. We begin to prove the main theorem in this section and complete it in Section 4 which consists of three lemmas relating the topological parameters of the embedding $G \hookrightarrow \Sigma$ with the combinatorial parameters of its matroids. We conclude in Section 5 with a discussion on the relation of the Krushkal polynomial with the Bollobás-Riordan polynomial of ribbon graphs.

This work has been done as a part of the Summer 2010 undergraduate research working group

1. Matroids and the Las Vergnas polynomial

Besides the classical paper of H. Whitney [Whi], for general background on matroids we refer to a remarkable book [Wel].

Definition 1.1. *A matroid is a finite set M and a function r, rank, assigning a number to a subset of M and satisfying the axioms:*

(**R1**) *The rank of an empty subset is zero.*
(**R2**) *For any subset H and any element $y \notin H$,*

$$r(H \cup \{y\}) = \left\{ \begin{array}{ll} r(H) \,, & or \\ r(H) + 1 \,. & \end{array} \right.$$

(**R3**) *For any subset H and two elements y,z not in H, if $r(H \cup y) = r(H \cup z) = r(H)$, then $r(H \cup \{y, z\}) = r(H)$.*

There are two major examples of matroids that we will be focusing on, although these examples do not exhaust the collection of matroids. That is to say, there are matroids that do not represent either of the following situations.

The first example is the cycle matroid $\mathcal{C}(G)$ of a graph G. The underlying set M is the set of edges $E(G)$ and the rank function is given by $r(H) := v(G) - c(H)$, where $v(G)$ is the number of vertices of G and $c(H)$ is the number of connected components of the spanning subgraph of G consisting of all the vertices of G and edges of H.

The second example is a finite set of vectors in a vector space. We may think about them as column vectors of a matrix. The rank function is the dimension of the subspace spanned by the subset of vectors, or the rank of the corresponding submatrix.

A subset $H \subset M$ is called *independent* if $r(H) = |H|$. In the first example, the independent subsets are those subsets of edges which do not contain cycles. In the second, independent subsets correspond to linearly independent subsets of vectors. A *basis* of a matroid is a maximal independent set.

A subset $H \subset M$ is called a *circuit* if $r(H) = |H| - 1$. In the first example, this new notion of a circuit and the traditional notion of a circuit in a graph match. In the second example a circuit is a subset of vectors with precisely one linear relation between them.

Given any matroid M, there is a dual matroid M^* with the same underlying set and with the rank function given by $r_{M^*}(H) := |H| + r_M(M \setminus H) - r(M)$. In particular $r(M) + r(M^*) = |M|$. Any basis of M^* is a complement to a basis of M.

The dual matroid to the cycle matroid of a graph G is called the *bond matroid* of G: $\mathcal{B}(G) := (\mathcal{C}(G))^*$. The circuits of $\mathcal{B}(G)$ are the minimal edge cuts, also known as the *bonds* of G. These are those subsets of the edges of G which, when removed from G, increase the number of connected components. The Whitney planarity criteria [Whi] says that a graph G is planar if and only if its bond matroid $\mathcal{B}(G)$ is the cycle matroid of some graph. In this case, it will be the cycle matroid of the dual graph, $\mathcal{B}(G) = (\mathcal{C}(G))^* = \mathcal{C}(G^*)$.

Definition 1.2 ([LV1, LV2, LV3]). For two matroids M and M', a bijection $M \to M'$ is called a *matroid perspective* if any circuit of M is mapped to a union of circuits of M'. Equivalently,

$$r_M(X) - r_M(Y) \geqslant r_{M'}(X) - r_{M'}(Y) \qquad \text{for all} \quad Y \subseteq X \,,$$

where r_M and $r_{M'}$ are the rank functions of matroids M an M'.

Definition 1.3 ([LV1, LV2, LV3])**.** The *Tutte polynomial* of a matroid perspective $M \to M'$ is the polynomial

$$(1) \quad T_{M \to M'}(x, y, z) := \sum_{H \subseteq M} (x-1)^{r(M')-r_{M'}(H)}(y-1)^{n_M(H)} z^{(r(M)-r_M(H))-(r(M')-r_{M'}(H))} ,$$

where $n_M(H) := |H| - r_M(H)$ is the nullity in M.

Properties. The usual Tutte polynomial of matroid M and M' can be recovered from the Tutte polynomial of matroid perspective in the following ways:

$T_M(x, y) = T_{M \to M}(x, y, z)$;
$T_M(x, y) = T_{M \to M'}(x, y, x-1)$;
$T_{M'}(x, y) = (y-1)^{r(M)-r(M')} T_{M \to M'}(x, y, \frac{1}{y-1})$.

For graphs G and G^* dually cellularly embedded in a surface Σ, the natural map of the bond matroid of G^* onto the cycle matroid of G, $\mathcal{B}(G^*) \to \mathcal{C}(G)$, is a matroid perspective. We call the Tutte polynomial of this matroid perspective, $T_{\mathcal{B}(G^*) \to \mathcal{C}(G)}(x, y, z)$, the *Las Vergnas polynomial* of the graph G on the surface Σ, and denote it $LV_{G,\Sigma}(x, y, z)$.

In this paper we assume that Σ is orientable.

Example 1.4. Let G be a graph with one vertex and two loops embedded into a torus as shown. Then G^* is the similar graph.

In this case the bond matroid $M = \mathcal{B}(G^*)$ has rank 2, and the cycle matroid $M' = \mathcal{C}(G)$ has rank 0. For any subset H: $r_M(H) = |H|$, $n_M(H) = 0$, and $r_{M'}(H) = 0$. We have

$$LV_{G,\Sigma}(x, y, z) = z^2 + 2z + 1 .$$

2. KRUSHKAL POLYNOMIAL

V. Krushkal came to his polynomial in research relating topological quantum field theories and algebraic and combinatorial properties of models of statistical mechanics, as well as generalizing the Tutte and Bollobás–Riordan polynomials [Kru].

Definition 2.1 ([Kru])**.** For a graph G embedded into a (not necessarily connected, but orientable) surface Σ,

$$(2) \qquad P_{G,\Sigma}(X, Y, A, B) := \sum_{H \subseteq G} X^{c(H)-c(G)} Y^{k(H)} A^{s(H)/2} B^{s^\perp(H)/2},$$

where
- $c(H)$ is the number of connected components of the spanning subgraph H;
- the restriction of the embedding $G \hookrightarrow \Sigma$ to H induces a map on the first homology groups and we define

$$k(H) := \dim(\ker(H_1(H; \mathbb{R}) \to H_1(\Sigma; \mathbb{R}))) ;$$

- $s(H)$ is equal to twice the genus of a regular neighborhood of the spanning subgraph H in Σ (the neighborhood is a surface with boundary, and its genus is defined as the genus of the closed surface obtained by attaching a disk to each boundary circle);
- $s^\perp(H)$ is equal to twice the genus of the surface obtained by removing a regular neighborhood of H from Σ.

Remark. V. Krushkal indicates in his paper [Kru] that the parameters $s(H)$ and $s^\perp(H)$ have the following interpretation in terms of the symplectic bilinear form on the vector space $H_1(\Sigma; \mathbb{R})$ giving by the intersection number. For a given spanning subgraph H, let V be its image in the homology group:

$$H_1(\Sigma; \mathbb{R}) \supset V := V(H) := \mathrm{im}(H_1(H; \mathbb{R}) \to H_1(\Sigma; \mathbb{R})) \ .$$

For the subspace V we can define its orthogonal complement $V^\perp$ in $H_1(\Sigma; \mathbb{R})$ with respect to the symplectic intersection form. Then

$$s(H) = \dim(V/(V \cap V^\perp)) \ , \qquad s^\perp(H) = \dim(V^\perp/(V \cap V^\perp)) \ .$$

Example 2.2. Continuing with example 1.4, for the graph G on the torus Σ, the map $H_1(G; \mathbb{R}) \to H_1(\Sigma; \mathbb{R})$ is not degenerate. Therefore $k(H) = 0$ for any subset H. Also $c(H) = 1$ for any H. If $H \neq G$, then its regular neighborhood has genus 0, so for such H, $s(H) = 0$, while $s(G) = 2$. Similarly, if $H \neq \emptyset$, then the regular neighborhood of its complement has genus 0, so for such H, $s^\perp(H) = 0$, while $s^\perp(\emptyset) = 2$. Combining all this we get

$$P_{G,\Sigma}(X, Y, A, B) = B + 2 + A \ .$$

3. MAIN THEOREM

Theorem 3.1. *Suppose G is cellularly embedded in an orientable surface Σ of genus g. Then*

(3) $$LV_{G,\Sigma}(x, y, z) = z^g P_{G,\Sigma}(x - 1, y - 1, z^{-1}, z).$$

Proof. The summands corresponding to each subgraph H of G will be shown to be equal. Applying the substitution (3) in (2) we arrive at summands of the form

$$(x - 1)^{c(H) - c(G)}(y - 1)^{k(H)} z^{g - s(H)/2 + s^\perp(H)/2}.$$

The cycle matroid rank of a graph is given by $r(M') = v(G) - c(G)$ and that of a subgraph $H \subseteq G$ by $r_{M'}(H) = v(G) - c(H)$. So $r(M') - r_{M'}(H) = c(H) - c(G)$ and hence the powers of the $(x - 1)$ factor coincide.

It remains to prove the equality of the exponents of $(y - 1)$ and z:

- $k(H) = n_M(H)$
- $g - s(H)/2 + s^\perp(H)/2 = r_M(G) - r_{M'}(G) - r_M(H) + r_{M'}(H)$

These will be proved separately in the next section. □

Corollary 3.2 ([LV1, LV2, LV3])**.**

$$LV_{G^*,\Sigma}(x, y, z) = z^{2g} LV_{G,\Sigma}(y, x, z^{-1}) \ .$$

This follows from Krushkal's formula [Kru]: $P_{G^*,\Sigma}(X, Y, A, B) = P_{G,\Sigma}(Y, X, B, A)$ for cellular embeddings $G \hookrightarrow \Sigma$.

4. MATROIDAL COMBINATORICS AND COMBINATORIAL TOPOLOGY

In this section we relate the topological parameters of the embedding $G \hookrightarrow \Sigma$ with the combinatorial parameters of the matroid perspective $\mathcal{B}(G^*) \to \mathcal{C}(G)$.

Lemma 4.1. $k(H) = n_M(H)$.

Proof. Let $N := \mathcal{C}(G^*) = (\mathcal{B}(G^*))^* = M^*$. The rank of a dual matroid can be defined in terms of the rank of a matroid by $r_M(H) = |H| + r_N(E \setminus H) - r_N(N)$, where H is a subset of edges of G^* that can be naturally identified with the corresponding subset of edges of G. Since the nullity is defined as the number of edges minus the rank we have that $n_M(H) = r_N(N) - r_N(G^* \setminus H)$, where, by $G^* \setminus H$ we mean the spanning subgraph of G^* consisting of the edges of H and all vertices of G^*. Thus, because N is the cycle matroid of the graph G^*, we have

$$n_M(H) = (v(G^*) - c(G^*)) - (v(G^*) - c(G^* \setminus H)) = c(G^* \setminus H) - c(G^*) \ .$$

Obviously, $c(G^*) = c(G)$ and is equal to the number of connected component $c(\Sigma)$ of Σ.

Now we consider H as a spanning subgraph of G. We want to remove its regular neighborhood from the surface Σ and count the number of connected components $c(\Sigma \setminus H)$. First we remove small discs around all vertices of H, i.e./ all vertices of G. These are exactly the faces of G^*. So we are left with a regular neighborhood of G^* in Σ. Secondly, removing neighborhoods of the edges of $H \subset G$ from Σ will give us the same surface, topologically, as deleting the corresponding neighborhoods of the edges of $H \subset G^*$, because these edges are transverse to each other. In other words $c(G^* \setminus H)$ is the number of components of $\Sigma \setminus H$. Hence

$$n_M(H) = c(\Sigma \setminus H) - c(G) \ ,$$

where H is regarded as a spanning subgraph of G.

Let us turn our attention to the number that we want $n_M(H)$ to be equal to. Denote by $i_* : H_1(H; \mathbb{R}) \to H_1(\Sigma; \mathbb{R})$ the linear map induced by the composition of embeddings $H \hookrightarrow G \hookrightarrow \Sigma$. We have $k(H) = \dim(\ker(i_*))$.

The topological pair (Σ, H) gives us a long exact sequence of homology groups

$$\cdots \to H_2(H) \to H_2(\Sigma) \to H_2(\Sigma, H) \xrightarrow{\delta} H_1(H) \xrightarrow{i_*} H_1(\Sigma) \to \cdots \ .$$

$H_2(H)$ is trivial as H is one dimensional, $H_2(\Sigma)$ has dimension equal to the number of components of Σ. and $H_2(\Sigma, H)$ has dimension equal to the number of components of $\Sigma \setminus H$.

So if we turn our attention to the short exact sequence

$$0 \to \mathbb{R}^{c(\Sigma)} \to H_2(\Sigma, H) \to \operatorname{im} \delta \to 0 \ ,$$

wherein $\operatorname{im} \delta = \ker i_*$, we see that $\dim \ker i_* = c(\Sigma \setminus H) - c(\Sigma) = c(\Sigma \setminus H) - c(G) = n_M(H)$. $\quad\square$

Thus the $(y-1)$ powers in the main theorem coincide.

In fact, the argument with the long exact sequence was used by V. Krushkal [Kru, end of proof of Theorem 3.1] where he essentially proved that $k(H) = c(\Sigma \setminus H) - c(\Sigma)$ using a slightly different terminology. However, the relation of this parameter to the matroidal $n_M(H)$ was not addressed there.

Lemma 4.2. $2g = r_M(G) - r_{M'}(G)$.

Proof. $r_{M'}(G) = v(G) - c(G) = e(T)$, where T is a spanning forest of G. The bond rank of a graph G^*, is the maximal number of edges that one can delete from it without increasing the number of connected components, i.e. all edges but a spanning forest, T^*, of the graph G^*. So $r_M(G) = e(G^*) - e(T^*) = e(G) - e(T^*)$ and $r_M(G) - r_{M'}(G) = e(G) - e(T^*) - e(T)$. But the number of edges in a spanning forest of G is equal to the number of vertices minus the number of components. Similarly, $e(T^*)$, the number of edges in a spanning forest of G^*, is the number of faces, f, of G minus the number of components. So $r_M(G) - r_{M'}(G) = e - (f - c(\Sigma)) - (v - c(\Sigma)) = e - f - v + 2c(\Sigma) = 2g(\Sigma) = 2g$. $\quad\square$

Lemma 4.3. $g + s(H)/2 - s^\perp(H)/2 = r_M(H) - r_{M'}(H)$.

Proof. Consider $r_M(H) - r_{M'}(H)$. For either matroid, M or M', rank is equal to the number of edges minus nullity. So $r_M(H) - r_{M'}(H) = n_{M'}(H) - n_M(H)$. We have seen in Lemma 4.1 that $n_M(H) = k(H)$. Formula (2.5) in Krushkal's paper [Kru] tells us that this nullity

$$n_{M'}(H) = k(H) + g + s(H)/2 - s^\perp(H)/2 \ .$$

So we have $r_M(H) - r_{M'}(H)$ equaling

$$k(H) + g + \frac{s(H)}{2} - \frac{s^\perp(H)}{2} - k(H) = g + \frac{s(H)}{2} - \frac{s^\perp(H)}{2} \ .$$

$\quad\square$

Lemma 4.2 is a particular case of Lemma 4.3 where we take $H = G$. Indeed, in this case we have $s(G) = 2g$ and $s^\perp(G) = 0$.

Lemmas 4.2 and 4.3 together imply

$$g - s(H)/2 + s^\perp(H)/2 = r_M(G) - r_{M'}(G) - r_M(H) + r_{M'}(H) \ .$$

Thus the z powers in the main theorem coincide. This completes the proof of the main theorem.

5. KRUSHKAL AND BOLLOBÁS-RIORDAN POLYNOMIALS

A cellular embedding $G \hookrightarrow \Sigma$ may be studied in terms of a *ribbon graphs* which represents a regular neighborhood $\mathcal{G}$ of G in Σ. Working backwards, starting with a ribbon graph $\mathcal{G}$, we can construct a surface Σ by capping all boundary components of $\mathcal{G}$ by discs. Then the core graph G of $\mathcal{G}$, obtained by contracting all edge-ribbons to their central lines and all vertex-discs to their central points, can be cellularly embedded in Σ. For ribbon graphs, we have the Bollobás-Riordan polynomial [BR] defined as

$$BR_{\mathcal{G}}(X,Y,Z) := \sum_{\mathcal{H} \subseteq \mathcal{G}} (X-1)^{c(\mathcal{H})-c(\mathcal{G})} Y^{n(\mathcal{H})} Z^{c(\mathcal{H})-bc(\mathcal{H})+n(\mathcal{H})} \ ,$$

where $bc(\mathcal{H})$ is the number of boundary components of the spanning ribbon subgraph $\mathcal{H}$. Note that the exponent $c(\mathcal{H}) - bc(\mathcal{H}) + n(\mathcal{H})$ is equal to $2g(\mathcal{H})$ for oriented ribbon graphs.

V. Krushkal proved [Kru, Lemma 4.1] that

(4) $$BR_{\mathcal{G}}(X,Y,Z) = Y^g P_{G,\Sigma}(X-1, Y, YZ^2, Y^{-1}) \ .$$

It was proved in [BR, Theorem 2] that $BR_{\mathcal{G}}$ is universal in the class of polynomials satisfying the contraction/deletion property. The Krushkal polynomial also satisfies *a* contraction/deletion property [Kru, Lemma 2.1]. Based on that V. Krushkal wrote that $BR_{\mathcal{G}}$ and $P_{G,\Sigma}$ carry equivalent information. However this is not the case as the contraction/deletion properties for $BR_{\mathcal{G}}$ and for $P_{G,\Sigma}$ are not quite the same. The problem arises when deletion of an edge of a ribbon graph changes its genus. The genus might decrease by 1 with removal of an edge. For example, if we delete a loop e from the ribbon graph $\mathcal{G}$ corresponding to G from Example 1.4, then the resulting graph with a single loop will have genus zero. So, while in the Bollobás-Riordan approach it is considered as a graph embedded to a sphere, in the Krushkal approach it is still embedded into the torus. We cannot apply the substitution (4) to that graph since its embedding on the torus is no longer cellular. Thus the Krushkal polynomial does not satisfy the contraction/deletion property in the sense of Bollobás and Riordan.

We also find that the Las Vergnas polynomial $LV_{G,\Sigma}(x,y,z)$ does not satisfy the contraction/deletion property in the sense of Bollobás and Riordan either.

Example 5.1. This is an example of a calculation of the three polynomials. Here G is a graph on torus with two vertices and three edges a, b, and c. Its dual G^* has one vertex and three loops. The ribbon graph corresponding to G is denoted $\mathcal{G}$. We use the same symbols a, b, c to denote the corresponding edges in all three graphs.
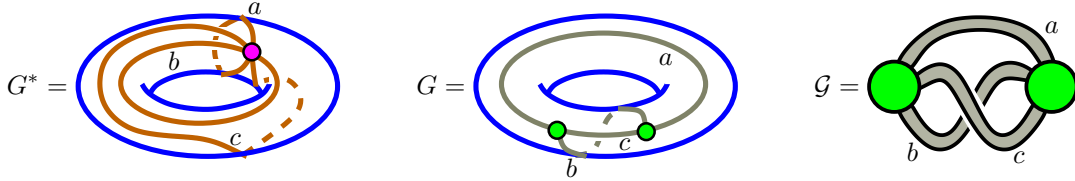
The matroid $M' = \mathcal{C}(G)$ is of rank 1, and for any nonempty subset H, $r_{M'}(H) = 1$. The cycle matroid $\mathcal{C}(G^*)$ of the dual graph is of rank zero because G^* has only loops. So its dual $M = \mathcal{B}(G^*)$

has rank 3, all subsets H are independent and $r_M(H) = |H|$. The next table shows the value of various parameters and contributions of all eight subsets $H \subseteq \{a, b, c\}$ to the three polynomials.

	H	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$	$\{c\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
Krushkal	$c(H)$	2	1	1	1	1	1	1	1
	$k(H)$	0	0	0	0	0	0	0	0
	$s(H)$	0	0	0	0	0	0	0	2
	$s^{\perp}(H)$	2	2	2	0	2	0	0	0
	$P_{G,\Sigma}$	XB	B	B	1	B	1	1	A
Las Vergnas	$r_M(H)$	0	1	1	2	1	2	2	3
	$r_{M'}(H)$	0	1	1	1	1	1	1	1
	$n_M(H)$	0	0	0	0	0	0	0	0
	$LV_{G,\Sigma}$	$(x-1)z^2$	z^2	z^2	z	z^2	z	z	1
Bollobás Riordan	$c(\mathcal{H})$	2	1	1	1	1	1	1	1
	$n(\mathcal{H})$	0	0	0	1	0	1	1	2
	$bc(\mathcal{H})$	2	1	1	2	1	2	2	1
	$BR_{\mathcal{G}}$	$(X-1)$	1	1	Y	1	Y	Y	Y^2Z^2

Thus

$$P_{G,\Sigma} = 3 + 3B + XB + A, \qquad LV_{G,\Sigma} = 3z + 3z^2 + (x-1)z^2 + 1, \qquad BR_{\mathcal{G}} = 3 + 3Y + (X-1) + Y^2Z^2.$$

One can readily confirm the relations (3) and (4) from here.

Now if we contract the edge c, the graph G/c still will be cellularly embedded into the same torus Σ, and its regular neighborhood coincides with the ribbon graph $\mathcal{G}/c$. Examples 1.4 and 2.2 and the right part of the table above give the following polynomials:

$$P_{G/c,\Sigma} = B + 2 + A, \qquad LV_{G/c,\Sigma} = z^2 + 2z + 1, \qquad BR_{\mathcal{G}/c} = 1 + 2Y + Y^2Z^2.$$

Meanwhile if we delete the edge c, then

$$P_{G-c,\Sigma} = XB + 2B + 1 .$$

But the graph $G - c$ is not cellularly embedded into the torus Σ any more. Thus the Las Vergnas and the Bollobás-Riordan polynomials are not defined for it. Its regular neighborhood gives the ribbon graph $\mathcal{G} - c$ which, after capping the discs to its two boundary components, results in the sphere S^2. Thus the graph $G - c$ embeds cellularly into the the sphere S^2. For this embedding we have

$$P_{G-c,S^2} = X + 2 + Y, \qquad LV_{G-c,S^2} = (x-1) + 2 + (y-1), \qquad BR_{\mathcal{G}-c} = (X-1) + 2 + Y.$$

Therefore

$$P_{G,\Sigma} = P_{G-c,\Sigma} + P_{G/c,\Sigma} \qquad \text{and} \qquad BR_{\mathcal{G}} = BR_{\mathcal{G}-c} + BR_{\mathcal{G}/c} ,$$

but

$$P_{G,\Sigma} \neq P_{G-c,S^2} + P_{G/c,\Sigma} \qquad \text{and} \qquad LV_{G,\Sigma} \neq LV_{G-c,S^2} + LV_{G/c,\Sigma} .$$

Currently, according to relations (3) and (4), the Krushkal polynomial is the most general polynomial of graphs on surfaces and so it clearly deserves further research. Also, because the two relations look quite different, the Las Vergnas and the Bollobás-Riordan polynomials seem to be independent. Very recently the Krushkal polynomial was generalized to higher dimensional simplicial complexes [KR]. It is related to matroids as well but in a very different way.

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