MOTIVIC ZETA FUNCTIONS FOR DEGENERATIONS OF ABELIAN VARIETIES AND CALABI-YAU VARIETIES

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1. Introduction

Let $f \in \mathbb{Z}[x_1, \ldots, x_n]$ be a non-constant polynomial, and let p be a prime. Igusa's p-adic zeta function $Z_f^p(s)$ is a meromorphic function on the complex plane that encodes the number of solutions of the congruence $f \equiv 0$ modulo powers p^m of the prime p. Igusa's *p-adic monodromy conjecture* predicts in a precise way how the singularities of the complex hypersurface defined by the equation $f = 0$ influence the poles of $Z_f^p(s)$ and thus the asymptotic behaviour of this number of solutions as m tends to infinity. The conjecture states that, when p is sufficiently large, poles of $Z_f^p(s)$ should correspond to local monodromy eigenvalues of the polynomial map $\mathbb{C}^n \to \mathbb{C}$ defined by f. We refer to Section 3 for a precise formulation.

Starting in the mid-nineties, J. Denef and F. Loeser developed the theory of motivic integration, which had been introduced by M. Kontsevich in his famous lecture at Orsay in 1995. Denef and Loeser used this theory to construct a motivic object $Z_f^{mot}(s)$ that interpolates the p-adic zeta functions $Z_f^p(s)$ for $p \gg 0$ and captures their geometric essence. This object is called the *motivic zeta function* of f. Denef and Loeser also formulated a motivic upgrade of the monodromy conjecture (Conjecture 4.6.2). Its precise relation with the p-adic monodromy conjecture is explained in Section 4.7.

The aim of this paper is to present a *global* version of Denef and Loeser's motivic zeta functions. Let X be a Calabi-Yau variety over a complete discretely valued field K (i.e., a smooth, proper and geometrically connected variety with trivial canonical sheaf). We'll define the motivic zeta function $Z_X(T)$ of X. This is a formal power series with coeffients in a certain localized Grothendieck ring of varieties over the residue field k of K. We'll show that $Z_X(T)$ has properties analogous to Denef and Loeser's zeta function, and we'll prove a global version of the motivic monodromy conjecture when X is an abelian variety, under a certain tameness condition on X (Theorem 5.5.1).

The link between Denef and Loeser's motivic zeta function $Z_f^{mot}(s)$ and our global variant is an alternative interpretation of $Z_f^{mot}(s)$ in terms of non-archimedean geometry, due to J. Sebag and the second author [NS07]. This interpretation is based on the theory of motivic integration on rigid varieties developed by F. Loeser and J. Sebag [LS03], which explains how one can associate a *motivic volume* to a gauge form on a smooth rigid variety over a complete discretely valued field. J. Sebag and the second author constructed the *analytic Milnor fiber* of a hypersurface singularity, a non-archimedean model for the classical Milnor

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fibration in the complex analytic setting. The analytic Milnor fiber is a smooth rigid variety over a field K of Laurent series. The motivic zeta function can be realized as a generating series whose coefficients are motivic volumes of a so-called *Gelfand-Leray* form on the analytic Milnor fiber over finite totally ramified extension of the base field K. This is explained in detail in Section 4.

This interpretation of the motivic zeta function admits a natural generalization to the global case, where we replace the analytic Milnor fiber by a Calabi-Yau variety X over a complete discretely valued field K and the Gelfand-Leray form by a suitably normalized gauge form ω on X. The zeta function $Z_X(T)$ is studied in Section 5 when X is an abelian variety and in Section 6 in the general case. We raise the question whether there exists a relation between the poles of $Z_X(T)$ and the monodromy eigenvalues of X as predicted by the monodromy conjecture in the case of hypersurface singularities (Question 6.4.2).

We studied motivic zeta functions of abelian varieties in detail in the papers [HN10a, HN10c, HN10d]. Section 5 gives an overview of the results and methods used in those papers. A powerful and central tool is the Néron model of an abelian K-variety A, which is the "minimal" extension of A to a smooth group scheme over R. The Néron model $\mathcal{A}$ of A comes equipped with much interesting structure, such as the Chevalley decomposition of the identity component of its special fiber, the Lie algebra $\mathrm{Lie}(\mathcal{A})$ and the component group Φ_A. The key point in the study of $Z_A(T)$ is to understand how these objects change under ramified extensions of K.

Our main result is Theorem 5.5.1, which states that if A is a tamely ramified abelian K-variety, then $Z_A(\mathbb{L}^{-s})$ is rational with a unique pole at $s = c(A)$, where $c(A)$ denotes Chai's base change conductor of A [Ch00]. Moreover, for every embedding of $\mathbb{Q}_\ell$ in $\mathbb{C}$, the complex number $\exp(2\pi c(A)i)$ is an eigenvalue of the monodromy transformation on $H^g(A \times_K K^t, \mathbb{Q}_\ell)$, where K^t denotes a tame closure of K, and where g is the dimension of A. This shows that a global version of Denef and Loeser's motivic monodromy conjecture holds for tamely ramified abelian varieties.

The situation for general Calabi-Yau varieties is at the moment far less clear than in the abelian case. Our proofs for abelian varieties rely heavily on the theory of Néron models, and these methods do not extend to the general case. However, if we restrict ourselves to equal characteristic zero, there is still much that can be said, and we present some of our results under this assumption in Section 6.

A particular advantage in characteristic zero, and the basis for many applications, is that we can find an *sncd*-model of X, i.e., a regular proper R-model $\mathcal{X}$ whose special fiber $\mathcal{X}_s$ is a divisor with strict normal crossings. We explain in Section 6 how the results in [NS07] yield an explicit expression for $Z_X(T)$ in terms of the model $\mathcal{X}$. This expression shows that $Z_X(T)$ is rational, and yields a finite subset of $\mathbb{Q}$ that contains all the poles of $Z_X(\mathbb{L}^{-s})$. However, due to cancellations in the formula, it is often difficult to use this description to determine the precise set of poles.

The opposite of the largest pole of $Z_X(\mathbb{L}^{-s})$ turns out to be an interesting invariant of X, we call it the log canonical threshold $lct(X)$ of X. It can be easily computed on the model $\mathcal{X}$. We can show that $lct(X)$ corresponds to a monodromy eigenvalue on the degree $\dim(X)$ cohomology of X. The value $lct(X)$ is a global version of the log canonical threshold for complex hypersurface singularities, we explain the precise relationship in Section 6.3. Since we know that for an abelian

K-variety A, the base change conductor $c(A)$ is the unique pole of $Z_A(\mathbb{L}^{-s})$, we find that $lct(A) = -c(A)$. This yields an interesting relation between the Néron model of A and the birational geometry of *sncd*-models of A. Our explicit expression for the zeta function allows to compute many other arithmetic invariants of A on an *sncd*-model, in particular the number of connected components of the Néron model. This generalizes the results that were known for elliptic curves. Conversely, we can use the zeta function to extend many interesting invariants of abelian K-varieties to arbitrary Calabi-Yau varieties.

2. Preliminaries

2.1. **Notation.** For every ring A, an A-variety is a reduced separated A-scheme of finite type. An algebraic group over a field F is a reduced group scheme of finite type over F. We denote by μ the profinite group scheme of roots of unity.

2.2. **Local monodromy eigenvalues.** Let k be a subfield of $\mathbb{C}$, let X be a k-variety, and let

$$f : X \to \mathbb{A}^1_k = \operatorname{Spec} k[t]$$

be a k-morphism. Let x be a point of $X(\mathbb{C})$ such that $f(x) = 0$. We denote by X^{an} the complex analytification of $X \times_k \mathbb{C}$, by $f^{\mathrm{an}} : X^{\mathrm{an}} \to \mathbb{C}$ the complex analytic map induced by f, and by X^{an}_s the zero locus of f^{an} in X. We say that a complex number α is a local monodromy eigenvalue of f at x if there exists an integer $j \geq 0$ such that α is an eigenvalue of the monodromy transformation on $R^j\psi_{f^{\mathrm{an}}}(\mathbb{C})_x$. Here

$$R\psi_{f^{\mathrm{an}}}(\mathbb{C}) \in D^b_c(X^{\mathrm{an}}_s, \mathbb{C})$$

denotes the complex of nearby cycles associated to f^{an} [Di04, §4.2]. If X is smooth at x, then the complex vector space $R^j\psi_{f^{\mathrm{an}}}(\mathbb{C})_x$ is isomorphic to the degree j singular cohomology space of the Milnor fiber of f^{an} at the point x.

If f is a polynomial in $k[x_1, \ldots, x_n]$, then we can speak of local monodromy eigenvalues of f by considering f as a morphism $\mathbb{A}^n_k \to \mathbb{A}^1_k$.

2.3. **The Bernstein-Sato polynomial.** Let k be a field of characteristic zero. Let X be a smooth irreducible k-variety of dimension n, endowed with a morphism

$$f : X \to \mathbb{A}^1_k = \operatorname{Spec} k[t].$$

Denote by X_s the fiber of f over the origin. For every closed point x of X_s, we denote by k_x the residue field at x and by $b_{f,x}(s)$ the Bernstein-Sato polynomial of the formal germ of f in $\widehat{\mathcal{O}}_{X,x} \cong k_x[[x_1, \ldots, x_n]]$ (see [Bj79, 3.3.6]). We call $b_{f,x}(s)$ the local Bernstein-Sato polynomial of f at x. If $k = \mathbb{C}$, then $b_{f,x}(s)$ coincides with the Bernstein polynomial of the analytic germ of f in $\mathcal{O}_{X^{\mathrm{an}},x}$, by [MN91, §4.2].

If $h : Y \to X$ is a morphism of smooth k-varieties and y is a closed point of Y such that $x = h(y)$ and h is étale at x, then the faithfully flat local homomorphism $\widehat{\mathcal{O}}_{X,x} \to \widehat{\mathcal{O}}_{Y,y}$ satisfies the conditions in [MN91, §4.2]. It follows that $b_{f,x}(s) = b_{f\circ h,y}(s)$. The same argument shows that $b_{f,x}(s)$ is invariant under arbitrary extensions of the base field k. If $k = \mathbb{C}$, then Kashiwara has shown that the roots of $b_{f,x}(s)$ are rational numbers [Ka76]. By [Sa94], they lie in the interval $]-n, 0[$. Invoking the Lefschetz principle, we see that these properties hold for arbitrary k.

If $k = \mathbb{C}$, then it was proven by Malgrange [Ma83] that, for every root α of the local Bernstein-Sato polynomial $b_{f,x}(s)$, the value $\exp(2\pi i\alpha)$ is a local monodromy eigenvalue of f at some point of X_s^{an}. Moreover, if we allow x to vary in the zero locus of f, all local monodromy eigenvalues arise in this way. Since $b_{f,x}(s)$ is invariant under extension of the base field k, this property still holds over all subfields k of $\mathbb{C}$.

By constructibility of the nearby cycles complex, the local monodromy eigenvalues of f form a finite set $\mathrm{Eig}(f)$. Thus, as x runs through the set of closed points of X_s, the polynomials $b_{f,x}(s)$ form a finite set, since they are all monic polynomials whose roots belong to the finite set of rational numbers α in $[-n,0[$ such that $\exp(2\pi i\alpha)$ lies in $\mathrm{Eig}(f)$. We call the least common multiple of the polynomials $b_{f,x}(s)$ the Bernstein-Sato polynomial of f, and we denote it by $b_f(s)$. If $X=\mathbb{A}^n_k$, then by [MN91, §4.2], this definition coincides with the usual definition of the Bernstein-Sato polynomial of an element f in $k[x_1,\ldots,x_n]$.

3. P-adic and motivic zeta functions

3.1. **The Poincaré series.** Let f be an element of $\mathbb{Z}[x_1,\ldots,x_n]\setminus\mathbb{Z}$, for some integer $n>0$, and let p be a prime number. For every integer $m\geq 0$, we denote by S_m the set of solutions of the congruence $f\equiv 0$ modulo p^{m+1}, i.e.,

$$S_m=\{a\in(\mathbb{Z}/p^{m+1}\mathbb{Z})^n\mid f(a)\equiv 0 \mod p^{m+1}\}.$$

We put $N_m=\sharp S_m$.

Definition 3.1.1. *The Poincaré series associated to f and p is the generating series*

$$P(T)=\sum_{m\geq 0}N_mT^m \quad \in\mathbb{Z}[[T]].$$

Example 3.1.2. If the closed subscheme X of $\mathbb{A}^n_{\mathbb{Z}_p}$ defined by the equation $f=0$ is smooth over $\mathbb{Z}_p$, then the Poincaré series $P(T)$ is easy to compute. For every integer $m\geq 0$, the set S_m is the set of $(\mathbb{Z}/p^{m+1}\mathbb{Z})$-valued points on X. Locally at every point, X admits an étale morphism to $\mathbb{A}^{n-1}_{\mathbb{Z}_p}$. The infinitesimal lifting criterion for étale morphisms implies that the map $S_{m+1}\to S_m$ is surjective, and that every fiber has cardinality p^{n-1}. In this way, we find that

$$P(T)=\frac{\sharp X(\mathbb{F}_p)}{1-p^{n-1}T}. \tag{3.1}$$

If X is not smooth over $\mathbb{Z}_p$, then the behaviour of the values N_m is much harder to understand. The following conjecture was mentioned in [BS66], Chapter 1, Section 5, Problem 9.

Conjecture 3.1.3. *The Poincaré series $P(T)$ is rational, i.e., it belongs to the subring $\mathbb{Q}(T)\cap\mathbb{Z}[[T]]$ of $\mathbb{Q}((T))$.*

3.2. **The p-adic zeta function.**

Definition 3.2.1. *We denote by $|\cdot|_p$ the p-adic absolute value on $\mathbb{Q}_p$. The p-adic zeta function of f is defined by*

$$Z_f^p(s)=\int_{\mathbb{Z}_p^n}|f(x)|_p^s\,dx$$

for every complex number s with $\Re(s)>0$.

The p-adic zeta function Z_f^p is an analytic function on the complex right half plane $\Re(s) > 0$. It was introduced by Weil, and systematically studied by Igusa. It can be defined in a much more general set-up, starting from a p-adic field K, an analytic function f on K^n, a Schwartz-Bruhat function Φ on K^n and a character χ of $\mathcal{O}_K^\times$. Moreover, one can formulate analogous definitions over the archimedean local fields $\mathbb{R}$ and $\mathbb{C}$. For a survey, we refer to [De91b] or [Ig00].

We can write $Z_f^p(s)$ as a power series in p^{-s}, in the following way:

$$Z_f^p(s) = \sum_{m \geq 0} \mu_{\mathrm{Haar}}\{a \in (\mathbb{Z}_p)^n \mid v_p(f(a)) = m\} p^{-ms}$$

where v_p denotes the p-adic valuation on $\mathbb{Z}_p$. Direct computation shows that, if we set $T = p^{-s}$, then $Z_f^p(s)$ is related to the Poincaré series $P(T)$ by the formula

$$P(p^{-n}T) = \frac{p^n(1 - Z_f^p(s))}{1 - T} \tag{3.2}$$

Thus the zeta function $Z_f^p(s)$ contains exactly the same information as the Poincaré series $P(T)$, namely, the values N_m for all $m \geq 0$.

Example 3.2.2. In the set-up of Example 3.1.2, we have

$$Z_f^p(s) = 1 - \sharp X(\mathbb{F}_p) p^{-(n-1)} \left(\frac{p^s - 1}{p^{s+1} - 1} \right).$$

Theorem 3.2.3 (Igusa [Ig74, Ig75])**.** *The p-adic zeta function $Z_f^p(s)$ is rational in the variable p^{-s}. In particular, it admits a meromorphic continuation to $\mathbb{C}$.*

By (3.2), this gives an affirmative answer to Conjecture 3.1.3:

Corollary 3.2.4. *The Poincaré series $P(T)$ is rational.*

Igusa proved Theorem 3.2.3 by taking an embedded resolution of singularities for the zero locus of f in the p-adic manifold $\mathbb{Z}_p^n$, and applying the change of variables formula for p-adic integrals to compute the p-adic zeta function locally on the resolution space. This essentially reduces the problem to the case where f is a monomial, in which case one can make explicit computations.

The poles of $P(T)$, or equivalently, $Z_f^p(s)$, contain information about the asymptotic behaviour of N_m as $m \to \infty$. Igusa's proof shows that there exists a finite subset $\mathscr{S}^p$ of $\mathbb{Q}_{<0}$ such that the set of poles of $Z_f^p(s)$ is given by

$$\{\alpha + \frac{2\pi i}{\ln p}\beta \mid \alpha \in \mathscr{S}^p,\ \beta \in \mathbb{Z}\}.$$

By Denef's explicit formula for the p-adic zeta function in [De91a], one can associate to every embedded resolution for f over $\mathbb{Q}$ a finite subset $\mathscr{S}$ of $\mathbb{Q}_{<0}$ such that $\mathscr{S}^p \subset \mathscr{S}$ for $p \gg 0$. The set $\mathscr{S}$ is computed from the so-called *numerical data* of the resolution (in the notation of [De91a], $\mathscr{S}$ is the set of values $-\nu_i/N_i$ with i in T). In general, many of the elements in $\mathscr{S}$ are not poles of $Z_f^p(s)$, due to cancellations in the formula for the zeta function. This phenomenon is related to the *Monodromy Conjecture*.

3.3. **Igusa's monodromy conjecture.** Example 3.2.2 suggests that the poles of the zeta function $Z_f^p(s)$ should be related to the singularities of the polynomial f. The relation is made precise by Igusa's Monodromy Conjecture.

Conjecture 3.3.1 (Igusa's Monodromy Conjecture, strong form). *If we denote by $b_f(s)$ the Bernstein-Sato polynomial of f, then for $p \gg 0$, the function $b_f(s)Z_f^p(s)$ is holomorphic at every point of $\mathbb{R}$.*

In other words, the conjecture states that for every pole α of $Z_f^p(s)$, the real part $\Re(\alpha)$ is a root of the Bernstein-Sato polynomial $b_f(s)$, and the order of the pole is at most the multiplicity of the root. The Monodromy Conjecture describes in a precise way how the singularities of f influence the asymptotic behaviour of the values N_m as $m \to \infty$, for $p \gg 0$.

Example 3.3.2. Assume that the closed subscheme of $\mathbb{A}^n_{\mathbb{Q}}$ defined by the equation $f = 0$ is smooth over $\mathbb{Q}$. Then the Bernstein-Sato polynomial $b_f(s)$ is equal to $s+1$. For $p \gg 0$, the closed subscheme of $\mathbb{A}^n_{\mathbb{Z}_p}$ defined by $f = 0$ is smooth, so that the zeta function $Z_f^p(s)$ has a unique real pole at $s = -1$, of order one, by Example 3.2.2.

Because of Kashiwara and Malgrange's result mentioned in Section 2.3, Conjecture 3.3.1 implies the following weaker statement.

Conjecture 3.3.3 (Igusa's Monodromy Conjecture, weak form). *For $p \gg 0$, the following holds: if α is a pole of $Z_f^p(s)$, then $\exp(2\pi\Re(\alpha)i)$ is an eigenvalue of the monodromy action on $R^j\psi_{f^{\mathrm{an}}}(\mathbb{C})_x$, for some integer $j \geq 0$ and some point x of $\mathbb{C}^n$ with $f^{\mathrm{an}}(x) = 0$.*

Several special cases of the Monodromy Conjecture have been proven, but the general case remains wide open. For a survey of known results and the relation with archimedean zeta functions over the local fields $\mathbb{R}$ and $\mathbb{C}$, we refer to [Ni10a].

3.4. **The motivic zeta function.** In the nineties, Denef and Loeser defined a "motivic" object $Z_f^{\mathrm{mot}}(s)$ that interpolates the p-adic zeta functions for $p \gg 0$. It captures the geometric nature of the p-adic zeta functions and explains their uniform behaviour in p. Denef and Loeser called $Z_f^{\mathrm{mot}}(s)$ the *motivic zeta function* associated to f. They showed that it is rational over an appropriate ring of coefficients, and they conjectured that its poles correspond to roots of the Bernstein-Sato polynomial as in Conjecture 3.3.1. We will refer to this conjecture as the Motivic Monodromy Conjecture. It will be discussed in more detail in Section 4.6. For a survey on motivic integration and motivic zeta functions, and the precise relation with p-adic zeta functions, we refer to [Ni10a].

Denef and Loeser defined the motivic zeta function by measuring spaces of the form

$$\{\psi \in (k[[t]]/t^{m+1})^n \mid f(\psi) \equiv 0 \mod t^{m+1}\} \tag{3.3}$$

with $m \geq 0$ and k a field of characteristic zero. In contrast with the p-adic case, the set (3.3) is no longer finite, because $k((t))$ is not a local field. Thus we cannot simply count points in (3.3). Instead, one shows that one can interpret (3.3) as the set of k-points on an algebraic $\mathbb{Q}$-variety, and one uses the *Grothendieck ring of varieties* to measure the size of an algebraic variety (see Section 4.1).

In the following section, we will explain an alternative interpretation of the motivic zeta function, due to J. Sebag and the second author [NS07][Ni09a], based on Loeser and Sebag's theory of motivic integration on non-archimedean analytic spaces [LS03]. This interpretation will eventually lead to the definition of the motivic zeta function of an abelian variety and, more generally, a Calabi-Yau variety over a complete discretely valued field.

4. Motivic integration on rigid varieties and the analytic Milnor fiber

4.1. **The Grothendieck ring of varieties.** Let F be a field. We denote by $K_0(\mathrm{Var}_F)$ the *Grothendieck ring of varieties over* F. As an abelian group, $K_0(\mathrm{Var}_F)$ is defined by the following presentation:

- *generators:* isomorphism classes $[X]$ of separated F-schemes of finite type X,
- *relations:* if X is a separated F-scheme of finite type and Y is a closed subscheme of X, then
$$[X] = [Y] + [X \setminus Y].$$
These relations are called *scissor relations*.

By the scissor relations, one has $[X] = [X_{\mathrm{red}}]$ for every separated F-scheme of finite type X, where X_{red} denotes the maximal reduced closed subscheme of X. We endow the group $K_0(\mathrm{Var}_F)$ with the unique ring structure such that
$$[X] \cdot [X'] = [X \times_F X']$$
for all F-varieties X and X'. The identity element for the multiplication is the class $[\mathrm{Spec}\, F]$ of the point. We denote by $\mathbb{L}$ the class $[\mathbb{A}^1_F]$ of the affine line, and by $\mathcal{M}_F$ the localization of $K_0(\mathrm{Var}_F)$ with respect to $\mathbb{L}$.

The scissor relations allow to cut an F-variety into subvarieties. For instance, we have
$$[\mathbb{P}^2_F] = \mathbb{L}^2 + \mathbb{L} + 1$$
in $K_0(\mathrm{Var}_F)$. Since these are the only relations that we impose on the isomorphism classes of F-varieties, taking the class of a variety in the Grothendieck ring should be viewed as the most general way to measure the size of the variety.

For technical reasons, we'll also need to consider the *modified Grothendieck ring of F-varieties* $K_0^{\mathrm{mod}}(\mathrm{Var}_F)$ [NS10a, § 3.8]. This is the quotient of $K_0(\mathrm{Var}_F)$ by the ideal $\mathcal{I}_F$ generated by elements of the form $[X] - [Y]$ where X and Y are separated F-schemes of finite type such that there exists a finite, surjective, purely inseparable F-morphism $Y \to X$.

If F has characteristic zero, then it is easily seen that $\mathcal{I}_F$ is the zero ideal [NS10a, 3.11]. It is not known if $\mathcal{I}_F$ is non-zero if F has positive characteristic. In particular, if F' is a non-trivial finite purely inseparable extension of F, it is not known whether $[\mathrm{Spec}\, F'] \neq 1$ in $K_0(\mathrm{Var}_F)$. With slight abuse of notation, we'll again denote by $\mathbb{L}$ the class of $\mathbb{A}^1_F$ in $K_0^{\mathrm{mod}}(\mathrm{Var}_F)$. We denote by $\mathcal{M}_F^{\mathrm{mod}}$ the localization of $K_0^{\mathrm{mod}}(\mathrm{Var}_F)$ with respect to $\mathbb{L}$.

For a detailed survey on the Grothendieck ring of varieties and some intriguing open questions, we refer to [NS10a].

4.2. **Motivic integration on rigid varieties.** Let R be a complete discrete valuation ring, with quotient field K and perfect residue field k. We fix an absolute value on K by assigning a value $|\pi| \in]0,1[$ to a uniformizer π of R. If R has equal characteristic, then we set $\mathcal{M}_k^R = \mathcal{M}_k$. If R has mixed characteristic, we set $\mathcal{M}_k^R = \mathcal{M}_k^{\mathrm{mod}}$.

If $\mathfrak{X}$ is a formal R-scheme of finite type, then we denote by $\mathfrak{X}_s = \mathfrak{X} \times_R k$ its special fiber (this is a k-scheme of finite type) and by $\mathfrak{X}_\eta$ its generic fiber (this is a quasi-compact and quasi-separated rigid K-variety; see [Ra74] or [BL93]).

Definition 4.2.1. *A rigid K-variety X is called* bounded *if there exists a quasi-compact open subvariety U of X such that $U(K') = X(K')$ for all finite unramified extensions K' of K.*

Definition 4.2.2. *Let X be a rigid K-variety. A weak Néron model for X is a smooth formal R-scheme of finite type $\mathfrak{X}$, endowed with an open immersion $\mathfrak{X}_\eta \to X$, such that $\mathfrak{X}_\eta(K') = X(K')$ for all finite unramified extensions K' of K.*

Note that, if X is separated, then $\mathfrak{X}$ will be separated by [BL93, 4.7].

Theorem 4.2.3 (Bosch-Schlöter)**.** *A quasi-separated smooth rigid K-variety X is bounded if and only if X admits a weak Néron model.*

Proof. Since the generic fiber of a formal R-scheme of finite type is quasi-compact, it is clear that the existence of a weak Néron model implies that X is bounded. The converse implication is [BS95, 3.3]. □

Proposition 4.2.4. *Let X be a bounded quasi-separated smooth rigid K-variety, and let U be as in Definition 4.2.1. If $\mathfrak{X}$ is a regular formal R-model of U, then the R-smooth locus* $\mathrm{Sm}(\mathfrak{X})$ *(endowed with the open immersion* $\mathrm{Sm}(\mathfrak{X})_\eta \hookrightarrow \mathfrak{X}_\eta = U \hookrightarrow X$*) is a weak Néron model for X.*

Proof. If R' is a finite unramified extension of R, with quotient field K', then the specialization map $\mathfrak{X}_\eta \to \mathfrak{X}$ induces a bijection $\mathfrak{X}_\eta(K') = \mathfrak{X}(R')$. Every R'-point on $\mathfrak{X}$ factors through $\mathrm{Sm}(\mathfrak{X})$, by [NS10b, 2.37]. □

A weak Néron model is far from unique, in general, as is illustrated by the following example.

Example 4.2.5. Consider the open unit disc

$$B(0,1^-) = \{z \in \mathrm{Sp}\, K\{x\} \mid |x(z)| < 1\}.$$

Let π be a uniformizer in R and K' a finite unramified extension of K. Then all K'-points in $B(0,1^-)$ are contained in the closed disc

$$B(0,|\pi|) = \{z \in \mathrm{Sp}\, K\{x\} \mid |x(z)| \leq |\pi|\,\}$$

because $|\pi|$ is the largest element in the value group $|(K')^*| = |K^*| = |\pi|^{\mathbb{Z}}$ that is strictly smaller than one. It follows that $B(0,1^-)$ is bounded, and that $\mathfrak{X} = \mathrm{Spf}\, R\{u\}$ is a weak Néron model for $B(0,1^-)$ with respect to the open immersion

$$\mathfrak{X}_\eta = \mathrm{Sp}\, K\{u\} \to B(0,1^-)$$

defined by $x \mapsto \pi^{-1}u$. This weak Néron model is not unique: one could also remark that all the unramified points in $B(0,1^-)$ lie on the union of the circle $|x(z)| = |\pi|$ and the closed disc $B(0,|\pi|^2)$. In this way, we get a weak Néron model $\mathfrak{X}'$ that is the disjoint union of $\mathrm{Spf}\, R\{u\}$ and $\mathrm{Spf}\, R\{v, v^{-1}\}$. Note that $\mathfrak{X}'$ can be obtained by blowing up $\mathfrak{X}$ at the origin of $\mathfrak{X}_s$ and taking the R-smooth locus.

The open annulus

$$A(0; 0^+, 1^-) = \{z \in \mathrm{Sp}\, K\{x\} \mid 0 < |x(z)| < 1\}.$$

is not bounded, since K-points can lie arbitrarily close to zero.

Let X be a smooth rigid K-variety of pure dimension m, and assume that X admits a weak Néron model $\mathfrak{X}$. Let ω be a gauge form on X, i.e., a nowhere vanishing differential form of maximal degree. Then for every connected component C of $\mathfrak{X}_s$, we can consider the order $\mathrm{ord}_C\omega$ of ω along C. It is the unique integer γ such that $\pi^{-\gamma}\omega$ extends to a generator of $\Omega^m_{\mathfrak{X}/R}$ at the generic point of C. In geometric terms, it is the order of the zero or minus the order of the pole of the form ω along C.

Theorem-Definition 4.2.6 (Loeser-Sebag)**.** *Let X be a separated, smooth and bounded rigid K-variety of pure dimension m, and let $\mathfrak{X}$ be a weak Néron model for X. Let ω be a gauge form on X. Then the expression*

$$\int_X |\omega| := \mathbb{L}^{-m} \sum_{C \in \pi_0(\mathfrak{X}_s)} [C]\mathbb{L}^{-\mathrm{ord}_C \omega} \quad \in \mathcal{M}_k^R \tag{4.1}$$

only depends on (X, ω), and not on the choice of weak Néron model $\mathfrak{X}$. We call it the motivic integral *or* motivic volume *of ω on X.*

Proof. This is a slight generalization of the result in [LS03, 4.3.1]. A proof can be found in [HN10c, 2.3]. □

In this way, we can measure the space of unramified points on a bounded separated smooth rigid K-variety X with respect to a motivic measure defined by a gauge form ω on X. Intuitively, one can view the set of unramified points on X as a family of open balls parameterized by the special fiber of a weak Néron model. The gauge form ω renormalizes the volume of each ball in such a way that the total volume of the family is independent of the chosen model. We refer to [NS10b] for more background and further results.

4.3. **The algebraic case.** One can also define the notion of weak Néron model in the algebraic setting. Let K^s be a separable closure of K. Denote by R^{sh} the strict henselization of R in K^s, and by K^{sh} its quotient field. The residue field k^s of R^{sh} is a separable closure of k.

Definition 4.3.1. *Let X be a smooth algebraic K-variety. A weak Néron model is a smooth R-variety $\mathfrak{X}$ endowed with an isomorphism*

$$\mathfrak{X} \times_R K \to X$$

such that the natural map

$$\mathfrak{X}(R^{sh}) \to \mathfrak{X}(K^{sh}) = X(K^{sh}) \tag{4.2}$$

is a bijection.

Note that any k^s-point on $\mathfrak{X}_s$ lifts to an R^{sh}-point on $\mathfrak{X}$, because $\mathfrak{X}$ is smooth and R^{sh} is henselian. Thus $\mathfrak{X}_s$ is empty if and only if $X(K^{sh})$ is empty.

Remark 4.3.2. Since R^{sh} is the direct limit of all finite unramified extensions of R inside K^s, and $\mathfrak{X}$ is of finite type over R, we have that (4.2) is a bijection if and only if $\mathfrak{X}(R') \to X(K')$ is a bijection for every finite unramified extension R' of R. Here K' denotes the quotient field of R'.

Proposition 4.3.3. *Let X be a smooth algebraic K-variety. Then X admits a weak Néron model $\mathfrak{X}$ iff the rigid analytification X^{rig} admits a weak Néron model, i.e., iff X^{rig} is bounded. In that case, the formal $\mathfrak{m}$-adic completion of $\mathfrak{X}$ is a weak Néron model for X^{rig}.*

Proof. This follows from [Ni10d, 3.15, 4.3 and 4.9]. □

In particular, if X is proper over K, then X^{rig} is quasi-compact, so that X admits a weak Néron model.

If X is a smooth K-variety with weak Néron model $\mathfrak{X}$, and ω is a gauge form on X, then one can define the order $\mathrm{ord}_C\omega$ of ω along a connected component C of $\mathfrak{X}_s$ exactly as in the formal-rigid case.

Definition 4.3.4. *Let X be a smooth algebraic K-variety of pure dimension such that the rigid analytification X^{rig} of X is bounded. Let ω be a gauge form on X, and denote by ω^{rig} the induced gauge form on X^{rig}. Then we set*

$$\int_X |\omega| = \int_{X^{\mathrm{rig}}} |\omega^{\mathrm{rig}}| \quad \in \mathcal{M}_k^R.$$

By Proposition 4.3.3, the motivic integral of ω on X can also be computed on a weak Néron model of X:

Proposition 4.3.5. *Let X be a smooth algebraic K-variety of pure dimension m, and assume that X admits a weak Néron model $\mathfrak{X}$. For every gauge form ω on X, we have*

$$\int_X |\omega| = \mathbb{L}^{-m} \sum_{C\in\pi_0(\mathfrak{X}_s)} [C]\mathbb{L}^{-\mathrm{ord}_C\omega}$$

in $\mathcal{M}_k^R$.

4.4. **The analytic Milnor fiber.** Let k be any field, and set $R = k[[t]]$ and $K = k((t))$. We fix a t-adic absolute value on K by choosing a value $|t| \in]0,1[$.

Let X be a k-variety, endowed with a flat morphism

$$f : X \to \mathbb{A}^1_k = \operatorname{Spec} k[t].$$

Let x be a closed point of the special fiber $X_s = f^{-1}(0)$ of f. Taking the completion of f at the point x, we obtain a morphism of formal schemes

$$\widehat{f}_x : \operatorname{Spf} \widehat{\mathcal{O}}_{X,x} \to \operatorname{Spf} R. \tag{4.3}$$

We consider the generic fiber $\mathscr{F}_x$ of $\widehat{f}_x$ in the sense of Berthelot [Bert96]. This is a separated rigid variety over the non-archimedean field K. It is bounded, by [NS08, 5.8]. If f is generically smooth (e.g., if k has characteristic zero and $X \setminus X_s$ is regular) then $\mathscr{F}_x$ is smooth over K.

Definition 4.4.1. *We call $\mathscr{F}_x$ the analytic Milnor fiber of f at the point x.*

Note that the construction of the analytic Milnor fiber is a non-archimedean analog of the definition of the classical Milnor fibration in complex singularity theory. For an explicit dictionary, see [NS10b, 6.1].

Example 4.4.2. Assume that x is k-rational and that X is smooth over k at x. By [EGA4a, 19.6.4], there exists an isomorphism of k-algebras

$$\widehat{\mathcal{O}}_{X,x} \cong k[[x_1, \ldots, x_n]]$$

where n is the dimension of X at x. Viewing f as an element of $k[[x_1, \ldots, x_n]]$, it defines an analytic function on the open unit polydisc

$$B^n(0, 1^-) = \{z \in \operatorname{Sp} K\{x_1, \ldots, x_n\} \mid |x_i(z)| < 1 \text{ for all } i\}.$$

The analytic Milnor fiber $\mathscr{F}_x$ is the closed subvariety of $B^n(0, 1^-)$ defined by the equation $f = t$. Note that, for every finite unramified extension K' of K, the K'-points on $\mathscr{F}_x$ are all contained in the closed polydisc

$$B^n(0, |\pi|) = \{z \in \operatorname{Sp} K\{x_1, \ldots, x_n\} \mid |x_i(z)| \leq |t| \text{ for all } i\}$$

by the same argument as in Example 4.2.5. This shows that $\mathscr{F}_x$ is bounded.

The following result follows immediately from [Berk96, 1.3 and 3.5].

Theorem 4.4.3 (Berkovich)**.** *Assume that k is algebraically closed. Let ℓ be a prime invertible in k, and denote by K^s a separable closure of K. Then for every integer $i \geq 0$, there exists a canonical $G(K^s/K)$-equivariant isomorphism*

$$H^i(\mathscr{F}_x \widehat{\times}_K \widehat{K^s}, \mathbb{Q}_\ell) \cong R^i\psi_f(\mathbb{Q}_\ell)_x. \tag{4.4}$$

In the left hand side of (4.4), we take Berkovich's étale cohomology for K-analytic spaces [Berk93]. In the right hand side, $R\psi_f(\mathbb{Q}_\ell)$ denotes the complex of étale ℓ-adic nearby cycles associated to f. If $k = \mathbb{C}$, then by Deligne's comparison theorem [SGA7b, Exp.XIV], there exists a canonical isomorphism

$$R^i\psi_f(\mathbb{Q}_\ell)_x \cong R^i\psi_{f^{\mathrm{an}}}(\mathbb{Q}_\ell)_x$$

where $f^{\mathrm{an}} : X^{\mathrm{an}} \to \mathbb{C}$ is the complex analytification of f and $\psi_{f^{\mathrm{an}}}$ is the complex analytic nearby cycles functor. Under this isomorphism, the action of the canonical generator of $G(K^s/K) = \mu(\mathbb{C})$ on $R^i\psi_f(\mathbb{Q}_\ell)_x$ corresponds to the monodromy transformation on $R^i\psi_{f^{\mathrm{an}}}(\mathbb{Q}_\ell)_x$. This means that we can read the local monodromy eigenvalues of f^{an} at x from the étale cohomology of the analytic Milnor fiber.

The following proposition shows that the analytic Milnor fiber $\mathscr{F}_x$ completely determines the formal germ $\widehat{f}_x$ of f at x, if X is normal at x. We denote by $\mathcal{O}(\mathscr{F}_x)$ the K-algebra of analytic functions on $\mathscr{F}_x$.

Proposition 4.4.4 (de Jong [dJ95], Prop. 7.4.1; see also [Ni09a], Prop. 8.8)**.** *If X is normal at x, then there exists a natural isomorphism of R-algebras*

$$\widehat{\mathcal{O}}_{X,x} \cong \{h \in \mathcal{O}(\mathscr{F}_x) \mid |h(z)| \leq 1 \textit{ for all } z \in \mathscr{F}_x\}.$$

There is an interesting relation between the Berkovich topology on $\mathscr{F}_x$ and the limit mixed Hodge structure on the nearby cohomology of f at x; see [Ni10c].

4.5. **The Gelfand-Leray form.** We keep the notations of Section 4.4, and we assume that k has characteristic zero and that X is smooth over k at the point x. Replacing X by an open neighbourhood of x, we can assume that f is smooth on $X^o = X \setminus X_s$, of relative dimension m, and that $\Omega^{m+1}_{X^o/k}$ is free, i.e., that X admits a gauge form ϕ. Then the exact complex

$$\Omega^{m-1}_{X^o/k} \xrightarrow{\wedge df} \Omega^{m}_{X^o/k} \xrightarrow{\wedge df} \Omega^{m+1}_{X^o/k} \longrightarrow 0$$

induces an isomorphism of sheaves

$$\Omega^m_{X^o/\mathbb{A}^1_k} \to \Omega^{m+1}_{X^o/k}$$

and thus an isomorphism of $\mathcal{O}(X^o)$-modules

$$\Omega^m_{X^o/\mathbb{A}^1_k}(X^o) \to \Omega^{m+1}_{X^o/k}(X^o).$$

The inverse image in $\Omega^m_{X^o/\mathbb{A}^1_k}(X^o)$ of the restriction of ϕ to X^o is called the *Gelfand-Leray form* on X^o associated to f and ϕ. It is denoted by ϕ/df. It induces a gauge form on the analytic Milnor fiber $\mathscr{F}_x$, since $\mathscr{F}_x$ is an open rigid sub-K-variety of the rigid analytification of $X \times_{k[t]} K$ [Bert96, 0.2.7 and 0.3.5]. We denote this gauge form again by ϕ/df. It can be constructed intrinsically on $\mathscr{F}_x$; see Proposition 4.4.4 and [Ni09a, § 7.3].

4.6. **The motivic zeta function and the motivic monodromy conjecture.** We keep the notations of Section 4.4. We assume that k has characteristic zero, and that X is smooth at x, of dimension n. For simplicity, we suppose that x is k-rational. Let ϕ be a gauge form on some open neighbourhood of x in X, and consider the Gelfand-Leray form ϕ/df on $\mathscr{F}_x$ constructed in Section 4.5. We set $\omega = t \cdot \phi/df$. This is a gauge form on $\mathscr{F}_x$. For every integer $d > 0$, we set $K(d) = k((\sqrt[d]{t}))$. This is a totally ramified extension of K of degree d. We set $\mathscr{F}_x(d) = \mathscr{F}_x \times_K K(d)$. For every differential form ω' on $\mathscr{F}_x$, we denote by $\omega'(d)$ the pullback of ω' to $\mathscr{F}_x(d)$.

We denote by $Z_{f,x}(T) \in \mathcal{M}_k[[T]]$ Denef and Loeser's local motivic zeta function of f at the point x (obtained from the zeta function in [DL01, 3.2.1] by taking the fiber at x and forgetting the μ-action). The reader who is unfamiliar with Denef and Loeser's definition may take the following theorem as a definition.

Theorem 4.6.1 (Nicaise-Sebag)**.** *We have*

$$Z_{f,x}(T) = \mathbb{L}^{n-1} \sum_{d>0} \left(\int_{\mathscr{F}_x(d)} |\omega(d)| \right) T^d \tag{4.5}$$

in $\mathcal{M}_k[[T]]$.

Proof. Note that, for every d in $\mathbb{N}$, we have

$$\int_{\mathscr{F}_x(d)} |\omega(d)| = \mathbb{L}^{-d} \int_{\mathscr{F}_x(d)} |(\phi/df)(d)|$$

in $\mathcal{M}_k$, since t has valuation d in $K(d)$. Thus the theorem is a reformulation of [Ni09a, 9.7], which is a consequence of the comparison theorem in [NS07, 9.10]. $\square$

The proof of Theorem 4.6.1 is based on an explicit construction of weak Néron models for the rigid varieties $\mathscr{F}_x(d)$, starting from an embedded resolution of singularities for (X, X_s). In this way, one obtains an explicit formula for the right hand side in (4.5) in terms of such a resolution, and one can compare this expression to the formula for $Z_{f,x}(T)$ obtained by Denef and Loeser [DL01, 3.3.1]. This formula implies in particular that $Z_{f,x}[T]$ is contained in the subring

$$\mathcal{M}_k \left[T, \frac{1}{1 - \mathbb{L}^a T^b} \right]_{a \in \mathbb{Z}_{<0},\, b \in \mathbb{Z}_{>0}}$$

of $\mathcal{M}_k[[T]]$.

If the residue field k_x of x is not k, one can adapt the construction as follows. Since k has characteristic zero, k_x is a separable extension of k so that the morphism $\widehat{f}_x$ from (4.3) factors through a morphism

$$\widehat{\mathcal{O}}_{X,x} \to \operatorname{Spf} k_x[[t]]$$

by [EGA4a, 19.6.2]. In this way, we can view $\mathscr{F}_x$ as a rigid variety over $K_x = k_x((t))$. Since K_x is separable over K, the natural morphism $\Omega^i_{\mathscr{F}_x/K_x} \to \Omega^i_{\mathscr{F}_x/K}$ is an isomorphism for all i, so that we can consider the Gelfand-Leray form ϕ/df as an element of $\Omega^m_{\mathscr{F}_x/K_x}$. Then the equality (4.5) holds in $\mathcal{M}_{k_x}[[T]]$.

Conjecture 4.6.2 (Motivic Monodromy Conjecture)**.** *Assume that k is a subfield of $\mathbb{C}$. There exists a finite subset $\mathscr{S}$ of $\mathbb{Z}_{<0} \times \mathbb{Z}_{>0}$ such that $Z_{f,x}(T)$ belongs to the subring*

$$\mathcal{M}_{k_x}\left[T, \frac{1}{1-\mathbb{L}^a T^b}\right]_{(a,b)\in\mathscr{S}}$$

of $\mathcal{M}_{k_x}[[T]]$, and such that for every couple (a,b) in $\mathscr{S}$, the quotient a/b is a root of the Bernstein-Sato polynomial $b_f(s)$ of f. In particular, there exists a point y of $X(\mathbb{C})$ such that $f(y) = 0$ and such that $\exp(2\pi i a/b)$ *is a local monodromy eigenvalue of f at the point y.*

Remark 4.6.3. One can drop the condition that k is a subfield of $\mathbb{C}$ by using ℓ-adic nearby cycles to define the notion of local monodromy eigenvalue. This does not yield a more general conjecture, since by the Lefschetz principle, one can always reduce to the case where k is a subfield of $\mathbb{C}$.

One needs to be careful when speaking about poles of the zeta function, since $\mathcal{M}_{k_x}$ is not a domain. A precise definition is given in [RV03]. The formulation in Conjecture 4.6.2 implies that, for any reasonable definition of pole in this context, the poles of $Z_{f,x}(\mathbb{L}^{-s})$ are of the form a/b, with $(a,b) \in \mathscr{S}$. With some additional work, one can define the order of a pole [RV03], and conjecture that the order of a pole of $Z_{f,x}(\mathbb{L}^{-s})$ is at most the multiplicity of the corresponding root of $b_f(s)$.

4.7. **Relation with the p-adic monodromy conjecture.** Let us explain the precise relation between Conjectures 4.6.2 and 3.3.1. In [DL01, 3.2.1], Denef and Loeser define the motivic zeta function $Z_f(T)$ (there denoted by $Z(T)$) associated to the morphism f. It carries more structure than we've considered so far: it is a formal power series with coefficients in the *equivariant* Grothendieck ring of X_s-varieties $\mathcal{M}^{\mu}_{X_s}$. The elements of this ring are virtual classes of X_s-varieties that carry an action of the profinite group scheme μ of roots of unity. The X_s-structure allows to consider various "motivic Schwartz-Bruhat functions" and the μ-action allows to twist the motivic zeta function by "motivic characters", like in the p-adic case; see [DL98]. The motivic zeta function that we've alluded to in Section 3.4 corresponds to the trivial character; it is the "naïve" motivic zeta function from [DL01, 3.2.1]. If $k = \mathbb{Q}$ and $X = \mathbb{A}^n_k$, then for $p \gg 0$, we can specialize the naïve motivic zeta function of f to the p-adic one, by counting rational points on the reductions of the coefficients modulo p. This is explained in [Ni10a, §5.3].

The local zeta function $Z_{f,x}(T)$ that we've considered above is obtained from $Z_f(T)$ by applying the morphism

$$\mathcal{M}^{\mu}_{X_s} \to \mathcal{M}_{k_x}$$

(base change to x and forgetting the μ-structure) to the coefficients of $Z_f(T)$. In an unpublished manuscript, the second author has shown that it is possible to recover the μ-structure on $Z_{f,x}(T)$ by considering the Galois action on the extensions $K(d)$ of K.

One can formulate Conjecture 4.6.2 for $Z_f(T)$ instead of $Z_{f,x}(T)$, as follows:

Conjecture 4.7.1 (Motivic Monodromy Conjecture II)**.** *Assume that k is a subfield of $\mathbb{C}$. There exists a finite subset $\mathscr{S}$ of $\mathbb{Z}_{<0}\times\mathbb{Z}_{>0}$ such that $Z_f(T)$ belongs to the subring*

$$\mathcal{M}^{\mu}_{X_s}\left[T,\frac{1}{1-\mathbb{L}^aT^b}\right]_{(a,b)\in\mathscr{S}}$$

of $\mathcal{M}^{\mu}_{X_s}[[T]]$, and such that for every couple (a,b) in $\mathscr{S}$, the quotient a/b is a root of the Bernstein-Sato polynomial $b_f(s)$ of f. In particular, there exists a point y of $X(\mathbb{C})$ such that $f(y)=0$ and such that $\exp(2\pi ia/b)$ *is a local monodromy eigenvalue of f at the point y.*

This conjecture implies

(1) Conjecture 4.6.2, since we can specialize $Z_f(T)$ to $Z_{f,x}(T)$ by taking fibers at x and forgetting the μ-structure,
(2) Conjecture 3.3.1, because $Z_f(T)$ can be specialized to the p-adic zeta function,
(3) more generally, the p-adic monodromy conjecture for zeta functions that are twisted by characters [DL98, §2.4].

Various weaker reformulations of Conjecture 4.7.1 have appeared in the literature (e.g. in [DL98, §2.4]). The formulation we use seems to be part of general folklore. We attribute it to Denef and Loeser.

5. The motivic zeta function of an abelian variety

5.1. **Some notation.** Let R be a complete discrete valuation ring with maximal ideal $\mathfrak{m}$, fraction field K and algebraically closed residue field k. We denote by p the characteristic exponent of k, and by $\mathbb{N}'$ the set of strictly positive integers that are prime to p. We fix a prime $\ell\neq p$ and a separable closure K^s of K. The Galois group $G(K^s/K)$ is called the *inertia group* of K.

A finite extension of K is called *tame* if its degree is prime to p. For every d in $\mathbb{N}'$, the field K admits a unique degree d extension $K(d)$ in K^s. It is obtained by joining a d-th root of a uniformizer to K. The extension $K(d)/K$ is Galois, with Galois group $\mu_d(k)$.

The union of the fields $K(d)$ is a subfield of K^s, called the *tame closure* K^t of K. The Galois group $G(K^t/K)$ is called the *tame inertia group* of K. It is canonically isomorphic to the procyclic group

$$\mu'(k)=\lim_{\substack{\longleftarrow\\ d\in\mathbb{N}'}}\mu_d(k)$$

where the elements in $\mathbb{N}'$ are ordered by divisibility and the transition morphisms in the projective system are given by

$$\mu_{de}(k)\to\mu_d(k):x\mapsto x^e$$

for all d,e in $\mathbb{N}'$. We call every topological generator of $G(K^t/K)$ a *tame monodromy operator*. The Galois group $P=G(K^s/K^t)$ is a pro-p-group which is called the *wild inertia subgroup* of $G(K^s/K)$. We have a short exact sequence

$$1 \to P \to G(K^s/K) \to G(K^t/K) \to 1.$$

5.2. **Néron models and semi-abelian reduction.** Let A be an abelian K-variety of dimension g. It is not always possible to extend A to an abelian scheme over R. However, there exists a canonical way to extend A to a smooth commutative group scheme over R, the so-called *Néron model* of A.

Definition 5.2.1. *A Néron model of A is a smooth R-scheme of finite type $\mathcal{A}$, endowed with an isomorphism*

$$\mathcal{A} \times_R K \to A,$$

such that the natural map

$$\mathrm{Hom}_R(T, \mathcal{A}) \to \mathrm{Hom}_K(T \times_R K, A) \tag{5.1}$$

is a bijection for every smooth R-scheme T.

Thus $\mathcal{A}$ is the minimal smooth R-model of A. The existence of a Néron model was first proved by A. Néron [Ne64]. For a modern scheme-theoretic treatment of the theory and an accessible proof of Néron's theorem, we refer to [BLR90]. The universal property of the Néron model implies that the Néron model $\mathcal{A}$ is unique up to unique isomorphism, and that the K-group structure on A extends uniquely to a commutative R-group structure on $\mathcal{A}$. Taking for T the spectrum of a finite unramified extension of R, we see that $\mathcal{A}$ is also a weak Néron model for A.

The special fiber $\mathcal{A}_s := \mathcal{A} \times_R k$ is a smooth commutative algebraic k-group. We denote by $\mathcal{A}_s^o$ the identity component of $\mathcal{A}_s$, i.e., the connected component containing the identity point for the group structure. The quotient $\Phi_A := \mathcal{A}_s/\mathcal{A}_s^o$ is called the *component group*. It is a finite étale group scheme over k whose group of k-points corresponds bijectively to the the set of connected components of $\mathcal{A}_s$. Since k is assumed to be algebraically closed, we will not distinguish between the group scheme Φ_A and the abstract group $\Phi_A(k)$.

The identity component $\mathcal{A}_s^o$ fits into a canonical short exact sequence of algebraic k-groups, the *Chevalley decomposition*,

$$0 \to T \times_k U \to \mathcal{A}_s^o \to B \to 0 \tag{5.2}$$

where B is an abelian variety, T is a torus and U is a unipotent group commonly referred to as the *unipotent radical* of $\mathcal{A}_s^o$. We call the dimension of T the *toric rank* of A, and the dimension of U the *unipotent rank* of A.

Definition 5.2.2. *We say that A has semi-abelian reduction if the unipotent rank of $\mathcal{A}_s^o$ is zero.*

A celebrated result by A. Grothendieck, the Semi-Stable Reduction Theorem for abelian varieties [SGA7a, IX.3.6], asserts that there exists a finite separable extension K'/K such that $A \times_K K'$ has semi-abelian reduction over the integral closure R' of R in K'. Inside our fixed separable closure K^s of K, there exists a unique minimal extension L with this property, and it is Galois over K. By [SGA7a, IX.3.8], L is the fixed field of the subgroup of $G(K^s/K)$ consisting of the elements that act unipotently on the ℓ-adic Tate module $T_\ell A$ of A. If L is a tame extension of K then we say that A is *tamely ramified*. Since the P-action on $T_\ell A$ factors through a finite quotient of P [LO85, p.3], A is tamely ramified if and only if P acts trivially on $T_\ell A$. In that case, P acts trivially on the ℓ-adic cohomology of A, and the natural morphism

$$H^i(A \times_K K^t, \mathbb{Q}_\ell) \to H^i(A \times_K K^s, \mathbb{Q}_\ell)$$

is an isomorphism for every i in $\mathbb{N}$.

5.3. **The base change conductor and the potential toric rank.** In [Ch00], Chai introduced an invariant that measures how far the abelian K-variety A is from having semi-abelian reduction. He called it the *base change conductor* of A and denoted it by $c(A)$. It is defined as follows. Let K'/K be a finite separable extension such that A acquires semi-abelian reduction over K', and let $\mathcal{A}'$ be the Néron model of $A \times_K K'$. By the universal property of the Néron model $\mathcal{A}'$, there is a unique morphism

$$h : \mathcal{A} \times_R R' \to \mathcal{A}' \tag{5.3}$$

that extends the canonical isomorphism between the generic fibers. The induced map

$$\mathrm{Lie}(h) : \mathrm{Lie}(\mathcal{A} \times_R R') \to \mathrm{Lie}(\mathcal{A}')$$

is an injective homomorphism of free R'-modules of rank g, so that $\mathrm{coker}(\mathrm{Lie}(h))$ is an R'-module of finite length. The rational number

$$c(A) := [K' : K]^{-1} \cdot \mathrm{length}_{R'}(\mathrm{coker}(\mathrm{Lie}(h)))$$

is independent of the choice of K'.

The importance of the Semi-Stable Reduction Theorem lies in the fact that, if A has semi-abelian reduction, then h is an open immersion, so that it induces an isomorphism between the identity components of $\mathcal{A} \times_R R'$ and $\mathcal{A}'$ [SGA7a, IX.3.3] (the number of connected components of $\mathcal{A}$ might still change, though; this will be discussed below). Thus $c(A)$ vanishes if A has semi-abelian reduction. Conversely, if $c(A) = 0$ then h must be étale, and the fact that h restricts to an isomorphism between the generic fibers then implies that h is an open immersion. Thus $c(A)$ is zero if and only if A has semi-abelian reduction.

Another invariant that will be important for us is the toric rank of $A \times_K K'$. We call this value the *potential toric rank* of A, and denote it by $t_{\mathrm{pot}}(A)$. Again, it is independent of the choice of K'. Moreover, it is the maximum of the toric ranks of the abelian varieties $A \times_K K''$ as K'' ranges over all the finite separable extensions of K. The potential toric rank is a measure for the potential degree of degeneration of A over the closed point of $\mathrm{Spec}\, R$; it vanishes if and only if, after a finite separable extension of the base field K, the abelian variety A extends to an abelian scheme over R. In this case, we say that A has *potential good reduction.* If $t_{pot}(A)$ has the largest possible value, namely, the dimension of A, then we say that A has *potential purely multiplicative reduction.* Thus A has potential purely multiplicative reduction if and only if the identity component of $\mathcal{A}'_s$ is a torus.

5.4. **The motivic zeta function of an abelian variety.** For every d in $\mathbb{N}'$, we set $A(d) = A \times_K K(d)$ and we denote by $\mathcal{A}(d)$ the Néron model of $A(d)$. For every gauge form ω on A, we denote by $\omega(d)$ its pullback to $A(d)$. We define the order $\mathrm{ord}_{\mathcal{A}(d)_s^o}\omega(d)$ of $\omega(d)$ along $\mathcal{A}(d)_s^o$ as in Section 4.2.

Definition 5.4.1. *A gauge form ω on A is* distinguished *if it is the restriction to A of a generator of the free rank one $\mathcal{O}_A$-module $\Omega^g_{\mathcal{A}/R}$.*

It is clear from the definition that a distinguished gauge form always exists, and that it is unique up to multiplication with a unit in R. Note that a gauge form ω on A is distinguished if and only if $\mathrm{ord}_{\mathcal{A}_s^o}\omega = 0$.

In general, a distinguished gauge form on A does not remain distinguished under base change to a finite tame extension $K(d)$ of K. To measure the defect, we introduce the following definition.

Definition 5.4.2. *Let A be an abelian K-variety, and let ω be a distinguished gauge form on A. The order function of A is the function*

$$\mathrm{ord}_A : \mathbb{N}' \to \mathbb{N} : d \mapsto -\mathrm{ord}_{\mathcal{A}(d)_s^o}\omega(d).$$

This definition does not depend on the choice of distinguished gauge form, since multiplying ω with a unit in R does not affect the order of $\omega(d)$ along $\mathcal{A}(d)_s^o$. The fact that ord_A takes its values in $\mathbb{N}$ follows easily from the existence of the morphism h in (5.3), for arbitrary finite extensions K' of K.

Definition 5.4.3. *Let A be an abelian K-variety, and let ω be a distinguished gauge form on A. We define the motivic zeta function $Z_A(T)$ of A as*

$$Z_A(T) = \sum_{d \in \mathbb{N}'} [\mathcal{A}(d)_s]\mathbb{L}^{\mathrm{ord}_A(d)} T^d \in \mathcal{M}_k[[T]].$$

The following proposition gives an interpretation of $Z_A(T)$ in terms of the volumes of the "motivic Haar measures" $|\omega(d)|$.

Proposition 5.4.4. *Let A be an abelian K-variety of dimension g, and let ω be a distinguished gauge form on A. The image of $Z_A(T)$ in the quotient ring $\mathcal{M}_k^R[[T]]$ of $\mathcal{M}_k[[T]]$ is equal to*

$$\mathbb{L}^g \sum_{d \in \mathbb{N}'} \left(\int_{A(d)} |\omega(d)| \right) T^d.$$

Proof. We've already observed that every Néron model is also a weak Néron model. The gauge form ω is translation-invariant, so that

$$\mathrm{ord}_C \omega(d) = \mathrm{ord}_{\mathcal{A}(d)_s^o}\omega(d)$$

for every connected component C of $\mathcal{A}(d)_s$. Now the result follows immediately from the definition of the motivic integral, and the fact that

$$[\mathcal{A}(d)_s] = \sum_{C \in \pi_0(\mathcal{A}(d)_s)} [C]$$

by the scissor relations in $\mathcal{M}_k$. □

5.5. **The monodromy conjecture.** Now we come to the formulation of the main result of [HN10c], which is a variant of Conjecture 3.3.3 for abelian varieties. For every integer $d \geq 0$, we denote by $\Phi_d(t) \in \mathbb{Z}[t]$ the cyclotomic polynomial whose roots are the primitive d-th roots of unity. For every rational number q, we denote by $\tau(q)$ its order in the group $\mathbb{Q}/\mathbb{Z}$.

Theorem 5.5.1 (Monodromy conjecture for abelian varieties)**.** *Let A be a tamely ramified abelian variety of dimension g, and let σ be a tame monodromy operator in $G(K^t/K)$.*

(1) *The motivic zeta function $Z_A(T)$ belongs to the subring*
$$\mathcal{M}_k\left[T, \frac{1}{1-\mathbb{L}^a T^b}\right]_{(a,b)\in\mathbb{N}\times\mathbb{Z}_{>0}, a/b=c(A)}$$
of $\mathcal{M}_k[[T]]$. The zeta function $Z_A(\mathbb{L}^{-s})$ has a unique pole at $s=c(A)$, of order $t_{pot}(A)+1$.

(2) *The cyclotomic polynomial $\Phi_{\tau(c(A))}(t)$ divides the characteristic polynomial of the tame monodromy operator σ on $H^g(A\times_K K^t,\mathbb{Q}_\ell)$. Thus for every embedding $\mathbb{Q}_\ell\hookrightarrow\mathbb{C}$, the value $\exp(2\pi c(A)i)$ is an eigenvalue of σ on $H^g(A\times_K K^t,\mathbb{Q}_\ell)$.*

We'll briefly sketch the main ideas of the proof. The first step is to refine the expression for the motivic zeta function, as follows. For every d in $\mathbb{N}'$, we denote by $t_A(d)$ and $u_A(d)$ the toric, resp. unipotent rank of $A(d)$, and we denote by $B_A(d)$ the abelian quotient in the Chevalley decomposition of $\mathcal{A}(d)^o_s$. Moreover, we denote by $\phi_A(d)=|\Phi_{A(d)}|$ the number of connected components of the k-group $\mathcal{A}(d)_s$.

Proposition 5.5.2. *For every abelian K-variety A, we have*
$$\begin{aligned} Z_A(T) &= \sum_{d\in\mathbb{N}'}[\mathcal{A}(d)_s]\mathbb{L}^{ord_A(d)}T^d \\ &= \sum_{d\in\mathbb{N}'}\left(\phi_A(d)\cdot(\mathbb{L}-1)^{t_A(d)}\cdot\mathbb{L}^{u_A(d)+ord_A(d)}\cdot[B_A(d)]T^d\right)\end{aligned}$$
in $\mathcal{M}_k[[T]]$.

Proof. The first equality is simply the definition of the zeta function. For every $d\in\mathbb{N}'$, the connected components of $\mathcal{A}(d)_s$ are all isomorphic to $\mathcal{A}(d)^o_s$, because k is algebraically closed. Thus by the scissor relations in the Grothendieck ring, one has
$$[\mathcal{A}(d)_s]=\phi_A(d)\cdot[\mathcal{A}(d)^o_s].$$
Now consider the Chevalley decomposition
$$0\to T_A(d)\times_k U_A(d)\to\mathcal{A}(d)^o_s\to B_A(d)\to 0$$
of $\mathcal{A}(d)^o_s$. The torus $T_A(d)$ is isomorphic to $\mathbb{G}_{m,k}^{t_A(d)}$, and $U_A(d)$ is a successive extension of additive groups $\mathbb{G}_{a,k}$. It follows that $\mathcal{A}(d)^o_s\to B_A(d)$ is a Zariski-locally trivial fibration. Moreover, as a k-variety, $U_A(d)$ is isomorphic to $\mathbb{A}_k^{u_A(d)}$. Thus
$$[\mathcal{A}(d)^o_s]=(\mathbb{L}-1)^{t_A(d)}\cdot\mathbb{L}^{u_A(d)}\cdot[B_A(d)]$$
in $\mathcal{M}_k$. □

Thus the study of $Z_A(T)$ can be split up into the following subproblems:

(1) How do $t_A(d)$, $u_A(d)$ and $B_A(d)$ vary with d?
(2) What is the shape of the order function ord_A?
(3) How does $\phi_A(d)$ vary with d?

Our main tool in the study of (1) was a theorem due to B. Edixhoven [Ed92], which says that for every $d\in\mathbb{N}'$, the Néron model $\mathcal{A}$ is canonically isomorphic to the fixed locus of the $G(K(d)/K)$-action on the Weil restriction of $\mathcal{A}(d)$ to R. This result enabled us to show that $t_A(d)$, $u_A(d)$ and $[B_A(d)]$ only depend on the residue class of d modulo e, where e is the degree of the minimal extension of K where A acquires semi-abelian reduction. In the same paper, Edixhoven constructs a filtration on $\mathcal{A}_s$ by closed algebraic subgroups, indexed by $\mathbb{Q}\cap[0,1]$. This filtration measures the behaviour of the Néron model of A under tamely ramified base change. The *jumps* of A are the indices where the subgroup changes. Edixhoven related these jumps to the Galois action of $G(K(e)/K) = \mu_e(k)$ on the k-vector space $\mathrm{Lie}(\mathcal{A}(e)^o_s)$. We deduced from Edixhoven's theory that $c(A)$ is the sum of the jumps of A and that on every residue class of $\mathbb{N}'$ modulo e, the function ord_A is affine with slope $c(A)$. The function ord_A is responsable for the pole of $Z_A(\mathbb{L}^{-s})$ at $s = c(A)$.

To control the behaviour of $\phi_A(d)$ turned out to be rather involved, we treated this in the separate paper [HN10a]. There, we used rigid uniformization of A in the sense of [BX96] to reduce to the case of tori and abelian varieties with potential good reduction, where more explicit methods could be used to describe the change in the component groups under ramified base extensions. In this way, we obtained the following result.

Theorem 5.5.3. *Let A be an abelian K-variety, and let e be the degree of the minimal extension of K where A acquires semi-abelian reduction. Denote by $t(A)$ the toric rank of A and by $\phi(A)$ the number of connected components of the Néron model of A. Assume either that A is tamely ramified or that A has potential purely multiplicative reduction. Then for every element d of $\mathbb{N}'$ that is prime to e, we have*

$$\phi_A(d) = \phi(A) \cdot d^{t(A)}.$$

This result was sufficient for our purposes. The behaviour of $\phi_A(d)$ is responsible for the order $t_{\mathrm{pot}}(A) + 1$ of the unique pole of the zeta function.

It remains to prove the relation between the base change conductor and the tame monodromy action on $H^g(A \times_K K^t, \mathbb{Q}_\ell)$. Here we again used Edixhoven's theory and we showed how to compute the eigenvalues of σ on $H^1(A \times_K K^t, \mathbb{Q}_\ell)$ from the Galois action of $\mu_e(k)$ on $\mathrm{Lie}(\mathcal{A}(e)^o_s)$.

5.6. **Strong version of the monodromy conjecture.** It is natural to ask for an analog of Conjecture 3.3.1 for abelian varieties. There is no good notion of Bernstein polynomials in this setting. However, the multiplicities of the roots of the Bernstein polynomial of a complex hypersurface singularity are closely related to the sizes of the Jordan blocks of the monodromy action on the cohomology of the Milnor fiber, so one may ask if the order of the pole of $Z_A(T)$ is related to Jordan blocks of the tame monodromy action on $H^g(A \times_K K^t, \mathbb{Q}_\ell)$. We've shown in [HN10d] that this is indeed the case.

Theorem 5.6.1. *Let A be a tamely ramified abelian K-variety of dimension g. For every tame monodromy operator σ in $G(K^t/K)$ and every embedding of $\mathbb{Q}_\ell$ in $\mathbb{C}$, the value $\alpha = \exp(2\pi c(A)i)$ is an eigenvalue of σ on $H^g(A \times_K K^t, \mathbb{Q}_\ell)$. Each Jordan block of σ on $H^g(A \times_K K^t, \mathbb{Q}_\ell)$ has size at most $t_{pot}(A) + 1$, and σ has a Jordan block with eigenvalue α on $H^g(A \times_K K^t, \mathbb{Q}_\ell)$ with size $t_{pot}(A) + 1$.*

In the case $K = \mathbb{C}((t))$, we also gave in [HN10d] a Hodge-theoretic interpretation of the jumps in Edixhoven's filtration, in terms of the limit mixed Hodge structure associated to A.

5.7. **Cohomological interpretation.** The motivic zeta function of an abelian K-variety admits a cohomological interpretation, by [Ni09b]. We consider the unique ring morphism

$$\chi : \mathcal{M}_k \to \mathbb{Z}$$

that sends the class of a k-variety X to the ℓ-adic Euler characteristic

$$\chi(X) = \sum_{i \geq 0} (-1)^i \dim H^i_c(X, \mathbb{Q}_\ell).$$

Since χ sends $\mathbb{L}$ to 1, the image of $Z_A(T)$ under the morphism $\mathcal{M}_k[[T]] \to \mathbb{Z}[[T]]$ induced by χ is equal to

$$\chi(Z_A(T)) = \sum_{d \in \mathbb{N}'} \chi(\mathcal{A}(d)_s) T^d.$$

Theorem 5.7.1. *Let A be a tamely ramified abelian K-variety. For every d in $\mathbb{N}'$, we have*

$$\sum_{i \geq 0} (-1)^i \mathrm{Trace}(\sigma^d \,|\, H^i(A \times_K K^t, \mathbb{Q}_\ell)) = \chi(\mathcal{A}(d)_s).$$

This value equals $\phi_A(d)$ if $A(d)$ has purely additive reduction (i.e, if $\mathcal{A}(d)^o_s$ is unipotent) and it equals zero in all other cases.

This result can be seen as a particular case of a more general theory that expresses a certain motivic measure for the number of rational points on a K-variety X in terms of the Galois action on the ℓ-adic cohomology of X; see [Ni09a, Ni10d, Ni10e]. For a similar formula for the zeta function of a hypersurface singularity, see [DL02, 1.1], [NS07, 9.12] and [Ni09a, 9.9].

6. Degenerations of Calabi-Yau varieties

We keep the notations from Section 5.1. To simplify the presentation, we assume that k has characteristic zero. Part of the theory below can be developed also in the case where k has positive characteristic; see [HN10b]. In particular, the definition of the zeta function remains valid.

6.1. **Motivic zeta functions of Calabi-Yau varieties.**

Definition 6.1.1. *A Calabi-Yau variety over a field F is a smooth, proper, geometrically connected F-variety with trivial canonical sheaf.*

For instance, every abelian variety is Calabi-Yau. By definition, every Calabi-Yau variety admits a gauge form. In the definition of a Calabi-Yau variety X, one often includes the additional condition that $h^{i,0}(X)$ vanishes for $0 < i < \dim X$. We do not impose this condition.

Let X be a Calabi-Yau variety over K. We will now define the motivic zeta function $Z_X(T)$ of X, in a way that generalizes our construction for abelian varieties. There is no canonical weak Néron model as in the abelian case, but we can generalize the expression for the zeta function in Proposition 5.4.4 in terms of the motivic volume of an appropriate gauge form on X. We first show how the notion of distinguished gauge form extends to Calabi-Yau varieties.

Proposition 6.1.2. *Let X be a Calabi-Yau variety over K, and let ω be a gauge form on X. Then for every weak Néron model $\mathfrak{X}$ of X, the value*

$$\mathrm{ord}(X, \omega) \quad := \quad \min\{ord_C(\omega) \,|\, C \in \pi_0(\mathfrak{X}_s)\} \quad \in \mathbb{Z} \cup \{-\infty\}$$

only depends on the pair (X, ω), and not on $\mathfrak{X}$. By convention, we set $\min \emptyset = -\infty$.

Proof. Since every connected component of $\mathfrak{X}_s$ has the same dimension as X, the value $\mathrm{ord}(X,\omega)$ is precisely minus the *virtual dimension* of the motivic integral

$$\int_X |\omega| = \mathbb{L}^{-\dim(X)} \sum_{C\in\pi_0(\mathcal{X}_s)} [C]\mathbb{L}^{-\mathrm{ord}_C\omega} \in \mathcal{M}_k.$$

The virtual dimension of an element α in $\mathcal{M}_k$ can be defined, for instance, as half of the degree of the *Poincaré polynomial* of α [Ni10d, §8]. $\square$

Note that $\mathrm{ord}(X,\omega) = -\infty$ if and only if $\mathfrak{X}_s$ is empty, i.e., if and only if $X(K)$ is empty.

Definition 6.1.3. *Let X be a Calabi-Yau variety over K. A distinguished gauge form on X is a gauge form ω such that* $\mathrm{ord}(X,\omega) = 0$.

Thus, a distinguished gauge form on X extends to a relative differential form on every weak Néron model, in a "minimal" way. It is clear that X admits a distinguished gauge form iff X has a K-rational point, and that a distinguished gauge form is unique up to multiplication with a unit in R.

Definition 6.1.4. *Let X be a Calabi-Yau variety over K, and assume that X has a K-rational point. Let ω be a distinguished gauge form on X. We define the motivic zeta function $Z_X(T)$ of X by*

$$Z_X(T) = \mathbb{L}^{\dim(X)} \sum_{d\in\mathbb{N}} \left(\int_{X(d)} |\omega(d)|\right) T^d \quad \in \mathcal{M}_k[[T]].$$

This definition only depends on X, and not on the choice of distinguished gauge form ω, since multiplying ω with a unit in R does not affect the motivic integral of ω on X. It follows from Proposition 5.4.4 that, when X is an abelian variety, Definition 6.1.4 is equivalent to Definition 5.4.3.

By embedded resolution of singularities, we can find an *sncd*-model $\mathcal{X}$ for X, i.e., a regular proper R-model such that $\mathcal{X}_s = \sum_{i\in I} N_i E_i$ is a divisor with strict normal crossings. For every $i \in I$, we define the order $\mu_i = \mathrm{ord}_{E_i}\omega$ of ω along E_i as in [NS07, 6.8]. These values do not depend on the choice of distinguished gauge form ω. For every non-empty subset J of I, we set

$$\begin{aligned} E_J &= \cap_{j\in J} E_j, \\ E_J^o &= E_J \setminus (\cup_{i\in I\setminus J} E_i). \end{aligned}$$

These are locally closed subsets of $\mathcal{X}_s$, and we endow them with the induced reduced structure. As J runs through the set of non-empty subsets of I, the subvarieties E_J^o form a partition of $\mathcal{X}_s$.

It follows from [NS07, 7.7] that the motivic zeta function $Z_X(T)$ can be expressed in the form

$$Z_X(T) = \sum_{\emptyset\neq J\subset I} (\mathbb{L}-1)^{|J|-1} [\widetilde{E}_J^o] \prod_{j\in J} \frac{\mathbb{L}^{-\mu_j} T^{N_j}}{1-\mathbb{L}^{-\mu_j} T^{N_j}} \quad \in \mathcal{M}_k[[T]] \tag{6.1}$$

where $\widetilde{E}_J^o$ is a certain finite étale cover of E_J. By [Ni10e, 2.2.2], one can construct $\widetilde{E}_J^o$ as follows: set

$$N_J = \gcd\{N_j \mid j \in J\},$$

choose a uniformizer π in R, and denote by $\mathcal{Y}$ the normalization of

$$\mathcal{X} \times_R (R[x]/(x^{N_J} - \pi)).$$

Then there is an isomorphism of E_J^o-schemes

$$\widetilde{E}_J^o \cong E_J^o \times_{\mathcal{X}} \mathcal{Y}.$$

In particular, one sees from (6.1) that $Z_X(T)$ is a rational function and that every pole of $Z_X(\mathbb{L}^{-s})$ is of the form $s = -\mu_i/N_i$ for some $i \in I$. Every irreducible component E_i of the special fiber yields in this way a "candidate pole" $-\mu_i/N_i$ of the zeta function. Since the expression in (6.1) is independent of the chosen normal crossings model $\mathcal{X}$, one expects in general that not all of these candidate poles are actual poles of $Z_X(T)$. But even candidate poles that appear in *every* model will not always be actual poles. To explain this phenomenon, we will propose in Section 6.4 a version of the Monodromy Conjecture for Calabi-Yau varieties.

Example 6.1.5. If X is an elliptic curve, then X admits a unique *minimal* regular model with strict normal crossings $\mathcal{X}$. It is not the case that all irreducible components of $\mathcal{X}_s$ give actual poles of the motivic zeta function. This can be seen by combining Theorem 5.5.1 with the Kodaira-Néron classification.

6.2. **Log canonical threshold.** Let X be a Calabi-Yau K-variety such that $X(K) \neq \emptyset$. From the formula in (6.1), we see that the poles of $Z_X(\mathbb{L}^{-s})$ form a finite subset of $\mathbb{Q}$. It turns out that the *largest* pole of $Z_X(T)$ is an interesting invariant for X, which can be read off from the numerical data associated to any *sncd*-model of X.

Choose a regular proper R-model $\mathcal{X}$ of X such that $\mathcal{X}_s$ is a strict normal crossings divisor $\mathcal{X}_s = \sum_{i \in I} N_i E_i$ and define the values μ_i, $i \in I$ as in Section 6.1. We put

$$\begin{aligned} lct(X) &= \min\{\mu_i/N_i \,|\, i \in I\}, \\ \delta(X) &= \max\{\, |J| \,|\, \emptyset \neq J \subset I,\ E_J \neq \emptyset,\ \mu_j/N_j = lct(X) \text{ for all } j \in J\} - 1. \end{aligned}$$

Definition 6.2.1. *We call $lct(X)$ the log canonical threshold of X, and $\delta(X)$ the degeneracy index of X.*

The following theorem shows that these values do not depend on the chosen model $\mathcal{X}$.

Theorem 6.2.2. *Let X be a Calabi-Yau variety with $X(K) \neq \emptyset$.*

(1) *The value $s = -lct(X)$ is the largest pole of the motivic zeta function $Z_X(\mathbb{L}^{-s})$, and its order equals $\delta(X)+1$. In particular, $lct(X)$ and $\delta(X)$ are independent of the model $\mathcal{X}$. For every integer $d > 0$, we have*

$$\begin{aligned} \delta(X \times_K K(d)) &= \delta(X), \\ lct(X \times_K K(d)) &= d \cdot lct(X). \end{aligned}$$

(2) *Assume moreover that $K = \mathbb{C}((t))$ and that X admits a projective model $\mathcal{Y}$ over the ring $\mathbb{C}\{t\}$ of germs of analytic functions at the origin of the complex plane. If we put $\alpha = lct(X)$, then $\exp(-2\pi i\alpha)$ is an eigenvalue of the action of the semi-simple part of monodromy on*

$$\mathrm{Gr}_F^m H^m(\mathcal{Y}_\infty, \mathbb{C}) := \mathrm{Gr}_F^m \mathbb{H}^m(\mathcal{Y}_s^{\mathrm{an}}, R\psi_{\mathcal{Y}}(\mathbb{C}))$$

where $m = \dim(X)$. In particular, for every embedding of $\mathbb{Q}_\ell$ in $\mathbb{C}$, $\exp(-2\pi i\alpha)$ is an eigenvalue of every tame monodromy operator σ on $H^m(X \times_K K^t, \mathbb{Q}_\ell)$.

In part (2) of Theorem 6.2.2 above, $H^m(\mathcal{Y}_\infty, \mathbb{C})$ denotes the limit cohomology at $t = 0$ associated to any projective model for $\mathcal{Y}$ over a small open disc around the origin of $\mathbb{C}$. It carries a natural mixed Hodge structure [St76, Na87], and $F^\bullet$ denotes the Hodge filtration. Note that $K^t = K^s$ since k has characteristic zero.

Comparing Theorem 5.5.1 and Theorem 6.2.2, we find:

Corollary 6.2.3. *If A is an abelian K-variety, then $lct(A) = -c(A)$ and $\delta(A) = t_{\mathrm{pot}}(A)$.*

The degeneracy index of a Calabi-Yau variety X over K is a measure for the potential degree of degeneration of X over the closed point of $\operatorname{Spec} R$. If A is an abelian variety, then by Corollary 6.2.3, the degeneracy index $\delta(A)$ is zero if and only if A has potential good reduction, and $\delta(A)$ reaches its maximal value $\dim(A)$ if and only if A has potential purely multiplicative reduction.

Looking at the expression for the zeta function of an abelian variety in Proposition 5.5.2, one sees that the zeta function of an abelian variety encodes many other interesting invariants of the abelian variety, such as the order function ord_A and the number of components $\phi_A(d)$ for every d in $\mathbb{N}$. Our motivic zeta function allows to generalize these invariants to Calabi-Yau varieties. Using the expression (6.1) for the zeta function in terms of an *sncd*-model, all these invariants can be explicitly computed on such a model. See [HN10b, §5].

6.3. **Comparison with the case of a hypersurface singularity.** Let us return for a moment to the set-up of Section 4.4, still assuming that k has characteristic zero. We can also apply Definition 6.2.1 to this situation, replacing X by the analytic Milnor fiber $\mathscr{F}_x$ of f at x and taking for ω the gauge form $t \cdot \phi/df$ on $\mathscr{F}_x$, where ϕ/df is a Gelfand-Leray form. In this way, we define the log-canonical threshold $lct_x(f)$ of f at x and the degeneracy index $\delta_x(f)$ of f at x. One can deduce from [Ni09a, 7.30] that $lct_x(f)$ coincides with the usual log-canonical threshold of f at x as it is defined in birational geometry. The results in Theorem 6.2.2 remain valid; in particular, using Theorem 4.6.1, we see that $s = -lct_x(f)$ is the largest pole of the motivic zeta function $Z_{f,x}(\mathbb{L}^{-s})$ of f at x. We refer to [HN10b] for details.

6.4. **Global Monodromy Property.** In the light of our results for abelian varieties, it is natural to wonder if there is a relation between poles of $Z_X(T)$ and monodromy eigenvalues for Calabi-Yau varieties X, similar to the one predicted by the motivic monodromy conjecture for hypersurface singularities (Conjecture 4.7.1).

Definition 6.4.1. *Let X be a Calabi-Yau variety with $X(K) \neq \emptyset$, and let σ be a topological generator of $G(K^t/K) = G(K^s/K)$. We say that X satisfies the Global Monodromy Property (GMP) if there exists a finite subset $\mathcal{S}$ of $\mathbb{Z} \times \mathbb{Z}_{>0}$ such that*

$$Z_X(T) \in \mathcal{M}_k \left[T, \frac{1}{1 - \mathbb{L}^a T^b} \right]_{(a,b) \in \mathcal{S}}$$

and such that for each $(a,b) \in \mathcal{S}$, the cyclotomic polynomial $\Phi_{\tau(a/b)}(t)$ divides the characteristic polynomial of the monodromy operator σ on $H^i(X \times_K K^t, \mathbb{Q}_\ell)$ for some $i \in \mathbb{N}$.

Recall that $\tau(a/b)$ denotes the order of a/b in $\mathbb{Q}/\mathbb{Z}$. By Theorem 5.5.1, every abelian K-variety satisfies the Global Monodromy Property.

Question 6.4.2. Is there a natural condition on X that guarantees that X satisfies the Global Monodromy Property (GMP)?

We don't know any example of a Calabi-Yau variety over K that does not satisfy the GMP. We would like to mention some work in progress where we can show that the GMP holds for certain types of varieties "beyond" abelian varieties.

Semi-abelian varieties. As a direct generalization of abelian varieties, it is natural to consider *semi-abelian* varieties, i.e., algebraic K-groups that are extensions of abelian varieties by tori. Néron models exist also for semi-abelian varieties, we refer to [BLR90] and [HN10c] for more details (the Néron model we consider is the maximal quasi-compact open subgroup scheme of the Néron lft-model from [BLR90]). We have generalized Theorem 5.5.1 to tamely ramified semi-abelian K-varieties, in arbitrary characteristic. The main complication is that one has to control the behaviour of the *torsion* part of the component group of the Néron lft-model under ramified base change.

$K3$-surfaces. Let X be a Calabi-Yau variety over K that admits a K-rational point. To show that X satisfies the Global Monodromy Property, one strategy would be to consider a regular proper model $\mathcal{X}$ whose special fiber has strict normal crossings. In principle, using the expression in (6.1), one can then determine the poles of $Z_X(T)$. The next step is to use A'Campo's formula (in the form of [Ni10e]) to compute the monodromy zeta function of X on the model $\mathcal{X}$ (the monodromy zeta function is the alternate product of the characteristic polynomials of the monodromy action on the cohomology spaces of X). In this way, one tries to show that the poles of $Z_X(T)$ correspond to monodromy eigenvalues. In practice, this kind of argumentation can be quite complicated. For one thing, when the dimension of X is greater than one, there is usually no distinguished *sncd*-model to work with, like the minimal *sncd*-model in the case of elliptic curves. And even when one has some more or less explicitly given model, the combinatorial and geometric complexity of the special fiber often make computations very hard: one needs to analyze the model in a very precise way to eliminate fake candidate poles and to find a sufficiently large list of monodromy eigenvalues. Worse, the monodromy zeta function might contain too little information to find all the necessary monodromy eigenvalues, due to cancellations in the alternate product.

There do however exist cases where this procedure leads to results. For instance, assume that X has dimension two and that it allows a triple-point-free degeneration. By this we mean that X has a proper regular model $\mathcal{X}/R$ where the special fiber $\mathcal{X}_s$ is a strict normal crossings divisor such that three distinct irreducible components of $\mathcal{X}_s$ never meet in one point. Such degenerate triple-point-free fibers have been classified by B. Crauder and D. Morrison [CM83]. In an ongoing project we use their classification to study the motivic zeta function of X, and we have been able to verify in almost all cases that the Global Monodromy Property holds.

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