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Associative dialgebras from a structural viewpoint

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In this note we study associative dialgebras proving that the most interesting such structures arise precisely when the algebra is not semiprime. In fact the presence of some "perfection" property (simpleness, primitiveness, primeness or semiprimeness) imply that the dialgebra comes from an associative algebra with both products $\dashv$ and $\vdash$ identified. We also describe the class of zero-cubed algebras and apply its study to that of dialgebras. Finally we describe two-dimensional associative dialgebras.

1. Introduction

The notion of Dialgebra is introduced and motivated by J. Loday in relation with Leibniz algebras. A Leibniz algebra is a kind of "non-commutative Lie algebra". To be more precise, a Leibniz algebra is an algebra A with product $[\ ,\ ]$ characterized by the so called Leibniz identity:

$$[[x, y], z] = [[x, z], y] + [x, [y, z]].$$

When the bracket $[\ ,\ ]$ is skew-symmetric, we get the definition of a Lie algebra. In the same way as any associative algebra gives rise to a Lie algebra by antisymmetrization $[x, y] := xy - yx$. we can associate to any associative dialgebra a Leibniz algebra structure by a new kind of antisymmetrization. The key point is to start with two distinct operations for the product xy and the product yx, so that the bracket is not necessarily skew-symmetric. So, we define an associative dialgebra as a vector space A provided with two associative operations $\dashv$ and $\vdash$, called respectively left and right product, satisfying the identities:

$$x \dashv (y \dashv z) = x \dashv (y \vdash z),$$

$$(x \vdash y) \dashv z = x \vdash (y \dashv z),$$

$$(x \dashv y) \vdash z = (x \vdash y) \vdash z.$$

Then one can easily check that the bracket $[x, y] := x \dashv y - y \vdash x$ defines a Leibniz algebra. Therefore any associative dialgebra gives rise to a Leibniz algebra. A classical example of dialgebra is constructed from a differential associative algebra (A, d) by defining $x \dashv y := xd(y)$ and $x \vdash y := d(x)y$. Then it is routinary to prove that $(A, \dashv, \vdash)$ is an associative dialgebra. There is a natural dialgebra structure on the de Rham complex of a manifold.

Dialgebras appear in the literature in quite different contexts. So for instance in [1], dialgebras are considered from the viewpoint of identities. However in [2] they are studied from the Yang-Baxter equation point of view. They can be related also to triple products as in [7] or to Leibniz algebras as previously mentioned (see [5]).

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2. *Preliminaries on zero-cubed associative algebras*

One easy example of dialgebra arises when one considers an associative algebra A with product denoted by juxtaposition. Then we can define a dialgebra by writing $\dashv := 0$ and defining $\vdash$ to be the product in A. However the identities for an associative dialgebra imply that A must be a nilpotent associative algebra of index tree, that is, one must have $A^3 = 0$ if the identities for an associative dialgebra are to be satisfied.

So we start our study describing those associative algebras A such that $A^3 = 0$. We call them *zero-cubed* algebras. Before this, we give a rather general example of associative algebra A satisfying $A^3 = 0$. Consider two vector spaces Z and X over the same field F and suppose that there is a bilinear map $f\colon X \times X \to Z$. Define then in the vector space $Z \oplus X$ the multiplication

$$(z + x)(z' + x') = f(x, x'), \qquad z, z' \in Z, x, x' \in X. \tag{1}$$

One can check immediately that $A := Z \oplus X$ is an associative algebra such that $A^3 = 0$. For any such algebra consider the annihilator Z defined as the set of all elements $x \in A$ such that $xA = Ax = 0$. In general, this is an ideal of the algebra. In our case we have also $A^2 \subset Z$. If we choose a complement X for Z in A we have a direct sum of vector subspaces $A = Z \oplus X$ and of course $X^2 \subset A^2 \subset Z$. Thus, there is a bilinear map $f\colon X \times X \to Z$ given by $f(a, b) := ab$.

Theorem 1. *For any associative zero-cubed algebra A (that is $A^3 = 0$) there are vector spaces Z and X over F and a bilinear map $f\colon X \times X \to Z$ such that $A = Z \oplus X$ with multiplication as in (1). Furthermore take any two F-algebras $A = Z_A \oplus X_A$ and $B = Z_B \oplus X_B$ such that $A^3 = 0$, $B^3 = 0$. Denote by $f_A\colon X_A \times X_A \to Z_A$ and $f_B\colon X_B \times X_B \to Z_B$ the bilinear maps involved in the products of A and B as in (1), then $A \cong B$ if and only if there are isomorphisms $\alpha\colon Z_A \to Z_B$ and $\beta\colon X_A \to X_B$ such that $f_B(\beta(x), \beta(x')) = \alpha(f_A(x, x'))$.*

Proof. It remains to prove the isomorphism condition. Thus suppose that $\omega\colon A \to B$ is an isomorphism. Then since Z_A is the annihilator of A and similarly for Z_B, we have $\omega(Z_A) = Z_B$. Thus define $\alpha = \omega|_{Z_A}$. Now consider the composition $X_A \stackrel{i}{\to} A \stackrel{\omega}{\to} B \stackrel{\pi_{X_B}}{\to} X_B$ where i is the inclusion map and π_{X_B} the canonical projection. Denote this composition by β. Let us check that β is an isomorphism. So take $x \in X_A$ and suppose that $\beta(x) = 0$. This means that $\omega(x) \in Z_B$ hence $x \in Z_A$ (but $x \in X_A$!). Thus $x = 0$ proving that β is a monomorphism. To see that it is also an epimorphism take $y \in X_B$. Then $\omega^{-1}(y) = x + z$ with $x \in X_A$ and $z \in Z_A$. Equivalently $\omega(x) = y - \omega(z)$ so that $\beta(x) = y$ hence β is an epimorphism. Finally if we take $x, x' \in X_A$ we know that $\omega(xx') = \omega(x)\omega(x')$. But $xx' = f_A(x, x') \in Z_A$ and $\omega(xx') = \alpha(f_A(x, x'))$. On the other hand $\omega(x) = y + z_1$ and $\omega(x') = y' + z_2$ where $y, y' \in X_B$, $z_1, z_2 \in Z_B$. This implies that $\omega(x)\omega(x') = f_B(y, y')$ and of course $y = \beta(x)$, $y' = \beta(x')$ so that we have proved $f_B(\beta(x), \beta(x')) = \alpha(f_A(x, x'))$ as required. The theorem is proved.

Consider the class $\mathcal{C}$ of all triples (Z, X, f) where Z and X are vector spaces over F and $f\colon X \times X \to Z$ a bilinear map. If (Z_A, X_A, f_A) and (Z_B, X_B, f_B) are two such triples we define the following relation: $(Z_A, X_A, f_A) \equiv (Z_B, X_B, f_B)$ if there are isomorphisms $\alpha\colon Z_A \to Z_B$ and $\beta\colon X_A \to X_B$ such that $f_B(\beta(x), \beta(x')) = \alpha(f_A(x, x'))$. Then the isomorphy classes of zero-cubed associative F-algebras are in one-to-one correspondence with the equivalence classes of $\mathcal{C}$ modulo the relation $\equiv$.

Recall that a semiprime (associative) algebra A is one in which for any ideal I of A if $I^2 = 0$ then $I = 0$. An easy observation is the following one:

Corollary 1. *Semiprime zero-cubed algebras do not exist.*

Proof. We are proving that if A is a semiprime zero-cubed algebra then $A = 0$. If $A \neq 0$ is a zero-cubed algebra $A = Z \oplus X$ where Z is the annihilator of A and

$X^2 \subset Z$. Since A is semiprime $A^2 \neq 0$ and since $Z^2 = 0$ we have $Z = 0$. But then $A = X$ and $A^2 = X^2 \subset Z = 0$ a contradiction. The corollary is proved.

Corollary 2. *Prime zero-cubed algebras do not exist, hence simple zero-cubed algebras do not exist and primitive zero-cubed algebras do not exist.*

Corollary 3. *Any zero-cubed two-dimensional associative F-algebra A is either a trivial algebra (that is $A^2 = 0$) or isomorphic to $F \times F$ with multiplication $(\lambda, \mu)(\lambda', \mu') := (\mu\mu', 0)$ for any $\lambda, \lambda', \mu, \mu' \in F$.*

Proof. If $Z = A$ then A is a trivial algebra of zero product. If $Z = 0$ since $A^2 \subset Z = 0$ we have again that A is a trivial algebra. If $\dim(Z) = 1$ then $\dim(X) = 1$ and taking generators $z \in Z$ and $x \in X$ we get a basis with $x^2 \in Fz$. So scaling z if necessary we may suppose $x^2 = z$ and the isomorphism follows directly.

3. Associative dialgebras

For a fixed field F an associative F-dialgebra is a vector space A provided with two binary operations $\dashv, \vdash : A \times A \to A$ such that $(A, \dashv)$ and $(A, \vdash)$ are associative algebras and

$$(x \dashv y) \dashv z \overset{(1)}{=} x \dashv (y \vdash z), \quad (x \vdash y) \dashv z \overset{(2)}{=} x \vdash (y \dashv z), \quad (x \dashv y) \vdash z \overset{(3)}{=} x \vdash (y \vdash z).$$

for any $x, y, z \in A$.

There are several trivial ways to construct associative dialgebras from associative algebras. For instance if A is an associative algebra with product denoted by juxtaposition, then we can define the dialgebra whose underlying vector space agree with that of A and whose products are $x \dashv y := xy =: x \vdash y$ for any $x, y \in A$. We shall call this the *associative dialgebra coming from the associative algebra A*. Often we shall rule out this algebra since it reduces to an associative algebra. Another trivial way to construct associative dialgebras is by considering one of the products to be zero. For instance if $\dashv = 0$ then $(A, \vdash)$ is an associative algebra and furthermore, it is a zero-cubed algebra which has been studied in previous sections of this work. The same applies if $\vdash = 0$.

Some standard definitions of annihilators in an associative dialgebra are the following

$$\begin{array}{ll} \mathrm{Rann}^{\dashv}(A) := \{x \in A \colon A \dashv x = 0\}, & \mathrm{Lann}^{\dashv}(A) := \{x \in A \colon x \dashv A = 0\} \\ \mathrm{Rann}^{\vdash}(A) := \{x \in A \colon A \vdash x = 0\}, & \mathrm{Lann}^{\vdash}(A) := \{x \in A \colon x \vdash A = 0\} \end{array}$$

As usual $\mathrm{Rann}^{\dashv}(A), \mathrm{Lann}^{\dashv}(A)$ are ideals of $(A, \dashv)$ while $\mathrm{Rann}^{\vdash}(A)$ and $\mathrm{Lann}^{\vdash}(A)$ are ideals of $(A, \vdash)$.

An interesting definition on associative dialgebras is that of *dual dialgebra*, A^{op}, of a given associative dialgebra A.

Definition 1. *For a dialgebra $(A, \dashv, \vdash)$ we denote by A^{op} the dialgebra $(A, \dashv', \vdash')$ whose underlying vector space agree with that of A and whose new products are $x \dashv' y := y \vdash x$ and $x \vdash' y := y \dashv x$ for any two elements $x, y \in A$. We will call A^{op} the* opposite algebra *of A.*

It is routinary to check that in case A is an associative dialgebra then A^{op} is also an associative dialgebra. Furthermore we have $\mathrm{Rann}^{\dashv}(A) = \mathrm{Lann}^{\vdash}(A^{\mathrm{op}})$, $\mathrm{Rann}^{\vdash}(A) = \mathrm{Lann}^{\dashv}(A^{\mathrm{op}})$, $\mathrm{Lann}^{\dashv}(A) = \mathrm{Rann}^{\vdash}(A^{\mathrm{op}})$ and $\mathrm{Lann}^{\vdash}(A) = \mathrm{Rann}^{\dashv}(A^{\mathrm{op}})$. In general we shall call *duality* to the fact of interpreting a result (proved for an associative dialgebra A) in the opposite algebra A^{op}.

Proposition 1. *In any associative dialgebra A the annihilators $Rann^{\dashv}(A)$ and $Lann^{\vdash}(A)$ are ideals of A.*

Proof. By duality we only need to prove that $\mathrm{Rann}^{\dashv}(A)$ is an ideal of A. Let us denote $R = \mathrm{Rann}^{\dashv}(A)$. We know that R is an ideal of $(A, \dashv)$ so we only need to check that $R \vdash A + A \vdash R \subset R$. Let us prove $A \vdash R \subset R$ and $R \vdash A \subset R$. For the first we must see that $A \dashv (A \vdash R) = 0$. But applying (1) one has $A \dashv (A \vdash R) = (A \dashv A) \dashv R \subset A \dashv R = 0$. To see that $R \vdash A \subset R$ we must prove $A \dashv (R \vdash A) = 0$. But applying again (1) we have $A \dashv (R \vdash A) = (A \dashv R) \dashv A = 0$ as we wanted to see. Thus we have proved

$$\mathrm{Rann}^{\dashv}(A) \triangleleft A.$$

Now that we know that $\mathrm{Rann}^{\dashv}(A)$ and $\mathrm{Lann}^{\vdash}(A)$ are ideals of A we can define the *annihilator* of A (denoted $\mathrm{Ann}(A)$) as the ideal

$$\mathrm{Ann}(A) := \mathrm{Rann}^{\dashv}(A) \cap \mathrm{Lann}^{\vdash}(A).$$

Remark 1. In a dialgebra A for any $y, z \in A$ we have $y \dashv z - y \vdash z \in \mathrm{Rann}^{\dashv}(A)$. Indeed, the identity (1) can be written as $x \dashv (y \dashv z - y \vdash z) = 0$ for any $x \in A$. Hence $y \dashv z - y \vdash z \in \mathrm{Rann}^{\dashv}(A)$. Of course by duality we also have $y \dashv z - y \vdash z \in \mathrm{Lann}^{\vdash}(A)$ hence $y \dashv z - y \vdash z \in \mathrm{Ann}(A)$ for any $y, z \in A$.

On the other hand we recall that a *bar-unit* of a dialgebra A is an element $e \in A$ such that $x \dashv e = x = e \vdash x$ for all $x \in A$.

Theorem 2. *If A is a dialgebra such that $A \dashv A = A$ (for instance if A has a bar-unit) then the quotient dialgebra $A/\,Rann^{\dashv}(A)$ has zero right annihilator:*

$$Rann^{\dashv}(A/\,Rann^{\dashv}(A)) = 0.$$

Similar statement follow by duality for $Lann^{\vdash}(A)$ and also $Ann(A/\,Ann(A)) = 0$.

Proof. Denote by $\bar{x} = x + \mathrm{Rann}^{\dashv}(A)$ the equivalence class of x, by $\bar{A}$ the quotient algebra $\bar{A} := A/\,\mathrm{Rann}^{\dashv}(A)$ and suppose $\bar{A} \dashv \bar{x} = \bar{0}$. Then $A \dashv x \in \mathrm{Rann}^{\dashv}(A)$ so that $A \dashv (A \dashv x) = 0$. Then $(A \dashv A) \dashv x = 0$ hence $A \dashv x = 0$ implying $x \in \mathrm{Rann}^{\dashv}(A)$ and $\bar{x} = \bar{0}$. The remaining assertions are proved in a similar way.

Theorem 3. *Let A be an associative dialgebra, then:*

a) If $Rann^{\dashv}(A) = A$ then A with the operation $\vdash$ is zero-cubed algebra (that is A can be described saying that $\dashv$ is the zero product and $(A, \vdash)$ is an in the previous section.

b) If $Rann^{\dashv}(A) = 0$ then $\dashv$ and $\vdash$ agree and A is the dialgebra coming from an associative algebra. If A has a bar unit e, then it is the dialgebra coming from an associative algebra with unit e.

Proof. If $\mathrm{Rann}^{\dashv}(A) = A$ then $A \dashv A = 0$ so the $\dashv$ product is null. Using (3) we get $A \vdash A \vdash A = 0$ so that $(A, \vdash)$ is a zero-cubed algebra. On the other hand if $\mathrm{Rann}^{\dashv}(A) = 0$ then by Remark 1 we have $y \dashv z - y \vdash z \in \mathrm{Rann}^{\dashv}(A) = 0$ hence both products coincide. So the dialgebra comes from an associative algebra A with product $\cdot$ defining both products $\dashv$ and $\vdash$ to be $\cdot$. If the dialgebra A has a bar unit then A is a unital associative algebra. The theorem is proved.

As we have seen the maximal and minimal cases $\mathrm{Rann}^{\dashv}(A) = A$ and $\mathrm{Rann}^{\dashv}(A) = 0$ are not interesting. The general situation will be the existence of a short exact sequence

$$0 \to \mathrm{Rann}^{\dashv}(A) \to A \to A/\,\mathrm{Rann}^{\dashv}(A) \to 0.$$

Thus, under the condition $A = A \dashv A$ we can say that A is a extension of a associative algebra $(A/\,\mathrm{Rann}^{\dashv}(A))$ by a zero-cubed algebra $(\mathrm{Rann}^{\dashv}(A))$.

Definition 2. *A dialgebra A with operations $\vdash$ and $\dashv$ as usual is said to be $\vdash$-simple if $(A,\vdash)$ is simple and is said to be $\dashv$-simple if $(A,\dashv)$ is it. The dialgebra A will be called simple iff it is both $\vdash$-simple and $\dashv$-simple.*

Theorem 4. *For an associative dialgebra A the following assertions are equivalent:*
i) A is $\vdash$-simple.
ii) A is $\dashv$-simple.
iii) A is simple.
iv) A is the dialgebra associated to a simple associative algebra.

Proof. First we observe that iv) implies i), ii) and iii). Next we prove that any of them implies iv). So suppose i), then $\text{Rann}^{\dashv}(A)$ is 0 or A. But if $\text{Rann}^{\dashv}(A) = A$ then $A \dashv A = 0$ hence $(A,\vdash)$ is a zero-cubed algebra by (3) which is impossible by Corollary 2. Thus necessarily $\text{Rann}^{\dashv}(A) = 0$ and iv) follows by Theorem 3. Suppose now ii), then again $\text{Rann}^{\dashv}(A) = 0$ or $\text{Rann}^{\dashv}(A) = A$. In this second case $A \dashv A = 0$ contradicting the fact that A is $\dashv$-simple. So necessarily $\text{Rann}^{\dashv}(A) = 0$ and we conclude as before. Finally iii) implies i) which implies iv). The theorem is proved.

Definition 3. *Given a property P of associative algebras and an associative dialgebra A, we will say that A has the $\vdash$-property P if $(A,\vdash)$ has the property P. Similarly we define dialgebras with the $\dashv$-property. We will say that A has the property P iff $(A,\vdash)$ and $(A,\dashv)$ have it. Thus we can consider $\vdash$-prime, $\vdash$-semiprime or $\vdash$-primitive dialgebras (and dually $\dashv$-prime, $\dashv$-semiprime or $\dashv$-primitive dialgebras). Also, with this meaning we consider semiprime, prime or primitive dialgebras.*

Theorem 5. *For an associative dialgebra A and being P the property of being semiprime, prime or primitive, the following assertions are equivalent:*
i) A has the $\vdash$-property P.
ii) A has the $\dashv$-property P.
iii) A has the property P.
iv) A is the dialgebra associated to an associative algebra satisfying P.

Proof. We can argue as before that iv) implies i), ii) and iii). Next, we must take into account that primitive $\Rightarrow$ prime $\Rightarrow$ semiprime. To see that i) or ii) implies iv), if $(A,\vdash)$ has the property P, since $\text{Lann}^{\vdash}(A)^2 = 0$ we have $\text{Lann}^{\vdash}(A) = 0$ hence A is the dialgebra coming from an associative algebra with both products $\vdash$ and $\dashv$ agreeing (this is the dual assertion of Theorem 3.b). Of course iii) implies i) hence iii) implies iv). Thus the theorem is proved.

4. Low dimensional associative dialgebras

In this section we classify 2-dimensional associative dialgebras which do not come from an associative algebra (that is both products $\vdash$ and $\dashv$ are different). Thus $\text{Ann}(A) \neq 0$ implying $\text{Rann}^{\dashv}(A) \neq 0$ (see Theorem 3). Of course we exclude also the trivial case $A = \text{Ann}(A)$. Thus $\dim(\text{Ann}(A)) = 1$ so that the annihilator is generated by an element (say r) and we can write $\text{Ann}(A) = Fr$. Choosing any complement s such that $A = Fr \oplus Fs$, the multiplication tables of A are

$\dashv$	r	s
r	0	$x_1 r$
s	0	$x_2 r + x_3 s$

$\vdash$	r	s
r	0	0
s	$x_4 r$	$x_5 r + x_6 s$

Imposing the associativity conditions to the products $\dashv$, $\vdash$ as well as the conditions (1), (2) and (3), we get the following equations:

$$0 = x_1x_2 = x_4x_5 = x_1(x_1 - x_3) = x_4(x_3 - x_4) = x_1(x_1 - x_6) = x_2(x_1 + x_3 - x_6) =$$

$$x_1x_5+x_2x_6-x_2x_4-x_3x_5 = x_3(x_3-x_6) = x_4(x_4-x_6) = x_6(x_3-x_6) = x_5(x_3-x_4-x_6)$$

and we analyze the different possibilities:

1) $x_1 = x_2 = 0$. Then if $x_3 = 0$ the product $\dashv$ is null and $(A, \vdash)$ is a zero-cubed algebra described in Corollary 3. So we suppose $x_3 \neq 0$ which implies $x_3 = x_6$. The identity $x_1x_5 + x_2x_6 - x_2x_4 - x_3x_5 = 0$ implies $x_5 = 0$ The previous list of identities reduces then to $0 = x_4(x_3 - x_4)$, $x_3 = x_6$, $x_5 = 0$. We have two subcases:

1.i) $x_4 = 0$, $x_3 = x_6 \neq 0$, $x_5 = 0$. Then $(A, \dashv)$ is given by $0 = r \dashv r = r \dashv s = s \dashv r$ and $s \dashv s = x_3s$. Defining $s' := x_3^{-1}s$ the multiplication table relative to the basis $\{r, s'\}$ is $r \dashv r = r \dashv s' = s' \dashv r = 0$ and $s' \dashv s' = x_3^{-2}x_3s = x_3^{-1}s = s'$. Moreover $r \vdash r = r \vdash s' = s' \vdash r = 0$ and $s' \vdash s' = s'$ as for the product $\dashv$. So in this case both operations $\dashv$ and $\vdash$ coincide and A is the associative dialgebra coming from the bidimensional associative algebra with multiplication table:

$\cdot$	r	s
r	0	0
s	0	s

1.ii) $x_3 = x_4 = x_6 \neq 0$, $x_5 = 0$. Defining again $s' = x_3^{-1}s$ we get as before $r \dashv r = r \dashv s' = s' \dashv r = 0$ and $s' \dashv s' = s'$. Moreover the product $\vdash$ satisfies $r \vdash s' = 0$, $s' \vdash r = r$ and $s' \vdash s' = s'$. Hence after scaling s if necessary we get the dialgebra with operations:

$\dashv$	r	s
r	0	0
s	0	s

$\vdash$	r	s
r	0	0
s	r	s

This algebra will be denoted by **I**.

2) $x_1 = 0$, $x_2 \neq 0$. The list of identities reduces to $0 = x_4x_5 = x_4(x_3 - x_4)$, $x_3 = x_6$, $x_2x_3 - x_2x_4 - x_3x_5 = 0$. We consider the following subcases:

2.i) $x_1 = 0$, $x_2 \neq 0$, $x_4 = 0$. Then we have $x_3 = x_6$ and $x_3(x_2 - x_5) = 0$ and there are again two possibilities:

2.i.a) $x_1 = 0$, $x_2 \neq 0$, $x_4 = 0$, $x_3 = x_6 = 0$. After scaling r if necessary we get the family of dialgebras with multiplications given by

$\dashv$	r	s
r	0	0
s	0	r

$\vdash$	r	s
r	0	0
s	0	kr

We shall denote this dialgebras by $\mathbf{II}_k$.

2.i.b) $x_1 = 0$, $x_2 \neq 0$, $x_4 = 0$, $x_3 = x_6 \neq 0$, $x_2 = x_5$. In this case again the product coincide and the algebra comes from an associative algebra.

2.ii) $x_1 = 0$, $x_2 \neq 0$, $x_4 \neq 0$. This implies $x_5 = 0$, $x_3 = x_4 = x_6 \neq 0$. Then replacing s with $s' := (x_2/x_3^2)r + x_3^{-1}s$ we get $r \dashv r = r \dashv s' = s' \dashv r = 0$ and $s' \dashv s' = s'$. On the other hand $r \vdash r = r \vdash s' = 0$ and $s' \vdash r = r$, $s' \vdash s' = s'$. But this algebra is isomorphic to **I**.

3) $x_1 \neq 0$, $x_2 = x_4 = 0$. The list of identities is then $x_1 = x_3 = x_6$. Thus necessarily $x_6 \neq 0$ and we can define $s' := x_6^{-2}x_5r + x_6^{-1}s$. Then replacing s with s' if necessary we get the dialgebra with multiplications:

$\dashv$	r	s
r	0	r
s	0	s

$\vdash$	r	s
r	0	0
s	0	s

We denote this algebra by **III**.

4) $x_1 \neq 0$, $x_2 = 0$, $x_4 \neq 0$, $x_5 = 0$. The list of identities is then $x_1 = x_3 = x_4 = x_6$. Replacing s with $s' := x_1^{-1} s$ if necessary we find the dialgebra with multiplications

$\dashv$	r	s
r	0	r
s	0	s

$\vdash$	r	s
r	0	0
s	r	s

We denote this algebra by **IV**.

Now we check that the algebras **I**, $\mathbf{II}_k$, **III** and **IV** are all non-isomorphic. Thus denote by $A^{2\ \dashv} := A \dashv A$ and similarly $A^{2\ \vdash} := A \vdash A$. Then we compute the dimension of these spaces in the following table:

A	$\dim(A^{2\ \dashv})$	$\dim(A^{2\ \vdash})$
I	1	2
$\mathbf{II}_k,\ (k \neq 0)$	1	1
III	2	1
IV	2	2

which proves that the algebras are not isomorphic except possibly for the family of algebras $\mathbf{II}_k$ $(k \neq 0)$. Let us now check that for these algebras, $\mathbf{II}_k \cong \mathbf{II}_{k'}$ if and only if $k = k'$. For this we shall need to compute the group of automorphisms of the two-dimensional algebra A with basis $\{r, s\}$ such that $r^2 = rs = sr = 0$ and $s^2 = r$. If $f \in \mathrm{aut}(A)$ then necessarily $f(r) = k_1 r$ for a nonzero scalar $k_1 \in F$. Then, since $s^2 = r$ we may write $f(s) = \alpha r + \beta s$ for some scalars $\alpha, \beta \in F$ and we have $(\alpha r + \beta s)^2 = k_1 r$ which implies $\beta^2 r = k_1 r$ so that $\beta^2 = k_1$. Thus

$$\begin{cases} f(r) = \beta^2 r \\ f(s) = \alpha r + \beta s. \end{cases}$$

As a corollary $\mathrm{aut}(A)$ is isomorphic to the subgroup of $\mathrm{GL}_2(F)$ of all matrices of the form

$$\begin{pmatrix} \beta^2 & 0 \\ \alpha & \beta \end{pmatrix}$$

where $\alpha \in F$, $\beta \in F^\times$. Thus if we have an isomorphism $f\colon \mathbf{II}_k \cong \mathbf{II}_{k'}$ then we have an isomorphism $f\colon (\mathbf{II}_k, \dashv) \cong (\mathbf{II}_{k'}, \dashv)$ which implies that f must have the canonical form given above. But on the other hand f is also an isomorphism $(\mathbf{II}_k, \vdash) \cong (\mathbf{II}_{k'}, \vdash)$ which implies $f(s \vdash s) = f(s) \vdash f(s)$ that is $kf(r) = (\alpha r + \beta s) \vdash (\alpha r + \beta s)$ or equivalently $k\beta^2 r = \beta^2 k' r$ which implies $k = k'$. Summarizing all the information in this section we claim:

Theorem 6. *Let A be a two-dimensional associative dialgebra over a field F. Then we have only one of the following possibilities for A:*

i) A is the dialgebra coming from an associative algebra.

ii) One (and only one) of the products $\dashv$ or $\vdash$ is null. Then A is a zero-cubed algebra with the nonzero product and has been described in Corollary 3.

iii) A is isomorphic to one of the algebras **I**, **II**$_k$ *($k \neq 0$),* **III** *or* **IV** *previously described. Any two of these algebras are not isomorphic.*

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