On backward stochastic differential equations approach to valuation of American options

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Abstract

We consider the problem of valuation of American (call and put) options written on a dividend paying stock governed by the geometric Brownian motion. We show that the value function has two different but related representations: by means of a solution of some nonlinear backward stochastic differential equation and weak solution to some semilinear partial differential equation.

1 Introduction

We consider a financial market model in which the price dynamics of a dividend paying stock $X^{s,x}$ evolves (under the equivalent martingale measure P) according to the stochastic differential equation (SDE) of the form

$$X_t^{s,x} = x + \int_s^t (r-d) X_\theta^{s,x}\, d\theta + \int_s^t \sigma X_\theta^{s,x}\, dW_\theta, \quad t \in [s,T]. \tag{1.1}$$

Here $x > 0$, W is a standard Wiener process, $d \geq 0$ is the dividend yield for the stock, $r \geq 0$ is the risk-free interest rate and $\sigma > 0$ is the volatility.

It is well known (see, e.g., [8, Section 2.5]) that the arbitrage-free value of an American option with payoff function $g : \mathbb{R} \to [0,\infty)$ and expiration time T is given by

$$V(s,x) = \sup_{s \leq \tau \leq T} E e^{-r(\tau - s)} g(X_\tau^{s,x}), \tag{1.2}$$

where E denotes the expectation with respect to P and the supremum is taken over all stopping times with respect to the standard augmentation $\{\mathcal{F}_t\}$ of the filtration generated by W. From [6] we know also that the optimal stopping problem and, a fortiori, the value function V, are related to the solution $(Y^{s,x}, Z^{s,x}, K^{s,x})$ of the reflected backward stochastic differential equation (RBSDE)

$$\begin{cases} Y_t^{s,x} = g(X_T^{s,x}) - \int_t^T r Y_\theta^{s,x}\, d\theta + K_T^{s,x} - K_t^{s,x} - \int_t^T Z_\theta^{s,x}\, dW_\theta, \quad t \in [s,T], \\ Y_t^{s,x} \geq g(X_t^{s,x}), \quad t \in [s,T], \\ K^{s,x} \text{ is increasing, continuous, } K_s^{s,x} = 0,\ \int_s^T (Y_t^{s,x} - g(X_t^{s,x}))\, dK_t^{s,x} = 0 \end{cases} \tag{1.3}$$

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via the equality

$$V(s,x)=Y^{s,x}_s,\quad (s,x)\in Q_T\equiv[0,T]\times\mathbb{R}. \tag{1.4}$$

Formula (1.4) when combined with general results on connections between RBSDEs and parabolic PDEs proved in [4] provides a probabilistic proof of the fact that $V=\{V(s,x);(s,x)\in Q_T\}$, where $V(s,x)$ is given by (1.2), is a viscosity solution of the obstacle problem (or, in another terminology, the quasi-variational inequality)

$$\begin{cases} \min(u(s,x)-g(x),-\mathcal{L}_{BS}u(s,x)+ru(s,x))=0, & (s,x)\in Q_T,\\ u(T,x)=g(x), & x\in\mathbb{R}, \end{cases} \tag{1.5}$$

where $\mathcal{L}_{BS}$ is the Black and Scholes differential operator defined by

$$\mathcal{L}_{BS}u=\partial_s u+(r-d)x\partial_x u+\frac{1}{2}\sigma^2x^2\partial^2_{xx}u.$$

In the present paper we concentrate on the American call and put options for which the payoff function is given by

$$g(x)=\begin{cases} (x-K)^+, & \text{call option,}\\ (K-x)^+, & \text{put option.}\end{cases}$$

We prove that in that case the process $K^{s,x}$ has the form

$$K^{s,x}_t=\begin{cases} \int_s^t(dX^{s,x}_\theta-rK)^+\mathbf{1}_{\{Y^{s,x}_\theta=g(X^{s,x}_\theta)\}}\,d\theta, & \text{call option,}\\ \int_s^t(rK-dX^{s,x}_\theta)^+\mathbf{1}_{\{Y^{s,x}_\theta=g(X^{s,x}_\theta)\}}\,d\theta, & \text{put option}\end{cases} \tag{1.6}$$

for $t\in[s,T]$, i.e. the first two components $(Y^{s,x},Z^{s,x})$ of the solution of (1.3) solve the usual (non-reflected) BSDE

$$Y^{s,x}_t=g(X^{s,x}_T)+\int_t^T(-rY^{s,x}_\theta+q(X^{s,x}_\theta,Y^{s,x}_\theta))\,d\theta-\int_t^T Z^{s,x}_\theta\,dW_\theta,\quad t\in[s,T], \tag{1.7}$$

where

$$q(x,y)=\begin{cases} (dx-rK)^+\mathbf{1}_{(-\infty,g(x)]}(y), & \text{call option,}\\ (rK-dx)^+\mathbf{1}_{(-\infty,g(x)]}(y), & \text{put option}\end{cases}$$

for $x,y\in\mathbb{R}$. The above result is in fact a reformulation of the representation for Snell envelope of the discounted payoff process $\xi_t=e^{-r(t-s)}g(X^{s,x}_t)$, $t\in[s,T]$ (see Section 3). Therefore our contribution here consists in providing new proof of the last statement and clarifying relations between (1.3) and (1.7). We hope also that our proof of the representation for Snell envelope for ξ will be of interest, because contrary to known to us proofs it avoids considering the parabolic free-boundary value problem associated with the optimal stopping problem (1.2).

Formula (1.6) has an analytical counterpart. Let $\varrho(x)=(1+|x|^2)^{-\alpha}$, $x\in\mathbb{R}$, where α is chosen so that $\int_{\mathbb{R}}\varrho^2(x)x^2\,dx<\infty$. By a solution of (1.5) we understand a pair (u,μ) consisting of a measurable function $u:Q_T\to\mathbb{R}$ possessing some regularity properties and a Radon measure μ on Q_T such that

$$\begin{cases} \mathcal{L}_{BS}u=ru-\mu,\\ u(T)=g,\quad u\geq g,\quad \int_{Q_T}(u-g)\varrho^2\,d\mu=0\end{cases} \tag{1.8}$$

(see Section 2 for details). We prove that (1.8) has a unique solution (u, μ) such that μ is absolutely continuous with respect to the Lebesgue measure and

$$d\mu(t,x) = q(x, u(t,x))\, dt\, dx. \tag{1.9}$$

Moreover, for each $(s,x) \in Q_T$ such that $x \neq 0$,

$$(Y_t^{s,x}, Z_t^{s,x}) = (u(s, X_t^{s,x}), \sigma x \partial_x u(t, X_t^{s,x})), \quad t \in [s,T], \quad P\text{-}a.s., \tag{1.10}$$

i.e. (1.3) provides probabilistic representation for the first component u of the solution of (1.8). In particular, $V = u$. Formula (1.9) is an analytical analogue of (1.6).

From (1.8), (1.9) it follows that V is a solution of the semilinear Cauchy problem

$$\mathcal{L}_{BS} u = ru - q(\cdot, u), \quad u(T, \cdot) = g. \tag{1.11}$$

The above problem was considered in [2, 3] as an alternative to the obstacle problem formulation (1.5) and the free boundary problem formulation (see, e.g., [8, Section 2.7]). In [2] it is shown that (1.11) has a unique viscosity solution (since q is discontinuous, the standard definition of a viscosity solution is modified appropriately) and $V = u$. Our approach to (1.5) via (1.8) shows that in fact (1.11) results from a better understanding of the nature of solutions of (1.5).

2 Obstacle problem for the Black and Scholes equation

In this section we prove existence, uniqueness and stochastic representation of solutions of the obstacle problem (1.8). We begin with the precise definition of solutions of (1.8).

Let $Q_{st} = [s,t] \times \mathbb{R}$, $Q_t = Q_{0t}$, and let $\mathcal{R}$ denote the space of all functions $\varrho : \mathbb{R} \to \mathbb{R}$ of the form $\varrho(x) = (1 + |x|^2)^{-\alpha}$, $x \in \mathbb{R}$, for some $\alpha \geq 0$. In the whole paper we will assume that $\int_{\mathbb{R}} \varrho^2(x) x^2\, dx < \infty$.

Given $\varrho \in \mathcal{R}$ we denote by $\mathbb{L}_{2,\varrho}(\mathbb{R})$ the Hilbert space of functions u on $\mathbb{R}$ such that $u\varrho \in \mathbb{L}_2(\mathbb{R})$ equipped with the inner product $\langle u, v \rangle_{2,\varrho} = \int_{\mathbb{R}} uv\varrho^2\, dx$. Similarly, by $\mathbb{L}_{2,\varrho}(Q_{st})$ we denote the Hilbert space of functions u on Q_{st} such that $u\varrho \in \mathbb{L}_2(Q_{st})$ with the inner product $\langle u, v \rangle_{2,\varrho,s,t} = \int_{Q_{st}} uv\varrho^2\, dx\, dt$. If $s = 0$ we drop the subscript s in the notation. $H_\varrho = \{\eta \in \mathbb{L}_{2,\varrho}(\mathbb{R}) : x\partial_x \eta(x) \in \mathbb{L}_{2,\varrho}(\mathbb{R})\}$, $W_\varrho = \{\eta \in \mathbb{L}_2(0,T; H_\varrho) : \partial_t \eta \in \mathbb{L}_2(0,T; H_\varrho^{-1})\}$, where H_ϱ^{-1} is the space dual to H_ϱ^{-1}. By $\langle \cdot, \cdot \rangle_{\varrho,T}$ we denote the duality pairing between $\mathbb{L}_2(0,T; H_\varrho)$ and $\mathbb{L}_2(0,T; H_\varrho^{-1})$. Finally, $V = W_\varrho \cap C(Q_T)$.

We say that a pair (u, μ), where $u \in V$ and μ is a Radon measure on Q_T, is a solution of the obstacle problem (1.8) if $(1.8)_2$ is satisfied and the equation $(1.8)_1$ is satisfied in the strong sense, i.e. for every $\eta \in C_0^\infty(Q_T)$,

$$\langle \partial_t u, \eta \rangle_{\varrho,T} + \langle L_{BS} u, \eta \rangle_{\varrho,T} = r \langle u, \eta \rangle_{2,\varrho,T} - \int_{Q_T} \eta \varrho^2\, d\mu,$$

where

$$\langle L_{BS} u, \eta \rangle_{\varrho,T} = \langle (r-d) x \partial_x u, \eta \rangle_{2,\varrho,T} - \frac{1}{2}\sigma^2 \langle \partial_x u, \partial_x (x^2 \eta \varrho^2) \rangle_{2,T}.$$

We say that a pair (u,μ) satisfies $(1.8)_1$ in the weak sense if μ is a Radon measure on Q_T, $u\in\mathbb{L}_2(0,T;H_\varrho)\cap C([0,T],\mathbb{L}_{2,\varrho}(\mathbb{R}))$ and for every $\eta\in C_0^\infty(Q_T)$,

$$\langle u,\partial_t\eta\rangle_{\varrho,T}-\langle L_{BS}u,\eta\rangle_{\varrho,T}=\langle h(T),\eta(T)\rangle_{2,\varrho}-\langle u(0),\eta(0)\rangle_{2,\varrho}-r\langle u,\eta\rangle_{2,\varrho,T}+\int_{Q_T}\eta\varrho^2\,d\mu.$$

Let $\{\mathcal{F}_t\}$ denote the standard augmentation of the natural filtration generated by W. By a solution of RBSDE (1.3) we understand a triple $(Y^{s,x},Z^{s,x},K^{s,x})$ of $\{\mathcal{F}_t\}$-progressively measurable process on $[s,T]$ such that

$$E\sup_{t\in[s,T]}|Y^{s,x}_t|^2<\infty,\quad E\int_s^T|Z^{s,x}_t|^2\,dt<\infty,\quad E|K^{s,x}_T|^2<\infty \tag{2.1}$$

and (1.3) is satisfied P-a.s.. A pair $(Y^{s,x},Z^{s,x})$ of $\{\mathcal{F}_t\}$-progressively measurable process is a solution of BSDE (1.7) if (1.7) holds P-a.s. and $Y^{s,x},Z^{s,x}$ satisfy the integrability conditions (2.1).

From general results proved in [4] it follows that (1.3) has a unique solution. We shall prove that the third component $K^{s,x}$ of the solution is absolutely continuous.

Proposition 2.1 *If $(Y^{s,x},Z^{s,x},K^{s,x})$ is a solution of RBSDE (1.3) then*

$$K^{s,x}_t-K^{s,x}_\tau\le\int_\tau^t\mathbf{1}_{\{Y^{s,x}_\theta=S_\theta\}}(dX^{s,x}_\theta-rK)^+\,d\theta,\quad s\le\tau\le t\le T. \tag{2.2}$$

Proof. We prove the theorem in the case of call option. The proof for put option is similar and therefore left to the reader.

Suppose that $(Y^{s,x},Z^{s,x},K^{s,x})$ is a solution of (1.3) and u is a viscosity solution of (1.5). By [4, Theorem 8.5],

$$Y^{s,x}_t=u(t,X^{s,x}_t),\quad t\in[s,T]. \tag{2.3}$$

Set $S_t=g(X^{s,x}_t)$, $t\in[s,T]$, and denote by $\{L^a_t(\xi);(t,a)\in[0,\infty)\times\mathbb{R}\}$ the local time of a continuous semimartingale ξ. By the Tanaka-Meyer formula, for every $t\in[s,T]$,

$$\begin{aligned}(X^{s,x}_t-K)^+&=\int_s^t\mathbf{1}_{(K,\infty)}(X^{s,x}_\theta)(r-d)X^{s,x}_\theta\,d\theta\\&\quad+\int_s^t\mathbf{1}_{(K,\infty)}(X^{s,x}_\theta)\sigma X^{s,x}_\theta\,dW_\theta+\frac12L^0_t(X^{s,x}-K)\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}0=(Y^{s,x}_t-S_t)^-&=-\int_s^t\mathbf{1}_{(-\infty,0]}(Y^{s,x}_\theta-S_\theta)\,dY^{s,x}_\theta+\int_s^t\mathbf{1}_{(-\infty,0]}(Y^{s,x}_\theta-S_\theta)\,dS_\theta\\&\quad+\frac12L^0_t(Y^{s,x}-S)\\&=\int_s^t\mathbf{1}_{\{Y^{s,x}_\theta=S_\theta\}}(-rY^{s,x}_\theta\,d\theta+dK^{s,x}_\theta-Z^{s,x}_\theta\,dW_\theta)\\&\quad+\int_s^t\mathbf{1}_{(K,\infty)}(X^{s,x}_\theta)\mathbf{1}_{\{Y^{s,x}_\theta=S_\theta\}}((r-d)X^{s,x}_\theta\,d\theta+\sigma X^{s,x}_\theta\,dW_\theta)\\&\quad+\frac12\int_s^t\mathbf{1}_{\{Y^{s,x}_\theta=S_\theta\}}\,dL^0_\theta(X^{s,x}-K)+\frac12L^0_t(Y^{s,x}-S).\end{aligned} \tag{2.5}$$

Write $I=\{u=g\}$ and observe that $(t,K)\notin I$ for all $t\in[0,T)$, because $u=V$ by [4, Proposition 2.3], and so u is strictly positive. Consequently,

$$\int_s^t \mathbf{1}_{\{Y^{s,x}_\theta=S_\theta\}}\,dL^0_\theta(X^{s,x}-K)=0.$$

Furthermore, from (2.4) and Proposition 4.2 and Remark 4.3 in [4] it follows that $\sigma X^{s,x}_t\mathbf{1}_{(K,\infty)}(X^{s,x}_t)=Z^{s,x}_t$ a.s. on $\{Y^{s,x}_t=S_t\}$. From (2.5) we therefore get

$$\begin{aligned}
&K^{s,x}_t-K^{s,x}_\tau+\frac{1}{2}L^0_t(Y^{s,x}-S)-\frac{1}{2}L^0_\tau(Y^{s,x}-S)\\
&\quad=\int_\tau^t r\mathbf{1}_{\{Y^{s,x}_\theta=S_\theta\}}S_\theta\,d\theta-\int_\tau^t\mathbf{1}_{\{Y^{s,x}_\theta=S_\theta\}}\mathbf{1}_{(K,\infty)}(X^{s,x}_\theta)(r-d)X^{s,x}_\theta\,d\theta\\
&\quad=\int_\tau^t\mathbf{1}_{\{Y^{s,x}_\theta=S_\theta\}}\mathbf{1}_{(K,\infty)}(X^{s,x}_\theta)((r-d)X^{s,x}_\theta-r(X^{s,x}_\theta-K)^+)^-\,d\theta.
\end{aligned}$$

Hence

$$K^{s,x}_t-K^{s,x}_\tau\le\int_\tau^t\mathbf{1}_{\{Y^{s,x}_\theta=S_\theta\}}\mathbf{1}_{(K,\infty)}(X^{s,x}_\theta)((r-d)X^{s,x}_\theta-r(X^{s,x}_\theta-K)^+)^-\,d\theta. \tag{2.6}$$

Since, by (2.3), $Y^{s,x}$ is strictly positive, $\{Y^{s,x}_t=g(X^{s,x}_t)\}\subset\{X^{s,x}_t>K\}$ and hence $K^{s,x}$ increases only on the set $\{X^{s,x}_t>K\}$. Therefore (2.6) forces (2.2). □

Proposition 2.2 *There exists at most one solution of the problem (1.8).*

Proof. Suppose that (u_1,μ_1), (u_2,μ_2) are solutions of (1.8). Write $u=u_1-u_2$, $\mu=\mu_1-\mu_2$. Then (u,μ) satisfies $(1.8)_1$ in the strong sense. Since by standard regularization arguments we can put u as a test function in $(1.8)_1$ and obviously $(1.8)_1$ is satisfied on Q_{tT} for any $t\in[0,T)$, we have

$$\begin{aligned}
\|u(t)\|_{2,\varrho}+\frac{1}{2}\sigma^2\|x\partial_xu\|^2_{2,\varrho,t,T}&=\langle(\mu-d)x\partial_xu,u\rangle_{2,\varrho,t,T}+\sigma^2\langle\partial_xu,xu\rangle_{2,\varrho,t,T}\\
&\quad+\sigma^2\langle\partial_xu,x^2u\partial_x\varrho,\varrho\rangle_{2,t,T}+r\|u\|^2_{2,\varrho,t,T}+\int_{Q_{tT}}u\varrho^2\,d\mu.
\end{aligned}$$

From the above, the fact that $\int_{Q_{tT}}u\varrho^2\,d\mu\le0$, $|\partial_x\varrho|\le C\varrho$ and the elementary inequality $ab\le\varepsilon a^2+\varepsilon^{-1}b^2$ we get

$$\|u(t)\|_{2,\varrho}\le C\int_t^T\|u(s)\|^2_{2,\varrho}\,ds,\quad t\in[0,T].$$

By Gronwall's lemma, $u=0$, and in consequence, $\mu=0$. □

Given $\delta>0$ write $D^+_\delta=(0,T)\times(\delta,+\infty)$, $D^-_\delta=(0,T)\times(-\infty,\delta)$ and $D^+=D^+_0$, $D^-=D^-_0$, $D=D^+\cup D^-$. Note that from the well known explicit formula for $X^{s,x}$ it follows that $X^{s,x}_t\in D^+$, $t\in[s,T]$, P-a.s. if $x>0$, and $X^{s,x}_t\in D^-$, $t\in[s,T]$, P-a.s. if $x<0$. Note also that if $x\neq0$ and $t>s$ then the density of the distribution of the random variable $X^{s,x}_t$ is given by the formula

$$p(s,x,t,y)=\frac{1}{y\sqrt{2\pi(t-s)}}\exp\Big(\frac{-(\ln\frac{y}{x}+(\frac{\sigma^2}{2}-r+d)(t-s))^2}{t-s}\Big)\mathbf{1}_{\{\frac{y}{x}>0\}}. \tag{2.7}$$

It follows in particular that for fixed $s \in [0,T)$, $x \neq 0$ and $\delta \in (0, T-s]$ the function $p(s,x,\cdot,\cdot)$ is bounded on $Q_{s+\delta,T}$.

Theorem 2.3 *(i) There exists a unique solution (u,μ) of the problem (1.8).*
(ii) Let $x \neq 0$ and let $(Y^{s,x}, Z^{s,x}, K^{s,x})$ be a solution of RBSDE (1.3). Then

$$(Y_t^{s,x}, Z_t^{s,x}) = (u(t, X_t^{s,x}), \sigma\partial_x u(t, X_t^{s,x})), \quad t \in [s,T], \quad P\text{-a.s.}$$

and for any $\eta \in C_0(Q_{sT})$,

$$E\int_s^T \eta(t, X_t)\, dK_t^{s,x} = \int_{Q_{sT}} \eta(t,y)p(s,x,t,y)\, d\mu(t,y). \tag{2.8}$$

Proof. Let u_n be a unique viscosity solution of the following penalized problem

$$\frac{\partial u_n}{\partial t} + L_{BS}u_n = ru_n - n(u_n - g)^-, \quad u_n(T) = g, \tag{2.9}$$

and for fixed (s,x) let $(Y^{s,x,n}, Z^{s,x,n})$ denote a solution of the BSDE

$$Y_t^{s,x,n} = g(X_T^{s,x}) - \int_t^T rY_\theta^{s,x,n}\, d\theta + \int_t^T n(Y_\theta^{s,x,n} - g(X_\theta^{s,x}))^-\, d\theta - \int_t^T Z_\theta^{s,x,n}\, d\theta.$$

Using standard arguments one can show that $x \mapsto EY_s^{s,x,n}$ is Lipschitz continuous uniformly in s. Therefore u_n has the same regularity, because by [4, Theorem 8.5], $Y_t^{s,x,n} = u_n(t, X_t^{s,x})$, $t \in [s,T]$, P-a.s., and hence $u_n(s,x) = EY_s^{s,x,n}$. Since the operator L_{BS} is uniformly elliptic on each domain D_δ^+, for each $\delta > 0$ there is a unique weak solution v_δ of the following terminal-boundary problem

$$\frac{\partial v_\delta}{\partial t} + L_{BS}v_\delta = rv_\delta - n(v_\delta - h)^-, \quad v_\delta(T) = g, \quad v_\delta(x) = u_n(x) \text{ on } (0,T) \times \{\delta\}$$

(see [10, 11]). Since v_δ is a viscosity solution of the above problem as well, $v_\delta = u_{n|D_\delta^+}$ by uniqueness. Using this, Lipschitz continuity of u_n and [7, Theorem 1.5.9] we conclude that $u_n \in C^{1,2}(D)$. Hence, by Proposition 1.2.3 and Theorem 2.2.1 in [12], $Y_t^{s,x,n} \in \mathbb{D}^{1,2}$ for every $(s,x) \in Q_T$ such that $x \neq 0$, where $\mathbb{D}^{1,2}$ is the domain of the derivative operator in $\mathbb{L}_2(\Omega)$ (see [12, Section 1.2] for a precise definition). Consequently, applying once again Proposition 1.2.3 and Theorem 2.2.1 in [12] and using the fact that g and $x \mapsto x^-$ are Lipschitz continuous functions we conclude that if $x \neq 0$ then $g(X_T^{s,x})$, $\int_t^T rY_\theta^{s,x,n}\, d\theta$, $\int_t^T n(Y_\theta^{s,x,n} - g(X_\theta^{s,x}))^-\, d\theta \in \mathbb{D}^{1,2}$. Moreover, by [12, Proposition 1.2.3] and [5, Lemma 5.1], there exists an adapted bounded process A such that for every $s < \tau \leq t$,

$$\begin{aligned} D_\tau Y_t^{s,x,n} = Z_\tau^{s,x,n} &+ \int_\tau^t D_\tau Z_\theta^{s,x,n}\, d\theta + r\int_\tau^t D_\tau Y_\theta^{s,x,n}\, d\theta \\ &- n\int_\tau^t A_\theta D_\tau(Y_\theta^{s,x,n} - h(X_\theta))\, d\theta, \end{aligned}$$

where D_τ denotes the derivative operator. From this it follows in particular that

$$D_t Y_t^{s,x,n} = Z_t^{s,x,n}, \quad P\text{-a.s.}$$

for every $t \in [s,T]$. On the other hand, by remarks following the proof of Proposition 2.2 and remark following the proof of [12, Proposition 1.2.3],

$$D_\tau Y_t^{s,x,n} = \partial_x u_n(t, X_t^{s,x}) D_\tau X_t^{s,x}, \quad P\text{-}a.s.$$

for every $r,t \in [s,T]$. Moreover, by [12, Theorem 2.2.1], $D_t X_t^{s,x} = \sigma X_t^{s,x}$. Thus, if $x \neq 0$, then

$$Z_t^{s,x,n} = \sigma X_t^{s,x} \partial_x u_n(t, X_t), \quad P\text{-}a.s..$$

By results from Section 6 in [4] and standard estimates for diffusions we have

$$\begin{aligned} E \sup_{s \le t \le T} |u_n(t, X_t^{s,x})|^2 + E \int_s^T |\sigma X_t^{s,x} \partial_x u_n(t, X_t^{s,x})|^2 \, dt \\ \le CE \sup_{s \le t \le T} |h(X_t^{s,x})|^2 \le C|x|^2. \end{aligned} \tag{2.10}$$

By the above and Proposition 5.1 in Appendix in [1] it follows that $u_n \in \mathbb{L}_2(0,T;H_\varrho)$. Accordingly, u_n is a weak solution of (2.9). Furthermore, from results proved in [4, Section 6] it follows that for every $(s,x) \in Q_T$,

$$\begin{aligned} E \sup_{s \le t \le T} |(u_n - u_m)(t, X_t^{s,x})|^2 + E \int_s^T |\sigma X_t^{s,x} \partial_x (u_n - u_m)(t, X_t^{s,x})|^2 \, dt \\ + E \sup_{s \le t \le T} |K_t^{s,x,n} - K_t^{s,x,m}|^2 \to 0 \end{aligned} \tag{2.11}$$

as $m,n \to \infty$. From (2.10), (2.11) and [9, Proposition 4.1] we conclude that there exists $u \in C(Q_T) \cap \mathbb{L}_2(0,T;H_\varrho)$ such that $u_n \to u$ uniformly on compact subsets of Q_T, $u_n \to u$ in $\mathbb{L}_2(0,T;H_\varrho)$ and $u_n \to u$ in $C([0,T], \mathbb{L}_{2,\varrho}(\mathbb{R}))$. Moreover, using (2.10) and [9, Proposition 4.1] we see that $\|u_n\|_{\mathbb{L}_2(0,T;H_\varrho)} \le C$. Therefore from (2.9) it follows that the sequence of measures $\{\mu_n\}$ defined by $d\mu_n = n(u_n - h)^- \, d\lambda$, $n \in \mathbb{N}$, where λ is the 2-dimensional Lebesgue measure, is tight. If $\mu_n \to \mu$ weakly, which we may assume, then letting $n \to \infty$ in (2.9) we conclude that the pair (u,μ) satisfies equation $(1.8)_1$ in the weak sense and that

$$u(t, X_t^{s,x}) = Y_t^{s,x},\ t \in [s,T],\ P\text{-}a.s., \quad Z_t^{s,x} = \sigma X_t^{s,x} \partial_x u(t, X_t^{s,x}),\ dt \otimes P\text{-}a.s.$$

because in [4, Section 6] it is proved that $Y_t^{s,x,n} \to Y_t^{s,x}$, $t \in [s,T]$, P-a.s. and $E\int_s^T |Z_t^{s,x,n} - Z_t^{s,x}|^2 \, dt \to 0$. In particular, it follows from the above that $u \ge g$. Let $\eta \in C_0(Q_T)$. Since $u_n \to u$ uniformly,

$$\int_{Q_T} (u_n - g)\eta \, d\mu_n \to \int_{Q_T} (u - g)\eta \, d\mu \ge 0.$$

On the other hand,

$$\int_{Q_T} (u_n - g)\eta \, d\mu_n = -\int_{Q_T} n((u_n - g)^-)^2 \, d\lambda \le 0.$$

From this we get $(1.8)_2$. Furthermore, if $x \neq 0$ then for any $\delta \in (0, T-s)$ and $\eta \in C_0(Q_{s+\delta,T})$ we have

$$E \int_s^T \eta(t, X_t^{s,x}) \, dK_t^{s,x,n} = \int_{Q_{sT}} \eta(t,y) p(s,x,t,y) \, d\mu_n(t,y). \tag{2.12}$$

Since it is known that $K_t^{s,x,n} \to K_t^{s,x}$ uniformly in $t \in [s,T]$ in probability (see [4, Section 6]), letting $n \to \infty$ in (2.12) and using (2.7), (2.11) we get (2.8) for $\eta \in C_0(Q_{s+\delta,T})$, and hence for any $\eta \in C_0(Q_{sT})$. In order to complete the proof we have to show that $u \in W_\varrho$. Since $p(s,x,\cdot,\cdot)$ is positive for every $(s,x) \in Q_T$ such that $x \neq 0$, it follows from (2.8) and Proposition 2.1 that $d\mu \le \mathbf{1}_{\{u=g\}}(t,x)(dx - rK)^+\,d\lambda$, i.e. for every $\eta \in C_0^+(Q_T)$,

$$\int_{Q_T} \eta(t,x)\,d\mu(t,x) \le \int_{Q_T} \eta(t,x)\mathbf{1}_{\{u=g\}}(t,x)(dx - rK)^+\,dx\,dt.$$

Hence there exists a measurable function α on Q_T such that $0 \le \alpha \le 1$ and

$$\frac{d\mu}{d\lambda}(t,x) = \alpha(t,x)\mathbf{1}_{\{u=g\}}(t,x)(dx - rK)^+. \tag{2.13}$$

This implies that $u \in W_\varrho$ and u satisfies $(1.8)_1$ in the strong sense, i.e. (u,μ) is a solution of (1.8). □

Remark 2.4 It is known that for each $t \in [0,T)$, $\partial_{\mathbb{R}}\{h = u(t)\}$ is a singleton (see, e.g., [2]). This implies that the Lebesgue measure of $\partial_{Q_T}\{u = h\}$ equals zero.

3 Linear RBSDEs and nonlinear BSDEs

We begin with proving the key formulas (1.6), (1.9). As the first application we will show the semimartingale representation for the Snell envelope of the discounted payoff process and the early exercise premium representation for V.

Theorem 3.1 *(i) If (u,μ) is a solution of the obstacle problem (1.8), then μ is given by (1.9).*
(ii) If $(Y^{s,x}, Z^{s,x}, K^{s,x})$ is a solution of RBSDE (1.3), then $K^{s,x}$ is given by (1.6).

Proof. We prove the theorem in the case of call option. The proof for put option requires only some obvious changes and is left to the reader.

Suppose that $(Y^{s,x}, Z^{s,x}, K^{s,x})$ is a solution of (1.3) and (u,μ) is a solution of (1.8). By (2.13), u solves the equation

$$\partial_t u + (r-d)x\partial_x u + \frac{1}{2}\sigma^2 x^2 \partial_{xx}^2 u = ru - \alpha(t,x)\mathbf{1}_{\{u=g\}}(t,x)(dx - rK)^+ \tag{3.1}$$

in the strong sense. By Remark 2.4, u is continuous. Let $I = \{u = h\}$ and $I_0 = \mathrm{Int}\, I$. By (3.1), for any $\eta \in C_0^\infty(I_0)$ we have

$$\begin{aligned}
\int_{Q_T} u(t,x)\partial_t\eta(t,x)\,dt\,dx &- \frac{1}{2}\int_{Q_T} \sigma^2 x^2 \partial_{xx}^2 u(t,x)\eta(t,x)\,dt\,dx \\
&- \int_{Q_T} (r-d)x\partial_x u(t,x)\eta(t,x)\,dt\,dx \\
= \int_{Q_T} &(-ru(t,x) + \alpha(t,x)\mathbf{1}_{\{u=g\}}(t,x)(dx - rK)^+)\eta(t,x)\,dt\,dx \\
+ \int_{\mathbb{R}} &g(x)\eta(T,x)\,dx - \int_{\mathbb{R}} u(0,x)\eta(0,x)\,dx.
\end{aligned}$$

Since $\operatorname{supp}\eta \subset I_0$ and g is regular on I_0, we deduce from the above that

$$\int_{I_0}(r-d)x\mathbf{1}_{[K,\infty)}(x)\eta(t,x)\,dt\,dx$$
$$=\int_{I_0} rg(x)\eta(t,x)\,dt\,dx-\int_{I_0}\alpha(t,x)\mathbf{1}_{\{u=g\}}(t,x)(dx-rK)^+\eta(t,x)\,dt\,dx.$$

Equivalently, we have

$$\int_{I_0} f(t,x)\eta(t,x)\,dt\,dx=\int_{I_0}\alpha(t,x)\mathbf{1}_{\{u=g\}}(t,x)\mathbf{1}_{[K,\infty)}(x)(dx-rK)^+\eta(t,x)\,dt\,dx,$$

where $f(t,x)=(r-d)x\mathbf{1}_{[K,\infty)}(x)-r(x-K)^+=(-dx+rK)\mathbf{1}_{[K,\infty)}(x)$. Since

$$\alpha(t,x)(dx-rK)^+=-\alpha(t,x)((r-d)x\mathbf{1}_{[K,\infty)}(x)-r(x-K)^+)^-=-\alpha(t,x)f^-(x)$$

on I_0, it follows that

$$\int_{I_0} f(t,x)\eta(t,x)\,dt\,dx=-\int_{I_0}\alpha(t,x)f^-(t,x)\eta(t,x)\,dt\,dx$$

for any $\eta\in C_0^\infty(I_0)$. Therefore, $f(t,x)=-\alpha(t,x)f^-(t,x)$ a.e. on I_0. Since $f=f^+-f^-$, we see that $f^+(t,x)=(1-\alpha(t,x))f^-(t,x)$, hence that $(1-\alpha(t,x))f^-(t,x)=0$ a.e. on I_0, i.e. $\alpha(t,x)(dx-rK)^+=(dx-rK)^+$ a.e. on I_0. Since the Lebesgue measure of ∂I equals zero (see Remark 2.4), the above equality holds a.e. on I, which in view of (2.13) completes the proof of (i).

In case $x=0$ part (ii) is trivial since in that case $X_t^{s,x}=K_t^{s,x}=0$, $t\in[s,T]$. In the case $x\neq 0$ part (ii) follows from part (i) and results proved in [9]. To see this, let us denote by X the canonical process on the space $C([0,T];\mathbb{R})$ of continuous functions on $[0,T]$, and by $P_{s,x}$ the law of $X^{s,x}$, i.e. $P_{s,x}=P\circ(X^{s,x})^{-1}$. We may and will assume that $X_s^{s,x}=x$, $t\in[0,s]$, and hence that $P_{s,x}$ is a measure on $C([0,T];\mathbb{R})$. Write

$$M_{s,t}=X_t-X_s-\int_s^t(r-d)X_\theta\,d\theta,\quad B_{s,t}=\int_s^t\frac{1}{\sigma X_\theta}\,dM_{s,\theta},\quad 0\le s\le t\le T$$

and observe that if $x\neq0$ then under $P_{s,x}$ the process $B_{s,\cdot}$ is a standard Wiener process on $[s,T]$ with respect to the natural filtration generated by X. Furthermore, set

$$K_{s,t}=u(s,X_s)-u(t,X_t)+\int_s^t ru(\theta,X_\theta)\,d\theta$$
$$+\int_s^t\sigma\partial_x u(\theta,X_\theta)\,dB_{s,\theta},\quad 0\le s<t\le T$$

and

$$\tilde K_{s,t}=\int_s^t(dX_\theta-rK)^+\mathbf{1}_{\{u(\theta,X_\theta)=g(X_\theta)\}}\,d\theta,\quad 0\le s<t\le T.$$

Let $(Y^{s,x},Z^{s,x},K^{s,x})$ be a solution of (1.3) and let $\tilde K^{s,x}$ denote the process defined by the right hand-side of (1.6). By Theorem 2.3, for every $(s,x)\in[0,T)\times\mathbb{R}$,

$$K_t^{s,x}-K_s^{s,x}=u(s,X_s^{s,x})-u(t,X_t^{s,x})+\int_s^t ru(\theta,X_\theta^{s,x})\,d\theta$$
$$+\int_s^t\sigma\partial_x u(\theta,X_\theta^{s,x})\,dW_\theta,\quad 0\le s<t\le T,\quad P\text{-}a.s..$$

From this and the fact that the law of $(X, B_{s,\cdot})$ under $P_{s,x}$ is equal to the law of $(X^{s,x}, W_\cdot - W_s)$ under P we conclude that the law of $K_{s,\cdot}$ under $P_{s,x}$ is equal to the law of $K^{s,x}$ under P. Consequently, by (2.8), for every $s \in [0,T)$, $x \neq 0$,

$$E_{s,x} \int_s^T \eta(t, X_t)\, dK_{s,t} = \int_{Q_{sT}} \eta(t,y) p(s,x,t,y)\, d\mu(t,y) \tag{3.2}$$

for all $\eta \in C_0(Q_{sT})$, where $E_{s,x}$ denotes expectation with respect to $P_{s,x}$. Thus, the additive functional $K = \{K_{s,t}; 0 \le s \le t \le T\}$ of the Markov family $\{(X, P_{s,x}); (s,x) \in [0,T) \times \mathbb{R}\}$ corresponds to the measure μ in the sense defined in [9]. Similarly, for every $s \in [0,T)$, $x \neq 0$ the law of $\tilde{K}_{s,\cdot}$ under $P_{s,x}$ is equal to the law of $\tilde{K}^{s,x}$ under P, and hence, by part (i), (3.2) is satisfied with K replaced by $\tilde{K}$, i.e. the additive functional $\tilde{K} = \{\tilde{K}_{s,t}; 0 \le s \le t \le T\}$ corresponds to μ, too. The proof of [9, Proposition 4.4] now shows that $P_{s,x}(K_{s,t} = \tilde{K}_{s,t}, t \in [s,T]) = 1$ for every $s \in [0,T)$, $x \neq 0$, hence that $P(K^{s,x}_t = \tilde{K}^{s,x}_t, t \in [s,T]) = 1$ for $s \in [0,T)$, $x \neq 0$, which completes the proof. □

Corollary 3.2 *If $(Y^{s,x}, Z^{s,x}, K^{s,x})$ is a solution of (1.3) then $(Y^{s,x}, Z^{s,x})$ is a solution of (1.7). Conversely, if $(Y^{s,x}, Z^{s,x})$ is a solution of (1.7) then $(Y^{s,x}, Z^{s,x}, K^{s,x})$ with $K^{s,x}$ defined by (1.6) is a solution of (1.3).*

Proof. The first part follows immediately from Theorem 3.1. The second part is a consequence of the first one and the fact that the solution of (1.7) is unique, because for every $x \in \mathbb{R}$ the function $y \mapsto q(x,y)$ is decreasing. □

Let ξ denote the discounted payoff process for the American option, i.e.

$$\xi_t = e^{-r(t-s)} g(X^{s,x}_t), \quad t \in [s,T].$$

By (1.7),

$$\begin{aligned} e^{r(t-s)} Y^{s,x}_t = e^{-r(T-s)} g(X^{s,x}_T) + \int_t^T e^{-r(\theta-s)} q(X^{s,x}_\theta, Y^{s,x}_\theta)\, d\theta \\ - \int_t^T e^{-r(\theta-s)} Z^{s,x}_\theta\, dW_\theta. \end{aligned}$$

From this and the fact that $V(t, X^{s,x}_t) = u(t, X^{s,x}_t) = Y^{s,x}_t$, $t \in [s,T]$, we obtain

Corollary 3.3 *The Snell envelope $\eta_t = e^{-r(t-s)} V(t, X^{s,x}_t)$, $t \in [s,T]$, of ξ admits the representation*

$$\eta_t = E\left(e^{-r(T-s)} g(X^{s,x}_T) + \int_t^T e^{-r(\theta-s)} q(X^{s,x}_\theta, Y^{s,x}_\theta)\, d\theta \,|\mathcal{F}_t \right). \tag{3.3}$$

From (3.3) we get immediately the early exercise premium representation for V. For instance, for American put option,

$$V(s,x) = Ee^{-r(T-s)} g(X^{s,x}_T) + E \int_s^T e^{-r(t-s)} (rK - dX^{s,x}_t)^+ \mathbf{1}_{\{V=g\}}(t, X^{s,x}_t)\, dt. \tag{3.4}$$

Representations (3.3), (3.4) are known (see [8, Corollary 7.11]). Up to our knowledge our proof is new. Let us stress, however, that we were influenced by results of [2].

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