REAL ANALYTIC APPROXIMATIONS WHICH ALMOST PRESERVE LIPSCHITZ CONSTANTS OF FUNCTIONS DEFINED ON THE HILBERT SPACE

D. AZAGRA, R. FRY, AND L. KEENER

ABSTRACT. Let X be a separable real Hilbert space. We show that for every Lipschitz function $f : X \to \mathbb{R}$, and for every $\varepsilon > 0$, there exists a Lipschitz, real analytic function $g : X \to \mathbb{R}$ such that $|f(x) - g(x)| \leq \varepsilon$ and $\mathrm{Lip}(g) \leq \mathrm{Lip}(f) + \varepsilon$.

In a recent paper [AFK1] we proved that for every separable Banach space X having a separating polynomial there exists a constant $C \geq 1$ such that, for every Lipschitz function $f : X \to \mathbb{R}$ and every $\varepsilon > 0$ there exists a Lipschitz, real analytic function $f : X \to \mathbb{R}$ such that $|f - g| \leq \varepsilon$ and $\mathrm{Lip}(g) \leq C\mathrm{Lip}(f)$. It is natural to wonder whether the constant C can be assumed to be 1 (as in the finite-dimensional case), or at least any number greater that 1. The aim of this note is to prove that the latter is indeed true in the case when X is a Hilbert space.

Theorem 1. *Let X be a separable real Hilbert space. For every Lipschitz function $f : X \to \mathbb{R}$, and for every $\varepsilon > 0$, there exists a Lipschitz, real analytic function $g : X \to \mathbb{R}$ such that $|f(x) - g(x)| \leq \varepsilon$ and $Lip(g) \leq Lip(f) + \varepsilon$.*

By using the main result of [AFK1] as well as the techniques developed for its proof, we have shown recently [AFK2] that, for such spaces X, for every C^1 function $f : X \to \mathbb{R}$ with a uniformly continuous derivative, and for every $\varepsilon > 0$ there exists a real analytic function g such that $|f - g| \leq \varepsilon$ and $\|f' - g'\| \leq \varepsilon$.

In fact, an analysis of the proof of [AFK2] (in view of Proposition 1 of [AFK1] and following the lines of Lemma 4 of [AFK1]) shows that the domain where a holomorphic extension of the function g is defined and ε-close to g only depends on $\|f\|_\infty$, on ε, and on the modulus of continuity of f'. Namely, we have the following sharp version of the main result of [AFK2].

Theorem 2 (AFK2). *Let X be a separable Banach space with a separating polynomial, and let $M, K, \varepsilon > 0$ be given. Then there exists an open neighborhood $\widetilde{U}$ of X in $\widetilde{X}$, depending only on M, K, ε, such that for every function $f \in C^{1,1}(X)$ with $\|f\|_\infty \leq M$ and $Lip(f') \leq K$ there exists a real*

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analytic function $g : X \to \mathbb{R}$, with holomorphic extension $\widetilde{g} : \widetilde{U} \to \mathbb{C}$, such that

(1) $|f(x) - g(x)| \leq \varepsilon$ *for all* $x \in X$.
(2) $|f'(x) - g'(x)| \leq \varepsilon$ *for all* $x \in X$.
(3) $|\widetilde{g}(x+z) - g(x)| \leq \varepsilon$ *for all* $x \in X$, $z \in \widetilde{X}$ *with* $x + z \in \widetilde{U}$.

(Here we use the same notation as in [AFK1, AFK2]; in particular $\widetilde{X}$ denotes the complexification of X, endowed with the Taylor norm.)

We will prove Theorem 1 by combining Theorem 2 with the Lasry-Lions sup-inf convolution regularization technique [LL], and with a refinement of Lemma 3 and Proposition 2 of [J] (alternatively, one can also use a refinement of the original tube and crown gluing method introduced in [AFK1, Lemma 5], but this produces a somewhat longer proof).

We start considering the special case when f is bounded.

Given an L-Lipschitz function $f : X \to [0, M]$ defined on a separable Hilbert space X, set

$$f_\lambda(x) = \inf\{f(u) + \frac{1}{2\lambda}|x-u|^2 \, : \, u \in X\} = \inf\{f(x-u) + \frac{1}{2\lambda}|u|^2 \, : \, u \in X\}$$

$$f^\mu(x) = \sup\{f(u) - \frac{1}{2\lambda}|x-u|^2 \, : \, u \in X\} = \sup\{f(x-u) - \frac{1}{2\lambda}|u|^2 \, : \, u \in X\}.$$

Since the supremum (and the infimum) of a family of L-Lipschitz functions is L-Lipschitz, it is clear that f_λ and f^μ are L-Lipschitz.

Now, since f is bounded and uniformly continuous, according to [LL], the function

$$g_{\lambda,\mu}(x) := (f_\lambda)^\mu(x) = \sup_{z \in X} \inf_{y \in X} \{f(y) + \frac{1}{2\lambda}|z-y|^2 - \frac{1}{2\mu}|x-z|^2\}$$

is well defined and has a Lipschitz derivative on X satisfying

$$\mathrm{Lip}(g'_{\lambda,\mu}) \leq \max\{\frac{1}{\mu}, \frac{1}{\lambda - \mu}\},$$

for all $0 < \mu < \lambda$ small enough, and converges to $f(x)$, uniformly on X, as $0 < \mu < \lambda \to 0$. In fact, as noted in [LL], the rate of convergence of $g_{\lambda,\mu}$ to f only depends on $\mathrm{Lip}(f)$, so for every $\varepsilon > 0$ there exists $\lambda_0 > 0$ (only depending on ε and L) so that $|g_{\lambda,\mu}(x) - f(x)| \leq \varepsilon/2$ for all $x \in X$, $0 < \mu < \lambda \leq \lambda_0$.

Also, according to the above observations, this function is L-Lipschitz. Therefore we have

$$\|g'_{\lambda,\mu}(x)\| \leq L, \text{ and } |f(x) - g_{\lambda,\mu}(x)| \leq \frac{\varepsilon}{2}$$

for all $x \in X$, for some $0 < \mu < \lambda$ small enough. Now fix λ, μ with

$$0 < \lambda < \lambda_0, \;\; \mu := \frac{\lambda}{2},$$

and apply Theorem 2 to obtain a real analytic function $g : X \to \mathbb{R}$ such that

$$|g_{\lambda,\mu}(x) - g(x)| \leq \frac{\varepsilon}{2} \text{ and } |g'_{\lambda,\mu}(x) - g'(x)| \leq \frac{\varepsilon}{2}$$

for all $x \in X$. By combining the last two inequalities we get $|f(x)-g(x)| \leq \varepsilon$ for all $x \in X$ and $\mathrm{Lip}(g) \leq L+\varepsilon$. Moreover $g_{\lambda,\mu}$ has a holomorphic extension to a neighborhood $\widetilde{U}$ of X in $\widetilde{X}$ which only depends on L, on $\mathrm{Lip}(g'_{\lambda,\mu})$, on M, and on ε. Since $\mathrm{Lip}(g'_{\lambda,\mu}) \leq \max\{1/\mu, 1/(\lambda-\mu)\} = 2/\lambda$ and in turn λ only depends on ε and on $\mathrm{Lip}(f) \leq L$, we have thus proved the following.

Proposition 1. *Let X be a separable Hilbert space. For every $L, M, \varepsilon > 0$ there exists a neighborhood $\widetilde{U} := \widetilde{U}_{L,M,\varepsilon}$ of X in $\widetilde{X}$ such that, for every L-Lipschitz function $f : X \to [0, M]$ there exists a real analytic function $g : X \to \mathbb{R}$, with holomorphic extension $\widetilde{g} : \widetilde{U} \to \mathbb{C}$, such that*

1. $|f(x) - g(x)| \leq \varepsilon$ *for all* $x \in X$.
2. g *is* $(L+\varepsilon)$*-Lipschitz.*
3. $|\widetilde{g}(x+iy) - g(x)| \leq \varepsilon$ *for all* $z = x+iy \in \widetilde{U}$.

Now fix $L = M = 1$, and $\varepsilon \in (0, 1/16)$. Let $\overline{\theta} : \mathbb{R} \to [0,1]$ be a C^∞ function such that:

1. $\overline{\theta}(t) = 0$ iff $t \in (-\infty, 4\varepsilon]$
2. $\overline{\theta}(t) = 1$ iff $t \in [1-4\varepsilon, \infty)$
3. $\overline{\theta}'(t) > 0$ iff $t \in (4\varepsilon, 1-4\varepsilon)$
4. $|\overline{\theta}(t) - t| \leq 5\varepsilon$ if $t \in [0,1]$
5. $\mathrm{Lip}(\overline{\theta}) \leq 1/(1-10\varepsilon)$.

Define $\widetilde{\theta}_\kappa : \mathbb{C} \to \mathbb{C}$ by

$$\widetilde{\theta}_\kappa(z) = a_\kappa \int_{\mathbb{R}} \overline{\theta}(t) e^{-\kappa(z-t)^2} dt, \quad \text{where} \quad a_\kappa := \frac{1}{\int_{\mathbb{R}} e^{-\kappa t^2} dt},$$

and denote by θ_κ the restriction of $\widetilde{\theta}$ to $\mathbb{R}$. It is clear that $\widetilde{\theta}$ is holomorphic in $\mathbb{C}$ and θ is real analytic, and $\mathrm{Lip}(\theta) = \mathrm{Lip}(\overline{\theta})$. Now, assume $z \in \mathbb{C}$ satisfies $|z| \leq \varepsilon$. Then, denoting $z = u + iv$, we can estimate

$$|\widetilde{\theta}_\kappa(z)| = |a_\kappa \int_{\mathbb{R}} \overline{\theta}(s) e^{-\kappa(u+iv-s)^2} ds| \leq a_\kappa e^{\kappa\varepsilon^2} \int_{4\varepsilon}^{\infty} e^{-\kappa(u-s)^2} ds =$$

$$a_\kappa e^{\kappa\varepsilon^2} \int_{3\varepsilon}^{\infty} e^{-\kappa t^2} dt = a_\kappa \int_{3\varepsilon}^{\infty} e^{-\kappa(t^2-\varepsilon^2)} dt = a_\kappa \int_{3\varepsilon}^{\infty} e^{-\kappa t^2/2} e^{-\kappa(t^2/2-\varepsilon^2)} dt \leq$$

$$a_\kappa e^{-\kappa((3\varepsilon)^2/2-\varepsilon^2)} \int_{3\varepsilon}^{\infty} e^{-\kappa t^2/2} dt \leq a_\kappa e^{-\kappa(9\varepsilon^2/2-\varepsilon^2)} \int_{-\infty}^{\infty} e^{-\kappa t^2/2} dt = \sqrt{2} e^{-7\kappa\varepsilon^2/2}.$$

On the other hand, θ_κ and θ'_κ uniformly converge on $\mathbb{R}$ to $\overline{\theta}$ and $\overline{\theta}'$, respectively, as $\kappa \to \infty$. Therefore, observing that $\lim_{\kappa\to\infty} \sqrt{2} e^{-7\kappa\varepsilon^2/2} = 0$, for every $n \in \mathbb{N}$ we can choose κ_n large enough so that, denoting $\widetilde{\theta}_n := \widetilde{\theta}_{\kappa_n}$ and $\theta_n := \theta_{\kappa_n}$, we have

1. $|\widetilde{\theta}_n(z)| \leq \varepsilon/2^{n+2}$ if $z \in \mathbb{C}$, $|z| \leq \varepsilon$
2. $\mathrm{Lip}(\theta_n) \leq 1/(1-10\varepsilon)$
3. $|\theta_n(t) - \overline{\theta}(t)| \leq \varepsilon/2^{n+2}$
4. $|\theta'_n(t)| \leq \varepsilon/2^{n+2}$ if $t \in (-\infty, 2\varepsilon] \cup [1-2\varepsilon, \infty)$.

Now, let $f : X \to [0, \infty)$ be a 1-Lipschitz function. Define, for every $n \in \mathbb{N}$, the function $f_n : X \to [0, 1]$ by

$$f_n(x) = \begin{cases} 0 & \text{if } f(x) \leq n-1, \\ f(x) & \text{if } n-1 \leq f(x) \leq n, \\ 1 & \text{if } n \leq f(x). \end{cases}$$

It is clear that the functions f_n are 1-Lipschitz, and we have

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

for every $x \in X$. In fact this sum is finite on every bounded set. By Proposition **??**, there exists an open neighborhood $\widetilde{U} := \widetilde{U}_{1,1,\varepsilon}$ of X in $\widetilde{X}$ and a collection of real analytic functions $g_n : X \to \mathbb{R}$ with holomorphic extensions $\widetilde{g}_n : \widetilde{U} \to \mathbb{C}$, such that

- $|f_n - g_n| \leq \varepsilon/2$
- $\mathrm{Lip}(g_n) \leq 1 + \varepsilon$
- $|\widetilde{g}(x+z) - g(x)| \leq \varepsilon/2$ for all $x \in X, z \in \widetilde{X}$ with $x + z \in \widetilde{U}$.

Define

$$\widetilde{g}(z) = \sum_{n=1}^{\infty} \widetilde{\theta}_n(\widetilde{g}_n(z)) \quad \text{for all} \quad z \in \widetilde{U}.$$

This function is well defined and holomorphic on $\widetilde{U}$. Indeed, for a given $x \in X$, there exists a unique $n_x \in \mathbb{N}$ such that $f(x) \in [n_x - 1, n_x)$, and in particular we have $f_n(x) = 0$ for all $n > n_x$. Therefore $|g_n(x)| \leq \varepsilon/2$, and

$$|\widetilde{g}_n(x+z)| \leq |\widetilde{g}_n(x+z) - g_n(x)| + |g_n(x)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

provided that $n > n_x, x + z \in \widetilde{U}$. By the first property of $\widetilde{\theta}_n$ noted above, this implies that

$$|\widetilde{\theta}_n(\widetilde{g}_n(x+z))| \leq \frac{\varepsilon}{2^{n+2}} \quad \text{if} \quad n > n_x, \, z \in \widetilde{X} \quad \text{with} \quad x + z \in \widetilde{U}.$$

Therefore the series of holomorphic functions $\sum_{n=1}^{\infty} \widetilde{\theta}_n \circ \widetilde{g}_n$ converges locally uniformly and absolutely on $\widetilde{U}$ and defines a holomorphic function $\widetilde{g}$ on $\widetilde{U}$, whose restriction to X will be denoted by g.

Let us now check that $|f(x) - g(x)| \leq 8\varepsilon$ for all $x \in X$. Indeed, if $f(x) \in [n_x - 1, n_x)$ then we have $f_n(x) = 1$ for $n < n_x$ and $f_n(x) = 0$ for $n > n_x$. Therefore

$$g_n(x) \geq 1 - \frac{\varepsilon}{2} \quad \text{if} \quad n < n_x;$$

$$|f_{n_x}(x) - g_{n_x}(x)| \leq \frac{\varepsilon}{2} \quad \text{and}$$

$$g_n(x) \leq \frac{\varepsilon}{2} \quad \text{if} \quad n > n_x,$$

which implies

$$\begin{aligned}
&|\theta_n(g_n(x)) - f_n(x)| = |\theta_n(g_n(x)) - 1| \\
&= |\theta_n(g_n(x)) - \overline{\theta}(g_n(x))| \leq \frac{\varepsilon}{2^{n+2}} \quad \text{if} \quad n < n_x; \\
&|\theta_{n_x}(g_{n_x}(x)) - f_{n_x}(x)| \leq \\
&|\theta_{n_x}(g_{n_x}(x)) - \theta_{n_x}(f_{n_x}(x))| + |\theta_{n_x}(f_{n_x}(x)) - f_{n_x}(x)| \leq \\
&\frac{1}{1-10\varepsilon}\frac{\varepsilon}{2} + 6\varepsilon \leq 7\varepsilon, \quad \text{and} \\
&|\theta_n(g_n(x)) - f_n(x)| = |\theta_n(g_n(x))| \leq \frac{\varepsilon}{2^{n+2}} \quad \text{if} \quad n > n_x.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&|\sum_{n=1}^{\infty} \theta_n(g_n(x)) - f(x)| = |\sum_{n=1}^{\infty} (\theta_n(g_n(x)) - f_n(x))\,| \leq \\
&\sum_{n=1}^{n_x-1} \frac{\varepsilon}{2^{n+2}} + 7\varepsilon + \sum_{n>n_x} \frac{\varepsilon}{2^{n+2}} \leq 8\varepsilon.
\end{aligned}$$

As for $\mathrm{Lip}(g)$, we observe that $g_n(x) \in (-\infty, \varepsilon/2] \cup [1-\varepsilon/2, 1+\varepsilon/2]$ if $n \neq n_x$, hence $|\theta_n'(g_n(x))| \leq \varepsilon/2^{n+2}$ if $n \neq n_x$, and

$$\|D(\theta \circ g_n)(x)\| = |\theta_n'(g_n(x))|\, \|Dg_n(x)\| \leq \frac{\varepsilon}{2^{n+2}}\mathrm{Lip}(g_n) \leq \frac{\varepsilon}{2^{n+2}}(1+\varepsilon) \text{ for } n \neq n_x.$$

Therefore

$$\begin{aligned}
&\|Dg(x)\| \leq \sum_{n=1}^{\infty} \|D(\theta_n \circ g_n)(x) \leq \sum_{n \neq n_x} \frac{\varepsilon}{2^{n+2}}(1+\varepsilon) + \mathrm{Lip}(\theta_{n_x})\mathrm{Lip}(g_{n_x}) \\
&\leq \varepsilon(1+\varepsilon) + \frac{1}{1-10\varepsilon}(1+\varepsilon) \leq \frac{1+3\varepsilon}{1-10\varepsilon},
\end{aligned}$$

which shows that $\mathrm{Lip}(g) \leq \frac{1+3\varepsilon}{1-10\varepsilon}$.

Since $\lim_{\varepsilon \to 0} \frac{1+3\varepsilon}{1-10\varepsilon} = 1$, up to a change of ε we have shown the following: for every 1-Lipschitz function $f : X \to [0, \infty)$ and for every $\varepsilon > 0$, there exists a $(1+\varepsilon)$-Lipschitz real analytic function $g : X \to [0, \infty)$ such that $|f - g| \leq \varepsilon$.

Now, if $f : X \to \mathbb{R}$ is 1-Lipschitz unbounded function, we have $f = f^+ - f^-$, where $f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$ are 1-Lipschitz and take values in $[0, \infty)$. According to what we have just proved, there are $(1+\varepsilon)$-Lipschitz, real analytic functions $g^+, g^- : X \to [0, \infty)$ such that $|f^{\pm} - g^{\pm}| \leq \varepsilon$ on X. Take a real analytic function $\alpha : \mathbb{R} \to \mathbb{R}$ such that

- $|\alpha(t) - t| \leq 3\varepsilon$ if $t \geq 0$
- $\mathrm{Lip}(\alpha) = 1$
- $|\alpha'(t)| \leq \varepsilon$ if $t \leq 3\varepsilon$.

Such a function can be defined, for instance, by

$$\alpha(t) = \frac{\int_{\mathbb{R}} \overline{\alpha}(s) e^{-\kappa(t-s)^2} ds}{\int_{\mathbb{R}} e^{-\kappa s^2} ds}$$

for κ large enough, where $\overline{\alpha}(s) = 0$ if $s \leq 2\varepsilon$ and $\overline{\alpha}(s) = s - 2\varepsilon$ if $s \geq 2\varepsilon$.

Define $g = \alpha \circ g^+ - \alpha \circ g^-$. It is clear that g is a real analytic function. Besides,

$$\begin{aligned} &|f - g| \leq \\ &|f^+ - \alpha \circ f^+| + |\alpha \circ (f^+ - g^+)| + |f^- - \alpha \circ f^-| + |\alpha \circ (f^- - g^-)| \leq \\ &3\varepsilon + 1\varepsilon + 3\varepsilon + 1\varepsilon = 8\varepsilon. \end{aligned}$$

On the other hand, if $f(x) \geq 0$ then $g^-(x) \leq \varepsilon$, so $|\alpha'(g^-(x))| \leq \varepsilon$, and $\|D(\alpha \circ g^-)(x)\| \leq \varepsilon(1+\varepsilon)$. Similarly, if $f(x) \leq 0$ then $\|D(\alpha \circ g^+)(x)\| \leq \varepsilon(1+\varepsilon)$. And in any case we also have $\|D(\alpha \circ g^{\pm})(x)\| \leq 1(1+\varepsilon)$. Therefore we can estimate

$$\|Dg(x)\| \leq \|D(\alpha \circ g^+)(x)\| + \|D(\alpha \circ g^-)(x)\| \leq 1(1+\varepsilon) + \varepsilon(1+\varepsilon) = (1+\varepsilon)^2.$$

Up to a change of ε this argument proves Theorem 1 in the case $\text{Lip}(f) \leq 1$. Finally, in the case of a function f with $\text{Lip}(f) := L \in (0, \infty)$, consider $F(x) = \frac{1}{\varepsilon} f(\frac{\varepsilon}{L} x)$, which is 1-Lipschitz. We can then find a $(1+\varepsilon)$-Lipschitz, real analytic function $G : X \to \mathbb{R}$ such that $|F - G| \leq 1$. If we define $g(x) = \varepsilon G(\frac{L}{\varepsilon} x)$, we get a real analytic function $g : X \to \mathbb{R}$ with $\text{Lip}(g) \leq (1+\varepsilon)\text{Lip}(f)$, and such that $|g - f| \leq \varepsilon$. This concludes the proof of Theorem 1 in the general case.

References

[AFK1] D. Azagra, R. Fry, and L. Keener, *Real analytic approximation of Lipschitz functions on Hilbert space and other Banach spaces*, preprint, arXiv:1005.1050.

[AFK2] D. Azagra, R. Fry, L. Keener, *Approximation of functions and their derivatives by analytic maps on certain Banach spaces,* preprint, arXiv:1011.4613.

[CH] M. Cepedello, and P. Hajek, *Analytic approximations of uniformly continuous functions in real Banach spaces.* J. Math. Anal. Appl. 256 (2001), no. 1, 80–98.

[DFH1] R. Deville, V. Fonf and P. Hajek, Analytic and C^k approximation of norms in separable Banach spaces, *Studia Math.* 120 (1996), 61-74.

[DFH1] R. Deville, V. Fonf and P. Hajek, Analytic and polyhedral approximation of convex bodies in separable polyhedral banach spaces, *Israel J. Math.* 105 (1998), 139-154.

[F1] R. Fry, *Analytic approximation on c_0.* J. Funct. Anal. 158 (1998), no. 2, 509–520.

[FK] R. Fry and L. Keener, *Approximation by Lipschitz, analytic maps on certain Banach spaces*, preprint, arXiv:0810.5600v2 [math.FA]

[JS] M. Jiménez-Sevilla, *A note on the range of the derivatives of analytic approximations of uniformly continuous functions on c_0.* J. Math. Anal. Appl. 348 (2008), no. 2, 573–580.

[J] M. Johanis, *A simple proof of the approximation by real analytic Lipschitz functions*, to appear in J. Math. Anal. Appl.

[K] J. Kurzweil, *On approximation in real Banach spaces*, Studia Math. 14 (1953), 214-231.

[LL] Lasry, J.-M.; Lions, P.-L., *A remark on regularization in Hilbert spaces.* Israel J. Math. 55 (1986), no. 3, 257–266.

[MST] G. Munoz, Y. Sarantopulos and A. Tonge, Complexifications of real Banach spaces, polynomials and multilinear maps, *Studia Math.* 134 (1999), 1-33.

[W] H. Whitney, Analytic extensions of differential functions in closed sets, *Trans. Amer. Math. Soc.* 36 (1934), 63-89.

ICMAT (CSIC-UAM-UC3-UCM), Departamento de Análisis Matemático, Facultad Ciencias Matemáticas, Universidad Complutense, 28040, Madrid, Spain
E-mail address: daniel_azagra@mat.ucm.es

Department of Mathematics and Statistics, Thompson Rivers University, Kamloops, B.C., Canada
E-mail address: Rfry@tru.ca

Department of Mathematics and Statistics, University of Northern British Columbia, Prince George, B.C., Canada
E-mail address: keener@unbc.ca