Completely positive mappings and mean matrices

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Abstract

Some functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ induce mean of positive numbers and the matrix monotonicity gives a possibility for means of positive definite matrices. Moreover, such a function f can define linear mapping $\beta_f : \mathbf{M}_n \to \mathbf{M}_n$ on matrices (which is basical in the constructions of monotone metrics). The present subject is to check the complete positivity of β_f in the case of a few concrete functions f. This problem has been motivated by applications in quantum information.

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1 Introduction

The matrix monotone function $f : \mathbb{R}^+ \to \mathbb{R}^+$ will be called *standard* if $f(1) = 1$ and $tf(t^{-1}) = f(t)$. Standard functions are used to define (symmetric) matrix means:

$$M_f(A, B) = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2},$$

see [8]. For numbers $m_f(x, y) = xf(y/x)$.

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It is well-known that if $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a standard matrix monotone function, then

$$\frac{2x}{x+1} \le f(x) \le \frac{x+1}{2}.$$

For example,

$$\frac{2x}{x+1} \le \sqrt{x} \le \frac{x-1}{\log x} \le \frac{x+1}{2}$$

they correspond to the harmonic, geometric, logarithmic and arithmetic mean. The matrix means have application in quantum theory and this paper is also motivated by that, see [4, 11, 13, 16].

Assume that a standard matrix monotone function f is given. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be strictly positive numbers. The positivity of the matrix $X \in \mathbf{M}_n$ defined as

$$X_{ij} = m_f(\lambda_i, \lambda_j) \tag{1}$$

is an interesting question. We call X *mean matrix*. Positivity of the mean matrix for all possibilities is equivalent to the positive definiteness of the kernel

$$F_f(x,y) := m_f(x,y).$$

(A stronger property than positivity is the so-called *infinite divisibility* [1], it is not studied here, but some results are used.)

The choice $\lambda_1 = 1$ and $\lambda_2 = x$ shows that

$$f(x) \le \sqrt{x}$$

is a necessary condition for the positivity of the mean matrix, in other words m_f should be smaller than the geometric mean. If $f(x) \ge \sqrt{x}$, then the matrix

$$T_{ij} = \frac{1}{m_f(\lambda_i, \lambda_j)} \tag{2}$$

can be positive. The matrix (2) was important in the paper [10] for the characterization of monotone metrics, see also [11, 13]. It will be shown that T is positive if and only if the linear mapping $A \mapsto A \circ C$ is completely positive. (Here $A \circ C$ is a notation for the Hadamard product.)

The subject of the paper is the study of the existence and description of this kind of completely positive mappings which are induced by a standard matrix monotone function. Examples of good matrix means are presented.

2 The positive operator $\mathbb{J}_D^f$

Let $D \in \mathbf{M}_n$ be a positive definite matrix and f be a standard matrix monotone function. A linear operator $\mathbb{J}_D^f : \mathbf{M}_n \to \mathbf{M}_n$ can be defined. If $D = \mathrm{Diag}\,(\lambda_1, \dots, \lambda_n)$, then

$$(\mathbb{J}_D^f A)_{ij} = A_{ij} m(\lambda_i, \lambda_j) \qquad (A \in \mathbf{M}_n).$$

Since

$$\operatorname{Tr} A^*(\mathbb{J}_D^f A) = \sum_{ij} |A_{ij}|^2 m(\lambda_i, \lambda_j) \geq 0,$$

the linear mapping $\mathbb{J}_D^f$ is positive with respect to the Hilbert-Schmidt inner product (for any $f \geq 0$). Another definition is

$$\mathbb{J}_D^f = f(\mathbb{L}_D \mathbb{R}_D^{-1}) \mathbb{R}_D \,, \tag{3}$$

where

$$\mathbb{L}_D(X) = DX \qquad \text{and} \qquad \mathbb{R}_D(X) = XD \,.$$

(The operator $\mathbb{L}_D \mathbb{R}_D^{-1}$ appeared in the modular theory of von Neumann algebras.)

The inverse of this mapping is

$$((\mathbb{J}_D^f)^{-1} B)_{ij} = B_{ij} \frac{1}{m(\lambda_i, \lambda_j)} \qquad (A \in \mathbf{M}_n).$$

and it appeared in [10] to describe the abstract quantum Fisher information, see also [13]. The linear mappings $(\mathbb{J}_D^f)^{-1} : \mathbf{M}_n \to \mathbf{M}_n$ have the monotonicity condition

$$\alpha^* (\mathbb{J}_{\alpha(D)}^f)^{-1} \alpha \leq (\mathbb{J}_D^f)^{-1} \tag{4}$$

for every completely positive trace preserving mapping $\alpha : \mathbf{M}_n \to \mathbf{M}_m$, if f is a standard matrix monotone function.

The linear transformation $(\mathbb{J}_D^f)^{-1}$ appeared also in the paper [16] in a different notation. There Ω_D^k is the same as $(\mathbb{J}_D^f)^{-1}$ with $f = 1/k$, see also [11, 13]. The complete positivity of the mapping $\beta := (\mathbb{J}_D^f)^{-1} : \mathbf{M}_n \to \mathbf{M}_n$ is a question in the paper [16].

The subject of this paper is to find functions f such that the mapping β is monotone (in the sense of (4)) and completely positive. The complete positivity is equivalent to the positivity of a matrix, see the next lemma.

If $D = \operatorname{Diag}(\lambda_1, \dots, \lambda_n)$, then

$$\beta(A)_{ij} = A_{ij} \frac{1}{m(\lambda_i, \lambda_j)},$$

where m is the mean corresponding to the function f. In the notation (2), we have $\beta(A) = A \circ T$, it is a Hadamard product.

Lemma 1 *The linear mapping $\beta : \mathbf{M}_n \to \mathbf{M}_n, \beta(A) = A \circ T$ is completely positive if and only if the matrices $T \in \mathbf{M}_n$ defined in (2) are positive.*

Proof: If β is completely positive, then $A \circ T \geq 0$ for every positive A. This implies the positivity of T.

The mapping β linearly depends on T. Therefore, it is enough to prove the complete positivity when $T_{ij} = \overline{\lambda}_i \lambda_j$. Then

$$\beta(A) = \mathrm{Diag}\,(\lambda_1, \lambda_2, \dots, \lambda_n)^* A \mathrm{Diag}\,(\lambda_1, \lambda_2, \dots, \lambda_n)$$

and the complete positivity is clear. □

Hence β is completely positive if and only if β is positive. The problem of [16] is equivalent to the positivity of the matrix T.

3 Completely positive mappings

In this section we analyze the complete positivity of $\beta = \mathbb{J}_D^f$ for several matrix monotone functions f. The first three examples are very simple and actually they are particular cases of Example 6.

Example 1 If $f(x) = \sqrt{x}$, then

$$\langle a, Ta \rangle = \sum_{ij} \frac{\overline{a}_i a_j}{m(\lambda_i, \lambda_j)} = \overline{\sum_i a_i \lambda_i^{-1/2}} \sum_j a_j \lambda_j^{-1/2} \geq 0,$$

so $T \geq 0$. In this case $\beta(A) = D^{-1/2} A D^{-1/2}$ and the complete positivity is obvious. Moreover, β^{-1} is completely positive as well. (This is the only example such that both β and β^{-1} are completely positive.) □

Example 2 If $f(x) = (1+x)/2$, the arithmetic mean, then T is the so-called Cauchy matrix,

$$T_{ij} = \frac{2}{\lambda_i + \lambda_j} = 2 \int_0^\infty e^{s\lambda_i} e^{s\lambda_j}\, ds,$$

which is positive. Therefore $\beta : A \mapsto A \circ T$ is completely positive. This can be seen also from the formula

$$\beta(A) = 2 \int_0^\infty \exp(-sD) A \exp(-sD)\, ds.$$

□

Example 3 The *logarithmic mean* corresponds to the function $f(x) = (x-1)/\log x$.

Let $D = \mathrm{Diag}\,(\lambda_1, \lambda_2, \dots, \lambda_n)$ be positive definite. The mapping

$$\beta : A \mapsto \int_0^\infty (D+t)^{-1} A (D+t)^{-1}\, dt$$

is a positive mapping. Since $\beta(A) = T \circ A$ is a Hadamard product with

$$T_{ij} = \frac{\log \lambda_i - \log \lambda_j}{\lambda_i - \lambda_j},$$

the positivity of the mapping β implies the positivity of T. Another proof comes from the formula

$$\frac{\log \lambda_i - \log \lambda_j}{\lambda_i - \lambda_j} = \int_0^\infty \frac{1}{(s+\lambda_i)(s+\lambda_j)}\, ds.$$

□

Example 4 Consider the mean

$$m(x,y) := \frac{1}{2}(x^t y^{1-t} + x^{1-t} y^t) \geq \sqrt{xy} \qquad (0 < t < 1)$$

(which is sometimes called *Heinz mean*). Let $D = \mathrm{Diag}\,(\lambda_1, \lambda_2, \dots, \lambda_n)$ be positive definite. The mapping

$$\alpha : A \mapsto \frac{1}{2}(D^t A D^{1-t} + D^{1-t} A D^t)$$

has the form $\alpha(A) = X \circ A$ with

$$X_{ij} = \frac{1}{2}(\lambda_i^t \lambda_j^{1-t} + \lambda_i^{1-t} \lambda_j^t).$$

The inverse of the mapping is denoted by β, it is the Hadamard product with

$$T_{ij} = \frac{2}{\lambda_i^t \lambda_j^{1-t} + \lambda_i^{1-t} \lambda_j^t}.$$

β is a positive mapping if and only $T \geq 0$.

To find the inverse of α, we should solve the equation

$$2A = D^t Y D^{1-t} + D^{1-t} Y D^t,$$

when $Y = \beta(A)$ is unknown. It has the form

$$2D^{-t} A D^{-t} = Y D^{1-2t} + D^{1-2t} Y$$

which is a Sylvester equation. The solution is

$$\beta(A) = Y = \int_0^\infty \exp(-sD^{1-2t})(2D^{-t} A D^{-t}) \exp(-sD^{1-2t})\, ds.$$

Therefore the mapping β is positive and the matrix T is positive as well. □

The function

$$f_t(x) = 2^{2t-1} x^t (1+x)^{1-2t} \tag{5}$$

is a kind of interpolation between the arithmetic mean ($t = 0$) and the harmonic mean ($t = 1$). This function appeared in the paper [3] and it is proven there that it is a standard matrix monotone function.

Theorem 1 *If $t \in (0, 1/2)$, then*

$$f_t(x) = 2^{2t-1} x^t (1+x)^{1-2t} \geq \sqrt{x}$$

and the matrix

$$T_{ij} = \frac{1}{m_{f_t}(\lambda_i, \lambda_j)} = \frac{2^{1-2t}}{(\lambda_i + \lambda_j)^{1-2t}} (\lambda_i \lambda_j)^{-t}$$

is positive and the corresponding mapping β is completely positive.

Proof: For $|x| < 1$ and $1 - 2t = \alpha > 0$ the binomial expansion yields

$$(1-x)^{-\alpha} = \sum_{k=0}^{\infty} a_k x^k,$$

where

$$a_k = (-1)^k \binom{-\alpha}{k} = (-1)^k \frac{(-\alpha-1)(-\alpha-2) \cdot \ldots \cdot (-\alpha-k+1)}{k!} > 0.$$

So that

$$\begin{aligned}
(\lambda_i + \lambda_j)^{-(1-2t)} &= \left(\left(\lambda_i + \frac{1}{2} \right) \left(\lambda_j + \frac{1}{2} \right) \left(1 - \frac{\left(\lambda_i - \frac{1}{2}\right)\left(\lambda_j - \frac{1}{2}\right)}{\left(\lambda_i + \frac{1}{2}\right)\left(\lambda_j + \frac{1}{2}\right)} \right) \right)^{-(1-2t)} \\
&= \left(\lambda_i + \frac{1}{2} \right)^{-(1-2t)} \left(\lambda_j + \frac{1}{2} \right)^{-(1-2t)} \sum_{k=0}^{\infty} a_k \left(\frac{\left(\lambda_i - \frac{1}{2}\right)\left(\lambda_j - \frac{1}{2}\right)}{\left(\lambda_i + \frac{1}{2}\right)\left(\lambda_j + \frac{1}{2}\right)} \right)^k \\
&= \sum_{k=0}^{\infty} a_k \frac{\left(\lambda_i - \frac{1}{2}\right)^k \left(\lambda_j - \frac{1}{2}\right)^k}{\left(\lambda_i + \frac{1}{2}\right)^{k+(1-2t)} \left(\lambda_j + \frac{1}{2}\right)^{k+(1-2t)}}.
\end{aligned}$$

Hence we have

$$T_{ij} = 2^{1-2t} \sum_{k=0}^{\infty} a_k \frac{\left(\lambda_i - \frac{1}{2}\right)^k}{\left(\lambda_i + \frac{1}{2}\right)^{k+(1-2t)} \lambda_i^t} \frac{\left(\lambda_j - \frac{1}{2}\right)^k}{\left(\lambda_j + \frac{1}{2}\right)^{k+(1-2t)} \lambda_j^t}$$

and T is the sum of positive-semidefinite matrices of rank one. □

If $t \in (1/2, 1)$ in (5), then

$$f_t(x) \leq \sqrt{x}$$

and the positivity of the matrix

$$Y_{ij} = m_{f_t}(\lambda_i, \lambda_j)$$

can be shown similarly to the above argument.

Example 5 The mean

$$m(x,y)=\frac{1}{2}\left(\frac{x+y}{2}+\frac{2xy}{x+y}\right)$$

is larger than the geometric mean. Indeed,

$$\frac{1}{2}\left(\frac{x+y}{2}+\frac{2xy}{x+y}\right)\ge\sqrt{\frac{x+y}{2}\frac{2xy}{x+y}}=\sqrt{xy}.$$

The numerical computation shows that in this case already the determinant of a 3×3 matrix T can be negative. This example shows that the corresponding mapping β is not completely positive. □

Next we consider the function

$$f_t(x)=t(1-t)\frac{(x-1)^2}{(x^t-1)(x^{1-t}-1)} \tag{6}$$

which was first studied in the paper [4]. If $0<t<1$, then the integral representation

$$\frac{1}{f_t(x)}=\frac{\sin t\pi}{\pi}\int_0^\infty d\lambda\,\lambda^{t-1}\int_0^1 ds\int_0^1 dr\frac{1}{x((1-r)\lambda+(1-s))+(r\lambda+s)} \tag{7}$$

shows that $f_t(x)$ is operator monotone. (Note that in the paper [15] the operator monotonicity was obtained for $-1\le t\le 2$.) The property $xf(x^{-1})=f(x)$ is obvious.

If $t=1/2$, then

$$f(x)=\left(\frac{1+\sqrt{x}}{2}\right)^2\ge\sqrt{x}$$

and the corresponding mean is called binomial or power mean. In this case we have

$$T_{ij}=\frac{4}{(\sqrt{\lambda_i}+\sqrt{\lambda_j})^2}.$$

The matrix

$$U_{ij}=\frac{1}{\sqrt{\lambda_i}+\sqrt{\lambda_j}}$$

is a kind of Cauchy matrix, so it is positive. Since $T=4U\circ U$, T is positive as well.

If $\gamma(A)=A\circ U$, then $\beta=4\gamma^2$. Since

$$\gamma(A)=\int_0^\infty \exp(-s\sqrt{D})A\exp(-s\sqrt{D})\,ds,$$

we have

$$\beta(A)=4\int_0^\infty\int_0^\infty \exp(-(s+r)\sqrt{D})A\exp(-(s+r)\sqrt{D})\,ds\,dr. \tag{8}$$

The complete positivity of β is clear from this formula.

For the other values of t in $(0,1)$ the proof is a bit more sophisticated.

Lemma 2 *If* $0 < t < 1$*, then* $f_t(x) \geq \sqrt{x}$ *for* $x > 0$.

Proof: It is enough to show that for $0 < t < 1$ and $x > 0$

$$t\frac{x-1}{x^t-1} \geq x^{\frac{1-t}{2}}, \tag{9}$$

since this implies

$$t\frac{x-1}{x^t-1}(1-t)\frac{x-1}{x^{1-t}-1} \geq x^{\frac{1-t}{2}}x^{\frac{t}{2}} = \sqrt{x}.$$

Denote

$$g(x) := t(x-1) + x^{\frac{1-t}{2}} - x^{\frac{1+t}{2}}.$$

Then inequality (9) reduces to $g(x) \geq 0$ for $x \geq 1$ and to $g(x) \leq 0$ for $0 < x \leq 1$. Since $g(1) = 0$ it suffices to verify that g is monotone increasing, in other words $g' \geq 0$. By simple calculation one obtains

$$g'(x) = t + \frac{1-t}{2}x^{\frac{-t-1}{2}} - \frac{1+t}{2}x^{\frac{t-1}{2}}$$

and

$$g''(x) = \frac{1-t^2}{4}x^{\frac{t-3}{2}} - \frac{1-t^2}{4}x^{\frac{-t-3}{2}},$$

which yields $g''(x) \leq 0$ for $0 < x < 1$ and $g''(x) \geq 0$ for $x \geq 1$. Thus, due to $g'(1) = 0$, $g' \geq 0$, the statement follows. $\square$

It follows from the lemma that the matrix

$$T_{ij} = t(1-t) \times \frac{\lambda_i^t - \lambda_j^t}{\lambda_i - \lambda_j} \times \frac{\lambda_i^{1-t} - \lambda_j^{1-t}}{\lambda_i - \lambda_j} \qquad (1 \leq i, j \leq m)$$

can be positive. It is a Hadamard product, so it is enough to see that

$$U_{ij}^{(t)} = \frac{\lambda_i^t - \lambda_j^t}{\lambda_i - \lambda_j} \qquad (1 \leq i, j \leq m)$$

is positive for $0 < t < 1$. It is a well-known fact that the function $g : \mathbb{R}^+ \to \mathbb{R}$ is matrix monotone if and only if the Löwner matrices

$$L_{ij} = \frac{g(\lambda_i) - g(\lambda_j)}{\lambda_i - \lambda_j} \qquad (1 \leq i, j \leq m)$$

are positive. The function $g(x) = x^t$ is matrix monotone for $0 < t < 1$ and the positivity of U and T follows. So we have:

Theorem 2 *For the function (6) the mapping* β *is completely positive if* $0 < t < 1$.

To see the explicit complete positivity of β, the mappings $\gamma_t(A) = A \circ U^{(t)}$ are useful, we have

$$\beta(A) = t(1-t)\gamma_t(\gamma_{1-t}(A)).$$

Instead of the Hadamard product, which needs the diagonality of D, we can use

$$\gamma_t(A) = \frac{\partial}{\partial x}(D + xA)^t\Big|_{x=0}.$$

We compute γ_t from

$$(D + xA)^t = \frac{\sin \pi t}{\pi}\int_0^\infty \left(I - s(D + xA + sI)^{-1}\right) s^{t-1}\, ds.$$

So we obtain

$$\gamma_t(A) = \frac{\sin \pi t}{\pi}\int_0^\infty s^t (D + sI)^{-1} A (D + sI)^{-1}\, ds$$

and

$$\begin{aligned}\beta(A) &= t(1-t)\frac{\sin \pi t\, \sin \pi(1-t)}{\pi^2} \\ &\int_0^\infty \int_0^\infty r^{1-t} s^t (D + rI)^{-1}(D + sI)^{-1} A (D + sI)^{-1}(D + rI)^{-1}\, ds\, dr. \quad (10)\end{aligned}$$

Example 6 The *power difference means* are determined by the functions

$$f_t(x) = \frac{t-1}{t}\frac{x^t - 1}{x^{t-1} - 1} \qquad (-1 \le t \le 2), \tag{11}$$

where the values $t = -1, 1/2, 1, 2$ correspond to the well-known means, harmonic, geometric, logarithmic and arithmetic. The functions (11) are operator monotone [2] and we show that for fixed $x > 0$ the value $f_t(x)$ is increasing function of t.

By substituting $x = e^{2\lambda}$ one has

$$f_t(e^{2\lambda}) = \frac{t-1}{t}\frac{e^{\lambda t}\frac{e^{\lambda t} - e^{-\lambda t}}{2}}{e^{\lambda(t-1)}\frac{e^{\lambda(t-1)} - e^{-\lambda(t-1)}}{2}} = e^\lambda \frac{t-1}{t}\frac{\sinh(\lambda t)}{\sinh(\lambda(t-1))}.$$

Since

$$\frac{d}{dt}\left(\frac{t-1}{t}\frac{\sinh(\lambda t)}{\sinh(\lambda(t-1))}\right) = \frac{\sinh(\lambda t)\sinh(\lambda(t-1)) - \lambda t(t-1)\sinh(\lambda)}{t^2\sinh^2(\lambda(t-1))},$$

it suffices to show that

$$g(t) = \sinh(\lambda t)\sinh(\lambda(t-1)) - \lambda t(t-1)\sinh(\lambda) \ge 0.$$

Observe that $\lim_{\pm\infty} g = +\infty$ thus g has a global minimum. By simple calculations one obtains

$$g'(t) = \lambda(\sinh(\lambda(2t-1)) - (2t-1)\sinh(\lambda)).$$

It is easily seen that the zeros of g' are $t = 0$, $t = 1/2$ and $t = 1$ hence $g(0) = g(1) = 0$ and $g(\frac{1}{2}) = \sinh^2(\frac{\lambda}{2}) + \frac{\lambda}{4}\sinh(\lambda) \geq 0$ implies that $g \geq 0$.

It follows that

$$\sqrt{x} \leq f_t(x) \leq \frac{1+x}{2}$$

when $1/2 \leq t \leq 2$. For these values of the parameter t the complete positivity holds. This follows from the next lemma which contains a bigger interval for t.

Lemma 3 *The matrix*

$$T_{ij} := \frac{t}{t-1}\frac{\lambda_i^{t-1} - \lambda_j^{t-1}}{\lambda_i^t - \lambda_j^t}$$

is positive if $\frac{1}{2} \leq t$.

Proof: For $t > 1$ the statement follows from the proof of Theorem 2, since

$$\frac{t}{t-1}\frac{\lambda_i^{t-1} - \lambda_j^{t-1}}{\lambda_i^t - \lambda_j^t} = \frac{t}{t-1}\frac{(\lambda_i^t)^{\frac{t-1}{t}} - (\lambda_j^t)^{\frac{t-1}{t}}}{\lambda_i^t - \lambda_j^t},$$

where $0 < \frac{t-1}{t} < 1$, further, for $t = 1$ the statement follows from Example 3. If $\frac{1}{2} \leq t < 1$ let $s := 1 - t$ where $0 < s \leq \frac{1}{2}$. Then

$$T_{ij} = \frac{t}{t-1}\frac{\lambda_i^{t-1} - \lambda_j^{t-1}}{\lambda_i^t - \lambda_j^t} = \frac{1-s}{-s}\frac{\lambda_i^{-s} - \lambda_j^{-s}}{\lambda_i^t - \lambda_j^t} = \frac{1-s}{s}\frac{(\lambda_i^t)^{\frac{s}{t}} - (\lambda_j^t)^{\frac{s}{t}}}{\lambda_i^t - \lambda_j^t}\frac{1}{\lambda_i^s\lambda_j^s}$$

so that T is the Hadamard product of U and V, where

$$U_{ij} = \frac{(\lambda_i^t)^{\frac{s}{t}} - (\lambda_j^t)^{\frac{s}{t}}}{\lambda_i^t - \lambda_j^t}$$

is positive due to $0 < \frac{s}{t} \leq 1$ and

$$V_{ij} = \frac{1-s}{s}\frac{1}{\lambda_i^s\lambda_j^s}$$

is positive, too. □

Example 7 Another interpolation between the arithmetic mean ($t = 1$) and the harmonic mean ($t = 0$) is the following:

$$f_t(x) = \frac{2(tx+1)(t+x)}{(1+t)^2(x+1)} \qquad (0 \leq t \leq 1).$$

First we compare this mean with the geometric mean:

$$f_t(x^2) - x = \frac{(x-1)^2(2tx^2 - (1-t)^2x + 2t)}{(1+t)^2(x^2+1)}$$

and the sign depends on

$$x^2 - \frac{(1-t)^2}{2t}x + 1 = \left(x - \frac{(1-t)^2}{4t}\right)^2 + 1 - \left(\frac{(1-t)^2}{4t}\right)^2 .$$

So the positivity condition is $(1-t)^2 \le 4t$ which gives $3-2\sqrt{2} \le t \le 3+2\sqrt{2}$. For these parameters $f_t(x) \ge \sqrt{x}$ and for $0 < t < 3-2\sqrt{2}$ the two means are not comparable.

For $3-2\sqrt{2} \le t \le 1$ the matrix monotonicity is rather straightforward:

$$f_t(x) = \frac{2}{(1+t)^2}\left(tx + t^2 - t + 1 - \frac{(t-1)^2}{x+1}\right)$$

However, the numerical computations show that $T \ge 0$ is not true. □

4 Some matrix monotone functions

First the *Stolarsky mean* is investigated [9, 14].

Theorem 3 *Let*

$$f_p(x) := \left(\frac{p(x-1)}{x^p - 1}\right)^{\frac{1}{1-p}}, \tag{12}$$

where $p \neq 1$. Then f_p is matrix monotone if $-2 \le p \le 2$.

Proof: First note that $f_2(x) = (x+1)/2$ is the arithmetic mean, the limiting case $f_0(x) = (x-1)/\log x$ is the logarithmic mean and $f_{-1}(x) = \sqrt{x}$ is the geometric mean, their matrix monotonicity is well-known. If $p = -2$ then

$$f_{-2}(x) = \frac{(2x)^{\frac{2}{3}}}{(x+1)^{\frac{1}{3}}}$$

which will be shown to be matrix monotone at the end of the proof.

Now let us suppose that $p \neq -2, -1, 0, 1, 2$. By Löwner's theorem f_p is matrix monotone if and only if it has a holomorphic continuation mapping the upper half plane into itself. We define $\log z$ as $\log 1 := 0$ then in case $-2 < p < 2$, since $z^p - 1 \neq 0$ in the upper half plane, the real function $p(x-1)/(x^p-1)$ has a holomorphic continuation to the upper half plane, moreover it is continuous in the closed upper half plane, further, $p(z-1)/(z^p-1) \neq 0$ $(z \neq 1)$ so f_p also has a holomorphic continuation to the upper half plane and it is also continuous in the closed upper half plane.

Assume $-2 < p < 2$ then it suffices to show that f_p maps the upper half plane into itself. We show that for every $\varepsilon > 0$ there is $R > 0$ such that the set $\{z : |z| \ge R, \operatorname{Im} z > 0\}$ is mapped into $\{z : 0 \le \arg z \le \pi + \varepsilon\}$, further, the boundary $(-\infty, +\infty)$ is mapped into the closed upper half plane. Then by the well-known fact that the image of a connected open set by a holomorphic function is either a connected open set or a single point it follows that the upper half plane is mapped into itself by f_p.

Clearly, $[0, +\infty)$ is mapped into $[0, \infty)$ by f_p.

Now first suppose $0 < p < 2$. Let $\varepsilon > 0$ be sufficiently small and $z \in \{z : |z| = R, \operatorname{Im} z > 0\}$ where $R > 0$ is sufficiently large. Then

$$\arg(z^p - 1) = \arg z^p \pm \varepsilon = p \arg z \pm \varepsilon,$$

and similarly $\arg z - 1 = \arg z \pm \varepsilon$ so that

$$\arg \frac{z-1}{z^p - 1} = (1-p) \arg z \pm 2\varepsilon.$$

Further,

$$\left| \frac{z-1}{z^p-1} \right| \geq \frac{|z|-1}{|z|^p+1} = \frac{R-1}{R^p+1},$$

which is large for $0 < p < 1$ and small for $1 < p < 2$ if R is sufficiently large, hence

$$\arg \left(\frac{z-1}{z^p-1} \right)^{\frac{1}{1-p}} = \frac{1}{1-p} \arg \left(\frac{z-1}{z^p-1} \right) \pm 2\varepsilon = \arg z \pm 2\varepsilon \frac{2-p}{1-p}.$$

Since $\varepsilon > 0$ was arbitrary it follows that $\{z : |z| = R, \operatorname{Im} z > 0\}$ is mapped into the upper half plane by f_p if $R > 0$ is sufficiently large.

Now, if $z \in [-R, 0)$ then $\arg(z-1) = \pi$, further, $p\pi \leq \arg(z^p - 1) \leq \pi$ for $0 < p < 1$ and $\pi \leq \arg(z^p - 1) \leq p\pi$ for $1 < p < 2$ whence

$$0 \leq \arg \left(\frac{z-1}{z^p-1} \right) \leq (1-p)\pi \quad \text{for} \quad 0 < p < 1,$$

and

$$(1-p)\pi \leq \arg \left(\frac{z-1}{z^p-1} \right) \leq 0 \quad \text{for} \quad 1 < p < 2.$$

Thus by

$$\pi \arg \left(\frac{z-1}{z^p-1} \right)^{\frac{1}{1-p}} = \frac{1}{1-p} \arg \left(\frac{z-1}{z^p-1} \right)$$

it follows that

$$0 \leq \arg \left(\frac{z-1}{z^p-1} \right)^{\frac{1}{1-p}} \leq \pi$$

so z is mapped into the closed upper half plane.

The case $-2 < p < 0$ can be treated similarly by studying the arguments and noting that

$$f_p(x) = \left(\frac{p(x-1)}{x^p-1} \right)^{\frac{1}{1-p}} = \left(\frac{|p| x^{|p|}(x-1)}{x^{|p|}-1} \right)^{\frac{1}{1+|p|}}.$$

Finally, we show that $f_{-2}(x)$ is matrix monotone. Clearly f_{-2} has a holomorphic continuation to the upper half plane (which is not continuous in the closed upper half plane). If $0 < \arg z < \pi$ then $\arg z^{\frac{2}{3}} = \frac{2}{3} \arg z$ and $0 < \arg(z+1) < \arg z$ so

$$0 < \arg \left(\frac{z^{\frac{2}{3}}}{(z+1)^{\frac{1}{3}}} \right) < \pi$$

thus the upper half plane is mapped into itself by f_{-2}. □

The limiting case $p = 1$ is the so-called *identric mean*:

$$f_1(x) = \frac{1}{e} x^{\frac{x}{x-1}} = \exp\left(\frac{x \log x}{x-1} - 1 \right).$$

It is not so difficult to show that f_1 is matrix monotone.

The inequality

$$\sqrt{x} \leq f_p(x) \leq \frac{1+x}{2}$$

holds if $p \in [-1, 2]$. It is proved in [1] that the matrix

$$T_{ij} = \left(\frac{\lambda_i^p - \lambda_j^p}{p(\lambda_i - \lambda_j)} \right)^{\frac{1}{1-p}}$$

is positive.

Corollary 1 *The mapping β induced by the Stolarsky mean is monotone and completely positive for $p \in [-1, 2]$.*

The *power* or *binomial mean*

$$m(a, b) = \left(\frac{a^p + b^p}{2} \right)^{\frac{1}{p}}$$

can be also a matrix monotone function:

Theorem 4 *The function*

$$f_p(x) = \left(\frac{x^p + 1}{2} \right)^{\frac{1}{p}} \tag{13}$$

is matrix monotone if and only if $-1 \leq p \leq 1$.

Proof: Observe that $f_{-1}(x) = 2x/(x+1)$ and $f_1(x) = (x+1)/2$, so f_p could be matrix monotone only if $-1 \leq p \leq 1$. We show that it is indeed matrix monotone. The case $p = 0$ is well-known. Further, note that if f_p is matrix monotone for $0 < p < 1$ then

$$f_{-p}(x) = \left(\left(\frac{x^{-p} + 1}{2} \right)^{\frac{1}{p}} \right)^{-1}$$

is also matrix monotone since x^{-p} is matrix monotone decreasing for $0 < p \leq 1$.

So let us assume that $0 < p < 1$. Then, since $z^p + 1 \neq 0$ in the upper half plane, f_p has a holomorphic continuation to the upper half plane (by defining $\log z$ as $\log 1 = 0$). By Löwner's theorem it suffices to show that f_p maps the upper half plane into itself. If $0 < \arg z < \pi$ then $0 < \arg(z^p + 1) < \arg z^p = p \arg z$ so

$$0 < \arg\left(\frac{z^p+1}{2}\right)^{\frac{1}{p}} = \frac{1}{p}\arg\left(\frac{z^p+1}{2}\right) < \arg z < \pi$$

thus z is mapped into the upper half plane. □

In the special case $p = \frac{1}{n}$,

$$f_p(x) = \left(\frac{x^{\frac{1}{n}}+1}{2}\right)^n = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}x^{\frac{k}{n}},$$

and it is well-known that x^α is matrix monotone for $0 < \alpha < 1$ thus f_p is also matrix monotone.

Since the power mean is infinitely divisible [1], we have:

Corollary 2 *The mapping β induced by the power mean is monotone and completely positive for $p \in [-1, 1]$.*

5 Discussion and conclusion

The complete positivity of some mappings $\beta_f : \mathbf{M}_n \to \mathbf{M}_n$ has been a question in physical applications when β is determined by a function $f : \mathbb{R}^+ \to \mathbb{R}^+$. The function f is in connection with means larger than the geometric mean. In the paper several concrete functions are studied, for example, Heinz mean, power difference means, Stolarsky mean and interpolations between some means. The complete positivity of β_f is equivalent with the positivity of a matrix. The analysis of the functions studied here is very concrete, general statement is not known.

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