Geometry of CR submanifolds of maximal CR dimension in complex space forms

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Summary

On real hypersurfaces in complex space forms many results are proven. In this paper we generalize some results concerning extrinsic geometry of real hypersurfaces, to CR submanifolds of maximal CR dimension in complex space forms.

Key words and phrases. Complex space form, CR submanifold of maximal CR dimension, shape operator, second fundamental form.
AMS Subject Classification. 53C15, 53C40, 53B20.

1 Introduction

Let $\overline{\mathbf{M}}$ be an $(n+p)$-dimensional complex space form, i.e. a Kaehler manifold of constant holomorphic sectional curvature $4c$, endowed with metric $\overline{g}$. Let $\mathbf{M}$ be an n-dimensional real submanifold of $\overline{\mathbf{M}}$ and J be the complex structure of $\overline{\mathbf{M}}$. For a tangent space $T_x(\mathbf{M})$ of $\mathbf{M}$ at x, we put $H_x(\mathbf{M}) = JT_x(\mathbf{M}) \cap T_x(\mathbf{M})$. Then, $H_x(\mathbf{M})$ is the maximal complex subspace of $T_x(\mathbf{M})$ and is called the holomorphic tangent space to $\mathbf{M}$ at x. If the complex dimension $dim_{\mathbf{C}} H_x(\mathbf{M})$ is constant over $\mathbf{M}$, $\mathbf{M}$ is called a Cauchy-Riemann submanifold or briefly a CR submanifold and the constant $dim_{\mathbf{C}} H_x(\mathbf{M})$ is called the CR dimension of $\mathbf{M}$. If, for any $x \in \mathbf{M}$, $H_x(\mathbf{M})$ satisfies $dim_{\mathbf{C}} H_x(\mathbf{M}) = \frac{n-1}{2}$, then $\mathbf{M}$ is called a CR submanifold of maximal CR dimension. It follows that there exists a unit vector field ξ normal to $\mathbf{M}$ such that $JT_x(\mathbf{M}) \subset T_x(\mathbf{M}) \oplus span\{\xi_x\}$, for any $x \in \mathbf{M}$.

A real hypersurface is a typical example of a CR submanifold of maximal CR dimension. The study of real hypersurfaces in complex space forms is a classical topic in differential geometry and the generalization of some results which are valid for real hypersurfaces to CR submanifolds of maximal CR dimension may be expected.

For instance, nonexistence of real hypersurfaces with the parallel shape operator ([1], [2]) and real hypersurfaces with the second fundamental form satisfying $h(JX,Y) - Jh(X,Y) = 0$ ([3]), in nonflat complex space forms, is proven.

In this paper we study the conditions that the shape operator of the distinguished vector field ξ is parallel and that the second fundamental form satisfies $h(JX,Y) - Jh(X,Y) = 0$, on CR submanifolds of maximal CR dimension in complex space forms.

The author wishes to express her gratitude to Professor Mirjana Djorić for her useful advice.

2 CR submanifolds of maximal CR dimension of a complex space form

Let $\overline{\mathbf{M}}$ be an $(n+p)$-dimensional complex space form with Kaehler structure $(J, \overline{g})$ and of constant holomorphic sectional curvature $4c$. Let $\mathbf{M}$ be an n-dimensional CR submanifold of maximal CR dimension in $\overline{\mathbf{M}}$ and $\iota : \mathbf{M} \to \overline{\mathbf{M}}$ immersion. Also, we denote by ι the differential of the immersion. The Riemannian metric g of $\mathbf{M}$ is induced from the Riemannian metric $\overline{g}$ of $\overline{\mathbf{M}}$ in such a way that $g(X,Y) = \overline{g}(\iota X, \iota Y)$, where $X, Y \in T(\mathbf{M})$. We denote by $T(\mathbf{M})$ and $T^{\perp}(\mathbf{M})$ the tangent bundle and the normal bundle of $\mathbf{M}$, respectively.

On $\overline{\mathbf{M}}$ we have the following decomposition into tangential and normal components:

$$J\iota X = \iota F X + u(X)\xi, \;\; X \in T(\mathbf{M}). \tag{1}$$

Here F is a skew-symmetric endomorphism acting on $T(\mathbf{M})$ and u in one-form on $\mathbf{M}$.

Since $T_1^{\perp}(\mathbf{M}) = \{\eta \in T^{\perp}(\mathbf{M}) | \overline{g}(\eta, \xi) = 0\}$ is J-invariant, from now on we will denote the orthonormal basis of $T^{\perp}(\mathbf{M})$ by $\xi, \xi_1, \cdots, \xi_q, \xi_{1^*}, \cdots, \xi_{q^*}$, where $\xi_{a^*} = J\xi_a$ and $q = \frac{p-1}{2}$. Also, $J\xi$ is the vector field tangent to $\mathbf{M}$ and we write

$$J\xi = -\iota U. \tag{2}$$

Furthermore, using (1), (2) and the Hermitian property of J implies

$$F^2 X = -X + u(X)U, \tag{3}$$

$$FU = 0, \tag{4}$$

$$g(X,U) = u(X). \tag{5}$$

Next, we denote by $\overline{\nabla}$ and ∇ the Riemannian connection of $\overline{\mathbf{M}}$ and $\mathbf{M}$, respectively, and by D the normal connection induced from $\overline{\nabla}$ in the normal bundle of $\mathbf{M}$. They are related by the following Gauss equation

$$\overline{\nabla}_{\iota X} \iota Y = \iota \nabla_X Y + h(X,Y), \tag{6}$$

where h denotes the second fundamental form, and by Weingarten equations

$$\begin{aligned}\overline{\nabla}_{\iota X}\xi &= -\iota AX + D_X\xi \\ &= -\iota AX + \sum_{a=1}^{q}\{s_a(X)\xi_a + s_{a^*}(X)\xi_{a^*}\},\end{aligned} \tag{7}$$

$$\begin{aligned}\overline{\nabla}_{\iota X}\xi_a = -\iota A_a X + D_X\xi_a = -\iota A_a X - s_a(X)\xi \\ + \sum_{b=1}^{q}\{s_{ab}(X)\xi_b + s_{ab^*}(X)\xi_{b^*}\},\end{aligned} \tag{8}$$

$$\begin{aligned}\overline{\nabla}_{\iota X}\xi_{a^*} = -\iota A_{a^*} X + D_X\xi_{a^*} = -\iota A_{a^*} X - s_{a^*}(X)\xi \\ + \sum_{b=1}^{q}\{s_{a^*b}(X)\xi_b + s_{a^*b^*}(X)\xi_{b^*}\},\end{aligned} \tag{9}$$

where the s's are the coefficients of the normal connection D and A, A_a, A_{a^*}; $a = 1, \cdots, q$, are the shape operators corresponding to the normals ξ, ξ_a, ξ_{a^*}, respectively. They are related to the second fundamental form by

$$\begin{aligned}h(X, Y) = g(AX, Y)\xi \\ + \sum_{a=1}^{q}\{g(A_a X, Y)\xi_a + g(A_{a^*}X, Y)\xi_{a^*}\}.\end{aligned} \tag{10}$$

Since the ambient manifold is a Kaehler manifold, using (1), (2), (8) and (9), it follows that

$$A_{a^*}X = FA_a X - s_a(X)U, \tag{11}$$

$$A_a X = -FA_{a^*}X + s_{a^*}(X)U, \tag{12}$$

$$s_{a^*}(X) = u(A_a X), \tag{13}$$

$$s_a(X) = -u(A_{a^*}X), \tag{14}$$

for all X, Y tangent to $\mathbf{M}$ and $a = 1, \cdots, q$.
Moreover, since F is skew-symmetric and A_a and A_{a^*}; $a = 1, \cdots, q$, are symmetric, (11) and (12) imply

$$g((A_a F + FA_a)X, Y) = u(Y)s_a(X) - u(X)s_a(Y), \tag{15}$$

$$g((A_{a^*}F + FA_{a^*})X, Y) = u(Y)s_{a^*}(X) - u(X)s_{a^*}(Y), \tag{16}$$

for all $a = 1, \cdots, q$.
Finally, the Codazzi equation for the distinguished vector field ξ becomes

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{u(X)FY - u(Y)FX - 2g(FX, Y)U\} \tag{17}$$
$$+ \sum_{a=1}^{q} \{s_a(X)A_aY - s_a(Y)A_aX\} + \sum_{a=1}^{q} \{s_{a^*}(X)A_{a^*}Y - s_{a^*}(Y)A_{a^*}X\},$$

for all X, Y tangent to $\mathbf{M}$.

3 Shape operator A is parallel

Here, we will give one well known result about hypersurfaces with the parallel shape operator.

Theorem 1. *[1], [2] Let $\boldsymbol{M}$ be an n-dimensional, where $n \geq 3$, hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Then the shape operator A of $\boldsymbol{M}$ cannot be parallel.*

We will study the same condition on CR submanifolds of maximal CR dimension in complex space forms. Therefore, we have the next two theorems.

Theorem 2. *Let $\boldsymbol{M}$ be an n-dimensional CR submanifold of maximal CR dimension in an $(n+p)$-dimensional complex space form $(\overline{\boldsymbol{M}}, J, \overline{g})$, where $n \geq 3$ and the constant holomorphic sectional curvature of $\overline{\boldsymbol{M}}$ equals $4c$. Let the distinguished vector field ξ be parallel with respect to the normal connection D and A be the shape operator of ξ. If $\nabla A = 0$ on $\boldsymbol{M}$, then $\overline{\boldsymbol{M}}$ is an Euclidean space.*

Proof. Putting $Y = U$ in the Codazzi equation (17), we get

$$(\nabla_X A)U - (\nabla_U A)X = -cFX + \sum_{a=1}^{q} \{s_a(X)A_aU - s_a(U)A_aX\} +$$
$$\sum_{a=1}^{q} \{s_{a^*}(X)A_{a^*}U - s_{a^*}(U)A_{a^*}X\}.$$

From the assumption of the Theorem 2 and the last equation we get

$$cFX = 0. \tag{18}$$

From the equation (18) we conclude that $c = 0$. □

Theorem 3. *Let $\boldsymbol{M}$ be an n-dimensional CR submanifold of maximal CR dimension in an $(n+p)$-dimensional complex space form $(\overline{\boldsymbol{M}}, J, \overline{g})$, where $n \geq 3$ and the constant holomorphic sectional curvature of $\overline{\boldsymbol{M}}$ equals $4c$. Let $p < n$ and A be the shape operator of the distinguished vector field ξ. If $\nabla A = 0$ on $\boldsymbol{M}$, then $\overline{\boldsymbol{M}}$ is an Euclidean space.*

Proof. After putting $Y = U$ in (17) and using the assumption of the Theorem 3, we get

$$-cFX + \sum_{a=1}^{q}\{s_a(X)A_aU - s_a(U)A_aX\}+$$
$$\sum_{a=1}^{q}\{s_{a^*}(X)A_{a^*}U - s_{a^*}(U)A_{a^*}X\} = 0. \tag{19}$$

Multiplying the equation (19) with an arbitrary vector field $Y \in T(\mathbf{M})$, we get

$$-cg(FX,Y) + \sum_{a=1}^{q}\{s_a(X)g(A_aU,Y) - s_a(U)g(A_aX,Y)\}+ \tag{20}$$
$$\sum_{a=1}^{q}\{s_{a^*}(X)g(A_{a^*}U,Y) - s_{a^*}(U)g(A_{a^*}X,Y)\} = 0.$$

Interchanging X and Y in (20) and subtracting (20) and the resulting equation, we get

$$-2cg(FX,Y) + \sum_{a=1}^{q}\{s_a(X)g(A_aU,Y) + s_{a^*}(X)g(A_{a^*}U,Y)\}-$$
$$\sum_{a=1}^{q}\{s_a(Y)g(A_aU,X) + s_{a^*}(Y)g(A_{a^*}U,X)\} = 0.$$

Now, using (5), (13) and (14), from the last equation it follows that

$$cFX = \sum_{a=1}^{q}\{s_a(X)A_aU + s_{a^*}(X)A_{a^*}U\}. \tag{21}$$

From (21) it follows that FX is a linear combination of A_aU and $A_{a^*}U$; $a = 1, \cdots, q$.
Since every tangent vector Y orthogonal to U can be expressed as $Y = FX$, for $n-1 > 2q = p-1$, i.e. $n > p$, it follows that there exists a unit vector field $Y = FX$ which is orthogonal to $span\{A_aU, A_{a^*}U\}$; $a = 1, \cdots, q$. For such $Y = FX$ it follows $s_a(Y) = 0 = s_{a^*}(Y)$; $a = 1, \cdots, q$, using (13) and (14). Consequently, using (21), we obtain

$$cF^2X = \sum_{a=1}^{q}\{s_a(FX)A_aU + s_{a^*}(FX)A_{a^*}U\}.$$

Finally, using (3), we conclude $c = 0$. □

4 CR submanifolds of maximal CR dimension satisfying $h(JX,Y) = Jh(X,Y)$

On real hypersurfaces the next theorem is proven.

Theorem 4. *[3] Let $\boldsymbol{M}$ be an n-dimensional, $n \geq 3$, real hypersurface in a complex space form $(\overline{\boldsymbol{M}}, J, \overline{g})$. If the second fundamental form satisfies condition $h(JX, Y) = Jh(X, Y)$; $X, Y \in T(\boldsymbol{M})$, then $\overline{\boldsymbol{M}}$ is an Euclidean space.*

In the next theorem we will see if the same result is true on CR submanifolds of maximal CR dimension.

Theorem 5. *Let $\boldsymbol{M}$ be an n-dimensional, $n \geq 3$, CR submanifold of maximal CR dimension in an $(n+p)$-dimensional complex space form $(\overline{\boldsymbol{M}}, J, \overline{g})$ and the constant holomorphic sectional curvature of $\overline{\boldsymbol{M}}$ equals $4c$. If the second fundamental form satisfies condition $h(JX, Y) = Jh(X, Y)$; $X, Y \in T(\boldsymbol{M})$, then $\overline{\boldsymbol{M}}$ is an Euclidean space.*

Proof. Using (10), we have the next two equations

$$h(JX, Y) = g(AJX, Y)\xi + \sum_{a=1}^{q}\{g(A_aJX, Y)\xi_a + g(A_{a^*}JX, Y)\xi_{a^*}\} \tag{22}$$

and

$$Jh(X, Y) = -g(AX, Y)\iota U + \sum_{a=1}^{q}\{g(A_aX, Y)\xi_{a^*} - g(A_{a^*}X, Y)\xi_a\}. \tag{23}$$

From (22), (23) and the assumption of the Theorem 5, we have

$$g(AX, Y)\iota U + g(JX, AY)\xi + \sum_{a=1}^{q}\{g(JX, A_aY) + g(A_{a^*}X, Y)\}\xi_a + \sum_{a=1}^{q}\{g(JX, A_{a^*}Y) - g(A_aX, Y)\}\xi_{a^*} = 0, \tag{24}$$

where we used the symmetry of the shape operators $A,\ A_a,\ A_{a^*};\ a = 1, \cdots, q$. From (1) and (24), we have

$$g(AX, Y)\iota U + g(AFX, Y)\xi + \sum_{a=1}^{q}\{g(A_aFX, Y) + g(A_{a^*}X, Y)\}\xi_a + \sum_{a=1}^{q}\{g(A_{a^*}FX, Y) - g(A_aX, Y)\}\xi_{a^*} = 0. \tag{25}$$

Because of the linear independence of the vectors

$$\iota U,\ \xi,\ \xi_a,\ \xi_{a^*};\ a = 1, \cdots, q,$$

from (25) we get the next equations

$$A = 0, \tag{26}$$

$$A_a F = -A_{a^*};\ \ a = 1, \cdots, q, \tag{27}$$

and

$$A_{a^*} F = A_a;\ \ a = 1, \cdots, q. \tag{28}$$

From the Codazzi equation (17) and (26), we conclude that

$$\begin{aligned} 0 &= c\{u(X)FY - u(Y)FX - 2g(FX,Y)U\} \\ &+ \sum_{a=1}^{q}\{s_a(X)A_aY - s_a(Y)A_aX\} + \sum_{a=1}^{q}\{s_{a^*}(X)A_{a^*}Y - s_{a^*}(Y)A_{a^*}X\}. \end{aligned} \tag{29}$$

Now, from the equations (11) and (27) we get

$$(FA_a + A_aF)X = s_a(X)U;\ a = 1, \cdots, q.$$

By scalar multiplication of the last equation with an arbitrary $Y \in T(\mathbf{M})$, and using (15), we get

$$s_a(Y) = 0;\ a = 1, \cdots, q. \tag{30}$$

From (12) and (28) we get

$$(A_{a^*}F + FA_{a^*})X = s_{a^*}(X)U;\ a = 1, \cdots, q.$$

By scalar multiplication of the last equation with an arbitrary $Y \in T(\mathbf{M})$, and using (16), we get

$$s_{a^*}(Y) = 0;\ a = 1, \cdots, q. \tag{31}$$

From (29), (30) and (31), we get

$$0 = c\{g(X,U)FY - g(Y,U)FX - 2g(FX,Y)U\}.$$

Multiplying the last equation with U, we get

$$0 = -2cg(FX,Y). \tag{32}$$

From (32) we conclude that $c = 0$. □
Now, using (30) and (31), we get the next proposition.

Proposition 1. *Let* $\boldsymbol{M}$ *be an* n*-dimensional,* $n \geq 3$*,* CR *submanifold of maximal* CR *dimension in an* $(n+p)$*-dimensional complex space form* $(\overline{\boldsymbol{M}}, J, \overline{g})$*. If on* $\boldsymbol{M}$ *the second fundamental form satisfies condition* $h(JX,Y) = Jh(X,Y)$*;* $X, Y \in T(\boldsymbol{M})$*, then the distinguished vector field* ξ *is parallel with respect to the normal connection* D*.*

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For instance, nonexistence of real hypersurfaces with the parallel shape operator ([1], [2]) and real hypersurfaces with the second fundamental form satisfying $h(JX,Y) - Jh(X,Y) = 0$ ([3]), in nonflat complex space forms, is proven.

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2 CR submanifolds of maximal CR dimension of a complex space form

Let $\overline{\mathbf{M}}$ be an $(n+p)$-dimensional complex space form with Kaehler structure $(J, \overline{g})$ and of constant holomorphic sectional curvature $4c$. Let $\mathbf{M}$ be an n-dimensional CR submanifold of maximal CR dimension in $\overline{\mathbf{M}}$ and $\iota : \mathbf{M} \rightarrow \overline{\mathbf{M}}$ immersion. Also, we denote by ι the differential of the immersion. The Riemannian metric g of $\mathbf{M}$ is induced from the Riemannian metric $\overline{g}$ of $\overline{\mathbf{M}}$ in such a way that $g(X,Y) = \overline{g}(\iota X, \iota Y)$, where $X, Y \in T(\mathbf{M})$. We denote by $T(\mathbf{M})$ and $T^{\perp}(\mathbf{M})$ the tangent bundle and the normal bundle of $\mathbf{M}$, respectively.

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Here F is a skew-symmetric endomorphism acting on $T(\mathbf{M})$ and $u : T(M) \rightarrow T^{\perp}(M)$.

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Furthermore, using (1), (2) and the Hermitian property of J implies

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Next, we denote by $\overline{\nabla}$ and ∇ the Riemannian connection of $\overline{\mathbf{M}}$ and $\mathbf{M}$, respectively, and by D the normal connection induced from $\overline{\nabla}$ in the normal bundle of $\mathbf{M}$. They are related by the following Gauss equation

$$\overline{\nabla}_{\iota X} \iota Y = \iota \nabla_X Y + h(X,Y), \tag{6}$$

where h denotes the second fundamental form, and by Weingarten equations

$$\begin{aligned}\overline{\nabla}_{\iota X}\xi &= -\iota AX + D_X\xi \\ &= -\iota AX + \sum_{a=1}^{q}\{s_a(X)\xi_a + s_{a^*}(X)\xi_{a^*}\},\end{aligned} \tag{7}$$

$$\begin{aligned}\overline{\nabla}_{\iota X}\xi_a = -\iota A_a X + D_X\xi_a &= -\iota A_a X - s_a(X)\xi \\ &+ \sum_{b=1}^{q}\{s_{ab}(X)\xi_b + s_{ab^*}(X)\xi_{b^*}\},\end{aligned} \tag{8}$$

$$\begin{aligned}\overline{\nabla}_{\iota X}\xi_{a^*} = -\iota A_{a^*} X + D_X\xi_{a^*} &= -\iota A_{a^*} X - s_{a^*}(X)\xi \\ &+ \sum_{b=1}^{q}\{s_{a^*b}(X)\xi_b + s_{a^*b^*}(X)\xi_{b^*}\},\end{aligned} \tag{9}$$

where the s's are the coefficients of the normal connection D and A, A_a, A_{a^*}; $a = 1, \cdots, q$, are the shape operators corresponding to the normals ξ, ξ_a, ξ_{a^*}, respectively. They are related to the second fundamental form by

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Since the ambient manifold is a Kaehler manifold, using (1), (2), (8) and (9), it follows that

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$$A_aX = -FA_{a^*}X + s_{a^*}(X)U, \tag{12}$$

$$s_{a^*}(X) = u(A_aX), \tag{13}$$

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for all X, Y tangent to $\mathbf{M}$ and $a = 1, \cdots, q$.
Moreover, since F is skew-symmetric and A_a and A_{a^*}; $a = 1, \cdots, q$, are symmetric, (11) and (12) imply

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$$+\sum_{a=1}^{q}\{s_a(X)A_aY - s_a(Y)A_aX\} + \sum_{a=1}^{q}\{s_{a^*}(X)A_{a^*}Y - s_{a^*}(Y)A_{a^*}X\},$$

for all X, Y tangent to $\mathbf{M}$.

3 Shape operator A is parallel

Here, we will give one well known result about hypersurfaces with the parallel shape operator.

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Proof. Putting $Y = U$ in Codazzi equation (17), we get

$$(\nabla_X A)Y - (\nabla_Y A)X = -cFX + \sum_{a=1}^{q}\{s_a(X)A_aU - s_a(U)A_aX\} +$$
$$\sum_{a=1}^{q}\{s_{a^*}(X)A_{a^*}U - s_{a^*}(U)A_{a^*}X\}.$$

From the assumption of the Theorem 2 and the last equation we get

$$cFX = 0. \tag{18}$$

From the equation (18) we conclude that $c = 0$. □

Theorem 3. *Let $\boldsymbol{M}$ be an n-dimensional CR submanifold of maximal CR dimension in an $(n+p)$-dimensional complex space form $(\overline{\boldsymbol{M}}, J, \overline{g})$, where $n \geq 3$ and the constant holomorphic sectional curvature of $\overline{\boldsymbol{M}}$ equals $4c$. Let $p < n$ and A be the shape operator of the distinguished vector field ξ. If $\nabla A = 0$ on $\boldsymbol{M}$, then $\overline{\boldsymbol{M}}$ is an Euclidian space.*

Proof. After putting $Y = U$ in (17) and using the assumption of the Theorem 3, we get

$$-cFX + \sum_{a=1}^{q}\{s_a(X)A_aU - s_a(U)A_aX\}+$$
$$\sum_{a=1}^{q}\{s_{a^*}(X)A_{a^*}U - s_{a^*}(U)A_{a^*}X\} = 0. \quad (19)$$

Multiplying the equation (19) with an arbitrary vector field $Y \in T(\mathbf{M})$, we get

$$-cg(FX,Y) + \sum_{a=1}^{q}\{s_a(X)g(A_aU,Y) - s_a(U)g(A_aX,Y)\}+ \quad (20)$$
$$\sum_{a=1}^{q}\{s_{a^*}(X)g(A_{a^*}U,Y) - s_{a^*}(U)g(A_{a^*}X,Y)\} = 0.$$

Interchanging X and Y in (20) and subtracting (20) and the resulting equation, we get

$$-2cg(FX,Y) + \sum_{a=1}^{q}\{s_a(X)g(A_aU,Y) + s_{a^*}(X)g(A_{a^*}U,Y)\}-$$
$$\sum_{a=1}^{q}\{s_a(Y)g(A_aU,X) + s_{a^*}(Y)g(A_{a^*}U,X)\} = 0.$$

Now, using (5), (13) and (14), from the last equation it follows that

$$cFX = \sum_{a=1}^{q}\{s_{a^*}(X)A_{a^*}U + s_a(X)A_aU\}. \quad (21)$$

On the other hand, if we put

$$0 = \sum_{a=1}^{q}\{c_{a^*}A_{a^*}U + c_aA_aU\}, \quad (22)$$

where c_{a^*} and c_a are constants; $a = 1, \cdots q$, by scalar multiplication of (22) with an arbitrary $X \in T(\mathbf{M})$, using $\overline{g}(\iota A_aX, \iota Y) = \overline{g}(h(X,Y), \xi_a)$, $\overline{g}(\iota A_{a^*}X, \iota Y) = \overline{g}(h(X,Y), \xi_{a^*})$; $a = 1, \cdots, q$, and (6), it follows that

$$0 = \sum_{a=1}^{q}\{c_{a^*}\overline{g}(\overline{\nabla}_{\iota U}\iota X, \xi_{a^*}) + c_a\overline{g}(\overline{\nabla}_{\iota U}\iota X, \xi_a)\},$$

i.e.

$$0 = \sum_{a=1}^{q}\{c_{a^*}\xi_{a^*} + c_a\xi_a\}.$$

From the last equation and the fact that ξ_{a^*}, ξ_a, $a = 1, \cdots q$, are linearly independent, it follows that $c_{a^*} = c_a = 0$; $a = 1, \cdots q$. Then, we can conclude that $A_{a^*}U$, A_aU; $a = 1, \cdots q$, are linearly independent vector fields.
It is known that $rankF = n - 1$ (see [1]), that's why from (21) it follows that there exist a vector field $Y \in T(\mathbf{M})$ such that $Y = FX$ and that Y is orthogonal to the vector fields A_aU, $A_{a^*}U$; $a = 1, \cdots, q$.
Multiplying (21) with $Y = FX$, we get

$$cg(FX, FX) = 0. \tag{23}$$

From (23) we conclude that $c = 0$. □

4 CR submanifolds of maximal CR dimension satisfying $h(JX, Y) = Jh(X, Y)$

On real hypersurfaces the next theorem is proven.

Theorem 4. *Let $\boldsymbol{M}$ be an n-dimensional, $n \geq 3$, real hypersurface in a complex space form $(\overline{\boldsymbol{M}}, J, \overline{g})$. If the second fundamental form satisfies condition $h(JX, Y) = Jh(X, Y)$; $X, Y \in T(\boldsymbol{M})$, then $\overline{\boldsymbol{M}}$ is an Euclidian space.*

In the next theorem we will see if the same result is true on CR submanifolds of maximal CR dimension.

Theorem 5. *Let $\boldsymbol{M}$ be an n-dimensional, $n \geq 3$, CR submanifold of maximal CR dimension in an $(n+p)$-dimensional complex space form $(\overline{\boldsymbol{M}}, J, \overline{g})$ and the constant holomorphic sectional curvature of $\overline{\boldsymbol{M}}$ equals $4c$. If the second fundamental form satisfies condition $h(JX, Y) = Jh(X, Y)$; $X, Y \in T(\boldsymbol{M})$, then $\overline{\boldsymbol{M}}$ is an Euclidian space.*

Proof. Using (10), we have the next two equations

$$h(JX, Y) = g(AJX, Y)\xi + \sum_{a=1}^{q}\{g(A_aJX, Y)\xi_a + g(A_{a^*}JX, Y)\xi_{a^*}\} \tag{24}$$

and

$$Jh(X, Y) = -g(AX, Y)\iota U + \sum_{a=1}^{q}\{g(A_aX, Y)\xi_{a^*} - g(A_{a^*}X, Y)\xi_a\}. \tag{25}$$

From (24), (25) and the assumption of the Theorem 5, we have

$$g(AX, Y)\iota U + g(JX, AY)\xi + \sum_{a=1}^{q}\{g(JX, A_aY) + g(A_{a^*}X, Y)\}\xi_a + \tag{26}$$
$$\sum_{a=1}^{q}\{g(JX, A_{a^*}Y) - g(A_aX, Y)\}\xi_{a^*} = 0,$$

where we used the symmetry of the shape operators $A,\ A_a,\ A_{a^*};\ \ a=1,\cdots,q$. From (1) and (26), we have

$$g(AX,Y)\iota U+g(AFX,Y)\xi+\sum_{a=1}^{q}\{g(A_aFX,Y)+g(A_{a^*}X,Y)\}\xi_a+ \qquad (27)$$
$$\sum_{a=1}^{q}\{g(A_{a^*}FX,Y)-g(A_aX,Y)\}\xi_{a^*}=0.$$

Because of the linear independence of the vectors

$$\iota U,\ \xi,\ \xi_a,\ \xi_{a^*};\ a=1,\cdots,q,$$

from (27) we get the next equations

$$A=0, \qquad (28)$$

$$A_aF=-A_{a^*};\ \ a=1,\cdots,q, \qquad (29)$$

and

$$A_{a^*}F=A_a;\ \ a=1,\cdots,q. \qquad (30)$$

From the Codazzi equation (17) and (28), we conclude that

$$0=c\{u(X)FY-u(Y)FX-2g(FX,Y)U\} \qquad (31)$$
$$+\sum_{a=1}^{q}\{s_a(X)A_aY-s_a(Y)A_aX\}+\sum_{a=1}^{q}\{s_{a^*}(X)A_{a^*}Y-s_{a^*}(Y)A_{a^*}X\}.$$

Now, from the equations (11) and (29) we get

$$(FA_a+A_aF)X=s_a(X)U;\ a=1,\cdots,q.$$

By scalar multiplication of the last equation with an arbitrary $Y\in T(\mathbf{M})$, and using (15), we get

$$s_a(Y)=0;\ a=1,\cdots,q. \qquad (32)$$

From (12) and (30) we get

$$(A_{a^*}F+FA_{a^*})X=s_{a^*}(X)U;\ a=1,\cdots,q.$$

By scalar multiplication of the last equation with an arbitrary $Y\in T(\mathbf{M})$, and using (16), we get

$$s_{a^*}(Y)=0;\ a=1,\cdots,q. \qquad (33)$$

From (31), (32) and (33), we get

$$0 = c\{g(X,U)FY - g(Y,U)FX - 2g(FX,Y)U\}.$$

Multiplying the last equation with U, we get

$$0 = -2cg(FX,Y). \tag{34}$$

From (34) we conclude that $c = 0$. □
Now, using (32) and (33), we get the next lemma.

Lemma 1. *Let $\boldsymbol{M}$ be an n-dimensional, $n \geq 3$, CR submanifold of maximal CR dimension in an $(n+p)$-dimensional complex space form $(\overline{\boldsymbol{M}}, J, \overline{g})$. If on $\boldsymbol{M}$ the second fundamental form satisfies condition $h(JX,Y) = Jh(X,Y)$; $X, Y \in T(\boldsymbol{M})$, then the distinguished vector field ξ is parallel with respect to the normal connection D.*

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