On equicontinuity of homeomorphisms with finite distortion in the plane

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Abstract

It is stated equicontinuity and normality of families $\mathfrak{F}^{\Phi}$ of the so–called homeomorphisms with finite distortion on conditions that $K_f(z)$ has finite mean oscillation, singularities of logarithmic type or integral constraints of the type $\int \Phi\left(K_f(z)\right) dx\, dy < \infty$ in a domain $D \subset \mathbb{C}$. It is shown that the found conditions on the function Φ are not only sufficient but also necessary for equicontinuity and normality of such families of mappings.

1 Introduction

In the theory of mappings called quasiconformal in the mean, conditions of the type

$$\int\limits_D \Phi(Q(z))\ dx\, dy\ <\ \infty \tag{1.1}$$

are standard for various characteristics Q of these mappings, see e.g. [1], [4], [8], [9], [14]–[18], [21], [22], [23] and [31]. The study of classes with the integral conditions (1.1) is also actual in the connection with the recent development of the theory of degenerate Beltrami equations and the so–called mappings with finite distortion, see e.g. related references in the monographs [11] and [20].

In the present paper we study the problems of equicontinuity and normality for wide classes of the homeomorphisms with finite distortion on conditions that $K_f(z)$ has finite mean oscillation, singularities of logarithmic type or integral constraints of the type (1.1) in a domain $D \subset \mathbb{C}$.

The concept of the generalized derivative was introduced by Sobolev in [29]. Given a domain D in the complex plane $\mathbb{C}$ the **Sobolev class** $W^{1,1}(D)$ consists of all functions $f : D \to \mathbb{C}$ in $L^1(D)$ with first partial generalized derivatives which are integrable

in D. A function $f : D \to \mathbb{C}$ belongs to $W^{1,1}_{\text{loc}}(D)$ if $f \in W^{1,1}(D_*)$ for every open set D_* with compact closure $\overline{D_*} \subset D$.

Recall that a homeomorphism f between domains D and D' in $\mathbb{C}$ is called of **finite distortion** if $f \in W^{1,1}_{\text{loc}}$ and

$$||f'(z)||^2 \leqslant K(z) \cdot J_f(z) \tag{1.2}$$

with a.e. finite function K where $||f'(z)||$ denotes the matrix norm of the Jacobian matrix f' of f at $z \in D$ and $J_f(z) = \det f'(z)$, see [11]. Later on, we use the notion $K_f(z)$ for the minimal function $K(z) \geqslant 1$ in (1.2). Note that $||f'(z)|| = |f_z| + |f_{\bar{z}}|$ and $J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2$ at the points of total differentiability of f. Thus, $K_f(z) = \frac{||f'(z)||^2}{|J_f(z)|} = \frac{|f_z|+|f_{\bar{z}}|}{|f_z|-|f_{\bar{z}}|}$ if $J_f(z) \neq 0$, $K_f(z) = 1$ if $f'(z) = 0$, i.e. $|f_z| = |f_{\bar{z}}| = 0$, and $K_f(z) = \infty$ at the rest points.

Recall that the **(conformal) modulus** of a family Γ of curves γ in $\mathbb{C}$ is the quantity

$$M(\Gamma) = \inf_{\rho \in \text{adm}\,\Gamma} \int_{\mathbb{C}} \rho^2(z) \ dx\,dy \tag{1.3}$$

where a Borel function $\rho : \mathbb{C} \to [0,\infty]$ is **admissible** for Γ, write $\rho \in \text{adm}\,\Gamma$, if

$$\int_{\gamma} \rho \ ds \geq 1 \qquad \forall \ \gamma \in \Gamma \ , \tag{1.4}$$

where s is a natural parameter of the length on γ.

One of the equivalent geometric definitions of $K-$**quasiconformal mappings** f with $K \in [1,\infty)$ given in a domain D in $\mathbb{C}$ is reduced to the inequality

$$M(f\Gamma) \leq K\,M(\Gamma) \tag{1.5}$$

that holds for an arbitrary family Γ of curves γ in the domain D.

Similarly, given a domain D in $\mathbb{C}$ and a (Lebesgue) measurable function $Q : D \to [1,\infty]$, a homeomorphism $f : D \to \overline{\mathbb{C}}$, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, is called $Q(z) -$ **homeomorphism** if

$$M(f\Gamma) \leq \int_{D} Q(z) \cdot \rho^2(z) \ dx\,dy \tag{1.6}$$

for every family Γ of curves γ in D and every $\rho \in \text{adm}\,\Gamma$, see e.g. [20].

In the case $Q(z) \leq K$ a.e., we again come to the inequality (1.5). In the general case, the latter inequality means that the conformal modulus of the family $f\Gamma$ is

estimated by the modulus M_Q of Γ with the weight Q, $M(f\Gamma) \leq M_Q(\Gamma)$, see e.g. [3]. The inequality of the type (1.6) was first stated by O. Lehto and K. Virtanen for quasiconformal mappings in the plane, see Section V.6.3 in [19].

Throughout this paper, $B(z_0, r) = \{z \in \mathbb{C} : |z_0 - z| < r\}$, $S(z_0, r) = \{z \in \mathbb{C} : |z_0 - z| = r\}$, $S(r) = S(0, r)$, $\mathbb{D} = B(0, 1)$, $R(r_1, r_2, z_0) = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$ and $S^2(x, r) = \{y \in \mathbb{R}^3 : |x - y| = r\}$. Let E, $F \subset \overline{\mathbb{C}}$ be arbitrary sets. Denote by $\Gamma(E, F, D)$ a family of all curves $\gamma : [a, b] \to \overline{\mathbb{C}}$ joining E and F in D, i.e. $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in D$ as $t \in (a, b)$.

The following notion generalizes and localizes the above notion of a Q–homeomorphism. It is motivated by the ring definition of Gehring for quasiconformal mappings, see e.g. [7], introduced first in the plane, see [27], and extended later on to the space case in [25], see also Chapters 7 and 11 in [20].

Given a domain D in $\mathbb{C}$ a (Lebesgue) measurable function $Q : D \to [0, \infty]$, $z_0 \in D$, a homeomorphism $f : D \to \overline{\mathbb{C}}$ is said to be a **ring Q–homeomorphism at the point** z_0 if

$$M\left(f\left(\Gamma\left(S_1, S_2, R(r_1, r_2, z_0)\right)\right)\right) \leq \int\limits_{R(r_1, r_2, z_0)} Q(z) \cdot \eta^2(|z - z_0|) \, dx \, dy \qquad (1.7)$$

for every ring $R(r_1, r_2, z_0)$ and the circles $S_i = S(z_0, r_i)$, where $0 < r_1 < r_2 < r_0 : = $ dist $(z_0, \partial D)$, and every measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r) \, dr \geq 1.$$

f is called a **ring Q–homeomorphism in the domain** D if f is a ring Q–homeomorphism at every point $z_0 \in D$. Note that, in particular, homeomorphisms $f : D \to \overline{\mathbb{C}}$ in the class $W_{loc}^{1,2}$ with $K_f(z) \in L_{loc}^1(D)$ are ring Q–homeomorphisms with $Q(z) = K_f(z)$, see e.g. Theorem 4.1 in [20]. A regular homeomorphism of the Sobolev class $W_{loc}^{1,1}$ in the plane is a ring Q-homeomorphism with $Q(z)$ is equal to the so-called tangential dilatation, see Theorem 3.1. in [28], cf. Lemma 20.9.1 in [2].

The notion of ring Q–homeomorphism can be extended in the natural way to ∞. More precisely, under $\infty \in D \subseteq \overline{\mathbb{C}}$ a homeomorphism $f : D \to \overline{\mathbb{C}}$ is called a **ring Q–homeomorphism at** ∞ if the mapping $\widetilde{f} = f\left(\frac{z}{|z|^2}\right)$ is a ring Q'–homeomorphism at the origin with $Q'(z) = Q\left(\frac{z}{|z|^2}\right)$. In other words, a mapping $f : \mathbb{C} \to \overline{\mathbb{C}}$ is a ring

Q–homeomorphism at ∞ iff

$$M\left(f\left(\Gamma\left(S(R_1), S(R_2), R(R_1, R_2, 0)\right)\right)\right) \leq \int\limits_{R(R_1, R_2, 0)} Q(w) \cdot \eta^2\left(|w|\right) du\, dv$$

holds for every ring $R(R_1, R_2, 0)$ in D with $0 < R_1 < R_2 < \infty$, $S(R_i)$ and for every measurable function $\eta : (R_1, R_2) \to [0, \infty]$ with $\int\limits_{R_1}^{R_2} \eta(r)\, dr \geq 1$.

A continuous mappings γ of an open subset Δ of the real axis $\mathbb{R}$ or a circle into D is called a **dashed line**, see e.g. 6.3 in [20]. The notion of the modulus of the family Γ of dashed lines γ can be given by analogy, see (1.3). We say that a property P holds for **a.e.** (almost every) $\gamma \in \Gamma$ if a subfamily of all lines in Γ for which P fails has the modulus zero, cf. [6]. Later on, we also say that a Lebesgue measurable function $\varrho : \mathbb{C} \to [0, \infty]$ is **extensively admissible** for Γ, write $\varrho \in \operatorname{ext} \operatorname{adm} \Gamma$, if (1.3) holds for a.e. $\gamma \in \Gamma$, see e.g. 9.2 in [20].

Given domains D and D' in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $z_0 \in \overline{D} \setminus \{\infty\}$, and a measurable function $Q : D \to (0, \infty)$, we say that a homeomorphism $f : D \to D'$ is a **lower Q-homeomorphism at the point** z_0 if

$$M(f\Sigma_\varepsilon) \;\geqslant\; \inf_{\varrho \in \operatorname{ext} \operatorname{adm} \Sigma_\varepsilon} \int\limits_{D \cap R(\varepsilon, \varepsilon_0, z_0)} \frac{\varrho^2(z)}{Q(z)}\, dx\, dy \tag{1.8}$$

for every ring $R(\varepsilon, \varepsilon_0, z_0)$, $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 \in (0, d_0)$, where $d_0 = \sup\limits_{z \in D} |z - z_0|$, and Σ_ε denotes the family of all intersections of the circles $S(z_0, r), r \in (\varepsilon, \varepsilon_0)$, with D.

The notion can be extended to the case $z_0 = \infty \in \overline{D}$ in the standard way by applying the inversion T with respect to the unit circle in $\overline{\mathbb{C}}$, $T(z) = z/|z|^2$, $T(\infty) = 0$, $T(0) = \infty$. Namely, a homeomorphism $f : D \to D'$ is a **lower Q-homeomorphism at $\infty \in \overline{D}$** if $F = f \circ T$ is a lower Q_*-homeomorphism with $Q_* = Q \circ T$ at 0. We also say that a homeomorphism $f : D \to \overline{\mathbb{C}}$ is a **lower Q-homeomorphism in ∂D** if f is a lower Q-homeomorphism at every point $z_0 \in \partial D$. Further we show that every homeomorphism of finite distortion in the plane is a lower Q-homeomorphism with $Q(z) = K_f(z)$ and, thus, the whole theory of the boundary behaviour in [13], see also Chapter 9 in [20] can be applied.

The following term was introduced in [10]. Let D be a domain in the complex plane $\mathbb{C}$. Recall that a function $\varphi : D \to \mathbb{R}$ has **finite mean oscillation at a point** $z_0 \in D$ if

$$\overline{\lim_{\varepsilon \to 0}} \; \fint_{D(z_0, \varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)|\, dx dy \; < \; \infty, \tag{1.9}$$

where

$$\overline{\varphi}_\varepsilon(z_0) = \fint_{D(z_0,\varepsilon)} \varphi(z) \, dxdy \ < \ \infty \qquad (1.10)$$

is the mean value of the function $\varphi(z)$ over the disk $D(z_0, \varepsilon)$. We also say that a function $\varphi : D \to \mathbb{R}$ is of **finite mean oscillation in D**, abbr. $\varphi \in \mathrm{FMO(D)}$ or simply $\varphi \in \mathrm{FMO}$, if φ has a finite mean oscillation at every point $z_0 \in D$.

2 Preliminaries

Recall that the **spherical (chordal) metric** $h(z', z'')$ in $\overline{\mathbb{C}}$ is equal to $|\pi(z') - \pi(z'')|$ where π is the stereographic projection of $\overline{\mathbb{C}}$ on the sphere $S^2(\frac{1}{2}e_3, \frac{1}{2})$ in $\mathbb{R}^3$, i.e., in the explicit form,

$$h(z', \infty) = \frac{1}{\sqrt{1 + |z'|^2}}, \quad h(z', z'') = \frac{|z' - z''|}{\sqrt{1 + |z'|^2}\sqrt{1 + |z''|^2}}, \quad z' \neq \infty \neq z''.$$

The **spherical diameter of a set** E in $\overline{\mathbb{C}}$ is the quantity $h(E) = \sup_{z', z'' \in E} h(z', z'')$.

A family $\mathfrak{F}$ of continuous mappings from $\mathbb{C}$ into $\overline{\mathbb{C}}$ is said to be a **normal** if every sequence of mappings f_m in $\mathfrak{F}$ has a subsequence f_{m_k} converging to a continuous mapping $f : \mathbb{C} \to \overline{\mathbb{C}}$ uniformly on each compact set $C \subset \mathbb{C}$. Normality is closely related to the following notion. A family $\mathfrak{F}$ of mappings $f : \mathbb{C} \to \overline{\mathbb{C}}$ is said to be **equicontinuous at a point** $z_0 \in \mathbb{C}$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that $h\left(f(z), f(z_0)\right) < \varepsilon$ for all $f \in \mathfrak{F}$ and $z \in \mathbb{C}$ with $|z - z_0| < \delta$. The family $\mathfrak{F}$ is called **equicontinuous** if $\mathfrak{F}$ is equicontinuous at every point $z_0 \in \mathbb{C}$. The following version of the Arzela – Ascoli theorem will be useful later on, see e.g. Section 20.4 in [30].

Proposition 2.1. *A family $\mathfrak{F}$ of mappings $f : \mathbb{C} \to \overline{\mathbb{C}}$ is normal if and only if $\mathfrak{F}$ is equicontinuous.*

For every non-decreasing function $\Phi : [0, \infty] \to [0, \infty]$, the **inverse function** $\Phi^{-1} : [0, \infty] \to [0, \infty]$ can be well defined by setting

$$\Phi^{-1}(\tau) \ = \ \inf_{\Phi(t) \geq \tau} t \, . \qquad (2.1)$$

As usual, here inf is equal to ∞ if the set of $t \in [0, \infty]$ such that $\Phi(t) \geq \tau$ is empty. Note that the function Φ^{-1} is non-decreasing, too.

Remark 2.1. Immediately by the definition it is evident that

$$\Phi^{-1}(\Phi(t)) \ \leq \ t \qquad \forall \, t \in [0, \infty] \qquad (2.2)$$

with the equality in (2.2) except intervals of constancy of the function $\Phi(t)$.

Since the mapping $t \mapsto t^p$ for every positive p is a sense–preserving homeomorphism $[0, \infty]$ onto $[0, \infty]$ we may rewrite Theorem 2.1 from [26] in the following form which is more convenient for further applications. Here, in (2.4) and (2.5), we complete the definition of integrals by ∞ if $\Phi_p(t) = \infty$, correspondingly, $H_p(t) = \infty$, for all $t \geq T \in [0, \infty)$. The integral in (2.5) is understood as the Lebesgue–Stieltjes integral and the integrals in (2.4) and (2.6)–(2.9) as the ordinary Lebesgue integrals.

Proposition 2.2. *Let* $\Phi : [0, \infty] \to [0, \infty]$ *be a non-decreasing function. Set*

$$H_p(t) \; = \; \log \Phi_p(t) \,, \qquad \Phi_p(t) = \Phi\left(t^p\right) \,, \quad p \in (0, \infty) \,. \tag{2.3}$$

Then the equality

$$\int_\delta^\infty H_p'(t) \, \frac{dt}{t} \; = \; \infty \tag{2.4}$$

implies the equality

$$\int_\delta^\infty \frac{dH_p(t)}{t} \; = \; \infty \tag{2.5}$$

and (2.5) is equivalent to

$$\int_\delta^\infty H_p(t) \, \frac{dt}{t^2} \; = \; \infty \tag{2.6}$$

for some $\delta > 0$, and (2.6) is equivalent to every of the equalities:

$$\int_0^\Delta H_p\left(\frac{1}{t}\right) \, dt \; = \; \infty \tag{2.7}$$

for some $\Delta > 0$,

$$\int_{\delta_*}^\infty \frac{d\eta}{H_p^{-1}(\eta)} \; = \; \infty \tag{2.8}$$

for some $\delta_* > H(+0)$,

$$\int_{\delta_*}^\infty \frac{d\tau}{\tau \Phi_p^{-1}(\tau)} \; = \; \infty \tag{2.9}$$

for some $\delta_* > \Phi(+0)$.

Moreover, (2.4) is equivalent to (2.5) and hence (2.4)–(2.9) are equivalent each to other if Φ is in addition absolutely continuous. In particular, all the conditions (2.4)–(2.9) are equivalent if Φ is convex and non–decreasing.

It is easy to see that conditions (2.4)–(2.9) become weaker as p increases, see e.g. (2.6). It is necessary to give one more explanation. From the right hand sides in the conditions (2.4)–(2.9) we have in mind $+\infty$. If $\Phi_p(t) = 0$ for $t \in [0, t_*]$, then $H_p(t) = -\infty$ for $t \in [0, t_*]$ and we complete the definition $H_p'(t) = 0$ for $t \in [0, t_*]$. Note, the conditions (2.5) and (2.6) exclude that t_* belongs to the interval of integrability because in the contrary case the left hand sides in (2.5) and (2.6) are either equal to $-\infty$ or indeterminate. Hence we may assume in (2.4)–(2.7) that $\delta > t_0$, correspondingly, $\Delta < 1/t_0$ where $t_0 := \sup_{\Phi_p(t)=0} t$, $t_0 = 0$ if $\Phi_p(0) > 0$.

3   The main results

Proposition 3.1. Let $f : D \to \mathbb{C}$ be a homeomorphism with finite distortion. Then f is a lower Q-homeomorphism at each point $z_0 \in \overline{D}$ with $Q(z) = K_f(z)$, see Theorem 3.1. in [12].

Proposition 3.2. Let D and D' be domains in $\mathbb{C}$, let $z_0 \in \overline{D} \setminus \{\infty\}$, and let $Q : D \to (0, \infty)$ be a measurable function. A homeomorphism $f : D \to D'$ is a lower Q-homeomorphism at z_0 if and only if

$$M(f\Sigma_\varepsilon) \geq \int_\varepsilon^{\varepsilon_0} \frac{dr}{\|Q\|_1(r)} \quad \forall\, \varepsilon \in (0, \varepsilon_0)\,, \quad \varepsilon_0 \in (0, d_0), \tag{3.1}$$

where

$$d_0 = \sup_{z \in D} |z - z_0|, \tag{3.2}$$

Σ_ε denotes the family of all the intersections of the circles $S(z_0, r)$, $r \in (\varepsilon, \varepsilon_0)$, with D, and

$$\|Q\|_1(r) = \int_{D(z_0,r)} Q(z)\, ds \tag{3.3}$$

is the L_1-norm of Q over $D(z_0, r) = \{z \in D : |z - z_0| = r\} = D \cap S(z_0, r)$. The infimum of the expression from the right-hand side in (1.8) is attained only for the function

$$\varrho_0(z) = \frac{Q(z)}{\|Q\|_1(|z|)},$$

see Theorem 2.1 in [13].

Proposition 3.3. Let D be a domain in $\mathbb{C}$ and $Q : D \to [0, \infty]$ a measurable function. A homeomorphism $f : D \to \mathbb{C}$ is a ring Q-homeomorphism at a point z_0 if and only if, for every $0 < r_1 < r_2 < d_0 = \text{dist}\,(z_0, \partial D)$,

$$M(\Delta(fS_1, fS_2, fD)) \leq \frac{2\pi}{I}, \tag{3.4}$$

where ω is the area of the unit circle in $\mathbb{C}$, $q_{z_0}(r)$ is the mean value of $Q(z)$ over the circle $|z - z_0| = r$, $S_j = S(z_0, r_j)$, $j = 1, 2$, and

$$I = I(r_1, r_2) = \int\limits_{r_1}^{r_2} \frac{dr}{r q_{z_0}(r)}$$

Moreover, the infimum from the right-hand side in (1.7) holds for the function

$$\eta_0(r) = \frac{1}{I r q_{z_0}(r)}, \tag{3.5}$$

see Theorem 3.15 in [25].

The above results now yield the following.

Lemma 3.1. Let D and D' be domains in $\mathbb{C}$, let $z_0 \in \overline{D} \setminus \{\infty\}$, and let $Q : D \to (0, \infty)$ be a measurable function. A homeomorphism $f : D \to D'$ is a lower Q-homeomorphism at z_0. Then f is a ring Q–homeomorphism at z_0.

Proof of Lemma 3.1. Denote by Σ_ε the family of all circles $S(z_0, r)$, $r \in (\varepsilon, \varepsilon_0)$, $\varepsilon_0 \in (0, d_0)$. By Theorem 3.13 in [32], we have

$$M\left(\Delta\left(fS_\varepsilon, fS_{\varepsilon_0}, f(D)\right)\right) \leq \frac{1}{M\left(f\Sigma_\varepsilon\right)} \leq \frac{2\pi}{\int\limits_\varepsilon^{\varepsilon_0} \frac{dr}{r q_{z_0}(r)}} \tag{3.6}$$

because $f\Sigma_\varepsilon \subset \Sigma\left(fS_\varepsilon, fS_{\varepsilon_0}\right)$, where $\Sigma\left(fS_\varepsilon, fS_{\varepsilon_0}\right)$ consists of all closed curves in $f(D)$ that separate fS_ε and fS_{ε_0}.

Proposition 3.1 and Lemma 3.1 imply the following result.

Theorem 3.1. Let $f : D \to \mathbb{C}$ be a homeomorphism with finite distortion. Then f is a ring Q-homeomorphism at each point $z_0 \in \overline{D}$ with $Q(z) = K_f(z)$.

4 Estimates of Distortion

The results of the following section can be obtained on the base of theorem 3.1 and the correspondent theorems of work [25].

Lemma 4.1. Let D be a domain in $\mathbb{C}$, let D' be a domain in $\overline{\mathbb{C}}$ with $h(\overline{\mathbb{C}} \setminus D') \geq \Delta > 0$, and let $f : D \to D'$ be a homeomorphism with finite distortion at a point $z_0 \in D$. If, for $0 < \varepsilon_0 < \mathrm{dist}(z_0, \partial D)$,

$$\int\limits_{\varepsilon < |z - z_0| \, < \, \varepsilon_0} K_f(z) \cdot \psi_\varepsilon^2(|z - z_0|) \, dx \, dy \; \leq \; c \cdot I^p(\varepsilon) \,, \qquad \varepsilon \in (0, \varepsilon_0), \tag{4.7}$$

where $p \leq 2$ and $\psi_\varepsilon(t)$ is nonnegative function on $(0, \infty)$ such that

$$0 \; < \; I(\varepsilon) \; = \; \int\limits_{\varepsilon}^{\varepsilon_0} \psi_\varepsilon(t) \, dt < \infty, \qquad \varepsilon \in (0, \varepsilon_0), \tag{4.8}$$

then

$$h(f(z), f(z_0)) \; \leq \; \frac{32}{\Delta} \, \exp \left\{ - \left(\frac{2\pi}{c} \right) I^{2-p}(|z - z_0|) \right\} \tag{4.9}$$

for all $z \in B(z_0, \varepsilon_0)$.

Corollary 4.1. Under the conditions of Lemma 4.1 and for $p = 1$,

$$h(f(z), f(z_0)) \; \leq \; \frac{32}{\Delta} \, \exp \left\{ - \left(\frac{2\pi}{c} \right) I(|z - z_0|) \right\}. \tag{4.10}$$

Theorem 4.1. Let D be a domain in $\mathbb{C}$, let D' be a domain in $\overline{\mathbb{C}}$ with $h(\overline{\mathbb{C}} \setminus D') \geq \Delta > 0$, and let $f : D \to D'$ be a homeomorphism with finite distortion at a point $z_0 \in D$. Then

$$h(f(z), f(z_0)) \; \leq \; \frac{32}{\Delta} \, \exp \left\{ - \int\limits_{|z - z_0|}^{\varepsilon(z_0)} \frac{dr}{r q_{z_0}(r)} \right\} \tag{4.11}$$

for $z \in B(z_0, \varepsilon(z_0))$, where $\varepsilon(z_0) < \mathrm{dist}(z_0, \partial D)$ and $q_{z_0}(r)$ is the mean integral value of $K_f(z)$ over the circle $|z - z_0| = r$.

Corollary 4.2. If

$$q_{z_0}(r) \leq \log \frac{1}{r} \tag{4.12}$$

for $r < \varepsilon(z_0) < \mathrm{dist}(z_0, \partial D)$, then

$$h(f(z), f(z_0)) \leq \frac{32}{\Delta} \frac{\log \frac{1}{\varepsilon_0}}{\log \frac{1}{|z - z_0|}} \tag{4.13}$$

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for all $z \in B(z_0, \varepsilon(z_0))$.

Corollary 4.3. If

$$K_f(z) \leq \log \frac{1}{|z - z_0|}, \qquad z \in B(z_0, \varepsilon(z_0)), \tag{4.14}$$

then (4.13) holds in the ball $B(z_0, \varepsilon(z_0))$.

Remark 4.1. If, instead of (4.12) and (4.14), we have the conditions

$$q_{z_0}(r) \leq c \cdot \log \frac{1}{r} \tag{4.15}$$

and, correspondingly,

$$K_f(z) \leq c \cdot \log \frac{1}{|z - z_0|}, \tag{4.16}$$

then

$$h(f(z), f(z_0)) \ \leq \ \frac{32}{\Delta} \left[\frac{\log \frac{1}{\varepsilon_0}}{\log \frac{1}{|z-z_0|}} \right]^{1/c}. \tag{4.17}$$

Choosing in Lemma 4.1 $\psi(t) = 1/t$ and $p = 1$, we also have the following conclusion.

Corollary 4.4. Let $f : \mathbb{D} \to \mathbb{D}$ be a homeomorphism with finite distortion such that $f(0) = 0$ and

$$\int\limits_{\varepsilon < |z| < 1} K_f(z) \ \frac{dx \, dy}{|z|^2} \ \leq \ c \ \log \frac{1}{\varepsilon}, \qquad \varepsilon \in (0, 1). \tag{4.18}$$

Then

$$|f(z)| \ \leq \ 64 \cdot |z|^{\frac{2\pi}{c}}. \tag{4.19}$$

Theorem 4.2. Let D be a domain in $\mathbb{C}$, let D' be a domain in $\overline{\mathbb{C}}$ with $h(\overline{\mathbb{C}} \backslash D') \geq \Delta > 0$, and let $f : D \to D'$ be a homeomorphism with finite distortion at a point $z_0 \in D$. If $K_f(z)$ has finite mean oscillation at the point $z_0 \in D$, then

$$h(f(z), f(z_0)) \leq \frac{32}{\Delta} \left\{ \frac{\log \frac{1}{\varepsilon_0}}{\log \frac{1}{|z-z_0|}} \right\}^{\beta_0} \tag{4.20}$$

for some $\varepsilon_0 < \operatorname{dist}(z_0, \partial D)$ and every $z \in B(z_0, \varepsilon_0)$, where $\beta_0 > 0$ depends only on the function K_f.

5 On Normal Families of homeomorphisms with finite distortion

The results stated bellow can be proved by theorem 3.1 and the correspondent criteria of normality from the paper [25].

Given a domain D in $\mathbb{C}$, let $\mathfrak{F}_{K_f,\Delta}(D)$ be the class of all homeomorphisms f with finite distortion K_f in D with $h(\overline{\mathbb{C}} \setminus f(D)) \geq \Delta > 0$.

Theorem 5.1. If $K_f \in$ FMO, then $\mathfrak{F}_{K_f,\Delta}(D)$ is a normal family.

Corollary 5.1. The class $\mathfrak{F}_{K_f,\Delta}(D)$ is normal if

$$\overline{\lim_{\varepsilon \to 0}} \fint_{B(z_0,\varepsilon)} K_f(z) \ dx\,dy \ < \ \infty \qquad \forall \ z_0 \in D. \tag{5.21}$$

Corollary 5.2. The class $\mathfrak{F}_{K_f,\Delta}(D)$ is normal if every $z_0 \in D$ is a Lebesgue point of $K_f(z)$.

Theorem 5.2. Let $\Delta > 0$ and let $Q : D \to [0,\infty]$ be a measurable function such that

$$\int\limits_0^{\varepsilon(z_0)} \frac{dr}{rq_{z_0}(r)} = \infty \tag{5.22}$$

holds at every point $z_0 \in D$, where $\varepsilon(z_0) = \text{dist}(z_0, \partial D)$ and $q_{z_0}(r)$ denotes the mean integral value of $K_f(z)$ over the circle $|z - z_0| = r$. Then $\mathfrak{F}_{K_f,\Delta}$ forms a normal family.

Corollary 5.3. The class $\mathfrak{F}_{K_f,\Delta}(D)$ is normal if $K_f(z)$ has singularities of the logarithmic type of order not greater than 1 at every point $z \in D$.

6 On some integral conditions

The following results can be found in [24].

Recall that a function $\Phi : [0,\infty] \to [0,\infty]$ is called **convex** if

$$\Phi(\lambda t_1 + (1 - \lambda)t_2) \ \leq \ \lambda \ \Phi(t_1) \ + \ (1 - \lambda) \ \Phi(t_2)$$

for all t_1 and $t_2 \in [0,\infty]$ and $\lambda \in [0,1]$.

In what follows, $\mathbb{R}(\varepsilon)$, $\varepsilon \in (0,1)$ denotes the ring in the space $\mathbb{C}$,

$$\mathbb{R}(\varepsilon) \ = \ R\,(\varepsilon,\,1,\,0). \tag{6.1}$$

The following statement is a generalization and strengthening of Lemma 3.1 from [26].

Lemma 6.1. *Let $Q : \mathbb{D} \to [0, \infty]$ be a measurable function and let $\Phi : [0, \infty] \to (0, \infty]$ be a non-decreasing convex function. Suppose that the mean value $M(\varepsilon)$ of the function $\Phi \circ Q$ over the ring $\mathbb{R}(\varepsilon)$, $\varepsilon \in (0, 1)$, is finite. Then*

$$\int\limits_{\varepsilon}^{1} \frac{dr}{rq^{\frac{1}{p}}(r)} \geq \frac{1}{2} \int\limits_{eM(\varepsilon)}^{\frac{M(\varepsilon)}{\varepsilon^2}} \frac{d\tau}{\tau \left[\Phi^{-1}(\tau)\right]^{\frac{1}{p}}} \qquad \forall \quad p \in (0, \infty) \qquad (6.2)$$

where $q(r)$ is the average of the function $Q(z)$ over the circle $|z| = r$.

Remark 6.1. Note that (6.2) is equivalent for each $p \in (0, \infty)$ to the inequality

$$\int\limits_{\varepsilon}^{1} \frac{dr}{rq^{\frac{1}{p}}(r)} \geq \frac{1}{2} \int\limits_{eM(\varepsilon)}^{\frac{M(\varepsilon)}{\varepsilon^2}} \frac{d\tau}{\tau \Phi_p^{-1}(\tau)} \;, \qquad \Phi_p(t) \; : \; = \; \Phi(t^p) \; . \qquad (6.3)$$

Note also that $M(\varepsilon)$ converges as $\varepsilon \to 0$ to the average of $\Phi \circ Q$ over the unit disk $\mathbb{B}$.

Corollary 6.1. *Let $\Phi : [0, \infty] \to (0, \infty]$ be a non-decreasing convex function, $Q : \mathbb{B} \to [0, \infty]$ a measurable function, $Q_*(z) = 1$ if $Q(z) < 1$ and $Q_*(z) = Q(z)$ if $Q(z) \geq 1$. Suppose that the mean $M_*(\varepsilon)$ of the function $\Phi \circ Q$ over the ring $\mathbb{R}(\varepsilon)$, $\varepsilon \in (0, 1)$, is finite. Then*

$$\int\limits_{\varepsilon}^{1} \frac{dr}{rq^{\frac{\lambda}{p}}(r)} \geq \frac{1}{2} \int\limits_{eM_*(\varepsilon)}^{\frac{M_*(\varepsilon)}{\varepsilon^2}} \frac{d\tau}{\tau \left[\Phi^{-1}(\tau)\right]^{\frac{1}{p}}} \qquad \forall \lambda \in (0, 1), \qquad p \in (0, \infty) \qquad (6.4)$$

where $q(r)$ is the average of the function $Q(z)$ over the circle $|z| = r$.

Indeed, let $q_*(r)$ be the average of the function $Q_*(z)$ over the circle $|z| = r$. Then $q(r) \leq q_*(r)$ and, moreover, $q_*(r) \geq 1$ for all $r \in (0, 1)$. Thus, $q^{\frac{\lambda}{p}}(r) \leq q_*^{\frac{\lambda}{p}}(r) \leq q_*^{\frac{1}{p}}(r)$ for all $\lambda \in (0, 1)$ and hence by Lemma 6.1 applied to $Q_*(z)$ we obtain (6.4).

Theorem 6.1. *Let $Q : \mathbb{D} \to [0, \infty]$ be a measurable function such that*

$$\int\limits_{\mathbb{B}} \Phi(Q(z)) \; dx \, dy \; < \; \infty \qquad (6.5)$$

where $\Phi : [0, \infty] \to [0, \infty]$ is a non-decreasing convex function such that

$$\int\limits_{\delta_0}^{\infty} \frac{d\tau}{\tau \left[\Phi^{-1}(\tau)\right]^{\frac{1}{p}}} \; = \; \infty \;, \qquad p \in (0, \infty) \;, \qquad (6.6)$$

for some $\delta_0 > \tau_0 := \Phi(0)$. *Then*

$$\int_0^1 \frac{dr}{rq^{\frac{1}{p}}(r)} = \infty \tag{6.7}$$

where $q(r)$ *is the average of the function* $Q(z)$ *over the circle* $|z| = r$.

Remark 6.2. Since $\left[\Phi^{-1}(\tau)\right]^{\frac{1}{p}} = \Phi_p^{-1}(\tau)$ where $\Phi_p(t) = \Phi(t^p)$, (6.6) implies that

$$\int_\delta^\infty \frac{d\tau}{\tau\Phi_p^{-1}(\tau)} = \infty \qquad \forall\, \delta \in [0, \infty) \tag{6.8}$$

but (6.8) for some $\delta \in [0, \infty)$, generally speaking, does not imply (6.6). Indeed, for $\delta \in [0, \delta_0)$, (6.6) evidently implies (6.8) and, for $\delta \in (\delta_0, \infty)$, we have that

$$0 \leq \int_{\delta_0}^\delta \frac{d\tau}{\tau\Phi_p^{-1}(\tau)} \leq \frac{1}{\Phi_p^{-1}(\delta_0)} \log \frac{\delta}{\delta_0} < \infty \tag{6.9}$$

because Φ_p^{-1} is non-decreasing and $\Phi_p^{-1}(\delta_0) > 0$. Moreover, by the definition of the inverse function $\Phi_p^{-1}(\tau) \equiv 0$ for all $\tau \in [0, \tau_0]$, $\tau_0 = \Phi_p(0)$, and hence (6.8) for $\delta \in [0, \tau_0)$, generally speaking, does not imply (6.6). If $\tau_0 > 0$, then

$$\int_\delta^{\tau_0} \frac{d\tau}{\tau\Phi_p^{-1}(\tau)} = \infty \qquad \forall\, \delta \in [0, \tau_0) \tag{6.10}$$

However, (6.10) gives no information on the function $Q(z)$ itself and, consequently, (6.8) for $\delta < \Phi(0)$ cannot imply (6.7) at all.

In view of (6.8), Theorem 6.1 follows immediately from Lemma 6.1.

Corollary 6.2. *If* $\Phi : [0, \infty] \to [0, \infty]$ *is a non-decreasing convex function and* Q *satisfies the condition (6.5), then each of the conditions (2.4)–(2.9) for* $p \in (0, \infty)$ *implies (6.7). Moreover, if in addition* $\Phi(1) < \infty$ *or* $q(r) \geq 1$ *on a subset of* $(0, 1)$ *of a positive measure, then each of the conditions (2.4)–(2.9) for* $p \in (0, \infty)$ *implies*

$$\int_0^1 \frac{dr}{rq^{\frac{\lambda}{p}}(r)} = \infty \qquad \forall\, \lambda \in (0, 1) \tag{6.11}$$

and also

$$\int_0^1 \frac{dr}{r^\alpha q^{\frac{\beta}{p}}(r)} = \infty \qquad \forall\, \alpha \geq 1, \ \beta \in (0, \alpha]. \tag{6.12}$$

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7 Sufficient conditions for equicontinuity

Let D be a fixed domain in the extended space $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Given a function $\Phi :$ $[0, \infty] \to [0, \infty]$, $M > 0$, $\Delta > 0$, $\mathfrak{F}_{M,\Delta}^{\Phi}$ denotes the collection of all homeomorphisms with finite distortion in D such that $h\left(\overline{\mathbb{C}} \setminus f(D)\right) \geq \Delta$ and

$$\int\limits_{D} \Phi\left(K_f(z)\right) \frac{dx\,dy}{\left(1 + |z|^2\right)^2} \leq M. \tag{7.1}$$

Theorem 7.1. *Let $\Phi : [0, \infty] \to [0, \infty]$ be non-decreasing convex function. If*

$$\int\limits_{\delta_0}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty \tag{7.2}$$

for some $\delta_0 > \tau_0 := \Phi(0)$, then the class $\mathfrak{F}_{M,\Delta}^{\Phi}$ is equicontinuous and, consequently, forms a normal family of mappings for every $M \in (0, \infty)$ and $\Delta \in (0, 1)$.

Remark 7.1. Note that the condition

$$\int\limits_{D} \Phi\left(K_f(z)\right) dx\,dy \leq M \tag{7.3}$$

implies (7.1). Thus, the condition (7.1) is more general than (7.3) and homeomorphisms with finite distortion satisfying (7.3) form a subclass of $\mathfrak{F}_{M,\Delta}^{\Phi}$. Conversely, if the domain D is bounded, then (7.1) implies the condition

$$\int\limits_{D} \Phi\left(K_f(z)\right) dx\,dy \leq M_* \tag{7.4}$$

where $M_* = M \cdot \left(1 + \delta_*^2\right)$, $\delta_* = \sup\limits_{z \in D} |z|$.

Corollary 7.1. *Each of the conditions (2.4)–(2.9) for $p \in (0, n-1]$ implies equicontinuity and normality of the classes $\mathfrak{F}_{M,\Delta}^{\Phi}$ for all $M \in (0, \infty)$ and $\Delta \in (0, 1)$.*

Given a function $\Phi : [0, \infty] \to [0, \infty]$, $M > 0$ and $\Delta > 0$, $S_{M,\Delta}^{\Phi}$ denotes the class of all homeomorphisms f of D in the Sobolev class $W_{loc}^{1,2}$ with a locally integrable $K_f(z)$ such that $h\left(\overline{\mathbb{C}} \setminus f(D)\right) \geq \Delta$ and (7.1) holds for $K_f(z)$. Note that if Φ is non-decreasing, convex and non–constant on $[0, \infty)$, then (7.1) itself implies that $K_f(z) \in L_{loc}^1$. Note also that $S_{M,\Delta}^{\Phi} \subset \mathfrak{F}_{M,\Delta}^{\Phi}$, see e.g. Theorem 4.1 in [20]. Thus, we have the following consequence.

Corollary 7.2. *Each of the conditions (2.4)–(2.9) for $p \in (0, 1]$ implies equicontinuity and normality of the class $S_{M,\Delta}^{\Phi}$ for all $M \in (0, \infty)$ and $\Delta \in (0, 1)$.*

8   Necessary conditions for equicontinuity

Theorem 8.1. *If the classes $S^\Phi_{M,\Delta} \subset \mathfrak{F}^\Phi_{M,\Delta}$ are equicontinuous (normal) for a non-decreasing convex function $\Phi : [0,\infty] \to [0,\infty]$, all $M \in (0,\infty)$ and $\Delta \in (0,1)$. Then*

$$\int\limits_{\delta_*}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty \tag{8.1}$$

for all $\delta_* \in (\tau_0, \infty)$ where $\tau_0 : = \Phi(0)$.

It is evident that the function $\Phi(t)$ in Theorem 8.1 cannot be constant because in the contrary case we would have no real restrictions for K_I except $\Phi(t) \equiv \infty$ when the classes $S^\Phi_{M,\Delta}$ are empty. Moreover, by the known criterion of convexity, see e.g. Proposition 5 in I.4.3 of [5], the slope $[\Phi(t) - \Phi(0)]/t$ is nondecreasing. Hence the proof of Theorem 8.1 follows from the next statement.

Lemma 8.1. *Let a function $\Phi : [0,\infty] \to [0,\infty]$ be non-decreasing and*

$$\Phi(t) \geq C \cdot t \qquad \forall\, t \in [T, \infty] \tag{8.2}$$

for some $C > 0$ and $T \in (0,\infty)$. If the classes $S^\Phi_{M,\Delta} \subset \mathfrak{F}^\Phi_{M,\Delta}$ are equicontinuous (normal) for all $M \in (0,\infty)$ and $\Delta \in (0,1)$, then (8.1) holds for all $\delta_* \in (\tau_0, \infty)$ where $\tau_0 : = \Phi(+0)$.

Remark 8.1. Theorem 8.1 shows that the condition (7.2) in Theorem 7.1 is not only sufficient but also necessary for equicontinuity (normality) of classes with the integral constraints of the type either (7.1) or (7.4) with a convex non-decreasing Φ. In view of Proposition 2.2, the same concerns to all the conditions (2.4)–(2.9) with $p = 1$.

Corollary 8.1. *The equicontinuity (normality) of the classes $S^\Phi_{M,\Delta} \subset \mathfrak{F}^\Phi_{M,\Delta}$ for $M \in (0,\infty)$, $\Delta \in (0,1)$ and non-decreasing convex Φ implies that*

$$\int\limits_{\delta}^{\infty} \log \Phi(t)\, \frac{dt}{t^2} = \infty \tag{8.3}$$

for all $\delta > t_0$ where $t_0 := \sup\limits_{\Phi(t)=0} t$, $t_0 = 0$ if $\Phi(0) > 0$.

The condition (8.3) is also sufficient for equicontinuity (normality) of the classes $S^\Phi_{M,\Delta}$ and $\mathfrak{F}^\Phi_{M,\Delta}$.

References

[1] Ahlfors L.: On quasiconformal mappings. J. Analyse Math. **3**, 1–58 (1953/54).

[2] Astala K., Iwaniec T., Martin G.: Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane. Princeton University Press (2009).

[3] Andreian Cazacu C.: On the length-area dilatation. Complex Var. Theory Appl. **50**:7–11, 765–776 (2005).

[4] Biluta P.A.: Extremal problems for mappings which are quasiconformal in the mean. Sib. Mat. Zh. **6**, 717–726 (1965).

[5] Bourbaki N.: Functions of a Real Variable. Springer, Berlin (2004).

[6] Fuglede B.: Extremal length and functional completion. Acta Math. **98**, 171–219 (1957).

[7] Gehring F.W.: Rings and quasiconformal mappings in space. Trans. Amer. Math. Soc. **103**, 353–393 (1962).

[8] Golberg A.: Homeomorphisms with finite mean dilatations. Contemporary Math. **382**, 177–186 (2005).

[9] Gutlyanskii V., Martio O., Sugawa T. and Vuorinen M.: On the degenerate Beltrami equation. Trans. Amer. Math. Soc. **357**, 875–900 (2005).

[10] Ignat'ev A, Ryazanov V.: Finite mean oscillation in the mapping theory. Ukrainian Math. Bull. **2** (3), 403–424 (2005).

[11] Iwaniec T. and Martin G.: Geometric Function Theory and Nonlinear Analysis. Clarendon Press, Oxford (2001).

[12] Kovtonyuk D., Petkov I. and Ryazanov V.: On homeomorphisms with finite distortion in the plane. ArXiv: 1011.3310v2 [math.CV], 16 p. (2010).

[13] Kovtonyuk D., Ryazanov V.: On the theory of lower Q-homeomorphisms. Ukr. Mat. Visn. **5** (2), 159–184 (2008) (in Russian); translated in Ukrainian Math. Bull. by AMS.

[14] Kruglikov V.I.: Capacities of condensors and quasiconformal in the mean mappings in space. Mat. Sb. **130** (2), 185–206 (1986).

[15] Krushkal' S.L.: On mappings that are quasiconformal in the mean. Dokl. Akad. Nauk SSSR. **157** (3), 517–519 (1964).

[16] Krushkal' S.L. and Kühnau R.: Quasiconformal mappings, new methods and applications. Novosibirsk, Nauka (1984) (in Russian).

[17] Kud'yavin V.S.: Behavior of a class of mappings quasiconformal in the mean at an isolated singular point. Dokl. Akad. Nauk SSSR. **277** (5), 1056–1058 (1984).

[18] Kühnau R.: Über Extremalprobleme bei im Mittel quasiconformen Abbildungen. Lecture Notes in Math. **1013**, 113–124 (1983) (in German).

[19] Lehto O. and Virtanen K.: Quasiconformal Mappings in the Plane. Springer, New York etc. (1973).

[20] Martio O., Ryazanov V., Srebro U. and Yakubov E.: Moduli in Modern Mapping Theory. Springer, New York (2009).

[21] Perovich M.: Isolated singularity of the mean quasiconformal mappings. Lect. Notes Math. **743**, 212–214 (1979).

[22] Pesin I.N.: Mappings quasiconformal in the mean. Dokl. Akad. Nauk SSSR. **187** (4), 740–742 (1969).

[23] Ryazanov V.I.: On mappings that are quasiconformal in the mean. Sibirsk. Mat. Zh. **37** (2), 378–388 (1996).

[24] Ryazanov V., Sevost'yanov E.: Equicontinuity of mappings quasiconformal in the mean. ArXiv: 1003.1199v3 [math.CV], 19 p., (2010).

[25] Ryazanov V., Sevostyanov E.: Toward the theory of ring Q – homeomorphisms. Israel J. Math. **168**, 101–118 (2008).

[26] Ryazanov V., Srebro U. and Yakubov E.: On integral conditions in the mapping theory. Ukrainian Math. Bull. **7**, 73–87 (2010).

[27] Ryazanov V., Srebro U. and Yakubov E.: On ring solutions of Beltrami equation. J. d'Analyse Math. **96**, 117–150 (2005).

[28] Salimov R.: On regular homeomorphisms in the plane. Ann. Acad. Sci. Fenn. Math. **35**, 285–289 (2010).

[29] Sobolev S.L.: Applications of functional analysis in mathematical physics. Izdat. Gos. Univ., Leningrad (1950); English transl, Amer. Math. Soc., Providence, R.I. (1963).

[30] Väisälä J.: Lectures on n–Dimensional Quasiconformal Mappings. Lecture Notes in Math. **229**. Springer–Verlag, Berlin etc. (1971).

[31] Ukhlov, A. and Vodopyanov, S. K.: Mappings associated with weighted Sobolev spaces. Complex Anal. Dynam. Syst. III, Contemp. Math. **455**, 369–382 (2008).

[32] Ziemer W.P.: Extremal length and conformal capacity. Trans. Amer. Math. Soc. **126**(3), 460–473 (1967).