Cohomology and L-values

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Abstract

In a paper published in 1959, Shimura presented an elegant calculation of the critical values of L-functions attached to elliptic modular forms using the first cohomology group. We will show that a similar calculation is possible for Hilbert modular forms over real quadratic fields using the second cohomology group. We present explicit numerical examples calculated by this method.

In a celebrated paper [Sh1] published in 1959, Shimura showed that ratios of critical values of the L-function attached to an elliptic modular form can be calculated explicitly using the cohomology group. This method was developed into the theory of modular symbols by Manin [Man]. Though there have been great advances during the next half century in understanding the relationship of automorphic forms and group cohomologies, it seems that no explicit calculations of L-values using cohomology groups were performed beyond the one dimensional case. The purpose of this paper is to show that we can use cohomology groups effectively for calculations of L-values even in higher dimensional cases.

To explain our ideas and results, it is best to review first the calculation in [Sh1]. Let $\mathfrak{H}$ be the complex upper half plane. Let Γ be a Fuchsian group and let Ω be a cusp form of weight $k \geq 2$ with respect to Γ. Put $l = k - 2$ and let ρ_l be the symmetric tensor representation of $\mathrm{GL}(2, \mathbf{C})$ of degree l on a vector space V. We regard V as a Γ-module. Put $\rho = \rho_l$. We consider a V-valued differential form on $\mathfrak{H}$:

$$\mathfrak{d}(\Omega) = \Omega(z) \begin{bmatrix} z \\ 1 \end{bmatrix}^l dz.$$

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Here $\begin{bmatrix} z \\ 1 \end{bmatrix}^l$ denotes the column vector of dimension $l+1$ whose components are $z^l,\ z^{l-1}, \ldots, 1$. We have $\mathfrak{d}(\Omega) \circ \gamma = \rho(\gamma)\mathfrak{d}(\Omega)$ for every $\gamma \in \Gamma$. Here $\mathfrak{d}(\Omega) \circ \gamma$ denotes the transform of $\mathfrak{d}(\Omega)$ by γ. Take a point of the complex upper half plane or a cusp of Γ and denote it by z_0. For $\gamma \in \Gamma$, we consider the integral

$$f(\gamma) = \int_{z_0}^{\gamma z_0} \mathfrak{d}(\Omega). \tag{1}$$

Then f satisfies the 1-cocycle condition:

$$f(\gamma_1\gamma_2) = f(\gamma_1) + \rho(\gamma_1)f(\gamma_2).$$

The cohomology class of f in $H^1(\Gamma, V)$ does not depend on the choice of z_0. Let $p \in \Gamma$ be a parabolic element and z_0' be the cusp fixed by p. Then we have

$$f(p) = (\rho(p) - 1) \int_{z_0'}^{z_0} \mathfrak{d}(\Omega).$$

Thus $f(p)$ looks like a coboundary, which is the parabolic condition on f.

Now let $\Gamma = \mathrm{SL}(2, \mathbf{Z})$ and $z_0 = i\infty$. Put

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then we find

$$f(\sigma\tau) = -\left(\int_0^{i\infty} \Omega(z) z^t dz \right)_{0 \le t \le l} = -\left(i^{t+1} R(t+1, \Omega) \right)_{0 \le t \le l}, \tag{2}$$

where $R(s, \Omega) = (2\pi)^{-s}\Gamma(s)L(s, \Omega)$ with the L-function $L(s, \Omega)$ of Ω. Since $(\sigma\tau)^3 = 1$, the 1-cocycle condition gives

$$[1 + \rho(\sigma\tau) + \rho((\sigma\tau)^2)]f(\sigma\tau) = 0. \tag{3}$$

In other words, $f(\sigma\tau)$ is annihilated by the element $1+\sigma\tau+(\sigma\tau)^2$ of the group ring $\mathbf{Z}[\mathrm{SL}(2, \mathbf{Z})]$. This gives a constraint on the critical values of $L(s, \Omega)$. For $k = 12$ and $\Omega = \Delta$, Shimura obtained that

$$R(8, \Delta) = \frac{5}{4} R(6, \Delta), \qquad R(10, \Delta) = \frac{12}{5} R(6, \Delta), \quad \text{etc.}$$

In this paper, we will treat the case of Hilbert modular forms over a real quadratic field F. Let $\mathcal{O}_F$ be the ring of integers of F and Γ be a congruence subgroup of $\mathrm{SL}(2, \mathcal{O}_F)$. Let Ω be a Hilbert modular cusp form of weight (k_1, k_2) with respect to Γ. We assume $2 \le k_2 \le k_1$ and put $l_i = k_i - 2$, $i = 1, 2$. The first step is to attach an explicitly given 2-cocycle of Γ to Ω. This is given in the author's book [Y3] as follows. Let $\rho = \rho_{l_1} \otimes \rho_{l_2}$ and V be the representation space of ρ. We consider a V-valued differential form on $\mathfrak{H}^2$:

$$\mathfrak{d}(\Omega) = \Omega(z) \begin{bmatrix} z_1 \\ 1 \end{bmatrix}^{l_1} \otimes \begin{bmatrix} z_2 \\ 1 \end{bmatrix}^{l_2} dz_1 dz_2, \qquad z = (z_1, z_2) \in \mathfrak{H}^2.$$

We have

$$\mathfrak{d}(\Omega) \circ \gamma = \rho(\gamma)\mathfrak{d}(\Omega), \qquad \gamma \in \Gamma. \tag{4}$$

Take a point $w = (w_1, w_2)$ on $\mathfrak{H}^2$. For γ_1, $\gamma_2 \in \Gamma$, we consider the integral

$$f(\gamma_1, \gamma_2) = \int_{\gamma_1\gamma_2 w_1}^{\gamma_1 w_1} \int_{w_2}^{\gamma_1' w_2} \mathfrak{d}(\Omega). \tag{5}$$

Here γ_1' denotes the conjugate of γ_1 by $\mathrm{Gal}(F/\mathbf{Q})$. Then f is a 2-cocycle of Γ taking values in V. The cohomology class of $f \in H^2(\Gamma, V)$ does not depend on the choice of w. Let $p \in \Gamma$ be a parabolic element and let (w_1^*, w_2^*) be the cusp fixed by p. Since Ω is a cusp form, we may replace w_2 by w_2^*. By this procedure, we find the parabolic condition satisfied by f.

Next let $\Gamma = \mathrm{SL}(2, \mathcal{O}_F)$ and let ϵ be the fundamental unit of F. We assume that $l_1 \equiv l_2 \mod 2$ and replace Γ by $\mathrm{PSL}(2, \mathcal{O}_F)$. Put

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mu = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}.$$

We choose $w_1 = i\epsilon^{-1}$, $w_2 = i\infty$. Then we have

$$f(\sigma, \mu) = f(\sigma, \sigma) = -\int_{i\epsilon^{-1}}^{i\epsilon} \int_0^{i\infty} \mathfrak{d}(\Omega).$$

For $0 \le s \le l_1$, $0 \le t \le l_2$, we put

$$P_{s,t} = \int_{i\epsilon^{-1}}^{i\epsilon} \int_0^{i\infty} \Omega(z) z_1^s z_2^t dz_1 dz_2.$$

The $(l_1 + 1)(l_2 + 1)$ components of $f(\sigma, \mu)$ are given by $-P_{s,t}$. We have

$$P_{m,m-(k_1-k_2)/2} = (-1)^{m+1} i^{-(k_1-k_2)/2} (2\pi)^{(k_1-k_2)/2} R(m+1, \Omega) \tag{6}$$

where $R(s,\Omega) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-(k_1-k_2)/2)L(s,\Omega)$ with the L-function $L(s,\Omega)$ of Ω. The formula (6) gives a generalization of (2); (5) and (6) were known to the author eight years ago.

The L-value $L(m,\Omega)$, $m \in \mathbf{Z}$ is a critical value if and only if

$$\frac{l_1-l_2}{2}+1 \le m \le \frac{l_1+l_2}{2}+1.$$

Since all of them appear as components of $f(\sigma,\mu)$, we expect that we can deduce information on critical values once we know the second cohomology group $H^2(\Gamma,V)$ well. Before to materialize this hope, we need to answer the following conceptual question: Can we annihilate the effect of adding a coboundary to f? We can give an affirmative answer to this question by using the parabolic condition. Put

$$P = \left\{ \begin{pmatrix} u & v \\ 0 & u^{-1} \end{pmatrix} \middle| \, u \in \mathcal{O}_F^\times, \ v \in \mathcal{O}_F \right\} / \{\pm 1_2\} \subset \Gamma.$$

Then we have

$$f(p\gamma_1,\gamma_2) = pf(\gamma_1,\gamma_2) \qquad \text{for every} \quad p \in P, \ \gamma_1,\gamma_2 \in \Gamma. \tag{7}$$

This is the parabolic condition on f when $\Gamma = \mathrm{PSL}(2,\mathcal{O}_F)$. A 2-cocycle which satisfies (7) will be called a *parabolic* 2-*cocycle*. In section 3, we will prove:

Theorem. *Let $i = 1$ or 2. Then*

$$\dim H^i(P,V) = \begin{cases} 0 & \text{if} \quad l_1 \neq l_2 \text{ or } N(\epsilon)^{l_1} = -1, \\ 1 & \text{if} \quad l_1 = l_2 \text{ and } N(\epsilon)^{l_1} = 1. \end{cases}$$

Now suppose that we add a coboundary to f keeping the parabolic condition (7). In section 4, using this theorem for the case $i = 1$, we will show: If $l_1 \neq l_2$, the components of $f(\sigma,\mu)$ related to the critical values do not change. If $l_1 = l_2$, the same assertion holds except for the critical values on the edges: $L(1,\Omega)$ and $L(l_1+1,\Omega)$. Therefore we can deduce information on critical values $L(m,\Omega)$ once we know a parabolic 2-cocycle corresponding to Ω.

The final step is to find constraints on $f(\sigma,\mu)$ which generalizes (3). This is technically the most difficult step. Let $\mathcal{O}_F = \mathbf{Z} + \mathbf{Z}\omega$ and put

$$\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \eta = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}.$$

It is known ([V]) that Γ is generated by σ, μ, τ and η. Let $\mathcal{F}$ be the free group on four letters $\widetilde{\sigma}$, $\widetilde{\mu}$, $\widetilde{\tau}$, $\widetilde{\eta}$. Define a surjective homomorphism $\pi : \mathcal{F} \longrightarrow \Gamma$ by $\pi(\widetilde{\sigma}) = \sigma$, $\pi(\widetilde{\mu}) = \mu$, $\pi(\widetilde{\tau}) = \tau$, $\pi(\widetilde{\eta}) = \eta$ and let R be the kernel of π. Then we have $\Gamma = \mathcal{F}/R$ and (cf. §1.4)

$$H^2(\Gamma, V) \cong H^1(R, V)^\Gamma / \mathrm{Im}(H^1(\mathcal{F}, V)). \tag{8}$$

Here we have

$$H^1(R, V)^\Gamma = \left\{ \varphi \in \mathrm{Hom}(R, V) \mid \varphi(grg^{-1}) = g\varphi(r), \quad g \in \mathcal{F}, r \in R \right\}.$$

We write

$$\epsilon^2 = A + B\omega, \qquad \epsilon^2\omega = C + D\omega.$$

We have relations: (i) $\sigma^2 = 1$. (ii) $(\sigma\tau)^3 = 1$. (iii) $(\sigma\mu)^2 = 1$. (iv) $\tau\eta = \eta\tau$. (v) $\mu\tau\mu^{-1} = \tau^A\eta^B$. (vi) $\mu\eta\mu^{-1} = \tau^C\eta^D$. For $t \in \mathcal{O}_F^\times$, we have

$$\sigma \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \sigma = \begin{pmatrix} 1 & -t^{-1} \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} -t & 1 \\ 0 & -t^{-1} \end{pmatrix}. \tag{vii}$$

The relation (ii) follows from (vii) by taking $t = 1$. We call the relation group R *minimal* if it is generated by the elements corresponding to (i) $\sim$ (vii) and their conjugates. We see that μ, τ and η genarate P and (iv) $\sim$ (vi) are their fundamental relations.

Now let $\varphi \in H^1(R, V)^\Gamma$ be a corresponding element to f. Adding an element of $\mathrm{Im}(H^1(\mathcal{F}, V))$, we may assume that $\varphi(\widetilde{\sigma}^2) = 0$. Then we find (cf. (5.3))

$$f(\sigma, \mu) = -\varphi((\widetilde{\sigma}\widetilde{\mu})^2).$$

Our problem is reduced to find constraints on $\varphi((\widetilde{\sigma}\widetilde{\mu})^2)$. We have an obvious constraint $\sigma\mu\varphi((\widetilde{\sigma}\widetilde{\mu})^2) = \varphi((\widetilde{\sigma}\widetilde{\mu})^2)$ but of course it is not enough.

To proceed further, we assume that l_1 and l_2 are even and change ρ to $\rho' = \rho'_{l_1} \otimes \rho'_{l_2}$ where $\rho'_l(g) = \rho_l(g)\det(g)^{-l/2}$ and regard V as a $\mathrm{PGL}(2, \mathcal{O}_F)$-module. The Γ-module structure does not change. We put

$$\nu = \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}, \qquad \delta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

These two elements act on Γ as outer automorphisms and induce automorphisms of $H^2(\Gamma, V)$ of order 2. Hence $H^2(\Gamma, V)$ decomposes into four pieces under their actions. Let Γ^* be the subgroup of $\mathrm{PGL}(2, \mathcal{O}_F)$ generated by Γ and ν. The transfer map gives an isomorphism of the plus part of $H^2(\Gamma, V)$ under the action of ν onto $H^2(\Gamma^*, V)$. For simplicity suppose that we can take $\omega = \epsilon$. Then σ, ν and τ generate Γ^*. Let $\mathcal{F}^*$ be the free group on three letters $\widetilde{\sigma}$, $\widetilde{\nu}$, $\widetilde{\tau}$. Define a surjective homorphism $\pi^* : \mathcal{F}^* \longrightarrow \Gamma^*$ by $\pi^*(\widetilde{\sigma}) = \sigma$, $\pi^*(\widetilde{\nu}) = \nu$, $\pi^*(\widetilde{\tau}) = \tau$ and let R^* be the kernel of π^*. Then we have $\Gamma^* = \mathcal{F}^*/R^*$ and

$$(8^*) \qquad H^2(\Gamma^*, V) \cong H^1(R^*, V)^{\Gamma^*}/\mathrm{Im}(H^1(\mathcal{F}^*, V)).$$

Let f^* be the transfer of f to Γ^* and let f^+ be the restriction of f^* to Γ. Then f^+ is the projection of f to the plus part. (We perform this procedure on the cocycle level.) We have

$$f^*(\sigma, \mu) = f^+(\sigma, \mu) = (1 + \nu)f(\sigma, \mu).$$

In Γ^*, σ, ν and τ satisfy the relations (i), (ii) and (iii*): $(\sigma\nu)^2 = 1$, (iv*): $\tau\nu\tau\nu^{-1} = \nu\tau\nu^{-1}\tau$, (v*): $\nu^2\tau\nu^{-2} = \tau^A(\nu\tau\nu^{-1})^B$. Let P^* be the subgroup of Γ^* generated by P and ν. We see that P^* is generated by ν and τ and (iv*) and (v*) are the fundamental relations between generators ν and τ. Let $\varphi^* \in H^1(R^*, V)^{\Gamma^*}$ be a corresponding element to f^*. By the parabolic condition on f, we may assume that φ^* vanishes on the elements of R^* corresponding to (iv*) and (v*). Adding an element of $\mathrm{Im}(H^1(\mathcal{F}^*, V))$, we may also assume that $\varphi^*(\widetilde{\sigma}^2) = 0$. Then we have (cf. (6.6))

$$f^*(\sigma, \mu) = -(1 + \nu^{-1})\varphi^*((\widetilde{\sigma}\widetilde{\nu})^2)$$

and two quantities

$$A = \varphi^*((\widetilde{\sigma}\widetilde{\nu})^2), \qquad B = \varphi^*((\widetilde{\sigma}\widetilde{\tau})^3)$$

remain to be determined. The Hecke operators act on $H^2(\Gamma^*, V)$. We can analyze its action on the right-hand side of (8*) and will give a simple formula for it. The quantity A is related to the critical values of $L(s, \Omega)$. We may assume that the class of f^* is in the plus space of $H^2(\Gamma^*, V)$ under the action of δ. Then A must satisfy the constraints

$$(9) \qquad (\sigma\nu - 1)A = 0, \qquad (\delta - 1)A = 0.$$

We will execute the determination of A for $F = \mathbf{Q}(\sqrt{5})$ and $F = \mathbf{Q}(\sqrt{13})$. First assume $F = \mathbf{Q}(\sqrt{5})$. In this case, we can show that R is minimal and that R^* is generated by the elements corresponding to the relations (i), (ii), (iii*), (iv*), (v*) and their conjugates. Calculating the action of the Hecke operator $T(2)$ on the right hand side of (8*), we find a certain element $x \in \mathcal{F}^*$ such that $\pi^*(x)^3 = 1$. We can give an explicit formula expressing $\varphi^*(x^3)$ in terms of A and B. In every case examined, we find by numerical computations that we may assume that $B = 0$ by adding an element of $\mathrm{Im}(H^1(\mathcal{F}^*, V))$. Therefore

$$(x-1)\varphi^*(x^3) = 0 \tag{10}$$

gives a new constraint on A. Let Z_A^+ be the subspace of V consisting of all A which satisfy (9) and (10) and let B_A^+ be the subspace of Z_A^+ which represents the contribution from $\mathrm{Im}(H^1(\mathcal{F}^*, V))$. Again in every case examined, we find by numerical computations that $\dim S_{l_1+2,l_2+2}(\Gamma) = \dim Z_A^+/B_A^+$. If this is one dimensional, we can deduce information on L-values by calculating Z_A^+. In general, calculating the action of $T(2)$ on Z_A^+/B_A^+ and taking eigenvectors, we can obtain many examples on L-values. Actually by considering f^+, we are losing half of the information on critical values (cf. §5.6). To treat all critical values, we need to consider f^-, the projection of f to the minus part of $H^2(\Gamma, V)$ under the action of ν. To handle f^- is a somewhat more complicated task and we leave the explanation of it to the text. Next let $F = \mathbf{Q}(\sqrt{13})$. The procedure is almost the same. Let $\mathfrak{p}$ be the prime ideal generated by $4 - \sqrt{13}$. Calculating the action of the Hecke operator $T(\mathfrak{p})$, we obtain a certain element $x \in \mathcal{F}^*$ such that $\pi^*(x)$ is of order 3. Then the constraint $(x-1)\varphi^*(x^3) = 0$ obtained from x is sufficient. Here remarkably we can perform rigorous calculations without proving that R is minimal. (This is actually true also for the case $F = \mathbf{Q}(\sqrt{5})$.) We have used Pari [PARI] for the numerical calculations in sections 6 and 7.

To calculate the ratios of critical values of L-functions, there is another method initiated by Shimura [Sh3] which employs the Rankin-Selberg convolution and differential operators. A comparison of this method and the cohomological method will be discussed in section 8.

Now let us explain the organization of this paper briefly. In section 1, we will review several facts on cohomology of a group which will be repeatedly used in later sections. In section 2, we will review Hilbert modular forms. We will prove (5) and (6). In section 3, we will study cohomology groups of P and will prove the theorem stated above. In section 4, we will examine the parabolic condition on a cocycle applying results in section 3. We will prove the nonvanishing of the cohomology class of f under mild conditions. In section 5, we will study the decomposition of $H^2(\Gamma, V)$ under the action of outer automorphisms of Γ. It decomposes into four pieces under this action. In section 6, we will study the case $F = \mathbf{Q}(\sqrt{5})$ in detail and will give many examples. In section 7, we will study the case $F = \mathbf{Q}(\sqrt{13})$. We devote section 8 to the comparison of two methods mentioned above. In Appendix, we will prove that the relation group R is minimal when $F = \mathbf{Q}(\sqrt{5})$.

Notation. For an associative ring A with identity element, $A^\times$ denotes the group of all invertible elements of A. Let R be a commutative ring with identity element. We denote by $M(n, R)$ the ring of all $n \times n$ matrices with entries in R. We define $\mathrm{GL}(n, R) = M(n, R)^\times$, $\mathrm{SL}(n, R) = \{g \in \mathrm{GL}(n, R) \mid \det g = 1\}$. The quotient group of $\mathrm{GL}(n, R)$ (resp. $\mathrm{SL}(n, R)$) by its center is denoted by $\mathrm{PGL}(n, R)$ (resp. $\mathrm{PSL}(n, R)$). Let G be a group. The subgroup of G generated by $g_1, \ldots, g_n \in G$ is denoted by $\langle g_1, \ldots, g_n \rangle$. When G acts on a module M, M^G denotes the submodule of M consisting of all elements fixed by G. For an algebraic number field F, $\mathcal{O}_F$ denotes the ring of integers of F. For $a \in \mathcal{O}_F$, the ideal $a\mathcal{O}_F$ generated by a is denoted by (a). We denote by E_F the group of units of F, i.e., $E_F = \mathcal{O}_F^\times$. When F is totally real and $\alpha \in F$, $\alpha \gg 0$ means that α is totally positive. We denote by $\mathfrak{H}$ the complex upper half plane. The set of all positive real numbers is denoted by $\mathbf{R}_+$.

§1. Preparations on cohomology groups

In this section, we will review group cohomology. Most of the results, except for the results presented in subsection 1.5, can be found in standard text books such as Cartan–Eilenberg [CE], Serre [Se1], Suzuki[Su].

1.1. Let G be a group, M be a left G-module. We set $C^0(G, M) = M$, and for $0 < n \in \mathbf{Z}$, let $C^n(G, M)$ be the abelian group consisting of all mappings of G^n into M. We define the coboundary operator $d_n : C^n(G, M) \longrightarrow C^{n+1}(G, M)$ by the usual formula

$$(1.1)\qquad \begin{aligned}(d_n f)(g_1, \ldots, g_{n+1}) &= g_1 f(g_2, \ldots, g_{n+1}) + (-1)^{n+1} f(g_1, \ldots, g_n) \\ &\quad + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}).\end{aligned}$$

We set

$$Z^n(G, M) = \mathrm{Ker}(d_n), \qquad B^n(G, M) = \mathrm{Im}(d_{n-1}).$$

Here we understand $B^0(G, M) = \{0\}$. An element of $C^n(G, M)$ (resp. $Z^n(G, M)$, resp. $B^n(G, M)$) is called an n-cochain (resp. n-cocycle, resp. n-coboundary). The cohomology group $H^n(G, M)$ is that of the complex $\{C^n(G, M), d_n\}$, i.e., $H^n(G, M) = Z^n(G, M)/B^n(G, M)$.

Let G' be a group and M' be a left G'-module. Let $\varphi : G \longrightarrow G'$ be a group homomorphism and $\psi : M' \longrightarrow M$ be a homomorphism of abelian groups. We assume that φ and ψ are compatible, that is

$$\psi(\varphi(g)m') = g(\psi(m')), \qquad m' \in M', \quad g \in G.$$

For $f \in C^n(G', M')$, define $\omega_n f \in C^n(G, M)$ by the formula

$$(1.2)\qquad (\omega_n f)(g_1, g_2, \ldots, g_n) = \psi(f(\varphi(g_1), \varphi(g_2), \ldots, \varphi(g_n))).$$

Then we can check easily that the following diagram is commutative.

$$\begin{array}{ccc} C^n(G', M') & \xrightarrow{\omega_n} & C^n(G, M) \\ \Big\downarrow{\scriptstyle d_n} & & \Big\downarrow{\scriptstyle d_n} \\ C^{n+1}(G', M') & \xrightarrow{\omega_{n+1}} & C^{n+1}(G, M). \end{array}$$

Therefore ω_n sends $Z^n(G', M')$ (resp. $B^n(G', M')$) into $Z^n(G, M)$ (resp. $B^n(G, M)$) and induces a homomorphism $H^n(G', M') \longrightarrow H^n(G, M)$.

Now let N be a subgroup of G. Let $g \in G$. We define

$$\varphi(n) = gng^{-1}, \quad n \in g^{-1}Ng, \qquad \psi(m) = g^{-1}m, \quad m \in M.$$

Then φ is an isomorphism of $g^{-1}Ng$ onto N; φ and ψ are compatible. Hence we obtain an isomorphism of $H^p(N, M)$ onto $H^p(g^{-1}Ng, M)$, which is induced by sending $f \in Z^p(N, M)$ to $f' \in Z^p(g^{-1}Ng, M)$:

$$f'(n_1, n_2, \ldots, n_p) = g^{-1}f(gn_1g^{-1}, gn_2g^{-1}, \ldots, gn_pg^{-1}). \tag{1.3}$$

1.2. Let H be a subgroup of G of finite index. We are going to consider an explicit form of the transfer map $H^n(H, M) \longrightarrow H^n(G, M)$ (cf. Eckmann [E]). To this end, it is convenient to go back to a more conceptual definition of group cohomology:

$$H^n(G, M) = \mathrm{Ext}^n_G(\mathbf{Z}, M).$$

The right-hand side can be computed as follows. We take a resolution of $\mathbf{Z}$ by projective G-modules P_n. [1]

$$\cdots \longrightarrow P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \xrightarrow{\epsilon} \mathbf{Z} \longrightarrow 0.$$

Then we obtain a complex

$$0 \longrightarrow \mathrm{Hom}_G(P_0, M) \xrightarrow{d_0^*} \mathrm{Hom}_G(P_1, M) \xrightarrow{d_1^*} \mathrm{Hom}_G(P_2, M) \xrightarrow{d_2^*} \cdots$$

which gives the cohomology group $H^n(G, M) = \mathrm{Ker}(d_n^*)/\mathrm{Im}(d_{n-1}^*)$. (If $n = 0$, we understand $\mathrm{Im}(d_{n-1}^*) = 0$.) As is well known, an explicit form of a resolution can be given as follows. Let P_n be the free abelian group on the base G^{n+1}. We give P_n a G-module structure by

$$g(g_0, g_1, \ldots, g_n) = (gg_0, gg_1, \ldots, gg_n), \qquad g \in G$$

[1] The d_n in the diagram below should not be confused with d_n in (1.1).

and define $d_{n-1} : P_n \longrightarrow P_{n-1}$ by

$$d_{n-1}(g_0, g_1, \dots, g_n) = \sum_{i=0}^{n} (-1)^i (g_0, \dots, g_{i-1}, g_{i+1}, \dots, g_n), \qquad n \geq 1.$$

We set $\epsilon(g_0) = 1$. We have

$$\mathrm{Hom}_G(P_n, M) = \left\{\varphi : G^{n+1} \longrightarrow M \mid \varphi(gg_0, \dots, gg_n) = g\varphi(g_0, \dots, g_n)\right\}.$$

An element of $\mathrm{Ker}(d_n^*)$ is called a *homogeneous n-cocycle*. To $\varphi \in \mathrm{Hom}_G(P_n, M)$, we let $f \in C^n(G, M)$ correspond by the formula

(1.4) $$f(g_1, g_2, \dots, g_n) = \varphi(1, g_1, g_1 g_2, \dots, g_1 g_2 \cdots g_n).$$

Then $\varphi \mapsto f$ gives an isomophism of abelian groups. The coboundary operator d_n^* induces d_n on $C^n(G, M)$ which is given by (1.1).

Now let H be a subgroup of G of finite index r and let

$$G = \sqcup_{i=1}^{r} x_i H$$

be a coset decomposition. For G-modules A and B, we define a homomorphism $t : \mathrm{Hom}_H(B, A) \longrightarrow \mathrm{Hom}_G(B, A)$ by

$$(t\varphi)(b) = \sum_{i=1}^{r} x_i \varphi(x_i^{-1} b).$$

We denote P_n for G (resp. H) by P_n^G (resp. P_n^H). Since P_n^G is a free H-module, the complex $\{\mathrm{Hom}_H(P_n^G, M), d_n^*\}$ gives the cohomology group $H^n(H, M)$. Define $P_n^G \longrightarrow P_n^H$ by $(g_0, \dots, g_n) \mapsto (h_0, \dots, h_n)$ where h_i is determined by

(1.5) $$g_i = h_i x_{j(i)}^{-1}, \qquad h_i \in H, \quad 0 \leq i \leq n.$$

This is an H-homomorphism and commutes with d_n; it induces an isomophism between the cohomology groups of complexes $\{\mathrm{Hom}_H(P_n^H, M), d_n^*\}$ and $\{\mathrm{Hom}_H(P_n^G, M), d_n^*\}$. Now the following diagram is commutative.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_G(P_0^G, M) & \xrightarrow{d_0^*} & \mathrm{Hom}_G(P_1^G, M) & \xrightarrow{d_1^*} & \mathrm{Hom}_G(P_2^G, M) & \xrightarrow{d_2^*} & \cdots \\
 & & \uparrow t & & \uparrow t & & \uparrow t & & \\
0 & \longrightarrow & \mathrm{Hom}_H(P_0^G, M) & \xrightarrow{d_0^*} & \mathrm{Hom}_H(P_1^G, M) & \xrightarrow{d_1^*} & \mathrm{Hom}_H(P_2^G, M) & \xrightarrow{d_2^*} & \cdots
\end{array}$$

Hence it gives the transfer map $T : H^n(H, M) \longrightarrow H^n(G, M)$.

Take $\varphi \in \mathrm{Hom}_H(P_n^H, M)$. The corresponding element $\varphi^* \in \mathrm{Hom}_H(P_n^G, M)$ to φ is given by

$$\varphi^*(g_0, g_1, \ldots, g_n) = \varphi(g_0 x_{j(0)}, g_1 x_{j(1)}, \ldots, g_n x_{j(n)}),$$

where $x_{j(i)}$ is determined by (1.5). Then $\tilde{\varphi} = t\varphi^* \in \mathrm{Hom}_G(P_n^G, M)$ is given by

$$\tilde{\varphi}(g_0, g_1, \ldots, g_n) = \sum_{j=1}^{r} x_j \varphi^*(x_j^{-1} g_0, x_j^{-1} g_1, \ldots, x_j^{-1} g_n).$$

Therefore we obtain the following result.

Proposition 1.1. *Let G be a group, H be a subgroup of finite index and M be a left G-module. Let $G = \sqcup_{i=1}^r x_i H$ be a coset decomposition. Let $c \in H^n(H, M)$ and let $\varphi \in \mathrm{Hom}_H(P_n^H, M)$ be a homogeneous n-cocycle representing c. Then a homogeneous n-cocycle $\tilde{\varphi} \in \mathrm{Hom}_G(P_n^G, M)$ which represents $T(c)$ is given by*

$$\tilde{\varphi}(g_0, g_1, \ldots g_n) = \sum_{i=1}^{r} x_i \varphi(x_i^{-1} g_0 x_{j_i(0)}, x_i^{-1} g_1 x_{j_i(1)}, \ldots, x_i^{-1} g_n x_{j_i(n)}).$$

Here $x_{j_i(k)}$ is chosen so that $x_i^{-1} g_k x_{j_i(k)} \in H$.

Then using (1.4), we immediately deduce:

Proposition 1.2. *Let the notation be the same as in Proposition 1.1. Let $f \in Z^n(H, M)$ be an n-cocycle representing $c \in H^n(H, M)$. Then an n-cocycle $\tilde{f} \in Z^n(G, M)$ which represents $T(c) \in H^n(G, M)$ is given by*

$$\tilde{f}(g_1, g_2, \ldots, g_n) = \sum_{i=1}^{r} x_i f(x_i^{-1} g_1 x_{p_i(1)}, x_{p_i(1)}^{-1} g_2 x_{p_i(2)}, \ldots, x_{p_i(n-1)}^{-1} g_n x_{p_i(n)}).$$

Here $x_{p_i(l)}$ is chosen so that

$$x_i^{-1} g_1 x_{p_i(1)} \in H, \qquad x_{p_i(l-1)}^{-1} g_l x_{p_i(l)} \in H, \quad 2 \le l \le n.$$

Let $\mathrm{Res} : H^n(G, M) \longrightarrow H^n(H, M)$ be the restriction homomorphism. Then we have the well-known result:

$$T \circ \mathrm{Res}(c) = [G : H]c, \qquad c \in H^n(G, M). \tag{1.6}$$

1.3. We are going to consider the action of Hecke operators on cohomology groups. Let $\widetilde{G}$ be a group and G be a subgroup. Let M be a $\widetilde{G}$-module.

We assume that G and tGt^{-1} are commensurable for every $t \in \tilde{G}$. For $t \in \widetilde{G}$, we put

$$G_t = G \cap t^{-1}Gt.$$

Let

$$\mathrm{conj} : H^n(G, M) \longrightarrow H^n(t^{-1}Gt, M)$$

be the isomorphism induced by (1.3). Let Res be the restriction map from $H^n(t^{-1}Gt, M)$ to $H^n(G_t, M)$ and let $T : H^n(G_t, M) \longrightarrow H^n(G, M)$ be the transfer map. Then we define

$$[GtG] = T \circ \mathrm{Res} \circ \mathrm{conj}. \tag{1.7}$$

(It is not difficult to check that the right-hand side of (1.7) depends only on the double coset GtG and that (1.7) defines a homomorphism of the Hecke ring $H(G, \widetilde{G})$ into $\mathrm{End}(H^n(G, M))$.) Let us write an explicit form of this operator when $n = 2$, which will be necessary for our later computation. Let

$$G = \sqcup_{i=1}^d G_t \alpha_i$$

be a coset decomposition. Then we have

$$GtG = \sqcup_{i=1}^d Gt\alpha_i.$$

Put $\beta_i = t\alpha_i$. Let $c \in H^2(G, M)$ and let $f \in Z^2(G, M)$ be a 2-cocycle which represents c. By (1.3), $\mathrm{conj}(c)$ is represented by the 2-cocycle $f'(g_1, g_2) = t^{-1}f(tg_1t^{-1}, tg_2t^{-1})$. By Proposition 1.2, $[GtG](c)$ is represented by the 2-cocycle

$$f''(g_1, g_2) = \sum_{i=1}^{d} \alpha_i^{-1} f'(\alpha_i g_1 \alpha_{j(i)}^{-1}, \alpha_{j(i)} g_2 \alpha_{k(j(i))}^{-1}),$$

since $G = \sqcup_{i=1}^d \alpha_i^{-1} G_t$. Here, for $1 \le i \le d$, we choose $j(i)$ and $k(i)$ so that

$$\alpha_i g_1 \alpha_{j(i)}^{-1} \in G_t, \qquad \alpha_i g_2 \alpha_{k(i)}^{-1} \in G_t.$$

Writing the result in terms of β_i, we obtain the following proposition.

Proposition 1.3. *Let $c \in H^2(G, M)$ and let $f \in Z^2(G, M)$ be a 2-cocycle representiong c. Let $GtG = \sqcup_{i=1}^d G\beta_i$ be a coset decomposition. Then a 2-cocycle $h \in Z^2(G, M)$ representing $[GtG](c)$ is given by*

$$h(g_1, g_2) = \sum_{i=1}^{d} \beta_i^{-1} f(\beta_i g_1 \beta_{j(i)}^{-1}, \beta_{j(i)} g_2 \beta_{k(j(i))}^{-1}).$$

Here, for $1 \leq i \leq d$, we choose $j(i)$ and $k(i)$ so that

$$\beta_i g_1 \beta_{j(i)}^{-1} \in G, \qquad \beta_i g_2 \beta_{k(i)}^{-1} \in G.$$

1.4. Let G be a group and M be a left G-module. Let N be a normal subgroup of G. Then we have the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(G/N, H^q(N,M)) \Longrightarrow H^n(G,M). \tag{1.8}$$

In low dimensions, this gives an exact sequence

$$\begin{aligned} 0 \longrightarrow H^1(G/N, M^N) \longrightarrow H^1(G,M) \longrightarrow H^1(N,M)^{G/N} \\ \longrightarrow H^2(G/N, M^N) \longrightarrow H^2(G,M). \end{aligned} \tag{1.9}$$

Now we are going to describe a method to calculate $H^2(G,M)$, which is originally due to MacLane (cf. [K], §50). Taking a free group $\mathcal{F}$, we write $G = \mathcal{F}/R$. Let $\pi : \mathcal{F} \longrightarrow G$ be the canonical homomorphism such that $\mathrm{Ker}(\pi) = R$. We regard M as an $\mathcal{F}$-module by $gm = \pi(g)m$, $g \in \mathcal{F}$, $m \in M$. Since

$$H^i(\mathcal{F}, M) = 0, \qquad i \geq 2, \tag{1.10}$$

(1.9) yields an exact sequence

$$0 \longrightarrow H^1(G,M) \longrightarrow H^1(\mathcal{F},M) \longrightarrow H^1(R,M)^G \longrightarrow H^2(G,M) \longrightarrow 0.$$

Therefore we have

$$H^2(G,M) \cong H^1(R,M)^G/\mathrm{Im}(H^1(\mathcal{F},M)). \tag{1.11}$$

Since R acts on M trivially, we have $B^1(R,M) = 0$ and $H^1(R,M) = \mathrm{Hom}(R,M)$. Therefore we have

$$H^1(R,M)^G = \{\varphi \in \mathrm{Hom}(R,M) \mid \varphi(grg^{-1}) = g\varphi(r), \quad g \in \mathcal{F}, r \in R\}.$$

The isomorphism (1.11) is explicitly given as follows. For $g \in \mathcal{F}$, we put $\pi(g) = \bar{g}$. Take a 2-cocycle $f \in Z^2(G,M)$. The mapping $(g_1, g_2) \longrightarrow f(\bar{g}_1, \bar{g}_2)$ is an M-valued 2-cocycle of $\mathcal{F}$. By (1.10), there exists a 1-cochain $a \in C^1(\mathcal{F}, M)$ such that

$$f(\bar{g}_1, \bar{g}_2) = g_1 a(g_2) + a(g_1) - a(g_1 g_2), \qquad g_1, g_2 \in \mathcal{F}. \tag{1.12}$$

Let $\varphi = a|R$, the restriction of a to R. We may assume that f is normalized, i.e.,

$$f(1,g) = f(g,1) = 0 \qquad \text{for all } g \in G.$$

If r_1, $r_2 \in R$, then, by (1.12), we have

$$a(r_2) + a(r_1) - a(r_1 r_2) = 0.$$

Therefore we get $\varphi \in Z^1(R, M) = \mathrm{Hom}(R, M)$. By (1.12), we have

$$a(gr) = ga(r) + a(g), \qquad g \in \mathcal{F},\ r \in R. \tag{1.13}$$

Again by (1.12), we have

$$\begin{aligned} a(grg^{-1}) &= gra(g^{-1}) + a(gr) - f(\bar{g}, \bar{g}^{-1}) \\ &= ga(g^{-1}) + ga(r) + a(g) - f(\bar{g}, \bar{g}^{-1}) \end{aligned}$$

for $g \in \mathcal{F}$, $r \in R$. Using (1.12) with $g_1 = g$, $g_2 = g^{-1}$ and noting $a(1) = 0$, we obtain

$$\varphi(grg^{-1}) = g\varphi(r), \qquad g \in \mathcal{F}, r \in R. \tag{1.14}$$

This formula shows that φ belongs to $H^1(R, M)^G$. Suppose that a' is another 1-cochain satisfying (1.12). Put $\varphi' = a'|R$, $a' = a + b$. Then $b \in Z^1(\mathcal{F}, M)$. Hence the classes of φ and φ' in $H^1(R, M)^G/\mathrm{Im}(H^1(\mathcal{F}, M))$ are the same. Suppose that we add the coboundary of a 1-cochain c to f. Then (1.12) holds when we replace $a(g)$ by $a(g) + c(\bar{g})$. Then $a|R$ does not change. Thus we have defined a homomorphism

$$\omega : H^2(G, M) \longrightarrow H^1(R, M)^G/\mathrm{Im}(H^1(\mathcal{F}, M)).$$

Next suppose that $\varphi \in H^1(R, M)^G$. Take a coset decomposition $\mathcal{F} = \sqcup_i f_i R$. We assume that if $f_i R = R$, then $f_i = 1$. We extend φ to a mapping from $\mathcal{F}$ to M as follows. Choose $a(f_i) \in M$ in arbitrary way. Then put

$$a(f_i r) = f_i \varphi(r) + a(f_i), \qquad r \in R. \tag{1.15}$$

For $g_1 = f_i r_1$, $g_2 = f_j r_2$, r_1, $r_2 \in R$, a direct calculation shows that

$$g_1 a(g_2) + a(g_1) - a(g_1 g_2) = f_i a(f_j) + a(f_i) - a(f_k) - \varphi(f_i f_j f_k^{-1}).$$

Here $f_i f_j = f_k r_3$, $r_3 \in R$. Note that f_k does not depend on r_1 and r_2. Therefore we can define a 2-cochain $f \in C^2(G, M)$ by (1.12) Then it is immediate to see that $f \in Z^2(G, M)$ and that f is normalized (see Lemma 1.4 below). When we add an element of $\mathrm{Im}(H^1(\mathcal{F}, M))$ to φ, the cohomology class of f does not change. Thus we have defined a homomorphism

$$\eta : H^1(R, M)^G/\mathrm{Im}(H^1(\mathcal{F}, M)) \longrightarrow H^2(G, M).$$

We can check easily that ω and η are inverse mappings to each other. This finishes an explicit description of the isomorphism (1.11).

1.5. Let $f \in Z^2(G,M)$ be a normalized cocycle. Take $a \in C^1(\mathcal{F},M)$ which satisfies (1.12) and put $\varphi = a|R \in H^1(R,M)^G$. For every $g \in G$, we choose $\widetilde{g} \in \mathcal{F}$ such that $\pi(\widetilde{g}) = g$. The formula (1.12) can be written as

$$f(g_1,g_2) = g_1 a(\widetilde{g}_2) + a(\widetilde{g}_1) - a(\widetilde{g}_1\widetilde{g}_2), \qquad g_1, g_2 \in G.$$

By (1.13), we have

$$a(\widetilde{g_1g_2}(\widetilde{g_1g_2})^{-1}\widetilde{g}_1\widetilde{g}_2) = g_1g_2\varphi((\widetilde{g_1g_2})^{-1}\widetilde{g}_1\widetilde{g}_2) + a(\widetilde{g_1g_2}).$$

Then, using (1.14), we have

$$a(\widetilde{g}_1\widetilde{g}_2) = a(\widetilde{g_1g_2}) + \varphi(\widetilde{g}_1\widetilde{g}_2(\widetilde{g_1g_2})^{-1}).$$

Therefore we obtain

$$f(g_1,g_2) = g_1 a(\widetilde{g}_2) + a(\widetilde{g}_1) - a(\widetilde{g_1g_2}) - \varphi(\widetilde{g}_1\widetilde{g}_2(\widetilde{g_1g_2})^{-1}), \qquad g_1, g_2 \in G. \tag{1.16}$$

This formula shows that, adding a coboundary to f, we may assume that

$$f(g_1,g_2) = -\varphi(\widetilde{g}_1\widetilde{g}_2(\widetilde{g_1g_2})^{-1}). \tag{1.17}$$

Conversely we note the following Lemma.

Lemma 1.4. *Let $\varphi \in H^1(R.M)^G$. For $g_1,\ g_2 \in G$, define $f(g_1,g_2)$ by (1.17). Then $f \in Z^2(G,M)$. If $\widetilde{1} = 1$, f is normalized.*

Proof. The cocycle condition is

$$g_1 f(g_2,g_3) - f(g_1g_2,g_3) + f(g_1,g_2g_3) - f(g_1,g_2) = 0.$$

We have

$$\begin{aligned}
&g_1\varphi(\widetilde{g}_2\widetilde{g}_3(\widetilde{g_2g_3})^{-1}) - \varphi(\widetilde{g_1g_2}\widetilde{g}_3(\widetilde{g_1g_2g_3})^{-1}) + \varphi(\widetilde{g}_1\widetilde{g_2g_3}(\widetilde{g_1g_2g_3})^{-1})\\
&- \varphi(\widetilde{g}_1\widetilde{g}_2(\widetilde{g_1g_2})^{-1})\\
&= \varphi(\widetilde{g}_1\widetilde{g}_2\widetilde{g}_3(\widetilde{g_2g_3})^{-1}\widetilde{g}_1^{-1}) + \varphi(\widetilde{g}_1\widetilde{g_2g_3}(\widetilde{g_1g_2g_3})^{-1}) + \varphi(\widetilde{g_1g_2g_3}\widetilde{g}_3^{-1}(\widetilde{g_1g_2})^{-1})\\
&+ \varphi(\widetilde{g_1g_2}\widetilde{g}_2^{-1}\widetilde{g}_1^{-1})\\
&= \varphi(\widetilde{g}_1\widetilde{g}_2\widetilde{g}_3(\widetilde{g_2g_3})^{-1}\widetilde{g}_1^{-1}) + \varphi(\widetilde{g}_1\widetilde{g_2g_3}\widetilde{g}_3^{-1}(\widetilde{g_1g_2})^{-1}) + \varphi(\widetilde{g_1g_2}\widetilde{g}_2^{-1}\widetilde{g}_1^{-1})\\
&= \varphi(\widetilde{g}_1\widetilde{g}_2(\widetilde{g_1g_2})^{-1}) + \varphi(\widetilde{g_1g_2}\widetilde{g}_2^{-1}\widetilde{g}_1^{-1}) = 0.
\end{aligned}$$

Hence the cocycle condition holds. The latter assertion is obvious. This completes the proof.

We are going to write the action of Hecke operators on the right-hand side of (1.11) explicitly. Let the notation be the same as in subsections 1.3 and 1.4. Let $f \in Z^2(G, M)$ be a normalized 2-cocycle of the cohomology class c. Let h be the 2-cocycle given by Proposition 1.3 which represents the class $[GtG](c)$. Clearly h is normalized. There exists a 1-cochain $b \in C^1(\mathcal{F}, M)$ such that

$$h(\bar{g}_1, \bar{g}_2) = g_1 b(g_2) + b(g_1) - b(g_1 g_2), \qquad g_1, g_2 \in \mathcal{F}.$$

Proposition 1.5. *Let $\varphi \in H^1(R, M)^G$ and let a normalized 2-cocycle $f \in Z^2(G, M)$ be given by (1.17). Suppose $g_j \in G$ are given for $1 \leq j \leq m$. For every j, we define a permutation on d letters $p_j \in S_d$ by*

$$\beta_i g_j \beta_{p_j(i)}^{-1} \in G, \qquad 1 \leq i \leq d.$$

We define $q_j \in S_d$ inductively by

$$q_1 = p_1, \qquad q_k = p_k q_{k-1}, \quad 2 \leq k \leq m.$$

We assume that $b(\widetilde{g}_j) = 0$ for $1 \leq j \leq m$. Then we have

$$\begin{aligned} &b(\widetilde{g}_1 \widetilde{g}_2 \cdots \widetilde{g}_m) \\ &= \sum_{i=1}^{d} \beta_i^{-1} \varphi(\widetilde{\beta_i g_1 \beta_{q_1(i)}^{-1}}\, \widetilde{\beta_{q_1(i)} g_2 \beta_{q_2(i)}^{-1}} \cdots \widetilde{\beta_{q_{m-1}(i)} g_m \beta_{q_m(i)}^{-1}} (\widetilde{\beta_i g_1 g_2 \cdots g_m \beta_{q_m(i)}^{-1}})^{-1}). \end{aligned} \tag{1.18}$$

Proof. If $m = 1$, the left-hand side of (1.18) is 0 and the right-hand side is 0 since $\varphi(1) = 0$. We assume that $m \geq 2$ and the formula is valid for $m - 1$. Then, by Proposition 1.3 and (1.17), we have

$$\begin{aligned} &b(\widetilde{g}_1 \widetilde{g}_2 \cdots \widetilde{g}_{m-1} \widetilde{g}_m) \\ &= g_1 g_2 \cdots g_{m-1} b(\widetilde{g}_m) + b(\widetilde{g}_1 \widetilde{g}_2 \cdots \widetilde{g}_{m-1}) - h(g_1 \cdots g_{m-1}, g_m) \\ &= \sum_{i=1}^{d} \beta_i^{-1} \varphi(\widetilde{\beta_i g_1 \beta_{q_1(i)}^{-1}} \cdots \widetilde{\beta_{q_{m-2}(i)} g_{m-1} \beta_{q_{m-1}(i)}^{-1}} (\widetilde{\beta_i g_1 g_2 \cdots g_{m-1} \beta_{q_{m-1}(i)}^{-1}})^{-1}) \\ &+ \sum_{i=1}^{d} \beta_i^{-1} \varphi(\widetilde{\beta_i g_1 g_2 \cdots g_{m-1} \beta_{q_{m-1}(i)}^{-1}}\, \widetilde{\beta_{q_{m-1}(i)} g_m \beta_{q_m(i)}^{-1}} (\widetilde{\beta_i g_1 g_2 \cdots g_m \beta_{q_m(i)}^{-1}})^{-1}) \\ &= \sum_{i=1}^{d} \beta_i^{-1} \varphi(\widetilde{\beta_i g_1 \beta_{q_1(i)}^{-1}} \cdots \widetilde{\beta_{q_{m-1}(i)} g_m \beta_{q_m(i)}^{-1}} (\widetilde{\beta_i g_1 g_2 \cdots g_m \beta_{q_m(i)}^{-1}})^{-1}) \end{aligned}$$

since $b(\widetilde{g}_m) = 0$. This completes the proof.

We have

$$b(g_1g_2) = g_1b(g_2) + b(g_1) - h(\bar{g}_1, \bar{g}_2), \qquad g_1, g_2 \in \mathcal{F}.$$

We may take $b(g) = 0$ for a fixed set of generators of $\mathcal{F}$ and we can apply the above formula to determine the value of $b(g)$ according to the length of $g \in \mathcal{F}$. But Proposition 1.5 is useful beyond this case as will be seen after section 5.

§2. Hilbert modular forms

2.1. In this subsection, we follow the exposition given in Shimura [Sh4]. Let F be a totally real algebraic number field of degree n. Let $\mathfrak{d}_F$ denote the different of F over $\mathbf{Q}$ and let $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ be the set of all isomorphisms of F into $\mathbf{R}$. For $\xi \in F$, we put $\xi^{(\nu)} = \xi^{\sigma_\nu}$. For $z = (z_1, z_2, \ldots, z_n) \in \mathfrak{H}^n$, we put

$$\mathbf{e}_F(\xi z) = \exp(2\pi i \sum_{\nu=1}^{n} \xi^{(\nu)} z_\nu).$$

Let $k = (k_1, k_2, \ldots, k_n) \in \mathbf{Z}^n$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbf{R})_+$ and $z \in \mathfrak{H}$, we put $gz = (az+b)/(cz+d)$, $j(g,z) = cz+d$, where $\mathrm{GL}(2,\mathbf{R})_+ = \{g \in \mathrm{GL}(2,\mathbf{R}) \mid \det g > 0\}$; $\mathrm{GL}(2,\mathbf{R})_+^n$ acts on $\mathfrak{H}^n$. For a function Ω on $\mathfrak{H}^n$, $g = (g_1, \ldots, g_n) \in \mathrm{GL}(2,\mathbf{R})_+^n$ and $z = (z_1, \ldots, z_n) \in \mathfrak{H}^n$, we define a function $\Omega|_k\, g$ on $\mathfrak{H}^n$ by the formula

$$(\Omega|_k\, g)(z) = \prod_{\nu=1}^{n} \det(g_\nu)^{k_\nu/2} j(g_\nu, z_\nu)^{-k_\nu} \Omega(gz).$$

We embed $\mathrm{GL}(2,F)$ into $\mathrm{GL}(2,\mathbf{R})^n$ by

$$\mathrm{GL}(2,F) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\begin{pmatrix} a^{(1)} & b^{(1)} \\ c^{(1)} & d^{(1)} \end{pmatrix}, \ldots, \begin{pmatrix} a^{(n)} & b^{(n)} \\ c^{(n)} & d^{(n)} \end{pmatrix} \right) \in \mathrm{GL}(2,\mathbf{R})^n.$$

Let Γ be a congruence subgroup of $SL(2, \mathcal{O}_F)$. A holomorphic function Ω on $\mathfrak{H}^n$ is called a Hilbert modular form of weight k with respect to Γ if

$$\Omega|_k\, \gamma = \Omega$$

holds for every $\gamma \in \Gamma$, and usual conditions at cusps when $F = \mathbf{Q}$. For every $g \in \mathrm{SL}(2,F)$, $\Omega|_k\, g$ has a Fourier expansion of the form $(\Omega|_k\, g)(z) = \sum_{\xi \in L} a_g(\xi) \mathbf{e}_F(\xi z)$, where L is a lattice in F. We have $a_g(\xi) = 0$ if $\xi \neq 0$ is not totally positive. We call Ω *a cusp form* if the constant term $a_g(0)$ vanishes for every $g \in \mathrm{SL}(2, F)$. We denote the space of Hilbert modular forms (resp. cusp forms) of weight k with respect to Γ by $M_k(\Gamma) = M_{k_1,k_2,\dots,k_n}(\Gamma)$ (resp. $S_k(\Gamma) = S_{k_1,k_2,\dots,k_n}(\Gamma)$).

Hereafter until the end of this subsection, we assume that $\Gamma = \mathrm{SL}(2, \mathcal{O}_F)$ and $0 \neq \Omega \in S_k(\Gamma)$. The Fourier expansion of Ω takes the form

$$\Omega(z) = \sum_{0 \ll \xi \in \mathfrak{d}_F^{-1}} a(\xi) \mathbf{e}_F(\xi z). \tag{2.1}$$

Since $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \in \Gamma$ for $u \in E_F$, we have

$$u^k \sum_{0 \ll \xi \in \mathfrak{d}_F^{-1}} a(\xi) \mathbf{e}_F(\xi u^2 z) = \sum_{0 \ll \xi \in \mathfrak{d}_F^{-1}} a(\xi) \mathbf{e}_F(\xi z),$$

where we put $u^k = \prod_{\nu=1}^n (u^{(\nu)})^{k_\nu}$. Therefore we have

$$a(u^2 \xi) = u^k a(\xi), \qquad u \in E_F. \tag{2.2}$$

In particular, taking $u = -1$, we have

$$\sum_{\nu=1}^n k_\nu \equiv 0 \mod 2. \tag{2.3}$$

For the sake of simplicity, we assume that

$$u^k > 0 \qquad \text{for every } u \in E_F. \tag{A}$$

Put

$$k_0 = \max(k_1, k_2, \dots, k_n), \qquad k'_\nu = k_0 - k_\nu, \qquad k' = (k'_1, k'_2, \dots, k'_n).$$

We define the L-function of Ω by

$$L(s, \Omega) = \sum_{\xi E_F^2} a(\xi) \xi^{k'/2} N(\xi)^{-s}, \qquad \xi^{k'/2} = \prod_{\nu=1}^n (\xi^{(\nu)})^{k'_\nu/2}. \tag{2.4}$$

Here the summation extends over all cosets ξE_F^2 with ξ satisfying $0 \ll \xi \in \mathfrak{d}_F^{-1}$. By (2.2) and (A), we see that the sum is well defined. The series (2.4) converges when $\Re(s)$ is sufficiently large. We put

$$R(s, \Omega) = (2\pi)^{-ns} \prod_{\nu=1}^n \Gamma(s - \frac{k'_\nu}{2}) L(s, \Omega). \tag{2.5}$$

By the standard calculation, we obtain the integral representation

$$\int_{\mathbf{R}_+^n/E_F^2} \Omega(iy_1, iy_2, \ldots, iy_n) \prod_{\nu=1}^{n} y_\nu^{s-k'_\nu/2-1} dy_\nu = (2\pi)^{\sum_{\nu=1}^n k'_\nu/2} R(s,\Omega) \tag{2.6}$$

when $\Re(s)$ is sufficiently large. By a suitable transformation of this integral, we can show that $R(s,\Omega)$ is an entire function of s and satisfies the functional equation

$$R(s,\Omega) = (-1)^{\sum_{\nu=1}^n k_\nu/2} R(k_0 - s, \Omega). \tag{2.7}$$

2.2. In [Y3], Chapter V, §5, we gave an explicit method to attach a cohomology class to a Hilbert modular form. We will review it in this subsection. For $0 \le l \in \mathbf{Z}$ and $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathbf{C}^2$, put

$$\begin{bmatrix} u \\ v \end{bmatrix}^l = {}^t(u^l \ u^{l-1}v \ldots uv^{l-1} \ v^l).$$

Define a representation $\rho_l : GL(2,\mathbf{C}) \longrightarrow GL(l+1,\mathbf{C})$ by

$$\rho_l(g) \begin{bmatrix} u \\ v \end{bmatrix}^l = (g \begin{bmatrix} u \\ v \end{bmatrix})^l.$$

Let Γ be a congruence subgroup of $SL(2,\mathcal{O}_F)$. Let $l_1, l_2, \ldots, l_n$ be nonnegative integers. Let V be the representation space of $\rho_{l_1} \otimes \rho_{l_2} \otimes \cdots \otimes \rho_{l_n}$. Let $\Omega \in M_{l_1+2, l_2+2, \ldots, l_n+2}(\Gamma)$ be a Hilbert modular form of weight $(l_1+2, l_2+2, \ldots, l_n+2)$. Define a holomorphic V-valued n-form $\mathfrak{d}(\Omega)$ on $\mathfrak{H}^n$ by

$$\mathfrak{d}(\Omega) = \Omega(z) \begin{bmatrix} z_1 \\ 1 \end{bmatrix}^{l_1} \otimes \begin{bmatrix} z_2 \\ 1 \end{bmatrix}^{l_2} \otimes \cdots \otimes \begin{bmatrix} z_n \\ 1 \end{bmatrix}^{l_n} dz_1 dz_2 \cdots dz_n. \tag{2.8}$$

We put $\rho = \rho_{l_1} \otimes \rho_{l_2} \otimes \cdots \otimes \rho_{l_n}$.

Let $g = (g_1, \ldots, g_n) \in \mathrm{GL}(2,\mathbf{R})_+^n$. Under the action of g on $\mathfrak{H}^n$, $\mathfrak{d}(\Omega)$ transforms to $\mathfrak{d}(\Omega) \circ g$, where

$$\mathfrak{d}(\Omega) \circ g = \Omega(g(z)) \begin{bmatrix} g_1 z_1 \\ 1 \end{bmatrix}^{l_1} \otimes \cdots \otimes \begin{bmatrix} g_n z_n \\ 1 \end{bmatrix}^{l_n} (dz_1 \circ g_1) \cdots (dz_n \circ g_n).$$

Since

$$\begin{bmatrix} g_\nu z_\nu \\ 1 \end{bmatrix}^{l_\nu} = j(g_\nu, z_\nu)^{-l_\nu} \rho_{l_\nu}(g_\nu) \begin{bmatrix} z_\nu \\ 1 \end{bmatrix}^{l_\nu},$$

$$dz_\nu \circ g_\nu = (\det g_\nu) j(g_\nu, z_\nu)^{-2} dz_\nu,$$

we obtain

$$(2.9_a)\quad \mathfrak{d}(\Omega) \circ g = \prod_{\nu=1}^{n} (\det g_\nu)^{-l_\nu/2} \rho(g) \mathfrak{d}(\Omega|_k\, g), \quad g \in \mathrm{GL}(2, \mathbf{R})_+^n \cap \mathrm{GL}(2, F).$$

In particular, we have

$$(2.9_b)\qquad \mathfrak{d}(\Omega) \circ \gamma = \rho(\gamma) \mathfrak{d}(\Omega), \qquad \gamma \in \Gamma.$$

We are going to discuss the case $n = 2$ in detail. Take $w = (w_1, w_2) \in \mathfrak{H}^2$. For $z = (z_1, z_2) \in \mathfrak{H}^2$, we put

$$(2.10)\qquad F(z) = \int_{w_1}^{z_1} \int_{w_2}^{z_2} \mathfrak{d}(\Omega),$$

a period integral of Eichler–Shimura type. Let $\mathcal{H}$ denote the vector space of all V-valued holomorphic functions on $\mathfrak{H}^2$. For $\varphi \in \mathcal{H}$ and $\gamma \in \Gamma$, we define a function $\gamma\varphi$ on $\mathfrak{H}^2$ by

$$(2.11)\qquad (\gamma\varphi)(z) = \rho(\gamma)\varphi(\gamma^{-1}z).$$

Then $\mathcal{H}$ becomes a left Γ-module. Since

$$\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}(\gamma F - F) = 0,$$

we can write

$$\gamma F - F = g(\gamma; z_1) + h(\gamma; z_2),$$

where $g(\gamma; z_1) \in \mathcal{H}$ (resp. $h(\gamma; z_2) \in \mathcal{H}$) is a function which depends only on z_1 (resp. z_2) (cf. [Y3], p. 208, Lemma 5.1). We regard g and h as 1-cochains in $C^1(\Gamma, \mathcal{H})$. Then clearly we have (d_1 in §1.1 is abbreviated to d)

$$dg(\gamma_1, \gamma_2; z_1) + dh(\gamma_1, \gamma_2; z_2) = 0.$$

Put

$$f(\Omega)(\gamma_1, \gamma_2) = dg(\gamma_1, \gamma_2; z_1).$$

We abbreviate $f(\Omega)$ to f. We see that $f(\gamma_1, \gamma_2) \in V$ is a constant. Furthermore, in $\mathcal{H}$, f is a coboundary. Hence f satisfies the cocycle condition

$$(2.12)\qquad \gamma_1 f(\gamma_2, \gamma_3) - f(\gamma_1\gamma_2, \gamma_3) + f(\gamma_1, \gamma_2\gamma_3) - f(\gamma_1, \gamma_2) = 0.$$

The 2-cocycle f determines a cohomology class in $H^2(\Gamma, V)$.

Let us give an explicit formula for f. For $x \in F$, let x' denote the conjugate of x over $\mathbf{Q}$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, let $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. We regard γ, $\gamma' \in SL(2, \mathbf{R})$. Then, for $\gamma \in \Gamma$, we have

$$
\begin{aligned}
&F(\gamma(z)) = F(\gamma z_1, \gamma' z_2) = \int_{w_1}^{\gamma z_1} \int_{w_2}^{\gamma' z_2} \mathfrak{d}(\Omega) \\
=&\int_{\gamma w_1}^{\gamma z_1} \int_{\gamma' w_2}^{\gamma' z_2} \mathfrak{d}(\Omega) + \int_{\gamma w_1}^{\gamma z_1} \int_{w_2}^{\gamma' w_2} \mathfrak{d}(\Omega) + \int_{w_1}^{\gamma w_1} \int_{w_2}^{\gamma' z_2} \mathfrak{d}(\Omega) \\
=&(\rho_{l_1}(\gamma) \otimes \rho_{l_2}(\gamma'))F(z) + \int_{\gamma w_1}^{\gamma z_1} \int_{w_2}^{\gamma' w_2} \mathfrak{d}(\Omega) + \int_{w_1}^{\gamma w_1} \int_{w_2}^{\gamma' z_2} \mathfrak{d}(\Omega),
\end{aligned}
$$

Substituting z by $\gamma^{-1} z$ in this formula, we get
(2.13)

$$(\rho_{l_1}(\gamma) \otimes \rho_{l_2}(\gamma'))F(\gamma^{-1} z) - F(z) = -\int_{\gamma w_1}^{z_1} \int_{w_2}^{\gamma' w_2} \mathfrak{d}(\Omega) - \int_{w_1}^{\gamma w_1} \int_{w_2}^{z_2} \mathfrak{d}(\Omega).$$

We may take

$$g(\gamma; z_1) = -\int_{\gamma w_1}^{z_1} \int_{w_2}^{\gamma' w_2} \mathfrak{d}(\Omega), \tag{2.14}$$

$$h(\gamma; z_2) = -\int_{w_1}^{\gamma w_1} \int_{w_2}^{z_2} \mathfrak{d}(\Omega). \tag{2.15}$$

For γ_1, $\gamma_2 \in \Gamma$, we have

$$f(\gamma_1, \gamma_2) = (\gamma_1 g)(\gamma_2; z_1) - g(\gamma_1 \gamma_2; z_1) + g(\gamma_1; z_1), \tag{2.16}$$

$$f(\gamma_1, \gamma_2) = -\{(\gamma_1 h)(\gamma_2; z_2) - h(\gamma_1 \gamma_2; z_2) + h(\gamma_1; z_2)\}. \tag{2.17}$$

By (2.14) and (2.16), we have

$$
\begin{aligned}
f(\gamma_1,\gamma_2) =& (\rho_{l_1}(\gamma_1)\otimes\rho_{l_2}(\gamma_1'))g(\gamma_2;\gamma_1^{-1}z_1) - g(\gamma_1\gamma_2;z_1) + g(\gamma_1;z_1)\\
=& -(\rho_{l_1}(\gamma_1)\otimes\rho_{l_2}(\gamma_1'))\int_{\gamma_2 w_1}^{\gamma_1^{-1}z_1}\int_{w_2}^{\gamma_2' w_2}\mathfrak{d}(\Omega)\\
&+\int_{\gamma_1\gamma_2 w_1}^{z_1}\int_{w_2}^{\gamma_1'\gamma_2' w_2}\mathfrak{d}(\Omega) - \int_{\gamma_1 w_1}^{z_1}\int_{w_2}^{\gamma_1' w_2}\mathfrak{d}(\Omega)\\
=& -\int_{\gamma_1\gamma_2 w_1}^{z_1}\int_{\gamma_1' w_2}^{\gamma_1'\gamma_2' w_2}\mathfrak{d}(\Omega) + \int_{\gamma_1\gamma_2 w_1}^{z_1}\int_{w_2}^{\gamma_1'\gamma_2' w_2}\mathfrak{d}(\Omega) - \int_{\gamma_1 w_1}^{z_1}\int_{w_2}^{\gamma_1' w_2}\mathfrak{d}(\Omega)\\
=& \int_{\gamma_1\gamma_2 w_1}^{z_1}\int_{w_2}^{\gamma_1' w_2}\mathfrak{d}(\Omega) - \int_{\gamma_1 w_1}^{z_1}\int_{w_2}^{\gamma_1' w_2}\mathfrak{d}(\Omega)\\
=& \int_{\gamma_1\gamma_2 w_1}^{\gamma_1 w_1}\int_{w_2}^{\gamma_1' w_2}\mathfrak{d}(\Omega)
\end{aligned}
$$

using (2.9_b). Thus we obtain an explicit formula

$$
f(\gamma_1,\gamma_2) = \int_{\gamma_1\gamma_2 w_1}^{\gamma_1 w_1}\int_{w_2}^{\gamma_1' w_2}\mathfrak{d}(\Omega). \tag{2.18}
$$

By (2.9_b), (2.18) can be written as

$$
f(\gamma_1,\gamma_2) = (\rho_{l_1}(\gamma_1)\otimes\rho_{l_2}(\gamma_1'))\int_{\gamma_2 w_1}^{w_1}\int_{\gamma_1'^{-1} w_2}^{w_2}\mathfrak{d}(\Omega). \tag{2.19}
$$

Suppose that w_1 is replaced by w_1^*, w_2 remaining the same. Then $g(\gamma;z_1)$ changes to $g(\gamma,z_1)+a(\gamma)$, where

$$
a(\gamma) = \int_{\gamma w_1}^{\gamma w_1^*}\int_{w_2}^{\gamma' w_2}\mathfrak{d}(\Omega).
$$

Hence $f(\gamma_1,\gamma_2)$ changes to $f(\gamma_1,\gamma_2)+\gamma_1 a(\gamma_2)-a(\gamma_1\gamma_2)+a(\gamma_1)$. Suppose that w_2 is replaced by w_2^*, w_1 remaining the same. Then $h(\gamma;z_2)$ changes to $h(\gamma,z_2)+b(\gamma)$, where

$$
b(\gamma) = \int_{w_1}^{\gamma w_1}\int_{w_2}^{w_2^*}\mathfrak{d}(\Omega).
$$

By (2.17), $f(\gamma_1,\gamma_2)$ changes to $f(\gamma_1,\gamma_2)-\gamma_1 b(\gamma_2)+b(\gamma_1\gamma_2)-b(\gamma_1)$. Therefore the cohomology class of f does not depend on the choice of the "base points" w_1, w_2.

Put $\overline{\Gamma} = \Gamma/(\{\pm 1_2\} \cap \Gamma)$. By (2.18), we see that f can be regarded as a 2-cocycle of $\overline{\Gamma}$ taking values in V. Depending on the context, we consider f as a 2-cocycle on $\overline{\Gamma}$. We see that the cocycle f is normalized, i.e.,

$$f(1,\gamma) = f(\gamma,1) = 0 \qquad \text{for every} \quad \gamma \in \overline{\Gamma}. \tag{2.20}$$

Now assume that Ω is a cusp form. Then the cocycle $f = f(\Omega)$ satisfies the "parabolic condition". Namely let $q \in \Gamma$ be a parabolic element and $w^* = (w_1^*, w_2^*)$ be the fixed point of q'. Since f is a cusp form, we may replace w_2 by w_2^*. [2] Let f^* be the cocycle obtained from (w_1, w_2^*). We have

$$f^*(\gamma_1,\gamma_2) = f(\gamma_1,\gamma_2) - \gamma_1 b(\gamma_2) + b(\gamma_1\gamma_2) - b(\gamma_1)$$

with a 1-cochain b and $f^*(q,\gamma) = 0$. Therefore

$$f(q,\gamma) = qb(\gamma) - b(q\gamma) + b(q), \qquad \gamma \in \Gamma,$$

i.e., $f(q,\gamma)$ is of the form of coboundary whenever q is parabolic. Similar argument applies to $f(\gamma,q)$.

2.3. We are going to investigate closely the relation between the critical values of L-function $L(s,\Omega)$ and the cocycle $f(\Omega)$. Until the end of this subsection, we assume $\Gamma = SL(2,\mathcal{O}_F)$. Let ϵ be the fundamental unit of F. We put

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mu = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}.$$

We regard σ and μ as elements of $\overline{\Gamma}$. Taking $\gamma_1 = \gamma_2 = \gamma_3 = \sigma$ in (2.12), we obtain

$$\sigma f(\sigma,\sigma) = f(\sigma,\sigma) \tag{2.21}$$

in view of (2.20). As the base points, we choose

$$w_1 = i\epsilon^{-1}, \qquad w_2 = i\infty.$$

By (2.18), we get

$$f(\sigma,\mu) = f(\sigma,\sigma) = -\int_{i\epsilon^{-1}}^{i\epsilon}\int_0^{i\infty} \mathfrak{d}(\Omega). \tag{2.22}$$

[2] For every $g \in \mathrm{SL}(2,F)$, we have the Fourier expansion $(\Omega|_k g)(z) = \sum_{0\ll\xi\in L} a_g(\xi)\mathbf{e}_F(\xi z)$ where L is a lattice in F. We have the estimate $|a_g(\xi)| \leq M\xi^{k_1/2}\xi'^{k_2/2}$ with a positive constant M depending on Ω and g (cf. [Sh7], p. 280, Proposition A6.4). Using this estimate, it is not difficult to check the absolute convergence of the integral (2.10) defining $F(z)$ when w_2 is replaced by w_2^*.

Put

$$P = \left\{ \begin{pmatrix} u & v \\ 0 & u^{-1} \end{pmatrix} \middle| \ u \in E_F, \ v \in \mathcal{O}_F \right\} \subset \Gamma.$$

By (2.18), we get

$$f(p, \gamma) = 0 \qquad \text{for every} \quad p \in P, \ \gamma \in \Gamma \tag{2.23}$$

since we have $pw_2 = w_2$ for $p \in P$. Taking $\gamma_1 = p \in P$ in (2.12), we obtain

$$f(p\gamma_1, \gamma_2) = pf(\gamma_1, \gamma_2) \qquad \text{for every} \quad p \in P, \ \gamma_1, \gamma_2 \in \Gamma. \tag{2.24}$$

This is the parabolic condition for $\Gamma = \mathrm{SL}(2, \mathcal{O}_F)$ and will play a crucial role in the succeeding sections.

For $0 \leq s \leq l_1$, $0 \leq t \leq l_2$, we put

$$P_{s,t} = \int_{i\epsilon^{-1}}^{i\epsilon} \int_0^{i\infty} \Omega(z) z_1^s z_2^t dz_1 dz_2. \tag{2.25}$$

The components of $f(\sigma, \sigma)$ are given by $-P_{s,t}$. The condition $\sigma f(\sigma, \sigma) = f(\sigma, \sigma)$ is equivalent to

$$P_{s,t} = (-1)^{l_1+l_2-s-t} P_{l_1-s, l_2-t}. \tag{2.26}$$

Put $k_1 = l_1 + 2$, $k_2 = l_2 + 2$. By (2.3), we have

$$l_1 \equiv l_2 \mod 2. \tag{2.27}$$

We assume that $l_1 \geq l_2$. Then we have

$$k_0 = k_1, \qquad k_1' = 0, \qquad k_2' = k_1 - k_2.$$

Since $E_F^2 = \langle \epsilon^2 \rangle$, a fundamental domain of $\mathbf{R}_+^2 / E_F^2$ is given by $[\epsilon^{-1}, \epsilon] \times \mathbf{R}_+$. By (2.6), we obtain

$$\int_{\epsilon^{-1}}^{\epsilon} \int_0^{\infty} \Omega(iy_1, iy_2) y_1^{s-1} y_2^{s-(k_1-k_2)/2-1} dy_1 dy_2 = (2\pi)^{(k_1-k_2)/2} R(s, \Omega) \tag{2.28}$$

when $\Re(s)$ is sufficiently large. We can verify that the integral converges locally uniformly for $s \in \mathbf{C}$. Take $m \in \mathbf{Z}$ and put $s = m$, $t = m - (k_1 - k_2)/2$. Then $0 \leq s \leq l_1$, $0 \leq t \leq l_2$ hold if and only if

$$\frac{k_1 - k_2}{2} \leq m \leq \frac{k_1 + k_2}{2} - 2. \tag{2.29}$$

For an integer m in this range, we have

$$\begin{aligned}P_{m,m-(k_1-k_2)/2} &= \int_{i\epsilon^{-1}}^{i\epsilon}\int_0^{i\infty}\Omega(z)z_1^m z_2^{m-(k_1-k_2)/2}dz_1dz_2\\ &= i^{2m-(k_1-k_2)/2+2}\int_{\epsilon^{-1}}^{\epsilon}\int_0^{\infty}\Omega(iy_1,iy_2)y_1^m y_2^{m-(k_1-k_2)/2}dy_1dy_2.\end{aligned}$$

Therefore we obtain

$$P_{m,m-(k_1-k_2)/2} = (-1)^{m+1}i^{-(k_1-k_2)/2}(2\pi)^{(k_1-k_2)/2}R(m+1,\Omega) \tag{2.30}$$

by (2.28). By the functional equation (2.7), this is equal to

$$(-1)^{m+1}i^{-(k_1-k_2)/2}(2\pi)^{(k_1-k_2)/2}(-1)^{(k_1+k_2)/2}R(k_1-m-1,\Omega).$$

Since k_1-m-2 satisfies (2.29), we obtain

$$P_{m,m-(k_1-k_2)/2} = (-1)^{(k_1-k_2)/2}P_{k_1-m-2,(k_1+k_2)/2-m-2} \tag{2.31}$$

using (2.30). We see that (2.31) is consistent with (2.26). Note that (2.29) is the condition for $L(m+1,\Omega)$ to be a critical value (cf. [Sh4], (4.14)).

2.4. Let $\Omega \in M_k(\Gamma)$ and $f = f(\Omega) \in Z^2(\Gamma, V)$ be the 2-cocycle attached to Ω defined by (2.18). In this subsection, we will write the action of Hecke operators on the cohomology class of $f(\Omega)$ explicitly. We denote $f(\Omega)$ also by f_Ω.

Let F be a totally real number field of degree n and Γ be a congruence subgroup of $\mathrm{SL}(2,\mathcal{O}_F)$. Let ϖ be a totally positive element of F and let

$$\Gamma\begin{pmatrix}1 & 0\\ 0 & \varpi\end{pmatrix}\Gamma = \sqcup_{i=1}^d \Gamma\beta_i \tag{2.32}$$

be a coset decomposition. Let $\Omega \in M_k(\Gamma)$. We define the Hecke operator $T(\varpi)$ by

$$\Omega \mid T(\varpi) = N(\varpi)^{k_0/2-1}\sum_{i=1}^d \Omega|_k\,\beta_i. \tag{2.33}$$

Clearly $T(\varpi)$ does not depend on the choice of the coset decomposition (2.32). We have $\Omega|T(\varpi) \in M_k(\Gamma)$; it is a cusp form if Ω is. By (2.9_a), we have

$$\mathfrak{d}(\Omega \mid T(\varpi)) = \prod_{\nu=1}^n(\varpi^{(\nu)})^{(k_0+k_\nu)/2-2}\sum_{i=1}^d \rho(\beta_i)^{-1}(\mathfrak{d}(\Omega)\circ\beta_i). \tag{2.34}$$

Put

$$(2.35) \qquad c = \prod_{\nu=1}^{n} (\varpi^{(\nu)})^{(k_0+k_\nu)/2-2}.$$

Until the end of this section, we assume that $n = 2$. We define (cf. (2.10))

$$(2.36) \qquad F_{\Omega|T(\varpi)}(z) = \int_{w_1}^{z_1} \int_{w_2}^{z_2} \mathfrak{d}(\Omega \mid T(\varpi)), \qquad z = (z_1, z_2).$$

By (2.34), we have

$$\begin{aligned} F_{\Omega|T(\varpi)}(z) &= c \sum_{i=1}^{d} \beta_i^{-1} \int_{w_1}^{z_1} \int_{w_2}^{z_2} \mathfrak{d}(\Omega) \circ \beta_i = c \sum_{i=1}^{d} \beta_i^{-1} \int_{\beta_i w_i}^{\beta_i z_1} \int_{\beta_i' w_2}^{\beta_i' z_2} \mathfrak{d}(\Omega) \\ &= c \sum_{i=1}^{d} \beta_i^{-1} \Big[\int_{w_1}^{\beta_i z_1} \int_{w_2}^{\beta_i' z_2} \mathfrak{d}(\Omega) - \int_{w_1}^{\beta_i z_1} \int_{w_2}^{\beta_i' w_2} \mathfrak{d}(\Omega) \\ &\qquad - \int_{w_1}^{\beta_i w_1} \int_{w_2}^{\beta_i' z_2} \mathfrak{d}(\Omega) + \int_{w_1}^{\beta_i w_1} \int_{w_2}^{\beta_i' w_2} \mathfrak{d}(\Omega) \Big]. \end{aligned}$$

Here, for simplicity, we write the action $\rho(\beta_i)^{-1}$ by β_i^{-1}. Therefore we obtain

$$(2.37) \qquad \begin{aligned} F_{\Omega|T(\varpi)}(z) = c \sum_{i=1}^{d} \beta_i^{-1} \Big[& F(\beta_i z) - F(\beta_i(z_1, w_2)) \\ & - F(\beta_i(w_1, z_2)) + F(\beta_i(w_1, w_2)) \Big]. \end{aligned}$$

Take $\gamma \in \Gamma$. We have

$$(\gamma F)(z) - F(z) = g_\Omega(\gamma; z_1) + h_\Omega(\gamma; z_2)$$

with $g_\Omega = g$ (resp. $h_\Omega = h$) defined by (2.14) (resp. (2.15)). Similarly to this formula, we have a decomposition

$$(2.38) \qquad (\gamma F_{\Omega|T(\varpi)})(z) - F_{\Omega|T(\varpi)}(z) = g_{\Omega|T(\varpi)}(\gamma; z_1) + h_{\Omega|T(\varpi)}(\gamma; z_2).$$

Here $g_{\Omega|T(\varpi)}(\gamma; z_1)$ (resp. $h_{\Omega|T(\varpi)}(\gamma; z_2)$) is a V-valued holomorphic function on $\mathfrak{H}^2$ which depends only on z_1 (resp. z_2). If (2.38) holds, then a 2-cocycle $f_{\Omega|T(\varpi)}$ attached to $\Omega \mid T(\varpi)$ is given by (cf. (2.16))
(2.39)

$$f_{\Omega|T(\varpi)}(\gamma_1, \gamma_2) = (\gamma_1 g_{\Omega|T(\varpi)})(\gamma_2; z_1) - g_{\Omega|T(\varpi)}(\gamma_1\gamma_2; z_1) + g_{\Omega|T(\varpi)}(\gamma_1; z_1).$$

Suppose that we have a decomposition

$$(2.40)\quad c\sum_{i=1}^{d}(\gamma\beta_i^{-1}F(\beta_i\gamma^{-1}z)-\beta_i^{-1}F(\beta_i z))=g^*_{\Omega|T(\varpi)}(\gamma;z_1)+h^*_{\Omega|T(\varpi)}(\gamma;z_2).$$

Then by (2.37), we see that $g_{\Omega|T(\varpi)}(\gamma;z_1)$ can be taken in the form

$$g_{\Omega|T(\varpi)}(\gamma;z_1)=g^*_{\Omega|T(\varpi)}(\gamma;z_1)+\gamma q(\gamma^{-1}z_1)-q(z_1)+x(\gamma).$$

Here $q(z_1)$ is a V-valued holomorphic function which depends only on z_1 and does not depend on γ and $x(\gamma)\in V$. Therefore, if we substract the coboundary which comes from $x(\gamma)$ from the cocycle $f_{\Omega|T(\varpi)}$, the resulting cocycle can be calculated using (2.39) with $g^*_{\Omega|T(\varpi)}(\gamma;z_1)$ in place of $g_{\Omega|T(\varpi)}(\gamma;z_1)$. Now we put

$$\beta_i\gamma=\delta_i\beta_{j(i)},\qquad 1\le i\le d,\quad \delta_i\in\Gamma.$$

Set

$$(2.41)\qquad g^*_{\Omega|T(\varpi)}(\gamma;z_1)=c\sum_{i=1}^{d}\beta_i^{-1}g_\Omega(\delta_i;\beta_i z_1),$$

$$(2.42)\qquad h^*_{\Omega|T(\varpi)}(\gamma;z_2)=c\sum_{i=1}^{d}\beta_i^{-1}h_\Omega(\delta_i;\beta'_i z_2).$$

Since $i\mapsto j(i)$ is a permutation on d letters, we have

$$\sum_{i=1}^{d}(\gamma\beta_{j(i)}^{-1}F(\beta_{j(i)}\gamma^{-1}z)-\beta_i^{-1}F(\beta_i z))=\sum_{i=1}^{d}(\beta_i^{-1}\delta_iF(\delta_i^{-1}\beta_i z)-\beta_i^{-1}F(\beta_i z))$$
$$=\sum_{i=1}^{d}\beta_i^{-1}(g_\Omega(\delta_i;\beta_i z_1)+h_\Omega(\delta_i;\beta'_i z_2)).$$

Hence we see that (2.40) holds.

For $\gamma_1,\ \gamma_2\in\Gamma$, we put

$$(2.43)\quad \beta_i\gamma_1=\delta_i^{(1)}\beta_{j(i)},\ \delta_i^{(1)}\in\Gamma,\quad \beta_i\gamma_2=\delta_i^{(2)}\beta_{k(i)},\ \delta_i^{(2)}\in\Gamma,\qquad 1\le i\le d.$$

We have

$$\beta_i\gamma_1\gamma_2=\delta_i^{(1)}\delta_{j(i)}^{(2)}\beta_{k(j(i))}.$$

Now we calculate $f_{\Omega|T(\varpi)}$ using (2.39) with $g^*_{\Omega|T(\varpi)}(\gamma; z_1)$ in place of $g_{\Omega|T(\varpi)}(\gamma; z_1)$. Then we have

$$\begin{aligned}
&f_{\Omega|T(\varpi)}(\gamma_1,\gamma_2) = \gamma_1 g^*_{\Omega|T(\varpi)}(\gamma_2;\gamma_1^{-1}z_1) - g^*_{\Omega|T(\varpi)}(\gamma_1\gamma_2;z_1) + g^*_{\Omega|T(\varpi)}(\gamma_1;z_1)\\
&= c\Big[\sum_{i=1}^{d} \gamma_1\beta_i^{-1} g_\Omega(\delta_i^{(2)};\beta_i\gamma_1^{-1}z_1) - \beta_i^{-1}g_\Omega(\delta_i^{(1)}\delta_{j(i)}^{(2)};\beta_i z_1) + \beta_i^{-1}g_\Omega(\delta_i^{(1)};\beta_i z_1)\Big]\\
&= c\Big[\sum_{i=1}^{d} \beta_i^{-1}\delta_i^{(1)} g_\Omega(\delta_{j(i)}^{(2)};\beta_{j(i)}\gamma_1^{-1}z_1) - \beta_i^{-1}g_\Omega(\delta_i^{(1)}\delta_{j(i)}^{(2)};\beta_i z_1) + \beta_i^{-1}g_\Omega(\delta_i^{(1)};\beta_i z_1)\Big]\\
&= c\sum_{i=1}^{d} \beta_i^{-1}\Big[\delta_i^{(1)} g_\Omega(\delta_{j(i)}^{(2)};(\delta_i^{(1)})^{-1}\beta_i z_1) - g_\Omega(\delta_i^{(1)}\delta_{j(i)}^{(2)};\beta_i z_1) + g_\Omega(\delta_i^{(1)};\beta_i z_1)\Big].
\end{aligned}$$

Therefore we obtain an explicit formula

$$f_{\Omega|T(\varpi)}(\gamma_1,\gamma_2) = c\sum_{i=1}^{d}\beta_i^{-1} f_\Omega(\delta_i^{(1)},\delta_{j(i)}^{(2)}). \tag{2.44}$$

This formula can be written as

$$f_{\Omega|T(\varpi)}(\gamma_1,\gamma_2) = c\sum_{i=1}^{d}\beta_i^{-1} f_\Omega(\beta_i\gamma_1\beta_{j(i)}^{-1},\beta_{j(i)}\gamma_2\beta_{k(j(i))}^{-1}) \tag{2.45}$$

and is consistent with Proposition 1.3.

2.5. Assume that the class number of F in the narrow sense is 1. Suppose that Ω is a Hecke eigenform. Then the L-function $L(s,\Omega)$ defined by (2.4) essentially coincides with the Euler product given in [Sh4] or in Jacquet-Langlands [JL] but there is a subtle diffence; we are going to explain it briefly for the reader's convenience.

We write $\mathfrak{d}_F = (\delta)$ with $\delta \gg 0$. Let $\Omega \in S_{k_1,k_2}(\Gamma)$, $\Gamma = \mathrm{SL}(2,\mathcal{O}_F)$ and let

$$\Omega(z) = \sum_{0\ll\alpha\in\mathcal{O}_F} c(\alpha)\mathbf{e}_F(\frac{\alpha}{\delta}z)$$

be the Fourier expansion. We have $a(\alpha/\delta) = c(\alpha)$ (cf. (2.1)). We set

$$\Delta = \{\alpha \in M(2,\mathcal{O}_F) \mid \det\alpha \gg 0\}.$$

Let $\mathfrak{m}$ be an integral ideal of F and take $m \gg 0$ so that $\mathfrak{m} = (m)$. Then we define

$$T(\mathfrak{m}) = \sum_{\alpha\in\Delta,\, \det\alpha = m} \Gamma\alpha\Gamma,$$

which is an element of the abstract Hecke ring $\mathcal{H}(\Gamma, \Delta)$ (cf. [Sh2], p. 54). Let $T(\mathfrak{m}) = \sqcup_{i=1}^{e} \Gamma\beta_i$ be a coset decomposition. Assume that $k_1 \geq k_2$. We define the action of $T(\mathfrak{m})$ on Ω by

$$\Omega \mid T(\mathfrak{m}) = N(\mathfrak{m})^{k_1/2-1} \sum_{i=1}^{e} \Omega|_k \beta_i.$$

Then $\Omega|T(\mathfrak{m}) \in S_k(\Gamma)$; we can verify easily that it does not depend on the choices of m and β_i. We have

$$T(\mathfrak{m}) = \sqcup_{(d), d \gg 0, ad=m} \ \sqcup_{b \bmod d} \ \Gamma \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

Calculating similarly to [Sh2], p. 79-80, we find that the Fourier expansion of $\Omega|T(\mathfrak{m})$ is given by

$$(\Omega|T(\mathfrak{m}))(z) = \sum_{0 \ll \alpha \in \mathcal{O}_F} c'(\alpha) \mathbf{e}_F(\frac{\alpha}{\delta} z),$$

$$(2.46) \qquad c'(\alpha) = (m^{(2)})^{(k_1-k_2)} \sum_{(a), a \gg 0, a|(m,\alpha)} N(a)^{k_1-1} (a^{(2)})^{k_2-k_1} c(\frac{m\alpha}{a^2}).$$

Now assume that Ω is a nonzero common eigenfunction for all Hecke operators $T(\mathfrak{m})$. We put

$$\Omega \mid T(\mathfrak{m}) = \lambda(\mathfrak{m})\Omega.$$

By (2.46), we find $c(1) \neq 0$. We assume that Ω is normalized so that $c(1) = 1$. Then we have

$$\lambda(\mathfrak{m}) = c(m)(m^{(2)})^{(k_1-k_2)/2}.$$

Using this formula and (2.46), we obtain, after routine computations, that

$$(2.47) \quad L(s, \Omega) = (\delta^{(2)})^{(k_1-k_2)/2} D_F^s \prod_{\mathfrak{p}} (1 - \lambda(\mathfrak{p}) N(\mathfrak{p})^{-s} + N(\mathfrak{p})^{k_1-1-2s})^{-1}.$$

Here $\mathfrak{p}$ extends over all prime ideals of F and $D_F = N(\delta)$ is the discriminant of F.

When $0 \ll \varpi \in \mathcal{O}_F$ generates a prime ideal $\mathfrak{p}$, we denote $T(\varpi)$ defined by (2.33) also by $T(\mathfrak{p})$.

§3. Cohomology of P

In this section, we will study cohomology groups of P. Main results are Theorems 3.7 and 3.9 which give the vanishing of $H^1(P, V)$ and $H^2(P, V)$

when $l_1 \neq l_2$. Hereafter in this paper, we assume that $[F : \mathbf{Q}] = 2$. We also assume $l_1 \equiv l_2 \mod 2$ and $l_2 \leq l_1$.

3.1. Put $\Gamma = \mathrm{PSL}(2, \mathcal{O}_F)$. In this section, we define subgroups P and U of Γ by

$$P = \left\{ \begin{pmatrix} t & 0 \\ u & t^{-1} \end{pmatrix} \middle| \, t \in E_F, \ u \in \mathcal{O}_F \right\} / \{\pm 1_2\},$$
$$U = \left\{ \begin{pmatrix} \pm 1 & 0 \\ u & \pm 1 \end{pmatrix} \middle| \, u \in \mathcal{O}_F \right\} / \{\pm 1_2\}.$$

We write $\mathcal{O}_F = \mathbf{Z} + \mathbf{Z}\omega$. Let ϵ be the fundamental unit of F and let

$$\epsilon^2 = A + B\omega, \qquad \epsilon^2\omega = C + D\omega.$$

Then we see that ϵ^2 is an eigenvalue of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and that $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z})$. We put

$$u_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad u_2 = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \in U, \qquad t = \begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & \epsilon \end{pmatrix}.$$

We have

$$tu_1t^{-1} = u_1^A u_2^B, \qquad tu_2t^{-1} = u_1^C u_2^D. \tag{3.1}$$

We put

$$\mathcal{Z} = \{(U_1, U_2) \in V \times V \mid (u_1 - 1)U_2 = (u_2 - 1)U_1\}. \tag{3.2}$$

It is easy to see that by the mapping

$$Z^1(U, V) \ni f \longrightarrow (f(u_1), f(u_2)) \in \mathcal{Z},$$

we have an isomorphism $Z^1(U, V) \cong \mathcal{Z}$. Put

$$\mathcal{B} = \{((u_1 - 1)\mathbf{b}, (u_2 - 1)\mathbf{b}) \mid \mathbf{b} \in V\}. \tag{3.3}$$

Then we have $B^1(U, V) \cong \mathcal{B} \subset \mathcal{Z}$.

We have $V = V_1 \otimes V_2$, $V_1 = \mathbf{C}^{l_1+1}$, $V_2 = \mathbf{C}^{l_2+1}$. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{l_1+1}\}$ (resp. $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_{l_2+1}\}$) be the standard basis of V_1 (resp. V_2).

Lemma 3.1. *We have* $\dim V^U = 1$ *and* V^U *is spanned by* $\mathbf{e}_{l_1+1} \otimes \mathbf{e}'_{l_2+1}$.

Proof. Put

$$\widetilde{U} = \left\{ (\begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ c_2 & 1 \end{pmatrix}) \,\middle|\, c_1, c_2 \in \mathbf{C} \right\} \subset \mathrm{SL}(2, \mathbf{C})^2,$$

$$\widetilde{U}_1 = \left\{ (\begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix}, 1_2) \,\middle|\, c_1 \in \mathbf{C} \right\}, \qquad \widetilde{U}_2 = \left\{ (1_2, \begin{pmatrix} 1 & 0 \\ c_2 & 1 \end{pmatrix}) \,\middle|\, c_2 \in \mathbf{C} \right\}.$$

Since U is Zariski dense in $\widetilde{U}$, we have $V^U = V^{\widetilde{U}}$. We also see easily that $V_1^{\widetilde{U}_1} = \mathbf{C}\mathbf{e}_{l_1+1}$, $V_2^{\widetilde{U}_2} = \mathbf{C}\mathbf{e}'_{l_2+1}$, $V^{\widetilde{U}} = V_1^{\widetilde{U}_1} \otimes V_2^{\widetilde{U}_2}$. This completes the proof.

Lemma 3.2. *Let* $g = (\begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ c_2 & 1 \end{pmatrix}) \in \mathrm{SL}(2, \mathbf{C})^2$. *We assume that* $c_1 \neq 0$, $c_2 \neq 0$. *Then the dimension of the subspace of* V *consisting of all vectors fixed by* g *is* $l_2 + 1$. *(Note that we have assumed* $l_1 \geq l_2$.*)*

Proof. By the definition of the symmetric tensor representation ρ_l of degree l, we have

$$\rho_l(\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix})\mathbf{e}_i = \sum_{k=i}^{l+1} \binom{k-1}{i-1} c^{k-i} \mathbf{e}_k. \tag{3.4}$$

Hence for $\rho = \rho_{l_1} \otimes \rho_{l_2}$, we have

$$\begin{aligned} &\rho(\begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ c_2 & 1 \end{pmatrix})(\mathbf{e}_i \otimes \mathbf{e}'_j) \\ =&\sum_{k=i}^{l_1+1} \sum_{l=j}^{l_2+1} \binom{k-1}{i-1} \binom{l-1}{j-1} c_1^{k-i} c_2^{l-j} (\mathbf{e}_k \otimes \mathbf{e}'_l). \end{aligned} \tag{3.5}$$

Put

$$N = \rho(\begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ c_2 & 1 \end{pmatrix}), \qquad n_{kl,ij} = \binom{k-1}{i-1} \binom{l-1}{j-1} c_1^{k-i} c_2^{l-j}.$$

Then (3.5) can be written as

$$N(\mathbf{e}_i \otimes \mathbf{e}'_j) = \sum_{k=i}^{l_1+1} \sum_{l=j}^{l_2+1} n_{kl,ij} (\mathbf{e}_k \otimes \mathbf{e}'_l).$$

The vector $\sum_{i=1}^{l_1+1} \sum_{j=1}^{l_2+1} x_{ij} (\mathbf{e}_i \otimes \mathbf{e}'_j)$ is annihilated by $N - \rho(1)$ if and only if

$$\sum_{i=1}^{k} \sum_{j=1, (i,j) \neq (k,l)}^{l} n_{kl,ij} x_{ij} = 0 \tag{3.6}$$

holds for every $1 \leq k \leq l_1 + 1$, $1 \leq l \leq l_2 + 1$. Note that $n_{kl,ij} \neq 0$ if $k \geq l$ and $l \geq j$ since $c_1 \neq 0$, $c_2 \neq 0$; $n_{kl,ij} = 0$ if $k < l$ or $l < j$.

First let $l = 1$ in (3.6). Then we have

$$\sum_{i=1}^{k-1} n_{k1,i1} x_{i1} = 0, \qquad 2 \leq k \leq l_1 + 1.$$

We successively obtain

$$x_{11} = x_{21} = \cdots = x_{l_1 1} = 0.$$

We are going to show that

$$x_{1l} = x_{2l} = \cdots = x_{l_1+1-l,l} = 0, \qquad 1 \leq l \leq l_2 + 1 \tag{3.7}$$

by induction on l. Assuming the hypothesis of induction, we obtain

$$\sum_{i=1}^{k-1} n_{kl,il} x_{il} = 0$$

when $k \leq l_1 + 2 - l$ from (3.6). Hence (3.7) follows. Now assume $k \geq l_1 + 3 - l$. We write (3.6) as

$$\sum_{j=1}^{l-1} \sum_{i=1}^{k} n_{kl,ij} x_{ij} + \sum_{i=1}^{k-1} n_{kl,il} x_{il} = 0.$$

Using (3.7), we see that this equation is equivalent to

$$\sum_{j=1}^{l-1} \sum_{i=l_1+2-j}^{k} n_{kl,ij} x_{ij} + \sum_{i=l_1+2-l}^{k-1} n_{kl,il} x_{il} = 0. \tag{3.8}$$

In (3.8), x_{il}, $1 \leq i \leq l_1$ are determined by $x_{\alpha\beta}$, $\beta \leq l - 1$ and x_{l_1+1l} is left undermined. Therefore when x_{l_1+1j}, $1 \leq j \leq l_2 + 1$ are given, the solution x_{ij} satisfying (3.8) exists and is unique. This completes the proof.

Lemma 3.3. *Let* $u = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$, $0 \neq c \in F$. *Then we have*

$$\mathbf{e}_i \otimes \mathbf{e}'_j \in \mathrm{Im}(u - 1), \qquad \text{for } 1 \leq j \leq l_2 + 1 \text{ if } i \geq l_2 + 3 - j.$$

Here $\mathrm{Im}(u - 1)$ *denotes the image of the linear mapping* $V \ni \mathbf{v} \mapsto (\rho(u) - \rho(1))\mathbf{v} \in V$.

Proof. We have

$$(u-1)(\mathbf{e}_i \otimes \mathbf{e}'_{l_2+1}) = (ic\mathbf{e}_{i+1} + \binom{i+1}{i-1} c^2 \mathbf{e}_{i+2} + \cdots) \otimes \mathbf{e}'_{l_2+1}.$$

By descending induction on i, we see that $\mathbf{e}_i \otimes \mathbf{e}'_{l_2+1} \in \mathrm{Im}(u-1)$, $i \geq 2$. This proves our assertion when $j = l_2 + 1$. When $j = l_2$, we have

$$(u-1)(\mathbf{e}_i \otimes \mathbf{e}'_{l_2}) = (ic\mathbf{e}_{i+1} + \cdots) \otimes \mathbf{e}'_{l_2} + (\mathbf{e}_i + ic\mathbf{e}_{i+1} + \cdots) \otimes l_2 c' \mathbf{e}'_{l_2+1}.$$

The second term belongs to $\mathrm{Im}(u-1)$ if $i \geq 2$, and by descending induction on i, we can show that $\mathbf{e}_i \otimes \mathbf{e}'_{l_2} \in \mathrm{Im}(u-1)$, $i \geq 3$. Proceeding similarly by induction on j, we see that the assertion holds.

Lemma 3.4. *We have*

$$\mathrm{Im}(u_1 - 1) + \mathrm{Im}(u_2 - 1) = (\oplus_{j=2}^{l_2+1} \mathbf{C}(\mathbf{e}_1 \otimes \mathbf{e}'_j)) \oplus (\oplus_{i=2}^{l_1+1} \oplus_{j=1}^{l_2+1} \mathbf{C}(\mathbf{e}_i \otimes \mathbf{e}'_j)).$$

In particular, $\dim(\mathrm{Im}(u_1 - 1) + \mathrm{Im}(u_2 - 1)) = \dim V - 1$.

Proof. Put $W = \mathrm{Im}(u_1 - 1) + \mathrm{Im}(u_2 - 1)$. Since it is clear that $\mathbf{e}_1 \otimes \mathbf{e}'_1 \notin \mathrm{Im}(u_1 - 1), \notin \mathrm{Im}(u_2 - 1)$, it suffices to show that $\mathbf{e}_i \otimes \mathbf{e}'_j \in W$ when $(i, j) \neq (1, 1)$. We have

$$(u_2 - 1)(\mathbf{e}_1 \otimes \mathbf{e}'_{l_2}) = (\omega\mathbf{e}_2 + \cdots) \otimes \mathbf{e}'_{l_2} + (\mathbf{e}_1 + \omega\mathbf{e}_2 + \cdots) \otimes l_2 \omega' \mathbf{e}'_{l_2+1}.$$

By Lemma 3.3, we have

$$\omega(\mathbf{e}_2 \otimes \mathbf{e}'_{l_2}) + l_2\omega'(\mathbf{e}_1 \otimes \mathbf{e}'_{l_2+1}) \in \mathrm{Im}(u_2 - 1).$$

Similarly we have

$$(\mathbf{e}_2 \otimes \mathbf{e}'_{l_2}) + l_2(\mathbf{e}_1 \otimes \mathbf{e}'_{l_2+1}) \in \mathrm{Im}(u_1 - 1).$$

Since $\det \begin{pmatrix} 1 & l_2 \\ \omega & l_2\omega' \end{pmatrix} \neq 0$, we obtain

$$\mathbf{e}_1 \otimes \mathbf{e}'_{l_2+1} \in W, \qquad \mathbf{e}_2 \otimes \mathbf{e}'_{l_2} \in W.$$

We are going to show that

$$(*) \quad \mathbf{e}_i \otimes \mathbf{e}'_j \in W,\ 1 \leq i \leq l_1 + 1, j \geq 2, \qquad \mathbf{e}_i \otimes \mathbf{e}'_{j-1} \in W,\ 2 \leq i \leq l_1 + 1, j \geq 2$$

by descending induction on j. By Lemma 3.3 and by what we have shown, $(*)$ holds when $j = l_2 + 1$. We assume that $(*)$ holds for $j + 1$. We have

$$(u_2 - 1)(\mathbf{e}_i \otimes \mathbf{e}'_{j-1}) = (i\omega\mathbf{e}_{i+1} + \cdots) \otimes \mathbf{e}'_{j-1} + (\mathbf{e}_i + i\omega\mathbf{e}_{i+1} + \cdots) \otimes ((j-1)\omega'\mathbf{e}'_j + \cdots).$$

Suppose that $i \geq 2$. Then the second term on the right-hand side belongs to W. By induction on i, we obtain

$$\mathbf{e}_i \otimes \mathbf{e}'_{j-1} \in W, \qquad i \geq 3.$$

If $i = 1$, we obtain

$$i\omega \mathbf{e}_2 \otimes \mathbf{e}'_{j-1} + (j-1)\omega' \mathbf{e}_1 \otimes \mathbf{e}'_j \in W.$$

Considering $u_1 - 1$, we obtain

$$i\mathbf{e}_2 \otimes \mathbf{e}'_{j-1} + (j-1)\mathbf{e}_1 \otimes \mathbf{e}'_j \in W.$$

Hence $(*)$ holds for j. This completes the proof.

By Lemma 3.1, we have

$$\dim \mathcal{B} = \dim V - 1. \tag{3.9}$$

Consider the surjective linear mapping

$$\mathcal{Z} \ni (U_1, U_2) \mapsto (u_2 - 1)U_1 \in \operatorname{Im}(u_1 - 1) \cap \operatorname{Im}(u_2 - 1).$$

The kernel of this mapping consists of (U_1, U_2) such that $U_1 \in \operatorname{Ker}(u_2 - 1)$, $U_2 \in \operatorname{Ker}(u_1 - 1)$. Hence by Lemma 3.2, we have

$$\dim \mathcal{Z} = \dim(\operatorname{Im}(u_1 - 1) \cap \operatorname{Im}(u_2 - 1)) + 2l_2 + 2. \tag{3.10}$$

By (3.9) and (3.10), we obtain

$$\dim H^1(U, V) = \dim(\operatorname{Im}(u_1 - 1) \cap \operatorname{Im}(u_2 - 1)) + 2l_2 + 3 - \dim V. \tag{3.11}$$

Lemma 3.5. *We have* $\dim H^1(U, V) = 2$.

Proof. We have

$$\begin{aligned} &\dim(\operatorname{Im}(u_1 - 1) \cap \operatorname{Im}(u_2 - 1)) \\ &= \dim(\operatorname{Im}(u_1 - 1)) + \dim(\operatorname{Im}(u_2 - 1)) - \dim(\operatorname{Im}(u_1 - 1) + \operatorname{Im}(u_2 - 1)). \end{aligned}$$

By Lemma 3.2, we have $\dim(\operatorname{Im}(u_i - 1)) = \dim V - (l_2 + 1)$, $i = 1, 2$. Then by Lemma 3.4, we get

$$\dim(\operatorname{Im}(u_1 - 1) \cap \operatorname{Im}(u_2 - 1)) = \dim V - 2l_2 - 1.$$

The assertion follows from (3.11).

3.2. In this subsection, we will prove the following theorems.

Theorem 3.6. *The eigenvalues of the action of t on $H^1(U,V)$ are $\epsilon^{l_1+2}(\epsilon')^{-l_2}$ and $\epsilon^{-l_1-2}(\epsilon')^{l_2}$. In particular, $H^1(U,V)^{P/U}=0$.*

Theorem 3.7.

$$\dim H^1(P,V)=\begin{cases}0 & \text{if} \quad l_1\neq l_2 \quad \text{or} \quad N(\epsilon)^{l_1}=-1,\\ 1 & \text{if} \quad l_1=l_2 \quad \text{and} \quad N(\epsilon)^{l_1}=1.\end{cases}$$

Here $N(\epsilon)$ denotes the norm of ϵ.

Taking $G=P$, $N=U$, $M=V$ in (1.9), we obtain the exact sequence

$$0 \longrightarrow H^1(P/U,V^U) \longrightarrow H^1(P,V) \longrightarrow H^1(U,V)^{P/U} \longrightarrow 0,$$

since $P/U\cong \mathbf{Z}$. Therefore Theorem 3.7 follows immediately from Theorem 3.6, since $\dim H^1(P/U,V^U)$ is easily seen to be equal to 0 (resp. 1) if $l_1\neq l_2$ or $N(\epsilon)^{l_1}=-1$ (resp. if $l_1=l_2$ and $N(\epsilon)^{l_1}=1$), in view of Lemma 3.1.

Proof of Theorem 3.6. First we recall the following fact on the action of t on $H^q(U,V)$ (cf. (1.3)). Let $f\in Z^q(U,V)$ and let $\bar f\in H^q(U,V)$ be the cohomology class represented by f. Put

$$g(n_1,n_2,\dots n_q)=t^{-1}f(tn_1t^{-1},tn_2t^{-1},\dots,tn_qt^{-1}),\qquad n_i\in U,\ 1\le i\le q.$$

Then $g\in Z^q(U,V)$ and $\bar f\mapsto \bar g$ is the action of t.

As in the proof of Lemma 3.4, let

$$W=\mathrm{Im}(u_1-1)+\mathrm{Im}(u_2-1)=(\oplus_{j=2}^{l_2+1}\mathbf{C}(\mathbf{e}_1\otimes\mathbf{e}'_j))\oplus(\oplus_{i=2}^{l_1+1}\oplus_{j=1}^{l_2+1}\mathbf{C}(\mathbf{e}_i\otimes\mathbf{e}'_j)).$$

We have

$$V=\mathbf{C}(\mathbf{e}_1\otimes\mathbf{e}'_1)\oplus W.$$

We may assume that $l_1>0$ since our assertion is clearly true if $l_1=l_2=0$.

Put $\mathbf{t}_1=\mathbf{e}_1\otimes\mathbf{e}'_{l_2+1}$. Let us show that for

$$\mathbf{t}_2=\omega(\mathbf{e}_1\otimes\mathbf{e}'_{l_2+1})+\sum_{i=2}^{l_1+1}x_i(\mathbf{e}_i\otimes\mathbf{e}'_{l_2+1})$$

with suitably chosen $x_i\in\mathbf{C}$, we have

$$(u_2-1)\mathbf{t}_1=(u_1-1)\mathbf{t}_2. \tag{3.12}$$

To this end, for $i\ge 1$, put

$$W_i=\oplus_{k=i}^{l_1+1}\mathbf{C}(\mathbf{e}_k\otimes\mathbf{e}'_{l_2+1}).$$

We have

$$(u_2 - 1)(\mathbf{e}_1 \otimes \mathbf{e}'_{l_2+1}) = (\omega \mathbf{e}_2 + \omega^2 \mathbf{e}_3 + \cdots) \otimes \mathbf{e}'_{l_2+1},$$
$$(u_1 - 1)(\mathbf{e}_i \otimes \mathbf{e}'_{l_2+1}) = (i\mathbf{e}_{i+1} + \binom{i+1}{i-1}\mathbf{e}_{i+2} + \cdots) \otimes \mathbf{e}'_{l_2+1}.$$

We see that

$$(u_2 - 1)\mathbf{t}_1 \equiv (u_1 - 1)\mathbf{t}_2 \mod W_3.$$

For $x_2 = (\omega^2 - \omega)/2$, we have

$$(u_2 - 1)\mathbf{t}_1 \equiv (u_1 - 1)\mathbf{t}_2 \mod W_4.$$

In this way, we can determine x_i successively so that (3.12) holds. Let $f_1 \in Z^1(U, V)$ be the 1-cocycle which corresponds to the point $(\mathbf{t}_1, \mathbf{t}_2) \in \mathcal{Z}$.

Put $\mathbf{t}_3 = \mathbf{e}_{l_1+1} \otimes \mathbf{e}'_1$. Similarly to the above, we can show that for

$$\mathbf{t}_4 = \omega'(\mathbf{e}_{l_1+1} \otimes \mathbf{e}'_1) + \sum_{j=2}^{l_2+1} y_j(\mathbf{e}_{l_1+1} \otimes \mathbf{e}'_j),$$

the relation

$$(u_2 - 1)\mathbf{t}_3 = (u_1 - 1)\mathbf{t}_4 \tag{3.13}$$

holds when y_j are suitably chosen. Let $f_2 \in Z^1(U, V)$ be the 1-cocycle which corresponds to the point $(\mathbf{t}_3, \mathbf{t}_4) \in \mathcal{Z}$.

Let $\bar{f}_i$ be the class of f_i in $H^1(U, V)$, $i = 1, 2$. Let us show that $\{\bar{f}_1, \bar{f}_2\}$ gives a basis of $H^1(U, V)$. To this end, assume that $\alpha f_1 + \beta f_2 \in B^1(U, V)$ for α, $\beta \in \mathbf{C}$. Then there exists $\mathbf{b} \in V$ such that

$$\alpha\mathbf{t}_1 + \beta\mathbf{t}_3 = (u_1 - 1)\mathbf{b}, \tag{i}$$

$$\alpha\mathbf{t}_2 + \beta\mathbf{t}_4 = (u_2 - 1)\mathbf{b} \tag{ii}$$

hold. Put

$$\mathbf{b} = \sum_{i=1}^{l_1+1}\sum_{j=1}^{l_2+1} x_{ij}(\mathbf{e}_i \otimes \mathbf{e}'_j).$$

On the left-hand side of (i), the coefficient of the tensor $\mathbf{e}_1 \otimes \mathbf{e}'_{l_2+1}$ is α and the coefficients of $\mathbf{e}_1 \otimes \mathbf{e}'_j$ are 0 for $1 \leq j \leq l_2$. We have

$$(u_1 - 1)(\mathbf{e}_1 \otimes \mathbf{e}'_j) = j(\mathbf{e}_1 \otimes \mathbf{e}'_{j+1}) + \sum_{l=j+2}^{l_2+1} z_l(\mathbf{e}_1 \otimes \mathbf{e}'_l) + A,$$

where $z_l \in \mathbf{Z}$ and A is a term which does not contain $\mathbf{e}_1 \otimes \mathbf{e}'_l$. Therefore we have $x_{11} = \cdots = x_{1l_2-1} = 0$. By comparing the coefficients of the tensor $\mathbf{e}_1 \otimes \mathbf{e}'_{l_2+1}$ on the both sides of (i), we obtain

$$\alpha = l_2 x_{1l_2}.$$

By comparing the coefficients of the tensor $\mathbf{e}_1 \otimes \mathbf{e}'_{l_2+1}$ on the both sides of (ii), we get

$$\alpha\omega = l_2 \omega' x_{1l_2}.$$

Hence we obtain $x_{1l_2} = 0$, $\alpha = 0$. Similarly by comparing the coefficients of the tensor $\mathbf{e}_{l_1+1} \otimes \mathbf{e}'_1$ for the both sides of (i) and (ii), we obtain $\beta = 0$.

Let f'_1 be the image of f_1 under the action of t and let $(U'_1, U'_2) \in \mathcal{Z}$ be the point corresponding to f'_1. Then we have

$$U'_1 = f'_1(u_1) = t^{-1} f_1(t u_1 t^{-1}) = t^{-1} f_1(u_1^A u_2^B) = t^{-1}[u_1^A f_1(u_2^B) + f_1(u_1^A)],$$

$$U'_2 = f'_1(u_2) = t^{-1} f_1(t u_2 t^{-1}) = t^{-1} f_1(u_1^C u_2^D) = t^{-1}[u_1^C f_1(u_2^D) + f_1(u_1^C)].$$

For $i = 1$, 2, we have $f_1(u_i) = \mathbf{t}_i$ and

$$f_1(u_i^n) = (1 + u_1 + \cdots + u_1^{n-1})\mathbf{t}_i \qquad \text{if} \quad n > 0. \tag{3.14}$$

$$f_1(u_i^{-n}) = -(u_1^{-1} + \cdots + u_1^{-n})\mathbf{t}_i \qquad \text{if} \quad n > 0. \tag{3.15}$$

From these formulas, we see easily that the coefficient of $\mathbf{e}_1 \otimes \mathbf{e}'_{l_2+1}$ in tU'_1 is $A + B\omega$. Hence the coefficient of $\mathbf{e}_1 \otimes \mathbf{e}'_{l_2+1}$ in U'_1 is $\epsilon^{l_1}(\epsilon')^{-l_2}(A + B\omega) = \epsilon^{l_1+2}(\epsilon')^{-l_2}$. Similarly we see that the coefficient of $\mathbf{e}_1 \otimes \mathbf{e}'_{l_2+1}$ in U'_2 is $\omega\epsilon^{l_1+2}(\epsilon')^{-l_2}$.

Now let

$$f'_1 \equiv \gamma f_1 + \delta f_2 \mod B^1(U, V)$$

with γ, $\delta \in \mathbf{C}$. Then there exists $\mathbf{c} \in V$ such that

$$\gamma \mathbf{t}_1 + \delta \mathbf{t}_3 - U'_1 = (u_1 - 1)\mathbf{c}, \tag{iii}$$

$$\gamma \mathbf{t}_2 + \delta \mathbf{t}_4 - U'_2 = (u_2 - 1)\mathbf{c}. \tag{iv}$$

Put $\mathbf{c} = \sum_{i=1}^{l_1+1} \sum_{j=1}^{l_2+1} y_{ij}(\mathbf{e}_i \otimes \mathbf{e}'_j)$. Comparing the coefficients of $\mathbf{e}_1 \otimes \mathbf{e}'_{l_2+1}$ on the both sides of (iii), we obtain

$$\gamma - \epsilon^{l_1+2}(\epsilon')^{-l_2} = l_2 y_{1l_2}.$$

Comparing the coefficients of $\mathbf{e}_1 \otimes \mathbf{e}'_{l_2+1}$ on the both sides of (iv), we obtain

$$(\gamma - \epsilon^{l_1+2}(\epsilon')^{-l_2})\omega = l_2 y_{1l_2} \omega'.$$

From these two formulas, we obtain $y_{1l_2} = 0$, $\gamma = \epsilon^{l_1+2}(\epsilon')^{-l_2}$. Similarly comparing the coefficients of $\mathbf{e}_{l_1+1} \otimes \mathbf{e}'_1$ on the both sides of (iii) and (iv), we obtain $\delta = 0$. Thus we have shown

$$f'_1 \equiv \epsilon^{l_1+2}(\epsilon')^{-l_2} f_1 \mod B^1(U, V). \tag{3.16}$$

Next let f'_2 be the image of f_2 under the action of t and let (U'_3, U'_4) be the point of $\mathcal{Z}$ corresponding to f'_2. Here $U'_3 = f'_2(u_1)$, $U'_4 = f'_2(u_2)$. Then we have

$$U'_3 = f'_2(u_1) = t^{-1} f_2(t u_1 t^{-1}) = t^{-1} f_2(u_1^A u_2^B) = t^{-1}[u_1^A f_2(u_2^B) + f_2(u_1^A)],$$

$$U'_4 = f'_2(u_2) = t^{-1} f_2(t u_2 t^{-1}) = t^{-1} f_2(u_1^C u_2^D) = t^{-1}[u_1^C f_2(u_2^D) + f_2(u_1^C)].$$

The coefficient of $\mathbf{e}_{l_1+1} \otimes \mathbf{e}'_1$ in tU'_3 is $A + B\omega' = \epsilon^{-2}$. The coefficient of $\mathbf{e}_{l_1+1} \otimes \mathbf{e}'_1$ in tU'_4 is $C + D\omega' = \epsilon^{-2}\omega'$. By a similar argument to the above, we obtain

$$f'_2 \equiv \epsilon^{-l_1-2}(\epsilon')^{l_2} f_2 \mod B^1(U, V). \tag{3.17}$$

This completes the proof of Theorem 3.6.

3.3. In this subsection, we will prove the following theorems.

Theorem 3.8. *We have* $\dim H^2(U, V) = 1$ *and* t *acts on it as the multiplication by* $\epsilon^{l_1}(\epsilon')^{l_2}$.

Theorem 3.9. *We have* $H^2(P, V) = 0$ *except for the case* $l_1 = l_2$ *and* $N(\epsilon)^{l_1} = 1$. *If* $l_1 = l_2$ *and* $N(\epsilon)^{l_1} = 1$, *then we have* $\dim H^2(P, V) = 1$.

First we will prove the part of Theorem 3.8 concerning the dimension.

Lemma 3.10. *We have* $\dim H^2(U, V) = 1$.

Proof. Let U_1 be the subgroup of U generated by u_1. We have the exact sequence

$$0 \longrightarrow U_1 \longrightarrow U \longrightarrow U_2 \longrightarrow 0 \tag{3.18}$$

and the associated spectral sequence (cf. (1.8))

$$E_2^{p,q} = H^p(U_2, H^q(U_1, V)) \Longrightarrow H^n(U, V). \tag{3.19}$$

Let $E^n = H^n(U, V)$ and $\{F^i\}$ denote the filtration on E^n induced by (3.19). We have $F^p(E^n)/F^{p+1}(E^n) \cong E_\infty^{p,n-p}$. Since $U_2 \cong \mathbf{Z}$, we have $E_2^{2,q} = E_\infty^{2,q} = 0$. Since $F^3(E^2) = 0$, we get $F^2(E^2) = 0$. Since $U_1 \cong \mathbf{Z}$, we have $E_2^{p,2} =$

$E_\infty^{p,2} = 0$. Hence we get $E^2/F^1(E^2) = 0$. We have $F^1(E^2)/F^2(E^2) \cong E_\infty^{1,1}$. Therefore it is sufficient to show that $\dim E_\infty^{1,1} = 1$.

We consider

$$E_2^{1,1} = H^1(U_2, H^1(U_1, V)).$$

The map $Z^1(U_1, V) \ni f \mapsto f(u_1) \in V$ induces the isomorphism

$$H^1(U_1, V) \cong V/\mathrm{Im}(u_1 - 1). \tag{3.20}$$

The action of $u \in U_2$ on the right-hand side of (3.20) is given by

$$V/\mathrm{Im}(u_1-1) \ni v \mod \mathrm{Im}(u_1-1) \longrightarrow u^{-1}v \mod \mathrm{Im}(u_1-1) \in V/\mathrm{Im}(u_1-1).$$

Since $\bar{u}_2 = u_2 \mod U_1$ is a generator of U_2, we have

$$H^1(U_2, H^1(U_1, V)) \cong (V/\mathrm{Im}(u_1-1))/\mathrm{Im}(\bar{u}_2-1) \cong V/(\mathrm{Im}(u_1-1)+\mathrm{Im}(u_2-1)).$$

By Lemma 3.4, we obtain

$$\dim H^1(U_2, H^1(U_1, V)) = \dim E_2^{1,1} = 1.$$

Since $E_2^{3,0} = 0$, we have $E_\infty^{1,1} = E_2^{1,1}$. This completes the proof.

Proof of Theorem 3.8. We set $\tau = u_1$, $\eta = u_2$. Let $\mathcal{F}$ be the free group on two free generators $\widetilde{\tau}$ and $\widetilde{\eta}$ and let $\pi : \mathcal{F} \longrightarrow U$ be the surjective homomorphism such that

$$\pi(\widetilde{\tau}) = \tau, \qquad \pi(\widetilde{\eta}) = \eta.$$

Let R be the kernel of π. For a, $b \in \mathcal{F}$, let $[a,b] = aba^{-1}b^{-1}$ be the commutator of a and b. We see easily that

$$R = \langle x[\widetilde{\tau}, \widetilde{\eta}]x^{-1} \mid x \in \mathcal{F}\rangle, \qquad R = [\mathcal{F}, \mathcal{F}].$$

We have the isomorphism (cf. (1.11))

$$H^2(U, V) \cong H^1(R, V)^U/\mathrm{Im}(H^1(\mathcal{F}, V)). \tag{3.21}$$

We have

$$H^1(R, V)^U = \{\varphi \in \mathrm{Hom}(R, V) \mid \varphi(grg^{-1}) = g\varphi(r), \ \ g \in \mathcal{F}, \ r \in R\}. \tag{3.22}$$

Hence $\varphi \in H^1(R, V)^U$ is completely determined by $\varphi([\widetilde{\tau}, \widetilde{\eta}])$. For $b \in H^1(\mathcal{F}, V)$, we have

$$b([\widetilde{\tau}, \widetilde{\eta}]) = (1 - \eta)b(\widetilde{\tau}) + (\tau - 1)b(\widetilde{\eta}).$$

Let W be the suspace of V as in the proof of Lemma 3.4. For $\varphi \in \operatorname{Im}(H^1(\mathcal{F}, V))$, the formula above shows that $\varphi([\widetilde{\tau}, \widetilde{\eta}])$ can take an arbitrary vector in W. In particular, it follows that $\dim H^2(U, V) \leq 1$. Since $\dim H^2(U, V) = 1$ by Lemma 3.10, we see that there exists $\varphi_1 \in H^1(R, V)^U$ such that $\varphi_1([\widetilde{\tau}, \widetilde{\eta}]) = \mathbf{e}_1 \otimes \mathbf{e}'_1$. This φ_1 corresponds to a generator of $H^2(U, V)$.

Let $f \in Z^2(U, V)$. For $g \in \mathcal{F}$, we put $\bar{g} = \pi(g)$. There exists $a \in C^1(\mathcal{F}, V)$ such that (cf. (1.12))

$$f(\bar{g}_1, \bar{g}_2) = g_1 a(g_2) + a(g_1) - a(g_1 g_2), \qquad g_1, g_2 \in \mathcal{F}. \tag{3.23}$$

The corresponding element $\varphi \in H^1(R, V)^U$ to f is obtained as the restriction of a to R. Now let ξ be an automorphism of $\mathcal{F}$. Since ξ stabilizes $R = [\mathcal{F}, \mathcal{F}]$, ξ induces an automorphism of $U = \mathcal{F}/R$, which we denote by $\bar{\xi}$. We have

$$\bar{\xi}(\bar{g}) = \overline{\xi(g)}, \qquad g \in \mathcal{F}.$$

From (3.23), we obtain

$$f(\bar{\xi}(\bar{g}_1), \bar{\xi}(\bar{g}_2)) = \xi(g_1) a(\xi(g_2)) + a(\xi(g_1)) - a(\xi(g_1)\xi(g_2)), \quad g_1, g_2 \in \mathcal{F}. \tag{3.24}$$

Lemma 3.11. *For* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z})$, *let* $\xi(\gamma)$ *be the automorphism of* U *defined by* $\xi(\gamma)(\tau) = \tau^a \eta^c$, $\xi(\gamma)(\eta) = \tau^b \eta^d$. *Then there exists an automorphism* $\widetilde{\xi(\gamma)}$ *of* $\mathcal{F}$ *such that* $\xi(\gamma) = \overline{\widetilde{\xi(\gamma)}}$. *Moreover* $\widetilde{\xi(\gamma)}$ *can be taken so that*

$$\varphi(\widetilde{\xi(\gamma)}(g)) \equiv \varphi(g) \mod W \tag{3.25}$$

holds for every $\varphi \in H^1(R, V)^U$ *and every* $g \in [\mathcal{F}, \mathcal{F}]$.

Proof. For γ_1, $\gamma_2 \in \mathrm{SL}(2, \mathbf{Z})$, we have $\xi(\gamma_1 \gamma_2) = \xi(\gamma_1)\xi(\gamma_2)$. For two automorphisms ξ_1, ξ_2 of $\mathcal{F}$, we have $\overline{\xi_1 \xi_2} = \bar{\xi}_1 \bar{\xi}_2$. Therefore to show the first assertion, it is sufficient to verify it for generators $\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of $\mathrm{SL}(2, \mathbf{Z})$. Clearly the formulas $\widetilde{\xi}(\gamma_1)(\widetilde{\tau}) = \widetilde{\tau}$, $\widetilde{\xi}(\gamma_1)(\widetilde{\eta}) = \widetilde{\tau}\widetilde{\eta}$, $\widetilde{\xi}(\gamma_2)(\widetilde{\tau}) = \widetilde{\eta}^{-1}$, $\widetilde{\xi}(\gamma_2)(\widetilde{\eta}) = \widetilde{\tau}$ define automorphisms $\widetilde{\xi}(\gamma_1)$ and $\widetilde{\xi}(\gamma_2)$ of $\mathcal{F}$ satisfying the requirements.

To show the latter assertion, we first note that

$$u\mathbf{v} \equiv \mathbf{v} \mod W \quad \text{for every } u \in U \text{ and every } \mathbf{v} \in V. \tag{3.26}$$

Let $\varphi \in H^1(R,V)^U$. Since $\widetilde{\xi}(\gamma)$ can be taken from the subgroup of $\mathrm{Aut}(\mathcal{F})$ generated by $\widetilde{\xi}(\gamma_1)$ and $\widetilde{\xi}(\gamma_2)$, it is sufficient to show (3.25) for these generators. Moreover since $\varphi(x[\widetilde{\tau},\widetilde{\eta}]x^{-1}) = x\varphi([\widetilde{\tau},\widetilde{\eta}])$ for $x \in \mathcal{F}$, it is enough to verify (3.25) for $g = [\widetilde{\tau},\widetilde{\eta}]$ in view of (3.26). For $\widetilde{\xi}(\gamma_1)$, we have

$$\varphi(\widetilde{\xi}(\gamma_1)([\widetilde{\tau},\widetilde{\eta}])) = \varphi(\widetilde{\tau}[\widetilde{\tau},\widetilde{\eta}]\widetilde{\tau}^{-1}) = \tau\varphi([\widetilde{\tau},\widetilde{\eta}]) \equiv \varphi([\widetilde{\tau},\widetilde{\eta}]) \mod W$$

by (3.26). For $\widetilde{\xi}(\gamma_2)$, we can check (3.25) similarly since $\widetilde{\xi}(\gamma_2)([\widetilde{\tau},\widetilde{\eta}]) = \widetilde{\eta}^{-1}[\widetilde{\tau},\widetilde{\eta}]\widetilde{\eta}$. This completes the proof of Lemma 3.11.

Applying Lemma 3.11 to $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we see that there exists an automorphism ξ_t of $\mathcal{F}$ such that (cf. (3.1))

$$\bar{\xi}_t(u) = tut^{-1}, \qquad u \in U.$$

Under the action of t, f is transformed to the 2-cocycle $f' \in Z^2(U,V)$ where

$$f'(h_1,h_2) = t^{-1}f(th_1t^{-1}, th_2t^{-1}), \qquad h_1,h_2 \in U.$$

By (3.24), we obtain

$$\begin{aligned} &t^{-1}f(t\bar{g}_1t^{-1}, t\bar{g}_2t^{-1}) \\ &= g_1t^{-1}a(\xi_t(g_2)) + t^{-1}a(\xi_t(g_1)) - t^{-1}a(\xi_t(g_1)\xi_t(g_2)), \qquad g_1,g_2 \in \mathcal{F}. \end{aligned} \tag{3.27}$$

This formula shows that 1-cochain $a' \in C^1(\mathcal{F},V)$ which splits f' is given by

$$a'(g) = t^{-1}a(\xi_t(g)), \qquad g \in \mathcal{F}.$$

Now suppose that f (resp. f') $\in Z^2(U,V)$ corresponds to φ (resp. φ') $\in H^1(R,V)^U$. We have

$$\varphi'([\widetilde{\tau},\widetilde{\eta}]) = t^{-1}\varphi(\xi_t([\widetilde{\tau},\widetilde{\eta}])). \tag{3.28}$$

We may assume that $\varphi = \varphi_1$, i.e., $\varphi([\widetilde{\tau},\widetilde{\eta}]) = \mathbf{e}_1 \otimes \mathbf{e}'_1$. Then by (3.25), we obtain

$$\varphi'([\widetilde{\tau},\widetilde{\eta}]) \equiv t^{-1}\varphi([\widetilde{\tau},\widetilde{\eta}]) \equiv \epsilon^{l_1}(\epsilon')^{l_2}\varphi([\widetilde{\tau},\widetilde{\eta}]) \mod W.$$

This completes the proof of Theorem 3.8.

Proof of Theorem 3.9. Set $T = P/U$. Then T is generated by t mod U. We consider the spectral sequence

$$E_2^{p,q} = H^p(T, H^q(U,V)) \Longrightarrow H^n(P,V). \tag{3.29}$$

Let $E^n = H^n(P, V)$ and $\{F^i\}$ denote the filtration induced by (3.29). Since $T \cong \mathbf{Z}$, we have $E_2^{p,q} = 0$ for $p \geq 2$, $q \geq 0$. Hence $F^2(E^2)/F^3(E^2) \cong E_\infty^{2,0} = 0$. Since $F^3(E^2) = 0$, we obtain $F^2(E^2) = 0$. By Theorem 3.6, we have $E_2^{1,1} = H^1(T, H^1(U, V)) = 0$. Hence we have $F^1(E^2)/F^2(E^2) \cong E_\infty^{1,1} = 0$. Therefore we obtain

$$\dim H^2(P, V) = \dim E^2/F^1(E^2) = \dim E_\infty^{0,2}. \tag{3.30}$$

Now assume $l_1 \neq l_2$ or $N(\epsilon)^{l_1} \neq 1$. By Theorem 3.8, we have $H^2(U, V)^T = 0$. Hence we get $E_2^{0,2} = E_\infty^{0,2} = 0$. Next assume that $l_1 = l_2$ and $N(\epsilon)^{l_1} = 1$. By Theorem 3.8, we have $\dim E_2^{0,2} = \dim H^2(U, V)^T = 1$. We clearly have $E_2^{0,2} \cong E_\infty^{0,2}$. This completes the proof.

§4. On the parabolic condition

In this section (in particular subsection 4.1), we will show that it is possible to deduce information on critical values of L-functions, once we know a corresponding 2-cocycle which satisfies the parabolic condition.

From this section until the end of the paper, we define subgroups of Γ by

$$P = \left\{ \begin{pmatrix} u & v \\ 0 & u^{-1} \end{pmatrix} \middle| \, u \in E_F, \ v \in \mathcal{O}_F \right\} / \{\pm 1_2\},$$
$$U = \left\{ \begin{pmatrix} \pm 1 & v \\ 0 & \pm 1 \end{pmatrix} \middle| \, v \in \mathcal{O}_F \right\} / \{\pm 1_2\}$$

restoring the notation to that of §2. We see that Theorems 3.7, 3.9 and the fact $H^1(P, V)^{P/U} = 0$ stated in Theorem 3.6 are valid, considering the isomorphism $P \ni p \mapsto {}^t p^{-1} \in {}^t P$ and noting that $g \mapsto \rho(g)$ and $g \mapsto \rho({}^t g^{-1})$ are equivalent as representations of $\mathrm{SL}(2, \mathbf{C})^2$.

4.1. Let V_1 (resp. V_2) be the representation space of ρ_{l_1} (resp. ρ_{l_2}). We take a basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{l_1+1}\}$ of V_1 so that $\rho_{l_1}(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})\mathbf{e}_i = a^{l_1+1-i}\mathbf{e}_i$. Similarly we take a basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_{l_2+1}\}$ of V_2 so that $\rho_{l_2}(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})\mathbf{e}'_i = a^{l_2+1-i}\mathbf{e}'_i$. We assume that $l_1 \geq l_2$, $l_1 \equiv l_2 \mod 2$. We put $k_1 = l_1 + 2$, $k_2 = l_2 + 2$, $k = (k_1, k_2)$. Let $\Omega \in S_k(\Gamma)$. We assume that l_1 is even if $N(\epsilon) = -1$. (This assumption is (A) in §1.)

We recall the formulas:

$$f(\gamma_1, \gamma_2) = \int_{\gamma_1\gamma_2 w_1}^{\gamma_1 w_1} \int_{w_2}^{\gamma'_1 w_2} \mathfrak{d}(\Omega), \qquad w_1 = i\epsilon^{-1}, w_2 = i\infty, \tag{4.1}$$

$$f(\sigma,\mu) = -\int_{i\epsilon^{-1}}^{i\epsilon}\int_0^{i\infty}\mathfrak{d}(\Omega). \tag{4.2}$$

The formula (2.30) shows that the coefficients of $\mathbf{e}_i \otimes \mathbf{e}'_{i-(l_1-l_2)/2}$ in $f(\sigma,\mu)$, $(l_1-l_2)/2+1 \le i \le (l_1+l_2)/2+1$ are related to the critical values of $L(s,\Omega)$.

The parabolic condition on the cocycle f is

$$f(p\gamma_1,\gamma_2) = pf(\gamma_1,\gamma_2) \qquad \text{for every} \quad p \in P,\ \gamma_1,\gamma_2 \in \Gamma. \tag{4.3}$$

Now suppose that we add the coboundary of $b \in C^1(\Gamma, V)$

$$b(\gamma_1\gamma_2) - \gamma_1 b(\gamma_2) - b(\gamma_1)$$

to f. We assume that the resulting 2-cocycle is normalized and still satisfies the parabolic condition (4.3). Then we see easily that b must satisfy the condition

$$b(p\gamma) = pb(\gamma) + b(p), \qquad p \in P,\ \gamma \in \Gamma. \tag{4.4}$$

Put $A = f(\sigma,\mu)$. After adding the cobounday of b, A changes to $A + b(\sigma\mu) - \sigma b(\mu) - b(\sigma)$. By (4.4), we have

$$b(\sigma\mu) = b(\mu^{-1}\sigma) = \mu^{-1}b(\sigma) + b(\mu^{-1}), \qquad b(\mu^{-1}) = -\mu^{-1}b(\mu).$$

Therefore A changes to

$$A + (\mu^{-1}-1)b(\sigma) - (\sigma + \mu^{-1})b(\mu).$$

By (4.4), we have $b|P \in Z^1(P,V)$. Suppose that $l_1 \neq l_2$. By Theorem 3.7, we have

$$b(\mu) = (\mu - 1)\mathbf{b}, \qquad \mathbf{b} \in V.$$

Since $(\sigma + \mu^{-1})(\mu - 1) = (\mu^{-1} - 1)(\sigma - 1)$, we see that A changes to

$$A + (\mu^{-1} - 1)[b(\sigma) + (1-\sigma)\mathbf{b}].$$

This formula shows that the components of A related to the critical values do not change by adding a coboundary, since $\mu^{-1}(\mathbf{e}_i \otimes \mathbf{e}_{i-(l_1-l_2)/2}) = \mathbf{e}_i \otimes \mathbf{e}_{i-(l_1-l_2)/2})$. Next suppose that $l_1 = l_2$. By Theorem 3.7 and by the exact sequence below it, we have

$$b(\mu) = (\mu - 1)\mathbf{b} + \mathbf{b}_0, \qquad \mathbf{b} \in V, \quad \mathbf{b}_0 \in V^U.$$

Hence A changes to

$$A+(\mu^{-1}-1)[b(\sigma)+(1-\sigma)\mathbf{b}]-(\sigma+\mu^{-1})\mathbf{b}_0.$$

Since $\mathbf{b}_0 \in V^U$, this formula shows that the components of A related to the critical values do not change except for two critical values $L(1,\Omega)$ and $L(l_1+1,\Omega)$ at the edges.

4.2. Let $\bar{Z}^2(\Gamma,V)$ be the subgroup of $Z^2(\Gamma,V)$ consisting of normalized 2-cocycles. Put

$$\bar{B}^2(\Gamma,V)=\{f=db \mid b\in C^1(\Gamma,V),\ b(1)=0\}.$$

Then we have

$$\bar{Z}^2(\Gamma,V)\cap B^2(\Gamma,V)=\bar{B}^2(\Gamma,V)$$

and therefore

$$\bar{Z}^2(\Gamma,V)/\bar{B}^2(\Gamma,V)\subset Z^2(\Gamma,V)/B^2(\Gamma,V).$$

Since every 2-cocycle can be normalized by adding a coboundary, we have

$$H^2(\Gamma,V)=\bar{Z}^2(\Gamma,V)/\bar{B}^2(\Gamma,V).$$

Put

(4.5)

$$Z^2_{\mathrm{P}}(\Gamma,V)=\{f\in\bar{Z}^2(\Gamma,V)\mid f \text{ satisfies the parabolic condition (4.3)}\},$$

$$\begin{aligned}B^2_{\mathrm{P}}(\Gamma,V)=&\{f\in\bar{B}^2(\Gamma,V)\mid f=db, b\in C^1(\Gamma,V),\\ &b(p\gamma)=pb(\gamma)+b(p),\quad p\in P,\gamma\in\Gamma\}.\end{aligned} \tag{4.6}$$

An element of $Z^2_P(\Gamma,V)$ is called a *normalized parabolic 2-cocycle*. The next lemma can easily be verified.

Lemma 4.1. *We have*

$$Z^2_{\mathrm{P}}(\Gamma,V)\cap\bar{B}^2(\Gamma,V)=B^2_{\mathrm{P}}(\Gamma,V).$$

By Lemma 4.1, we have

$$Z^2_{\mathrm{P}}(\Gamma,V)/B^2_{\mathrm{P}}(\Gamma,V)\subset\bar{Z}^2(\Gamma,V)/\bar{B}^2(\Gamma,V)=H^2(\Gamma,V).$$

We define the parabolic part $H^2_{\mathrm{P}}(\Gamma,V)$ of $H^2(\Gamma,V)$ by

$$H^2_{\mathrm{P}}(\Gamma,V)=Z^2_{\mathrm{P}}(\Gamma,V)/B^2_{\mathrm{P}}(\Gamma,V). \tag{4.7}$$

4.3. As another application of Theorem 3.7, we are going to show the nonvanishing of the cohomology class attached to a Hecke eigenform.

Lemma 4.2. *Assume l_1 is even if $N(\epsilon) = -1$. Let $f \in Z_P^2(\Gamma, V)$ be a normalized parabolic 2-cocycle. For $(l_1 - l_2)/2 + 1 \le i \le (l_1 + l_2)/2 + 1$, let c_i be the coefficient of $\mathbf{e}_i \otimes \mathbf{e}'_{i-(l_1-l_2)/2}$ in $f(\sigma, \mu)$. Assume that $c_i \neq 0$ for some i if $l_1 \neq l_2$ and that $c_i \neq 0$ for some $i \neq 1, l_1 + 1$ if $l_1 = l_2$. Then the cohomology class of f is non-trivial.*

Proof. Suppose that the cohomology class of f is trivial. Then there exists $b \in C^1(\Gamma, V)$ such that

$$f(\gamma_1, \gamma_2) = \gamma_1 b(\gamma_2) + b(\gamma_1) - b(\gamma_1\gamma_2), \qquad \gamma_1, \gamma_2 \in \Gamma.$$

From $f(1, 1) = 0$, we get $b(1) = 0$. Since f satisfies the parabolic condition, we have

$$p\gamma_1 b(\gamma_2) + b(p\gamma_1) - b(p\gamma_1\gamma_2) = p\gamma_1 b(\gamma_2) + pb(\gamma_1) - pb(\gamma_1\gamma_2)$$

for $p \in P$. Taking $\gamma_2 = \gamma_1^{-1}$ and writing γ_1 as γ, we find

$$b(p\gamma) = pb(\gamma) + b(p), \qquad p \in P,\ \gamma \in \Gamma.$$

Now

$$\begin{aligned} f(\sigma, \mu) &= \sigma b(\mu) + b(\sigma) - b(\sigma\mu) = \sigma b(\mu) + b(\sigma) - b(\mu^{-1}\sigma) \\ &= \sigma b(\mu) + b(\sigma) - \mu^{-1} b(\sigma) - b(\mu^{-1}). \end{aligned}$$

Since $b(\mu^{-1}) = -\mu^{-1} b(\mu)$, we obtain

$$f(\sigma, \mu) = (1 - \mu^{-1}) b(\sigma) + (\sigma + \mu^{-1}) b(\mu). \tag{4.8}$$

First we consider the case $l_1 \neq l_2$. Since $b|P \in Z^1(P, V)$ and $H^1(P, V) = 0$ (Theorem 3.7), there exists $\mathbf{b} \in V$ such that $b(\mu) = (\mu - 1)\mathbf{b}$. Then we have

$$f(\sigma, \mu) = (1 - \mu^{-1})[b(\sigma) + (1 - \sigma)\mathbf{b}].$$

We have $\mu^{-1}(\mathbf{e}_i \otimes \mathbf{e}'_{i-(l_1-l_2)/2}) = N(\epsilon)^{l_1}(\mathbf{e}_i \otimes \mathbf{e}'_{i-(l_1-l_2)/2})$. Hence c_i vanishes for all i. This is a contradiction and the proof is complete in this case.

Next we consider the case $l_1 = l_2$. By Theorem 3.7, there exist $\mathbf{b} \in V$ and $\mathbf{b}_0 \in V^U$ such that

$$b(\mu) = (\mu - 1)\mathbf{b} + \mathbf{b}_0.$$

Then we have

$$f(\sigma, \mu) = (1 - \mu^{-1})[b(\sigma) + (1 - \sigma)\mathbf{b}] + (\sigma + \mu^{-1})\mathbf{b}_0.$$

Since $\mathbf{b}_0 \in V^U$, this formula shows that $c_i = 0$ if $i \neq 1, l_1 + 1$. This is a contradiction and completes the proof.

Proposition 4.3. *Let $k = (k_1, k_2)$, $k_1 \geq k_2$, $k_1 \equiv k_2 \equiv 0 \mod 2$. Let $\Omega \in S_k(\Gamma)$ and let $f = f(\Omega)$ be the normalized parabolic 2-cocycle attached to Ω (cf. (4.1)). We assume that the class number of F in the narrow sense is 1 and that Ω is a nonzero Hecke eigenform. If $k_1 \neq k_2$, we assume $k_2 \geq 4$. If $k_1 = k_2$, we assume $k_2 \geq 6$. Then the cohomology class of f in $H^2(\Gamma, V)$ is non-trivial.*

Proof. Let $k_1 = l_1 + 2$, $k_2 = l_2 + 2$. By (2.30), we see that the coefficient c_i of $\mathbf{e}_i \otimes \mathbf{e}'_{i-(l_1-l_2)/2}$ in $f(\sigma, \mu)$ is $L(l_1 + 2 - i, \Omega)$ times a nonzero constant for $(l_1 - l_2)/2 + 1 \leq i \leq (l_1 + l_2)/2 + 1$. It is well known that $L(s, \Omega) \neq 0$ for $\Re(s) \geq (k_1 + 1)/2$ (cf. [Sh4], Proposition 4.16). For $i = (l_1 - l_2)/2 + 1$, c_i is nonzero times $L((k_1 + k_2)/2 - 1, \Omega)$. Since $(k_1 + k_2)/2 - 1 \geq (k_1 + 1)/2$ if $k_2 \geq 3$, our assertion follows from Lemma 4.2 if $k_1 \neq k_2$. Assume $k_1 = k_2$. For $i = 2$, c_i is nonzero times $L(k_1 - 2, \Omega)$. Since $k_1 - 2 \geq (k_1 + 1)/2$ if $k_1 \geq 5$, our assertion in this case also follows from Lemma 4.2.

4.4. With a free group $\mathcal{F}$ with finitely many generators, we write $\Gamma = \mathcal{F}/R$. Let $\pi : \mathcal{F} \longrightarrow \Gamma$ be the canonical homomorphism with $\mathrm{Ker}(\pi) = R$. For $g \in \mathcal{F}$, we put $\pi(g) = \bar{g}$. We regard V as an $\mathcal{F}$-module by $gv = \bar{g}v$, $g \in \mathcal{F}$, $v \in V$. By (1.11), we have

$$H^2(\Gamma, V) \cong H^1(R, V)^\Gamma/\mathrm{Im}(H^1(\mathcal{F}, V)). \tag{4.9}$$

We are going to examine the part of the right-hand side of (4.9) which corresponds to $H^2_{\mathrm{P}}(\Gamma, V)$. Put $\mathcal{P} = \pi^{-1}(P)$. Let $f \in Z^2_{\mathrm{P}}(\Gamma, V)$. Take a 1-cochain $a \in C^1(\mathcal{F}, V)$ which satisfies (1.12). Then we have

$$f(\bar{p}\bar{g}_1, \bar{g}_2) = pg_1a(g_2) + a(pg_1) - a(pg_1g_2), \qquad p \in \mathcal{P},\ g_1, g_2 \in \mathcal{F}.$$

By the parabolic condition on f, this is equal to

$$p(g_1a(g_2) + a(g_1) - a(g_1g_2)).$$

Hence we have

$$a(pg_1g_2) - a(pg_1) = pa(g_1g_2) - pa(g_1), \qquad p \in \mathcal{P},\ g_1, g_2 \in \mathcal{F}.$$

Taking $g_1 = g_2^{-1} = g$, we obtain

$$a(pg) = pa(g) + a(p), \qquad p \in \mathcal{P},\ g \in \mathcal{F}. \tag{4.10}$$

Conversely if a satisfies (4.10), then f satisfies the parabolic condition.

Let $\varphi = a|R$. We note that a satisfies (1.13) and $\varphi \in H^1(R, V)^\Gamma$. For every $s \in P$, we take an element $\widetilde{s} \in \mathcal{P}$ such that $\pi(\widetilde{s}) = s$. We fix the choice of $\widetilde{s}$. Then we write $a(\widetilde{s})$ as $\widetilde{a}(s)$. By (1.13), we have

$$a(\widetilde{s}r) = s\varphi(r) + \widetilde{a}(s), \qquad s \in P,\ r \in R. \tag{4.11}$$

Now for $s_1,\ s_2 \in P$ and $r_1,\ r_2 \in R$, we have

$$\begin{aligned} a(\widetilde{s}_1 r_1 \widetilde{s}_2 r_2) &= a((\widetilde{s_1 s_2})(\widetilde{s_1 s_2})^{-1}\widetilde{s}_1\widetilde{s}_2\widetilde{s}_2^{-1} r_1 \widetilde{s}_2 r_2) \\ &= s_1 s_2 \varphi((\widetilde{s_1 s_2})^{-1}\widetilde{s}_1\widetilde{s}_2\widetilde{s}_2^{-1} r_1 \widetilde{s}_2 r_2) + \widetilde{a}(s_1 s_2) \\ &= s_1 s_2 [\varphi(\widetilde{s}_2^{-1} r_1 \widetilde{s}_2) + \varphi(r_2) + \varphi((\widetilde{s_1 s_2})^{-1}\widetilde{s}_1\widetilde{s}_2)] + \widetilde{a}(s_1 s_2) \\ &= s_1 \varphi(r_1) + s_1 s_2 \varphi(r_2) + \varphi(\widetilde{s}_1\widetilde{s}_2(\widetilde{s_1 s_2})^{-1}) + \widetilde{a}(s_1 s_2), \end{aligned}$$

using (1.13), (1.14) and (4.11). On the other hand, by (4.10), we have

$$\begin{aligned} a(\widetilde{s}_1 r_1 \widetilde{s}_2 r_2) &= s_1 a(\widetilde{s}_2 r_2) + a(\widetilde{s}_1 r_1) \\ &= s_1(s_2\varphi(r_2) + \widetilde{a}(s_2)) + s_1\varphi(r_1) + \widetilde{a}(s_1). \end{aligned}$$

Comparing two results, we obtain

$$\varphi(\widetilde{s}_1\widetilde{s}_2(\widetilde{s_1 s_2})^{-1}) = s_1\widetilde{a}(s_2) + \widetilde{a}(s_1) - \widetilde{a}(s_1 s_2). \tag{4.12}$$

The condition (4.12) can be interpreted as follows. The group extension

$$1 \longrightarrow R \longrightarrow \mathcal{P} \longrightarrow P \longrightarrow 0.$$

defines the factor set

$$(s_1, s_2) \longrightarrow \widetilde{s}_1\widetilde{s}_2(\widetilde{s_1 s_2})^{-1} \tag{4.13}$$

of P taking values in R. Mapping this factor set by φ, we obtain a 2-cocycle of P taking values in V (cf. Lemma 1.4). Then (4.12) means that this 2-cocycle splits.

Next we consider the condition (4.10) on a double coset $\mathcal{P}\delta R$, where δ is an arbitrary element of $\mathcal{F}$. Since R is a normal subgroup of $\mathcal{P}$, we have

$$\mathcal{P}\delta R = \sqcup_{\widetilde{s}\in\mathcal{P}/R}\ \widetilde{s}\delta R.$$

We assume that $\mathcal{P}\delta R \neq \mathcal{P}R$. We write $a(\widetilde{s}\delta)$ as $\widetilde{a}(s\delta)$. By (1.13), we have

$$a(\widetilde{s}\delta r) = s\delta\varphi(r) + \widetilde{a}(s\delta), \qquad s \in P,\ r \in R. \tag{4.14}$$

Now for $s_1,\ s_2 \in P$ and $r_1,\ r_2 \in R$, we have

$$\begin{aligned} a(\widetilde{s}_1 r_1 \widetilde{s}_2 \delta r_2) &= a((\widetilde{s_1 s_2})\delta\delta^{-1}(\widetilde{s_1 s_2})^{-1}\widetilde{s}_1\widetilde{s}_2\delta\delta^{-1}\widetilde{s}_2^{-1} r_1 \widetilde{s}_2 \delta r_2) \\ &= s_1 s_2 \delta[\varphi(\delta^{-1}(\widetilde{s_1 s_2})^{-1}\widetilde{s}_1\widetilde{s}_2\delta^{-1}) + \varphi(\delta^{-1}\widetilde{s}_2^{-1} r_1 \widetilde{s}_2\delta) + \varphi(r_2)] + \widetilde{a}(s_1 s_2 \delta) \\ &= s_1 s_2 \varphi((\widetilde{s_1 s_2})^{-1}\widetilde{s}_1\widetilde{s}_2) + s_1\varphi(r_1) + s_1 s_2 \delta\varphi(r_2) + \widetilde{a}(s_1 s_2 \delta) \\ &= \varphi(\widetilde{s}_1\widetilde{s}_2(\widetilde{s_1 s_2})^{-1}) + s_1\varphi(r_1) + s_1 s_2\delta\varphi(r_2) + \widetilde{a}(s_1 s_2 \delta) \end{aligned}$$

using (1.13), (1.14) and (4.12). On the other hand, we have, using (4.10),

$$\begin{aligned} a(\widetilde{s}_1 r_1 \widetilde{s}_2 \delta r_2) &= s_1 a(\widetilde{s}_2 \delta r_2) + a(\widetilde{s}_1 r_1) \\ &= s_1[s_2 \delta \varphi(r_2) + a(\widetilde{s}_2 \delta)] + a(\widetilde{s}_1 r_1) \\ &= s_1 s_2 \delta \varphi(r_2) + s_1 \widetilde{a}(s_2 \delta) + s_1 \varphi(r_1) + \widetilde{a}(s_1). \end{aligned}$$

Comparing two formulas, we obtain

$$\varphi(\widetilde{s}_1 \widetilde{s}_2 (\widetilde{s_1 s_2})^{-1}) = s_1 \widetilde{a}(s_2 \delta) + \widetilde{a}(s_1) - \widetilde{a}(s_1 s_2 \delta). \tag{4.15}$$

Thus we have shown the following: For $f \in Z^2_{\mathrm{P}}(\Gamma, V)$, take $a \in C^1(\mathcal{F}, V)$ which satisfies (1.12). Define φ and $\widetilde{a}$ as above. Then (4.12) holds. Conversely take $\varphi \in H^1(R, V)^\Gamma$. Suppose that the 2-cocycle

$$(s_1, s_2) \longrightarrow \varphi(\widetilde{s}_1 \widetilde{s}_2 (\widetilde{s_1 s_2})^{-1})$$

of P taking values in V splits. There exists $\widetilde{a} \in C^1(P, V)$ with which (4.12) holds. Take a double coset decomposition $\mathcal{F} = \sqcup_\delta \mathcal{P} \delta R$. We put $\widetilde{a}(s\delta) = \widetilde{a}(s)$. Then we define $a \in C^1(\mathcal{F}, V)$ by the formula

$$a(\widetilde{s} \delta r) = s \delta \varphi(r) + \widetilde{a}(s \delta), \qquad r \in R.$$

Then a satisfies (4.10). Therefore the 2-cocycle f determined by (1.12) belongs to $Z^2_{\mathrm{P}}(\Gamma, V)$. Thus we have proved the following proposition.

Proposition 4.4. *On the right-hand side of (4.9), the subgroup which corresponds to $H^2_{\mathrm{P}}(\Gamma, V)$ consists of the class of $\varphi \in H^1(R, V)^\Gamma$ for which the 2-cocycle $(s_1, s_2) \mapsto \varphi(\widetilde{s}_1 \widetilde{s}_2 (\widetilde{s_1 s_2})^{-1})$ of P taking values in V splits.*

By Theorem 3.9, we have $H^2(P, V) = 0$ if $l_1 \neq l_2$. Hence the next proposition follows.

Proposition 4.5. *If $l_1 \neq l_2$, then we have $H^2(\Gamma, V) = H^2_{\mathrm{P}}(\Gamma, V)$.*

It is known that there are no holomorphic Eisenstein series of weight (k_1, k_2) if $k_1 \neq k_2$ ([Sh6], Proposition 2.1). We can interpret this proposition as the cohomological counter part of this fact.

Remark 4.6. In view of the results of Matsushima-Shimura [MS], Hida [Hi1], [Hi2] and Harder [Ha], we should be able to prove that $\dim H^2_P(\Gamma, V) = 4 \dim S_{l_1+2, l_2+2}(\Gamma)$. The author does not work out the details yet. The parabolic cohomology group is also discussed in [Hi2].

§5. Decompositions of $H^2(\Gamma, V)$

5.1. Let F be a real quadratic field and let $\Gamma = \mathrm{PSL}(2, \mathcal{O}_F)$. We define elements σ, μ, τ and η of Γ by

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mu = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \eta = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}.$$

Here we choose an ω so that $\mathcal{O}_F = \mathbf{Z} + \mathbf{Z}\omega$. Let $\mathcal{F}$ be the free group on four letters $\widetilde{\sigma}$, $\widetilde{\mu}$, $\widetilde{\tau}$, $\widetilde{\eta}$. Let $\pi : \mathcal{F} \longrightarrow \Gamma$ be the homomorphism such that

$$\pi(\widetilde{\sigma}) = \sigma, \quad \pi(\widetilde{\mu}) = \mu, \quad \pi(\widetilde{\tau}) = \tau, \quad \pi(\widetilde{\eta}) = \eta.$$

By Vaserštein [V], π is surjective. Let R be the kernel of π. For $\gamma \in \Gamma$, we choose a $\widetilde{\gamma} \in \mathcal{F}$ so that $\pi(\widetilde{\gamma}) = \gamma$. For $\gamma = \sigma$, μ, τ and η, we choose $\widetilde{\gamma}$ so that the notation to be consistent. We choose $\widetilde{1} = 1$. For other γ, we will specify the choice of $\widetilde{\gamma}$ later (cf. (5.2) and §6.2).

Let $f \in Z^2(\Gamma, V)$ be a normalized 2-cocycle. There exists $a \in C^1(\mathcal{F}, V)$ which satisfies

$$f(\gamma_1, \gamma_2) = \gamma_1 a(\widetilde{\gamma}_2) + a(\widetilde{\gamma}_1) - a(\widetilde{\gamma}_1\widetilde{\gamma}_2).$$

A corresponding element $\varphi \in H^1(R, V)^\Gamma$ to f is given by $\varphi = a|R$. As was shown in §1.5, adding a cobounday to f, we may assume that $f \in Z^2(\Gamma, V)$ is given by

$$f(\gamma_1, \gamma_2) = -\varphi(\widetilde{\gamma}_1\widetilde{\gamma}_2(\widetilde{\gamma_1\gamma_2})^{-1}), \qquad \gamma_1, \gamma_2 \in \Gamma. \tag{5.1}$$

Let $\mathcal{F}_P$ be the subgroup of $\mathcal{F}$ generated by $\widetilde{\mu}$, $\widetilde{\tau}$ and $\widetilde{\eta}$. Let π_P be the restriction of π to $\mathcal{F}_P$ and let R_P be the kernel of π_P. We see that R_P is generated by the elements corresponding to the relations (iv), (v), (vi) of Appendix and their conjugates. Suppose that f satisfies the parabolic condition (4.3). Then, by (4.12), we see that we may assume that $\varphi|R_P = 0$ in addition to (5.1), adding a cobounday to f if necessary.

Conversely assume that $\varphi|R_P = 0$. Take a complete set of representatives Δ for $P\backslash\Gamma$ and fix it. We have

$$\Gamma = \sqcup_{\delta\in\Delta} P\delta.$$

For $\gamma = p\delta$, $p \in P$, $\delta \in \Delta$, we define

$$\widetilde{\gamma} = \widetilde{p}\widetilde{\delta}. \tag{5.2}$$

In (5.1), write $\gamma_1 = p_1\delta_1$, $p_1 \in P$, $\delta_1 \in \Delta$, $\gamma_1\gamma_2 = p_2\delta_2$, $p_2 \in P$, $\delta_2 \in \Delta$. Let $p \in P$. Then we have

$$\widetilde{p\gamma_1} = \widetilde{pp_1}\widetilde{\delta}_1 = \widetilde{pp_1}(\widetilde{p}\widetilde{p}_1)^{-1}\widetilde{p}\widetilde{\gamma}_1, \qquad \widetilde{p\gamma_1\gamma_2} = \widetilde{pp_2}\widetilde{p}_2^{-1}\widetilde{\gamma_1\gamma_2}.$$

Hence, by (5.1), we have

$$\begin{aligned} f(p\gamma_1,\gamma_2) &= -\varphi(\widetilde{pp_1}(\widetilde{p}\widetilde{p}_1)^{-1}\widetilde{p}\widetilde{\gamma}_1\widetilde{\gamma}_2\{\widetilde{pp_2}(\widetilde{p}\widetilde{p}_2)^{-1}\widetilde{p}\widetilde{\gamma_1\gamma_2}\}^{-1}) \\ &= -\varphi(\widetilde{p}\widetilde{\gamma}_1\widetilde{\gamma}_2(\widetilde{\gamma_1\gamma_2})^{-1}\widetilde{p}^{-1}) = -p\varphi(\widetilde{\gamma}_1\widetilde{\gamma}_2(\widetilde{\gamma_1\gamma_2})^{-1}) = pf(\gamma_1,\gamma_2). \end{aligned}$$

Therefore f satisfies the parabolic condition (4.3).

The value $f(\sigma,\mu)$ of the cocycle is related to the critical values of the L-function. By (5.1), we have

$$f(\sigma,\mu) = -\varphi(\widetilde{\sigma}\widetilde{\mu}(\widetilde{\sigma\mu})^{-1}) = -\varphi(\widetilde{\sigma}\widetilde{\mu}(\widetilde{\mu^{-1}\sigma})^{-1}).$$

We assume that $\sigma \in \Delta$. Then we have

$$f(\sigma,\mu) = -\varphi(\widetilde{\sigma}\widetilde{\mu}\widetilde{\sigma}^{-1}(\widetilde{\mu^{-1}})^{-1}),$$

since $\widetilde{\mu^{-1}\sigma} = \widetilde{\mu^{-1}}\widetilde{\sigma}$. As $\widetilde{\mu^{-1}}\widetilde{\mu} \in R_P$, we have

$$\begin{aligned} f(\sigma,\mu) &= -\varphi(\widetilde{\sigma}\widetilde{\mu}\widetilde{\sigma}^{-1}\widetilde{\mu}) = -\varphi(\widetilde{\sigma}\widetilde{\mu}\widetilde{\sigma}^{-2}\widetilde{\sigma}\widetilde{\mu}) = -\varphi(\widetilde{\sigma}\widetilde{\mu}\widetilde{\sigma}^{-2}(\widetilde{\sigma}\widetilde{\mu})^{-1}\widetilde{\sigma}\widetilde{\mu}\widetilde{\sigma}\widetilde{\mu}) \\ &= -\sigma\mu\varphi(\widetilde{\sigma}^{-2}) - \varphi(\widetilde{\sigma}\widetilde{\mu}\widetilde{\sigma}\widetilde{\mu}). \end{aligned}$$

Therefore we obtain

$$f(\sigma,\mu) = -\varphi((\widetilde{\sigma}\widetilde{\mu})^2) + \sigma\mu\varphi(\widetilde{\sigma}^2). \tag{5.3}$$

5.2. Let us consider the action of Hecke operators. Let ϖ be a totally positive element of F. Let

$$\Gamma\begin{pmatrix}1 & 0\\ 0 & \varpi\end{pmatrix}\Gamma = \sqcup_{i=1}^{d}\Gamma\beta_i$$

be a coset decomposition. We put

$$c = \prod_{\nu=1}^{2}(\varpi^{(\nu)})^{(k_0+k_\nu)/2-2}.$$

Let $f \in Z^2(\Gamma, V)$ and put $g = cT(\varpi)f$. The explicit form of g is given as follows (cf. Proposition 1.3 and (2.45)). Let

$$\beta_i\gamma_1 = \delta_i^{(1)}\beta_{j(i)}, \quad \delta_i^{(1)} \in \Gamma, \qquad \beta_i\gamma_2 = \delta_i^{(2)}\beta_{k(i)}, \quad \delta_i^{(2)} \in \Gamma,$$

for $1 \le i \le d$. Here j and k are permutations on d letters. Then

$$g(\gamma_1,\gamma_2) = c\sum_{i=1}^{d}\beta_i^{-1}f(\beta_i\gamma_1\beta_{j(i)}^{-1}, \beta_{j(i)}\gamma_2\beta_{k(j(i))}^{-1}). \tag{5.4}$$

We assume that $f \in Z_P^2(\Gamma, V)$ and that it is given by (5.1) with $\varphi \in H^1(R, V)^\Gamma$ satisfying $\varphi | R_P = 0$. Then we have

$$g(\gamma_1, \gamma_2) = -c \sum_{i=1}^{d} \beta_i^{-1} \varphi(\widetilde{\beta_i \gamma_1 \beta_{j(i)}^{-1}} \widetilde{\beta_{j(i)} \gamma_2 \beta_{k(j(i))}^{-1}} (\widetilde{\beta_i \gamma_1 \gamma_2 \beta_{k(j(i))}^{-1}})^{-1}). \tag{5.5}$$

Let $\psi \in H^1(R, V)^\Gamma$ be a corresponding element to g. We are going to give an explicit form of ψ. There exists $b \in C^1(\mathcal{F}, V)$ such that

$$g(\bar{x}_1, \bar{x}_2) = x_1 b(x_2) + b(x_1) - b(x_1 x_2), \qquad x_1, x_2 \in \mathcal{F}$$

and ψ is given as the restriction of b to R. Here $\bar{x} = \pi(x)$, $x \in \mathcal{F}$. We assume that (ϖ) is a prime ideal. Then $d = N(\varpi) + 1$ and $\{\beta_i\}$ can be taken as

$$\left\{ \begin{pmatrix} 1 & u \\ 0 & \varpi \end{pmatrix}, u \mod \varpi, \quad \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Take $p \in P$ and let $\beta_i p \beta_{j(i)}^{-1} \in \Gamma$ for $1 \le i \le d$. Then we see easily that

$$\beta_i p \beta_{j(i)}^{-1} \in P, \qquad 1 \le i \le d. \tag{5.6}$$

By (5.5), (5.6) and $\varphi | R_P = 0$, we find

$$g(p_1, p_2) = 0, \qquad p_1, p_2 \in P. \tag{5.7}$$

We have

$$b(x_1 x_2) = x_1 b(x_2) + b(x_1) - g(\bar{x}_1, \bar{x}_2), \qquad x_1, x_2 \in \mathcal{F}$$

and we can use this formula to determine the value $b(x)$, $x \in \mathcal{F}$ by the induction on the length of the element x. As the initial conditions, we may assume that

$$b(\widetilde{\mu}) = 0, \quad b(\widetilde{\tau}) = 0, \quad b(\widetilde{\eta}) = 0, \quad b(\widetilde{\sigma}) = 0.$$

Then, by (5.7), we see that

$$b | \mathcal{F}_P = 0. \tag{5.8}$$

The next Proposition is a special case of Proposition 1.5.

Proposition 5.1. *Suppose $\gamma_j \in \Gamma$ are given for $1 \le j \le m$. For every j, we define $p_j \in S_d$ by*

$$\beta_i \gamma_j \beta_{p_j(i)}^{-1} \in \Gamma, \qquad 1 \le i \le d.$$

We define $q_j \in S_d$ inductively by

$$q_1 = p_1, \qquad q_k = p_k q_{k-1}, \quad 2 \le k \le m.$$

We assume that $\gamma_j \in P$ or $\gamma_j = \sigma$ for every j. Then we have

(5.9)

$$b(\widetilde{\gamma}_1 \widetilde{\gamma}_2 \cdots \widetilde{\gamma}_m)$$
$$= c \sum_{i=1}^{d} \beta_i^{-1} \varphi(\widetilde{\beta_i \gamma_1 \beta_{q_1(i)}^{-1}} \widetilde{\beta_{q_1(i)} \gamma_2 \beta_{q_2(i)}^{-1}} \cdots \widetilde{\beta_{q_{m-1}(i)} \gamma_m \beta_{q_m(i)}^{-1}} (\widetilde{\beta_i \gamma_1 \gamma_2 \cdots \gamma_m \beta_{q_m(i)}^{-1}})^{-1}).$$

5.3. For the practical computation, it is convenient to decompose $H^2(\Gamma, V)$ into a direct sum of subspaces under the action of the automorphisms of Γ. We put

$$Z = \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \middle| \, u \in E_F \right\},$$

which is the center of $\mathrm{GL}(2, \mathcal{O}_F)$. Then we have

$$Z \cdot \mathrm{SL}(2, \mathcal{O}_F)/Z \cong \mathrm{SL}(2, \mathcal{O}_F)/\{\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\} = \mathrm{PSL}(2, \mathcal{O}_F) = \Gamma.$$

By this isomorphism, we regard Γ as a subgroup of $\mathrm{PGL}(2, \mathcal{O}_F) = \mathrm{GL}(2, \mathcal{O}_F)/Z$. Hereafter we assume that l_1 and l_2 are even. When l is even, we define a representation ρ'_l of $\mathrm{GL}(2, \mathbf{C})$ by

$$\rho'_l(g) = \rho_l(g) \det(g)^{-l/2}, \qquad g \in \mathrm{GL}(2, \mathbf{C}).$$

Then ρ'_l is trivial on the center. We put $\rho' = \rho'_{l_1} \otimes \rho'_{l_2}$. By $gv = \rho'(g)v$, $g \in \mathrm{GL}(2, \mathcal{O}_F)$, $v \in V$, we regard V as a left $\mathrm{GL}(2, \mathcal{O}_F)$-module. Since $\rho'(z) = \mathrm{id}$, $z \in Z$, we can regard V as a $\mathrm{PGL}(2, \mathcal{O}_F)$-module. Since $\rho'|\Gamma = \rho|\Gamma$, the Γ-module structure of V is the same as before.

We have

$$\mathrm{PGL}(2, \mathcal{O}_F)/\mathrm{PSL}(2, \mathcal{O}_F) \cong E_F/E_F^2 \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}.$$

By conjugation, $\mathrm{PGL}(2, \mathcal{O}_F)$ acts on $H^2(\Gamma, V)$ and it decomposes into a direct sum of four subspaces. We put

$$\nu = \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}, \qquad \delta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We see that $\mathrm{PGL}(2, \mathcal{O}_F)$ is generated by ν and δ over $\mathrm{PSL}(2, \mathcal{O}_F)$. We first examine the action of ν. For $f \in Z^2(\Gamma, V)$, define $\widetilde{e}f \in Z^2(\Gamma, V)$ by (cf. (1.3))

$$\widetilde{e}f(\gamma_1, \gamma_2) = \nu^{-1} f(\nu\gamma_1\nu^{-1}, \nu\gamma_2\nu^{-1}), \qquad \gamma_1, \gamma_2 \in \Gamma. \tag{5.10}$$

Then $\widetilde{e}$ induces an automorphism e of $H^2(\Gamma, V)$. Since $\nu^2 = \mu$, $\widetilde{e}^2$ is obtained from the inner automorphism by μ. Hence $e^2 = 1$. By (5.10), we see that $\widetilde{e}f$ is a parabolic cocycle if f is parabolic. Therefore, by the action of e, we have the decompositions

$$H^2(\Gamma, V) = H^2(\Gamma, V)^+ \oplus H^2(\Gamma, V)^-, \quad H^2_P(\Gamma, V) = H^2_P(\Gamma, V)^+ \oplus H^2_P(\Gamma, V)^-.$$

Here we put

$$H^2(\Gamma, V)^\pm = \{c \in H^2(\Gamma, V) \mid ec = \pm c\},\ H^2_P(\Gamma, V)^\pm = \{c \in H^2_P(\Gamma, V) \mid ec = \pm c\}.$$

Explicitly the decomposition is given by

$$f = \frac{1}{2}\big[(1+\widetilde{e})f + (1-\widetilde{e})f\big], \qquad f \in Z^2(\Gamma, V).$$

Proposition 5.2. *Let $k = (k_1, k_2)$, $k_1 \geq k_2$, k_1 and k_2 are even. Let $\Omega \in S_k(\Gamma)$ and let $f = f(\Omega)$ be the normalized parabolic 2-cocycle attached to Ω by (4.1). We assume that the class number of F in the narrow sense is 1 and that Ω is a nonzero Hecke eigenform.*

(1) If $k_1 \neq k_2$, we assume $k_2 \geq 6$. If $k_1 = k_2$, we assume $k_2 \geq 8$. Then the cohomology class of $(1+\widetilde{e})f$ in $H^2(\Gamma, V)$ is non-trivial.

(2) If $k_1 \neq k_2$, we assume $k_2 \geq 4$. If $k_1 = k_2$, we assume $k_2 \geq 6$. Then the cohomology class of $(1-\widetilde{e})f$ in $H^2(\Gamma, V)$ is non-trivial.

Proof. We apply Lemma 4.2 in a similar way to the proof of Proposition 4.3. We use the same notation as there. By (5.10), we have

$$(\widetilde{e}f)(\sigma, \mu) = \nu^{-1} f(\nu\sigma\nu^{-1}, \nu\mu\nu^{-1}) = \nu^{-1} f(\mu\sigma, \mu) = \nu^{-1}\mu f(\sigma, \mu) = \nu f(\sigma, \mu).$$

We have

$$\nu(\mathbf{e}_i \otimes \mathbf{e}'_{i-(l_1-l_2)/2}) = N(\epsilon)^{l_1/2+1-i}(\mathbf{e}_i \otimes \mathbf{e}'_{i-(l_1-l_2)/2}) = N(\epsilon)^{k_1/2-i}(\mathbf{e}_i \otimes \mathbf{e}'_{i-(l_1-l_2)/2}).$$

By the assumption, we have $N(\epsilon) = -1$. The range of i is $\frac{k_1}{2} - \frac{l_2}{2} \leq i \leq \frac{k_1}{2} + \frac{l_2}{2}$. We see that $L(l_1+2-i, \Omega)$ is non-vanishing if $i \neq k_1/2$. To conclude the non-vanishing of the cohomology class of $(1+\widetilde{e})f$, it suffices to find an even integer j such that $0 < j \leq l_2/2$ if $k_1 \neq k_2$ and $0 < j \leq l_2/2 - 1$ if $k_1 = k_2$. Such a j exists under the condition stated in (1). To conclude the non-vanishing of the cohomology class of $(1-\widetilde{e})f$, it suffices to find an odd integer j such that $0 < j \leq l_2/2$ if $k_1 \neq k_2$ and $0 < j \leq l_2/2 - 1$ if $k_1 = k_2$. Such a j exists under the condition stated in (2). This completes the proof.

We put

$$\overline{\Gamma}^* = \{\gamma \in \mathrm{GL}(2, \mathcal{O}_F) \mid \det(\gamma) = \epsilon^n, n \in \mathbf{Z}\}, \qquad \Gamma^* = Z\overline{\Gamma}^*/Z.$$

Then Γ^* is generated by ν over Γ and we have $[\Gamma^* : \Gamma] = 2$. Let

$$\mathrm{Res} : H^2(\Gamma^*, V) \longrightarrow H^2(\Gamma, V), \qquad T : H^2(\Gamma, V) \longrightarrow H^2(\Gamma^*, V)$$

be the restriction map and the transfer map respectively.

Proposition 5.3. *We have*

(1) $\mathrm{Res}(H^2(\Gamma^*, V)) = H^2(\Gamma, V)^+$.
(2) $T(H^2(\Gamma, V)^+) = H^2(\Gamma^*, V)$.
(3) $\mathrm{Ker}(T) = H^2(\Gamma, V)^-$.

Proof. It is clear that $\mathrm{Res}(H^2(\Gamma^*, V)) \subseteq H^2(\Gamma, V)^+$. Let $f \in Z^2(\Gamma, V)$ and take a coset decomposition $\Gamma^* = \Gamma \sqcup \nu^{-1}\Gamma$. Then by Proposition 1.2, we have

$$\widetilde{T}(f)(\gamma_1, \gamma_2) = f(\gamma_1\nu^a, \nu^{-a}\gamma_2\nu^b) + \nu^{-1}f(\nu\gamma_1\nu^c, \nu^{-c}\gamma_2\nu^d)$$

for γ_1, $\gamma_2 \in \Gamma^*$. Here $\widetilde{T}(f)$ denotes a cocycle which represents the transfer of the class of f; a, b, c, $d = 0$ or -1 and they are determined so that all arguments on the right-hand side belong to Γ. In particular, if γ_1, $\gamma_2 \in \Gamma$, then we have

$$\widetilde{T}(f)(\gamma_1, \gamma_2) = f(\gamma_1, \gamma_2) + \nu^{-1}f(\nu\gamma_1\nu^{-1}, \nu\gamma_2\nu^{-1}).$$

Therefore we have

$$\mathrm{Res} \circ T = 1 + e$$

on $H^2(\Gamma, V)$. This formula combined with (1.6) shows that $T \circ \mathrm{Res}$ and $\mathrm{Res} \circ T$ are the multiplication by 2 on $H^2(\Gamma^*, V)$ and $H^2(\Gamma, V)^+$ respectively. Hence (1) and (2) follow. Then (3) follows since $H^2(\Gamma, V)^- \subset \mathrm{Ker}(T)$ and $T|H^2(\Gamma, V)^+$ is injective. This completes the proof.

5.4. We have

$$H^2(\Gamma, V) \cong H^1(R, V)^\Gamma/\mathrm{Im}(H^1(\mathcal{F}, V)).$$

Let us consider the action of e on the right-hand side under this isomorphism. We use the same notation as in §5.1. Let ξ be the automorphism of Γ defined by $\xi(\gamma) = \nu\gamma\nu^{-1}$, $\gamma \in \Gamma$. Put

$$\epsilon = A + B\omega, \qquad \epsilon\omega = C + D\omega.$$

Then we have $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}(2, \mathbf{Z})$. We have

$$\nu\sigma\nu^{-1} = \sigma\mu^{-1}, \quad \nu\mu\nu^{-1} = \mu, \quad \nu\tau\nu^{-1} = \tau^A\eta^B, \quad \nu\eta\nu^{-1} = \tau^C\eta^D.$$

Using Lemma 3.11, we can check that there exists an automorphism $\widetilde{\xi}$ of $\mathcal{F}$ which satisfies

$$\pi(\widetilde{\xi}(g)) = \xi(\pi(g)), \qquad g \in \mathcal{F}. \tag{5.11}$$

Now let $f \in Z^2(\Gamma, V)$ and take $a \in C^1(\mathcal{F}, V)$ so that

$$f(\pi(g_1), \pi(g_2)) = g_1 a(g_2) + a(g_1) - a(g_1 g_2), \qquad g_1, g_2 \in \mathcal{F}.$$

Then we have

$$\begin{aligned} &(\widetilde{e}f)(\pi(g_1), \pi(g_2)) = \nu^{-1} f(\xi(\pi(g_1)), \xi(\pi(g_2))) = \nu^{-1} f(\pi(\widetilde{\xi}(g_1)), \pi(\widetilde{\xi}(g_2)) \\ &= g_1\nu^{-1} a(\widetilde{\xi}(g_2)) + \nu^{-1} a(\widetilde{\xi}(g_1)) - \nu^{-1} a(\widetilde{\xi}(g_1 g_2)) \end{aligned}$$

for g_1, $g_2 \in \mathcal{F}$. Put

$$a'(g) = \nu^{-1} a(\widetilde{\xi}(g)), \qquad g \in \mathcal{F}.$$

Then we have

$$(\widetilde{e}f)(\pi(g_1), \pi(g_2)) = g_1 a'(g_2) + a'(g_1) - a'(g_1 g_2), \qquad g_1, g_2 \in \mathcal{F}.$$

Thus we obtain the following proposition.

Proposition 5.4. *Let $f \in Z^2(\Gamma, V)$ and let $\varphi \in H^1(R, V)^\Gamma$ be a corresponding element. Then a corresponding element ψ of $H^1(R, V)^\Gamma$ to $\widetilde{e}f$ is given by*

$$\psi(r) = \nu^{-1}\varphi(\widetilde{\xi}(r)), \qquad r \in R.$$

We can check easily that the map $\varphi \longrightarrow \psi$ induces a map from $H^1(R, V)^\Gamma/\mathrm{Im}(H^1(\mathcal{F}, V))$ to itself and gives an automorphism of order 2.

5.5. For the actual computation, the cohomology group $H^2(\Gamma^*, V)$ is easier to handle than $H^2(\Gamma, V)$. By the action of δ, we can further decompose $H^2(\Gamma^*, V)$ so that

$$H^2(\Gamma^*, V) = H^2(\Gamma^*, V)^+ \oplus H^2(\Gamma^*, V)^-.$$

Let $\widetilde{d}$ (resp. d) denote the action of δ on $Z^2(\Gamma^*, V)$ (resp. $H^2(\Gamma^*, V)$).

Proposition 5.5. *Let $k = (k_1, k_2)$, $k_1 \geq k_2$, k_1 and k_2 are even. Let $\Omega \in S_k(\Gamma)$ and let $f = f(\Omega)$ be the normalized parabolic 2-cocycle attached to Ω by (4.1). We assume that the class number of F in the narrow sense is 1 and that Ω is a nonzero Hecke eigenform. Take $f^* \in Z^2(\Gamma^*, V)$ so that $f^*|\Gamma = (1 + \widetilde{e})f$.* [3] *If $k_1 \neq k_2$, we assume $k_2 \geq 6$. If $k_1 = k_2$, we assume $k_2 \geq 8$. Then the cohomology class of $(1 + \widetilde{d})f^*$ in $H^2(\Gamma^*, V)$ is non-trivial.*

Proof. The proof is similar to that of Proposition 5.2. We consider the restriction f_0 of $(1 + \widetilde{d})f^*$ to Γ. Since δ commutes with σ and μ, we find

$$f_0(\sigma, \mu) = (1 + \delta)(1 + \nu)f(\sigma, \mu).$$

We have

$$\delta(\mathbf{e}_i \otimes \mathbf{e}'_{i-(l_1-l_2)/2}) = \mathbf{e}_i \otimes \mathbf{e}'_{i-(l_1-l_2)/2}.$$

Hence the assertion follows from Lemma 4.2 in the same way as Proposition 5.2.

Until the end of this subsection, we assume that σ, ν and τ generate Γ^*. (This assumption is satisfied if $\mathcal{O}_F = \mathbf{Z} + \mathbf{Z}\epsilon$.) Let $\mathcal{F}^*$ be the free group on three letters $\widetilde{\sigma}$, $\widetilde{\nu}$ and $\widetilde{\tau}$. We define a surjective homomorphism π^* of $\mathcal{F}^*$ onto Γ^* by

$$\pi^*(\widetilde{\sigma}) = \sigma, \qquad \pi^*(\widetilde{\nu}) = \nu, \qquad \pi^*(\widetilde{\tau}) = \tau$$

and let R^* be the kernel of π^*. We see that δ commutes with σ and ν and $\delta\tau\delta^{-1} = \tau^{-1}$. We can define an automorphism $x \mapsto x_\delta$ of $\mathcal{F}^*$ by $(\widetilde{\sigma})_\delta = \widetilde{\sigma}$, $(\widetilde{\nu})_\delta = \widetilde{\nu}$, $(\widetilde{\tau})_\delta = \widetilde{\tau}^{-1}$. Then we have

$$\pi^*(x_\delta) = \delta\pi^*(x)\delta^{-1}, \qquad x \in \mathcal{F}^*.$$

The following proposition can be shown in a similar manner to Proposition 5.4.

Proposition 5.6. *Let $f \in Z^2(\Gamma^*, V)$ and let $\varphi \in H^1(R^*, V)^{\Gamma^*}$ be a corresponding element. Then a corresponding element ψ of $H^1(R^*, V)^{\Gamma^*}$ to $\widetilde{d}f$ is given by*

$$\psi(r) = \delta^{-1}\varphi(r_\delta), \qquad r \in R^*.$$

Let $\varphi \in H^1(R^*, V)^{\Gamma^*}$. We define $\varphi_\delta \in H^1(R^*, V)^{\Gamma^*}$ by the formula

$$\varphi_\delta(r) = \delta^{-1}\varphi(r_\delta). \tag{5.12}$$

Then we can check easily that $(\varphi_\delta)_\delta = \varphi$ and $H^1(R^*, V)^{\Gamma^*}$ decomposes into a direct sum of ± 1 eigenspaces under the action of δ:

$$H^1(R^*, V)^{\Gamma^*} = H^1(R^*, V)^{\Gamma^*,+} \oplus H^1(R^*, V)^{\Gamma^*,-}. \tag{5.13}$$

[3] $f^* = \widetilde{T}(f)$ satisfies this condition.

5.6. Let l_1 and l_2 be nonnegative even integers. We assume that $l_1 \geq l_2$. Let $\Omega \in S_{l_1+2,l_2+2}(\Gamma)$. Define $L(s,\Omega)$ and $R(s,\Omega)$ by (2.4) and (2.5) respectively. The functional equation is (cf. (2.7))

$$R(s,\Omega) = (-1)^{(l_1+l_2)/2} R(l_1+2-s,\Omega).$$

For an integer m, $L(m,\Omega)$ is a critical value if and only if

$$\frac{l_1-l_2}{2}+1 \leq m \leq \frac{l_1+l_2}{2}+1. \tag{5.14}$$

The central critical value is $L(l_1/2+1,\Omega)$ which vanishes if $(l_1+l_2)/2$ is odd. By (2.30), we have

$$R(m,\Omega) = (-1)^m i^{(l_1-l_2)/2}(2\pi)^{(l_2-l_1)/2} P_{m-1,m-1-(l_1-l_2)/2}. \tag{5.15}$$

Here $P_{s,t}$ denotes the period integral given by (2.25). Let $f = f(\Omega) \in Z_P^2(\Gamma, V)$ be the parabolic 2-cocycle defined by (4.1). Then we have

$$f(\sigma,\mu) = -\int_{i\epsilon^{-1}}^{i\epsilon}\int_0^{i\infty} \mathfrak{d}(\Omega)$$

and $-P_{m-1,m-1-(l_1-l_2)/2}$ is equal to the coefficient of $\mathbf{e}_{l_1+2-m}\otimes\mathbf{e}'_{(l_1+l_2)/2+2-m}$ in $f(\sigma,\mu)$.

Using the operator $\widetilde{e}$ (cf. (5.10)), we define

$$f^+ = (1+\widetilde{e})f, \qquad f^- = (1-\widetilde{e})f.$$

We have $f^\pm \in Z_P^2(\Gamma,V)$. As was shown in the proof of Proposition 5.2, we have

$$f^+(\sigma,\mu) = (1+\nu)f(\sigma,\mu), \qquad f^-(\sigma,\mu) = (1-\nu)f(\sigma,\mu). \tag{5.16}$$

We have

$$\nu(\mathbf{e}_{l_1+2-m}\otimes\mathbf{e}'_{(l_1+l_2)/2+2-m}) = N(\epsilon)^{m-1-l_1/2}\mathbf{e}_{l_1+2-m}\otimes\mathbf{e}'_{(l_1+l_2)/2+2-m}. \tag{5.17}$$

Assume $N(\epsilon) = -1$. Suppose that $l_1/2$ is even. By (5.17), we see that $f^+(\sigma,\mu)$ contains information on $R(m,\Omega)$ for odd m and $f^-(\sigma,\mu)$ contains information on $R(m,\Omega)$ for even m. If $l_1/2$ is odd, then $f^+(\sigma,\mu)$ contains information on $R(m,\Omega)$ for even m and $f^-(\sigma,\mu)$ contains information on $R(m,\Omega)$ for odd m.

To treat f^- efficiently, we will need more techniques which will be explained in the next section.

§6. Numerical examples I

6.1. In this section, we assume that $F = \mathbf{Q}(\sqrt{5})$. (The formulas (6.1) $\sim$ (6.6) and those given in §6.5 are valid for any real quadratic field.) We use the notation of §5. The elements σ, ν and τ of Γ^* satisfy the relations

$$\text{(i}'\text{)} \qquad \sigma^2 = 1.$$

$$\text{(ii}'\text{)} \qquad (\sigma\tau)^3 = 1.$$

$$\text{(iii}'\text{)} \qquad (\sigma\nu)^2 = 1.$$

$$\text{(iv}'\text{)} \qquad \tau\nu\tau\nu^{-1} = \nu\tau\nu^{-1}\tau.$$

$$\text{(v}'\text{)} \qquad \nu^2\tau\nu^{-2} = \tau\nu\tau\nu^{-1}.$$

Theorem 6.1. *The fundamental relations satisfied by the generators σ, ν, τ of Γ^* are (i$'$) $\sim$ (v$'$).*

This theorem follows from Theorem A.1. We sketch a proof. We have $\mu = \nu^2$, $\eta = \nu\tau\nu^{-1}$. Then we can check easily that the relations (i) $\sim$ (vii) in Theorem A.1 follow from (i$'$) $\sim$ (v$'$). Suppose that

$$(*) \qquad u_1u_2\cdots u_m = 1$$

is a relation. Here u_i is one of σ, ν, ν^{-1}, τ, τ^{-1}. In $(*)$, we substitute ν^{-1} by $\mu^{-1}\nu$. Then we obtain a relation

$$(**) \qquad v_1v_2\cdots v_n = 1.$$

Here v_i is one of σ, ν, μ^{-1}, τ, τ^{-1}. The number of v_i such that $v_i = \nu$ is even. If this number is 0, then $(**)$ is the relation among the elements σ, μ and τ. If this number is positive, then in $(**)$, a term of the form $\nu X\nu$ is contained, where X is an expression which contains only σ, τ and μ. We may replace $\nu X\nu$ by $\nu X\nu^{-1}\mu$. By the relations

$$\nu\sigma\nu^{-1} = \sigma\nu^{-2} = \sigma\mu^{-1}, \qquad \nu\tau\nu^{-1} = \eta,$$

$\nu X\nu^{-1}$ is transformed to an expression which contains only σ, μ, τ, η and their inverses. Repeating this procedure, $(**)$ can be reduced to a relation among the elements σ, μ, τ and η. By Theorem A.1, this relation follows from the fundamental relations (i) $\sim$ (vii). Since (i) $\sim$ (vii) follow from (i′) $\sim$ (v′), our assertion is proved.

Let $\mathcal{F}^*$ be the free group on three letters $\widetilde{\sigma}$, $\widetilde{\nu}$, $\widetilde{\tau}$. We define a surjective homomorphism $\pi^* : \mathcal{F}^* \longrightarrow \Gamma^*$ by $\pi^*(\widetilde{\sigma}) = \sigma$, $\pi^*(\widetilde{\nu}) = \nu$, $\pi^*(\widetilde{\tau}) = \tau$. Let R^* be the kernel of π^*. We have $\Gamma^* = \mathcal{F}^*/R^*$. By Theorem 6.1, R^* is generated by the elements

(i*) $$\widetilde{\sigma}^2,$$

(ii*) $$(\widetilde{\sigma}\widetilde{\tau})^3,$$

(iii*) $$(\widetilde{\sigma}\widetilde{\nu})^2,$$

(iv*) $$\widetilde{\tau}\widetilde{\nu}\widetilde{\tau}\widetilde{\nu}^{-1}(\widetilde{\nu}\widetilde{\tau}\widetilde{\nu}^{-1}\widetilde{\tau})^{-1},$$

(v*) $$\widetilde{\nu}^2\widetilde{\tau}\widetilde{\nu}^{-2}(\widetilde{\tau}\widetilde{\nu}\widetilde{\tau}\widetilde{\nu}^{-1})^{-1}$$

and their conjugates.

Let P^* be the subgroup of Γ^* consisting of elements which can be represented by upper triangular matrices. Let $\mathcal{F}_{P^*}$ be the subgroup of $\mathcal{F}^*$ generated by $\widetilde{\nu}$ and $\widetilde{\tau}$. Then $\pi^*|\mathcal{F}_{P^*} : \mathcal{F}_{P^*} \longrightarrow P^*$ is surjective. Let R_{P^*} be the kernel of this homomorphism. We see that R_{P^*} is generated by (iv*) and (v*) and their conjugates.

We have $[\mathcal{F}^* : (\pi^*)^{-1}(\Gamma)] = 2$. The following lemma can be proved easily by applying the method of Reidemeister–Schreier (cf. Schreier [Sc], Suzuki [Su], §6).

Lemma 6.2. *The group $(\pi^*)^{-1}(\Gamma)$ is the free group on five elements $\widetilde{\sigma}$, $\widetilde{\nu}^2$, $\widetilde{\tau}$, $\widetilde{\nu}\widetilde{\sigma}\widetilde{\nu}^{-1}$ and $\widetilde{\nu}\widetilde{\tau}\widetilde{\nu}^{-1}$.*

We put $\widetilde{\nu}^2 = \widetilde{\mu}$, $\widetilde{\nu}\widetilde{\tau}\widetilde{\nu}^{-1} = \widetilde{\eta}$. Let $\mathcal{F}$ be the free group on four elements $\widetilde{\sigma}$, $\widetilde{\mu}$, $\widetilde{\tau}$ and $\widetilde{\eta}$. Then our notation becomes consistent with that given in the beginning of section 5. We have $\mathcal{F}R^* = (\pi^*)^{-1}(\Gamma)$.

6.2. For every $\gamma \in \Gamma^*$, we choose $\widetilde{\gamma} \in \mathcal{F}^*$ so that $\pi^*(\widetilde{\gamma}) = \gamma$. For explicit calculations, it is necessary to specify the choice of $\widetilde{\gamma}$. First let $p \in P$. We can write $p = \mu^a\tau^b\eta^c$ and this expression is unique. We put $\widetilde{p} = \widetilde{\mu}^a\widetilde{\tau}^b\widetilde{\eta}^c$. Next let $p \in P^*$. We have $p \in P$ or $p = \nu p_1$ with $p_1 \in P$. In the latter case, we put $\widetilde{p} = \widetilde{\nu}\widetilde{p}_1$.

Let Δ be a complete set of representatives for $P\backslash\Gamma$ as in §5.1. Then Δ is also a complete set of representatives for $P^*\backslash\Gamma^*$. For $\gamma \in \Gamma^*$, we write $\gamma = p\delta$

with $p \in P^*$, $\delta \in \Delta$ and put $\widetilde{\gamma} = \widetilde{p}\widetilde{\delta}$. Our task is to specify the choice of Δ and define $\widetilde{\delta}$ for $\delta \in \Delta$. To specify Δ is equivalent to choose one element from every coset $P\gamma$, $\gamma \in \Gamma$. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(1) The case where $P\gamma = P$. We take 1 as the representative. We take the identity element of $\mathcal{F}$ as $\widetilde{1}$.

(2) The case where $c \in E_F$. We can take an element of the form $\begin{pmatrix} 0 & -1 \\ 1 & d \end{pmatrix}$ as the representative. We define

$$\widetilde{\begin{pmatrix} 0 & -1 \\ 1 & d \end{pmatrix}} = \widetilde{\sigma}\widetilde{\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}}.$$

(3) [4] The case where $c \neq 0$ and $c \notin E_F$. We note that $\mathcal{O}_F$ is a Euclidean ring with respect to the absolute value of the norm (cf. [HW], Theorem 247, p. 213): For every x, $y \in \mathcal{O}_F$, $x \neq 0$, there exist q, $r \in \mathcal{O}_F$ such that

$$y = qx + r, \qquad |N(r)| < |N(x)|.$$

We have

$$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ua & ub \\ u^{-1}c & u^{-1}d \end{pmatrix}, \quad \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+tc & b+td \\ c & d \end{pmatrix}.$$

First mulplying γ on the left by $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$, $u \in E_F$, we normalize c so that

$$c \gg 0, \qquad 1 \leq c'/c < \epsilon^2.$$

Next mulplying γ on the left by $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $t \in \mathcal{O}_F$, we may assume that $|N(a)| < |N(c)|$ by the Euclidean algorithm. However to specify the choice of t is not necessarily easy. In other words, there can be many choices of such a's. We make the preference order of the choice of a as follows. Put $a = \alpha + \beta\epsilon$, α, $\beta \in \mathbf{Z}$.

1. $|\alpha| + |\beta|$ is minimum. 2. $|\alpha|$ is minimum. 3. $|\beta|$ is minimum. 4. $\alpha \geq 0$. 5. $\beta \geq 0$.

We define $\widetilde{\delta}$ for $\delta \in \Delta$ as follows. We put $\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and proceed by induction on $|N(c)|$. The case $|N(c)| = 0$ or 1 is settled by (1) and (2). By

[4] In this paper, this step will be used for the actual calculations only in the case $a \in E_F$. Since it will become necessary in future calculations, we write one (tentative) algorithm explicitly.

our choice of Δ, we have $|N(a)| < |N(c)|$. Put $\sigma^{-1}\delta = p_1\delta_1$, $p_1 \in P$, $\delta_1 \in \Delta$, $\delta_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$. We have $|N(c_1)| = |N(a)| < |N(c)|$. We define $\widetilde{\delta} = \widetilde{\sigma}\widetilde{p}_1\widetilde{\delta}_1$.

6.3. Let $f \in Z^2_P(\Gamma, V)$ be a normalized parabolic 2-cocycle. We first consider f^+ (cf. §5.6). We put $f^* = \widetilde{T}(f)$. Then $f^* \in Z^2(\Gamma^*, V)$ and $f^*|\Gamma = f^+$ (cf. §5.3). We can verify easily the parabolic condition

$$f^*(p\gamma_1, \gamma_2) = pf^*(\gamma_1, \gamma_2), \qquad p \in P^*,\ \gamma_1, \gamma_2 \in \Gamma^*. \tag{6.1}$$

We have

$$H^2(\Gamma^*, V) \cong H^1(R^*, V)^{\Gamma^*}/\mathrm{Im}(H^1(\mathcal{F}^*, V)). \tag{6.2}$$

Let $\varphi \in H^1(R^*, V)^{\Gamma^*}$ be a corresponding element to f^*. We recall that φ is obtained in the following way. There exists $a \in C^1(\mathcal{F}^*, V)$ such that

$$a(g_1g_2) = g_1a(g_2) + a(g_1) - f^*(\pi^*(g_1), \pi^*(g_2)), \qquad g_1, g_2 \in \mathcal{F}^*. \tag{6.3}$$

Then $\varphi = a|R^*$. We may regard (6.3) as a rule for determining the value $a(g)$ according to the length of a word $g \in \mathcal{F}^*$. We can take $a(\widetilde{\sigma}) = a(\widetilde{\nu}) = a(\widetilde{\tau}) = 0$. Then we have $a|\mathcal{F}_{P^*} = 0$, since (6.1) yields $f^*(p, \gamma) = 0$, $p \in P^*$, $\gamma \in \Gamma^*$. In particular, we have

$$\varphi|R_{P^*} = 0. \tag{6.4}$$

As shown in §1.5, we may assume that

$$f^*(\gamma_1, \gamma_2) = -\varphi(\widetilde{\gamma}_1\widetilde{\gamma}_2(\widetilde{\gamma_1\gamma_2})^{-1}) \tag{6.5}$$

adding a coboundary to f^*. By (6.4), we can check that f^* satisfies (6.1) in the same way as in §5.1. We have (cf. (5.3))

$$f^*(\sigma, \mu) = -\varphi((\widetilde{\sigma}\widetilde{\mu})^2) + \sigma\mu\varphi(\widetilde{\sigma}^2).$$

We have

$$\begin{aligned} &\varphi((\widetilde{\sigma}\widetilde{\mu})^2) = \varphi(\widetilde{\sigma}\widetilde{\nu}^2\widetilde{\sigma}\widetilde{\nu}^2) = \varphi(\widetilde{\sigma}\widetilde{\nu}\widetilde{\sigma}\widetilde{\nu}\widetilde{\nu}^{-1}\widetilde{\sigma}^{-1}\widetilde{\nu}\widetilde{\sigma}\widetilde{\nu}^2) \\ &= \varphi(\widetilde{\sigma}\widetilde{\nu}\widetilde{\sigma}\widetilde{\nu}) + \varphi(\widetilde{\nu}^{-1}\widetilde{\sigma}^{-2}\widetilde{\nu}) + \varphi(\widetilde{\nu}^{-1}\widetilde{\sigma}\widetilde{\nu}\widetilde{\sigma}\widetilde{\nu}^2) = (1+\nu^{-1})\varphi((\widetilde{\sigma}\widetilde{\nu})^2) - \nu^{-1}\varphi(\widetilde{\sigma}^2). \end{aligned}$$

Therefore we obtain

$$f^*(\sigma, \mu) = -(1+\nu^{-1})\varphi((\widetilde{\sigma}\widetilde{\nu})^2) + (\sigma\mu + \nu^{-1})\varphi(\widetilde{\sigma}^2). \tag{6.6}$$

Clearly φ is determined by its values on the elements (i*) ∼ (v*). By (6.4), φ takes the value 0 on the elements (iv*) and (v*). We have $\sigma\varphi(\widetilde{\sigma}^2) = \varphi(\widetilde{\sigma}^2)$.

Take $h \in H^1(\mathcal{F}^*, V)$ so that $h(\widetilde{\sigma}) = -\varphi(\widetilde{\sigma}^2)/2$, $h(\widetilde{\nu}) = 0$, $h(\widetilde{\tau}) = 0$. Adding $h|R^*$ to φ, we may assume that $\varphi(\widetilde{\sigma}^2) = 0$; φ still satisfies (6.4).

We analyze the process of adding $h|R^*$ to φ in more detail. For S, T, $U \in V$, we can find $h \in H^1(\mathcal{F}^*, V)$ such that

$$h(\widetilde{\sigma}) = S, \qquad h(\widetilde{\tau}) = T, \qquad h(\widetilde{\nu}) = U.$$

We find easily that the conditions that h vanishes on the elements (iv*) and (v*) are

$$(1 + \tau\nu - \nu - \nu\tau\nu^{-1})T + (\tau - 1)(1 - \nu\tau\nu^{-1})U = 0, \tag{6.7}$$

$$(\nu^2 - 1 - \tau\nu)T + (1 + \nu - \nu^2\tau\nu^{-1} - \tau)U = 0 \tag{6.8}$$

respectively. We have

$$h(\widetilde{\sigma}^2) = (1 + \sigma)S. \tag{6.9}$$

We put

$$A = \varphi((\widetilde{\sigma}\widetilde{\nu})^2), \qquad B = \varphi((\widetilde{\sigma}\widetilde{\tau})^3).$$

We note that

$$\sigma\nu A = A, \qquad \sigma\tau B = B. \tag{6.10}$$

Our objective is to determine A explicitly.

6.4. Let us consider the Hecke operators. We put $g^* = T(\varpi)f^*$ where g^* is defined by (5.4) with Γ^* in place of Γ. Let $\psi \in H^1(R^*, V)^{\Gamma^*}$ be a corresponding element to g^*. We see that Proposition 5.1 remains valid with Γ^* and P^* in place of Γ and P. In particular we may assume that ψ is given by the formula

(6.11)
$$\psi(\widetilde{\gamma}_1\widetilde{\gamma}_2\cdots\widetilde{\gamma}_m) = c\sum_{i=1}^{d} \beta_i^{-1}\varphi(\widetilde{\beta_i\gamma_1\beta_{q_1(i)}^{-1}}\,\widetilde{\beta_{q_1(i)}\gamma_2\beta_{q_2(i)}^{-1}}\cdots\widetilde{\beta_{q_{m-1}(i)}\gamma_m\beta_{q_m(i)}^{-1}}\,(\widetilde{\beta_i\gamma_1\gamma_2\cdots\gamma_m\beta_{q_m(i)}^{-1}})^{-1}).$$

Here $\gamma_j = \sigma$ or $\gamma_j \in P^*$ and $\gamma_1\gamma_2\cdots\gamma_m = 1$.

Example 6.3. Let us consider $T(2)$. We may take

$$\beta_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \beta_2 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \qquad \beta_3 = \begin{pmatrix} 1 & \epsilon \\ 0 & 2 \end{pmatrix},$$
$$\beta_4 = \begin{pmatrix} 1 & \epsilon^2 \\ 0 & 2 \end{pmatrix}, \qquad \beta_5 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

By (6.11), we find

$$\psi((\widetilde{\sigma}\widetilde{\tau})^3) = c(\beta_3^{-1}Z_3 + \beta_4^{-1}Z_4),$$

where

$$Z_3 = \varphi((\widetilde{\begin{pmatrix} \epsilon & -\epsilon^2 \\ 2 & -\epsilon^2 \end{pmatrix}}\widetilde{\tau})^3), \qquad Z_4 = \varphi((\widetilde{\begin{pmatrix} \epsilon^2 & -\epsilon^2 \\ 2 & -\epsilon \end{pmatrix}})^3). \tag{6.12}$$

We have

$$\widetilde{\begin{pmatrix} \epsilon & -\epsilon^2 \\ 2 & -\epsilon^2 \end{pmatrix}} = \widetilde{\sigma}\widetilde{\begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & \epsilon \end{pmatrix}}\widetilde{\begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}}^{-2}\widetilde{\sigma}\widetilde{\begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}}^{-1}.$$

Hence, using (6.4), we have

$$Z_3 = \varphi((\widetilde{\sigma}\widetilde{\begin{pmatrix} \epsilon^{-1} & -2 \\ 0 & \epsilon \end{pmatrix}}\widetilde{\sigma}\widetilde{\begin{pmatrix} 1 & -\epsilon^{-1} \\ 0 & 1 \end{pmatrix}})^3).$$

Similarly we obtain

$$Z_4 = \varphi((\widetilde{\sigma}\widetilde{\begin{pmatrix} \epsilon^{-2} & -2 \\ 0 & \epsilon^2 \end{pmatrix}}\widetilde{\sigma}\widetilde{\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}})^3).$$

6.5. In general, every element r of R^* can be written as

$$r = \widetilde{\sigma}\widetilde{p}_1\widetilde{\sigma}\widetilde{p}_2 \cdots \widetilde{\sigma}\widetilde{p}_m$$

with $p_i \in P^*$, $1 \le i \le m$ such that $\sigma p_1 \sigma p_2 \cdots \sigma p_m = 1$. We call such an element an m *terms relation*. Theorem 6.1 assures us that $\varphi(r)$ can be expressed by A and B. The following formulas can be proved easily.

$$\varphi((\widetilde{\sigma}\widetilde{\nu}^n)^2) = (1 + \nu^{-1} + \cdots + \nu^{1-n})A, \qquad n \ge 1, \tag{6.13a}$$

$$\varphi((\widetilde{\sigma}\widetilde{\nu}^{-n})^2) = -(\nu + \nu^2 + \cdots + \nu^n)A, \qquad n \ge 1, \tag{6.13b}$$

For $t \in E_F$, we put

$$B(t) = \varphi(\widetilde{\sigma}\widetilde{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}}\widetilde{\sigma}\widetilde{\begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix}}\widetilde{\sigma}\widetilde{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}}\widetilde{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}}).$$

Then we have $B(1) = B$,

$$B(-t) = -\sigma\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}B(t) - \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\varphi((\widetilde{\sigma}\widetilde{\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}})^2), \tag{6.14}$$

$$B(\epsilon t) = \nu^{-1} B(t) + \left[1 + \sigma \begin{pmatrix} 1 & \epsilon t \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & \epsilon^{-1} t^{-1} \\ 0 & 1 \end{pmatrix} - \sigma \begin{pmatrix} 1 & \epsilon t \\ 0 & 1 \end{pmatrix} \sigma \right] A, \tag{6.15}$$

$$B(t) = \sigma \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} B(t^{-1}) + \varphi((\widetilde{\sigma} \widetilde{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}})^2). \tag{6.16}$$

By these formulas, we can express $B(t)$ in terms of A and B explicitly. Using $B(t)$, we have an explicit formula for $\varphi(r)$ for a three terms relation r:

$$\begin{aligned} &\varphi(\widetilde{\sigma} \widetilde{\begin{pmatrix} u_1 & x_1 \\ 0 & 1 \end{pmatrix}} \widetilde{\sigma} \widetilde{\begin{pmatrix} u_2 & x_2 \\ 0 & 1 \end{pmatrix}} \widetilde{\sigma} \widetilde{\begin{pmatrix} u_3 & x_3 \\ 0 & 1 \end{pmatrix}}) \\ &= \begin{pmatrix} u_1^{-1} & 0 \\ 0 & 1 \end{pmatrix} B(u_1^{-1} x_1) + \varphi((\widetilde{\sigma} \widetilde{\begin{pmatrix} u_1 & 0 \\ 0 & 1 \end{pmatrix}})^2) \\ &+ \begin{pmatrix} u_3^{-1} & -u_3^{-1} x_3 \\ 0 & 1 \end{pmatrix} \sigma \varphi((\widetilde{\sigma} \widetilde{\begin{pmatrix} u_2 & 0 \\ 0 & 1 \end{pmatrix}})^2). \end{aligned} \tag{6.17}$$

For an m terms relation $r \in R^*$, $m \geq 4$, we may write $p_i = \begin{pmatrix} u_i & x_i \\ 0 & 1 \end{pmatrix}$, $u_i \in E_F$, $x_i \in \mathcal{O}_F$, $1 \leq i \leq m$. We see that $\varphi(r)$ reduces to an $(m-2)$ terms relation if $x_i = 0$ for some i. If $x_i \in E_F$ for some i, $\varphi(r)$ reduces to an $(m-1)$ terms relation. For example, if $x_1 \in E_F$ and $m \geq 4$, we have

$$\begin{aligned} &\varphi(\widetilde{\sigma} \widetilde{\begin{pmatrix} u_1 & x_1 \\ 0 & 1 \end{pmatrix}} \widetilde{\sigma} \widetilde{\begin{pmatrix} u_2 & x_2 \\ 0 & 1 \end{pmatrix}} \widetilde{\sigma} \widetilde{\begin{pmatrix} u_3 & x_3 \\ 0 & 1 \end{pmatrix}} \widetilde{\sigma} \cdots \widetilde{\sigma} \widetilde{\begin{pmatrix} u_m & x_m \\ 0 & 1 \end{pmatrix}}) \\ &= \begin{pmatrix} u_1^{-1} u^{-1} & -u_1^{-1} \\ 0 & u \end{pmatrix} \varphi(\widetilde{\sigma} \widetilde{\begin{pmatrix} 1 & -u^{-1} \\ 0 & 1 \end{pmatrix}} \widetilde{\begin{pmatrix} u_2 & x_2 \\ 0 & 1 \end{pmatrix}} \widetilde{\sigma} \widetilde{\begin{pmatrix} u_3 & x_3 \\ 0 & 1 \end{pmatrix}} \widetilde{\sigma} \\ &\cdots \widetilde{\sigma} \widetilde{\begin{pmatrix} u_m & x_m \\ 0 & 1 \end{pmatrix}} \widetilde{\begin{pmatrix} u_1^{-1} u^{-1} & -u_1^{-1} \\ 0 & u \end{pmatrix}}) \\ &+ \begin{pmatrix} u_1^{-1} & 0 \\ 0 & 1 \end{pmatrix} B(u) + \varphi((\widetilde{\sigma} \widetilde{\begin{pmatrix} u_1 & 0 \\ 0 & 1 \end{pmatrix}})^2). \end{aligned} \tag{6.18}$$

Here $u = u_1^{-1} x_1$. For a general m terms relation r, the explicit reduction of $\varphi(r)$ to A and B is a highly non-trivial problem. The author has an idea on a heuristic algorithm to solve this problem, but it will not be discussed in this paper. For our present purposes, the formulas (6.13a) $\sim$ (6.18) are sufficient.

6.6. For actual computations, it is convenient to use the decomposition (5.13). Proposition 5.5 shows that we will lose little information by assuming $\varphi \in H^1(R^*, V)^{\Gamma^*,+}$, so we do assume this. Then we have

$$\begin{aligned} &-\varphi((\widetilde{\sigma}\widetilde{\tau})^3) = \varphi(\widetilde{\tau}^{-1}\widetilde{\sigma}\widetilde{\tau}^{-1}\widetilde{\sigma}\widetilde{\tau}^{-1}\widetilde{\sigma}) = \tau^{-1}\varphi((\widetilde{\sigma}\widetilde{\tau}^{-1})^3) \\ &= \tau^{-1}\varphi(((\widetilde{\sigma}\widetilde{\tau})^3)_\delta) = \tau^{-1}\delta\varphi((\widetilde{\sigma}\widetilde{\tau})^3). \end{aligned}$$

Hence

$$(\delta\tau + 1)B = 0.$$

Similarly we obtain

$$(\delta - 1)A = 0.$$

Now we are ready to state explicit numerical examples. First by numerical computations, we have verified:

Fact 1. Suppose $0 \leq l_2 \leq l_1 \leq 20$. Then adding $h|R^*$, $h \in H^1(\mathcal{F}^*, V)$ to φ (keeping φ in the plus space under the action of δ), we may assume $B = 0$.

Therefore our task is to find constraints on $A = \varphi((\widetilde{\sigma}\widetilde{\nu})^2)$. Note that $(\sigma\nu - 1)A = 0$. We put $x = \begin{pmatrix} \epsilon & -\epsilon^2 \\ 2 & -\epsilon^2 \end{pmatrix} \tau$ and

$$Z_A^+ = \{\mathbf{v} \in V \mid (\sigma\nu - 1)\mathbf{v} = 0,\ (\delta - 1)\mathbf{v} = 0,\ xZ_3 = Z_3\}. \tag{6.19}$$

Here some explanation is called for on the meaning of $xZ_3 = Z_3$. First note that Z_3 is defined by (6.12); clearly we must have $xZ_3 = Z_3$. Using the formulas (6.13a) $\sim$ (6.18), we see that Z_3 can be expressed by A. Therefore $xZ_3 = Z_3$ gives a constraint on A. We define a linear mapping

$$\zeta^+ : Z_A^+ \longrightarrow \mathbf{C}^{l_2+1} \tag{6.20}$$

as follows. Let $\mathbf{v} \in Z_A^+$. We let the coefficient of $\mathbf{e}_{l_1+2-m} \otimes \mathbf{e}'_{(l_1+l_2)/2+2-m}$ in $(1 + \nu^{-1})\mathbf{v}$ be equal to the $(l_1 + l_2)/2 + 2 - m$-th coefficient of $\zeta^+(\mathbf{v})$, for $(l_1 - l_2)/2 + 1 \leq m \leq (l_1 + l_2)/2 + 1$ (cf. (6.6)).

Suppose that φ as above corresponds to a (nonzero) Hecke eigenform $\Omega \in S_{l_1+2,l_2+2}(\Gamma)$. Suppose that l_1 and l_2 are in the range of Fact 1. Then $\zeta^+(A) \neq 0$ if $l_2 \geq 4$ in the case $l_1 \neq l_2$, if $l_2 \geq 6$ in the case $l_1 = l_2$ by Proposition 5.5.

Example 6.4. We take $l_1 = 8$, $l_2 = 4$. Then $\dim S_{10,6}(\Gamma) = 1$. We find $\zeta^+(Z_A^+)$ is one dimensional and consists of scalar multiples of ${}^t(4, 0, 1, 0, 4)$. Hence we obtain

$$R(7, \Omega)/R(5, \Omega) = 4, \qquad \Omega \in S_{10,6}(\Gamma).$$

My computer calculates this example in six seconds.

Example 6.5. In the same way as in Example 6.4, we obtain the following numerical values.

$$R(9,\Omega)/R(7,\Omega) = 6, \qquad \Omega \in S_{14,6}(\Gamma).$$

$$R(6,\Omega)/R(4,\Omega) = \frac{25}{6}, \qquad \Omega \in S_{8,8}(\Gamma).$$

$$R(8,\Omega)/R(6,\Omega) = 7, \qquad \Omega \in S_{12,8}(\Gamma).$$

$$R(10,\Omega)/R(8,\Omega) = \frac{720}{11}, \qquad \Omega \in S_{12,10}(\Gamma).$$

The spaces of cusp forms appearing in this example are all one dimensional.

6.7. To deal with the case where $\dim S_{l_1+2,l_2+2}(\Gamma) > 1$, it is necessary to use the action of Hecke operators. To this end, we consider the contribution of $H^1(\mathcal{F}^*, V)$ to Z_A^+. Take $h \in H^1(\mathcal{F}^*, V)$ and put

$$h(\widetilde{\sigma}) = S, \qquad h(\widetilde{\nu}) = U, \qquad h(\widetilde{\tau}) = T.$$

We require that $h|R^*$ vanishes on the elements (i*), (ii*), (iv*), (v*). These conditions are equivalent to

$$(\sigma + 1)S = 0, \tag{6.21}$$

$$\{(\sigma\tau)^2 + \sigma\tau + 1\}(\sigma T + S) = 0 \tag{6.22}$$

and (6.7), (6.8). We have

$$h((\widetilde{\sigma}\widetilde{\nu})^2) = (\sigma\nu + 1)(\sigma U + S).$$

We also require that

$$(\delta - 1)(\sigma\nu + 1)(\sigma U + S) = 0. \tag{6.23}$$

Let B_A^+ be the subspace of V generated by $(\sigma\nu + 1)(\sigma U + S)$ when S, T, U extend over vectors of V satifying the relations (6.7), (6.8), (6.21), (6.22) and (6.23). We have $B_A^+ \subset Z_A^+$. As shown in §4.1, we have

$$\zeta^+(B_A^+) = \{0\} \quad \text{if } l_1 \neq l_2, \qquad \dim \zeta^+(B_A^+) \leq 1 \quad \text{if } l_1 = l_2. \tag{6.24}$$

By Proposition 5.5, we have

$$\dim Z_A^+/B_A^+ \geq \dim S_{l_1+2.l_2+2}(\Gamma) \qquad \text{if } l_2 \geq 4,\ l_1 \neq l_2 \text{ or if } \ l_1 = l_2,\ l_2 \geq 6.$$

Now by numerical computations, we have verified:

Fact 2. Suppose $0 \le l_2 \le l_1 \le 20$. Then $\dim S_{l_1+2,l_2+1}(\Gamma) = \dim Z_A^+/B_A^+$.

This fact means that the constraints posed on $A = \varphi((\widetilde{\sigma}\widetilde{\nu})^2)$ is enough.

Example 6.6. We take $l_1 = 12$, $l_2 = 8$. We have $\dim S_{14,10}(\Gamma) = 2$. Moreover we have $\zeta^+(Z_A^+) = 2$ in this case. Hence ζ^+ gives an isomorphism of Z_A^+/B_A^+ into $\mathbf{C}^{l_2+1}$. Calculating the action of $T(2)$ on Z_A^+/B_A^+ using (6.11), we find that the eigenvalues are $-2560 \pm 960\sqrt{106}$. Take an eigenvector in Z_A^+/B_A^+ and map it by ζ^+. Then we find

$$R(11,\Omega)/R(7,\Omega) = 1616 - 76\sqrt{106}, \qquad R(9,\Omega)/R(7,\Omega) = \frac{58}{3} - \frac{5}{6}\sqrt{106}$$

if $0 \ne \Omega \in S_{14,10}(\Gamma)$ satisfies $\Omega|T(2) = (-2560 + 960\sqrt{106})\Omega$. If $0 \ne \Omega \in S_{14,10}(\Gamma)$ satisfies $\Omega|T(2) = (-2560 - 960\sqrt{106})\Omega$, then we have

$$R(11,\Omega)/R(7,\Omega) = 1616 + 76\sqrt{106}, \qquad R(9,\Omega)/R(7,\Omega) = \frac{58}{3} + \frac{5}{6}\sqrt{106}.$$

Remark 6.7. The relation $\dim \zeta^+(Z_A^+) = \dim S_{l_1+2,l_2+2}(\Gamma)$ is rather accidental in the above example. It holds in many cases but we have $\dim S_{l_1+2,l_2+2}(\Gamma) > \dim \zeta^+(Z_A^+)$ in general. Even in the general case, we can obtain ratios of L-values by finding an eigenvector of Hecke operators in Z_A^+/B_A^+ and mapping it by ζ^+.

6.8. We next consider the 2-cocycle f^- (cf. §5.6). The technique of calculation is basically same as for f^+, but this case is somewhat more complicated. Put

(6.25)
$$H^1(R^*,V)^\Gamma = \{\varphi \in \mathrm{Hom}(R^*,V) \mid \varphi(grg^{-1}) = g\varphi(r),\ g \in \mathcal{F}, r \in R^*\}.$$

Let $\varphi \in H^1(R^*,V)^\Gamma$. We put

$$(e\varphi)(r) = \nu^{-1}\varphi(\widetilde{\nu} r \widetilde{\nu}^{-1}), \qquad r \in R^*.$$

Then we can verify easily that

$$e\varphi \in H^1(R^*,V)^\Gamma, \qquad e^2\varphi = \varphi.$$

Therefore $H^1(R^*,V)^\Gamma$ decomposes as

$$H^1(R^*,V)^\Gamma = H^1(R^*,V)^{\Gamma,+} \oplus H^1(R^*,V)^{\Gamma,-}, \tag{6.26}$$

where, for $\epsilon = \pm 1$,

$$H^1(R^*,V)^{\Gamma,\epsilon} = \{\varphi \in \mathrm{Hom}(R^*,V)^\Gamma \mid \varphi(\widetilde{\nu} r \widetilde{\nu}^{-1}) = \epsilon\nu\varphi(r), r \in R^*\}.$$

First we take an arbitrary normalized 2-cocycle $f \in Z^2(\Gamma, V)$. Since $\mathcal{F}R^*$ is a free group, there exists $a \in C^1(\mathcal{F}R^*, V)$ such that

$$f(\pi^*(g_1), \pi^*(g_2)) = g_1 a(g_2) + a(g_1) - a(g_1 g_2), \qquad g_1, g_2 \in \mathcal{F}R^*. \tag{6.27}$$

As shown in §1.4, we have

$$a(gr) = ga(r) + a(g), \qquad a(grg^{-1}) = ga(r), \qquad g \in \mathcal{F}R^*,\ r \in R^*.$$

Put $\varphi = a|R^*$. Then the above formulas imply $\varphi \in H^1(R^*, V)^\Gamma$. From the isomorphism $\Gamma \cong \mathcal{F}R^*/R^* (\cong \mathcal{F}/\mathcal{F} \cap R^* = \mathcal{F}/R)$, we obtain

$$H^2(\Gamma, V) \cong H^1(R^*, V)^\Gamma / \mathrm{Im}(H^1(\mathcal{F}R^*, V)) \tag{6.28}$$

and the procedure $f \mapsto \varphi$ described above gives an explicit form of the isomorphism (6.28). We consider the decomposition of $H^2(\Gamma, V)$ under the action of ν (cf. the formula below (5.10)). Then we have

$$H^2(\Gamma, V)^\pm \cong H^1(R^*, V)^{\Gamma,\pm} / (\mathrm{Im}(H^1(\mathcal{F}R^*, V)) \cap H^1(R^*, V)^{\Gamma,\pm})). \tag{6.29}$$

6.9. Now we consider the 2-cocycle f^-. Let $\varphi \in H^1(R^*, V)^{\Gamma,-}$ be a corresponding element. As for f^+, we may assume that

$$\varphi|R_{P^*} = 0, \tag{6.30}$$

$$f^-(\gamma_1, \gamma_2) = -\varphi(\widetilde{\gamma}_1 \widetilde{\gamma}_2 (\widetilde{\gamma_1 \gamma_2})^{-1}) \tag{6.31}$$

adding a coboundary to f^-. We put

$$A = \varphi((\widetilde{\sigma}\widetilde{\nu})^2), \qquad B = \varphi((\widetilde{\sigma}\widetilde{\tau})^3).$$

The formulas (6.13a) $\sim$ (6.18) hold with the following modifications.

$$\varphi((\widetilde{\sigma}\widetilde{\nu}^n)^2) = (1 - \nu^{-1} + \nu^{-2} + \cdots + (-1)^{1-n}\nu^{1-n})A, \qquad n \geq 1, \tag{6.13a$^-$}$$

$$\varphi((\widetilde{\sigma}\widetilde{\nu}^{-n})^2) = (\nu - \nu^2 + \nu^3 - \cdots + (-1)^{1-n}\nu^n)A, \qquad n \geq 1. \tag{6.13b$^-$}$$

We define $B(t)$, $t \in E_F$ by the same formula as before. In (6.15), the term $\nu^{-1}B(t)$ should be replaced by $-\nu^{-1}B(t)$; (6.14) and (6.16) hold without any change. For $u = \pm\epsilon^n \in E_F$, we define $\epsilon_0(u) = (-1)^n$. On the right-hand side of (6.17), the first term should be multiplied by $\epsilon_0(u_1)$ and the third term should be multiplied by $\epsilon_0(u_3)$. On the right-hand side of (6.18), both of the first and the second term should be multiplied by $\epsilon_0(u_1)$.

We may and do assume that f^- belongs to the plus subspace of $H^2(\Gamma, V)^-$ under the action of δ. Then we have

$$(\delta - 1)A = 0, \qquad (\delta\tau + 1)B = 0.$$

By numerical computations, we have verified

Fact 3. Suppose $0 \le l_2 \le l_1 \le 20$. Then adding $h|R^*$ for $h \in H^1(\mathcal{F}R^*, V)$ such that $h|R^* \in H^1(R^*, V)^{\Gamma,-}$ to φ (keeping φ in the plus space under the action of δ), we may assume $B = 0$.

Therefore our task is to find constraints on $A = \varphi((\widetilde{\sigma}\widetilde{\nu})^2)$. Note that $(\sigma\nu + 1)A = 0$. We put $x = \begin{pmatrix} \epsilon & -\epsilon^2 \\ 2 & -\epsilon^2 \end{pmatrix} \tau$ and

$$Z_A^- = \{\mathbf{v} \in V \mid (\sigma\nu + 1)\mathbf{v} = 0,\ (\delta - 1)\mathbf{v} = 0,\ xZ_3 = Z_3\}. \tag{6.32}$$

Here the meaning of the constraint $xZ_3 = Z_3$ is the same as for Z_A^+. We define a linear mapping

$$\zeta^- : Z_A^- \longrightarrow \mathbf{C}^{l_2+1}$$

as follows. Let $\mathbf{v} \in Z_A^-$. We let the coefficient of $\mathbf{e}_{l_1+2-m} \otimes \mathbf{e}'_{(l_1+l_2)/2+2-m}$ in $(1 - \nu^{-1})\mathbf{v}$ be equal to the $(l_1 + l_2)/2 + 2 - m$-th coefficient of $\zeta^-(\mathbf{v})$, for $(l_1 - l_2)/2 + 1 \le m \le (l_1 + l_2)/2 + 1$ (cf. (6.6)).

Example 6.8. We take $l_1 = 8$, $l_2 = 6$. Then $\dim S_{10,8}(\Gamma) = 1$. We find $\zeta^-(Z_A^-)$ is one dimensional and consists of scalar multiples of ${}^t(2, 0, 7/90, 0, -7/90, 0, -2)$. Hence we obtain

$$R(8, \Omega)/R(6, \Omega) = \frac{180}{7}, \qquad \Omega \in S_{10,8}(\Gamma).$$

Example 6.9. In the same way as in Example 6.8, we obtain the following numerical values.

$$R(9, \Omega)/R(7, \Omega) = \frac{70}{3}, \qquad \Omega \in S_{12,8}(\Gamma).$$

$$R(9, \Omega)/R(7, \Omega) = 42, \qquad \Omega \in S_{12,10}(\Gamma).$$

The spaces of cusp forms appearing in this example are all one dimensional.

6.10. To treat the case where $\dim S_{l_1+2,l_2+2}(\Gamma) > 1$, it is necessary to consider Hecke operators.

First let us write down $\mathrm{Im}(H^1(\mathcal{F}R^*, V)) \cap H^1(R^*, V)^{\Gamma,\pm}$ which appears on the right-hand side of (6.29), explicitly. Take $h \in Z^1(\mathcal{F}R^*, V)$. We put

$$(e_0 h)(x) = \nu^{-1} h(\widetilde{\nu} x \widetilde{\nu}^{-1}), \qquad x \in \mathcal{F}R^*. \tag{6.33}$$

We can check easily that $e_0 h \in Z^1(\mathcal{F}R^*, V)$ and that

$$(e_0^2 h)(x) = h(x) + (\nu^{-2} - x\nu^{-2})h(\widetilde{\nu}^2), \qquad x \in \mathcal{F}R^*.$$

If we restrict h to R^*, then the action e_0 coincides with the action of e defined in §6.8. We have $(e_0^2 h)|R^* = h|R^*$. We put

$$h^{\pm} = h \pm e_0 h.$$

A general element of $\mathrm{Im}(H^1(\mathcal{F}R^*, V)) \cap H^1(R^*, V)^{\Gamma,\pm}$ can be obtained as $h^{\pm}|R^*$ from a general element $h \in Z^1(\mathcal{F}R^*, V)$.

Let $Z^1(\mathcal{F}R^*, V)^{\pm}$ be the subgroup of $Z^1(\mathcal{F}R^*, V)$ consisting of all elements whose restrictions to R^* belong to $H^1(R^*, V)^{\Gamma,\pm}$. Take $\epsilon_1 = \pm 1$ and put $h^{\pm} = h + \epsilon_1 e_0 h$. For the free generators $\widetilde{\sigma}$, $\widetilde{\tau}$, $\widetilde{\nu}^2$, $\widetilde{\nu}\widetilde{\sigma}\widetilde{\nu}^{-1}$, $\widetilde{\nu}\widetilde{\tau}\widetilde{\nu}^{-1}$ of $\mathcal{F}R^*$, we put

$$h(\widetilde{\sigma}) = S_1, \quad h(\widetilde{\tau}) = T_1, \quad h(\widetilde{\nu}^2) = U, \quad h(\widetilde{\nu}\widetilde{\sigma}\widetilde{\nu}^{-1}) = V_1, \quad h(\widetilde{\nu}\widetilde{\tau}\widetilde{\nu}^{-1}) = W_1.$$

Then we find

$$\begin{aligned} h^{\pm}(\widetilde{\sigma}) &= S_1 + \epsilon_1 \nu^{-1} V_1, \\ h^{\pm}(\widetilde{\tau}) &= T_1 + \epsilon_1 \nu^{-1} W_1, \\ h^{\pm}(\widetilde{\nu}^2) &= (1 + \epsilon_1 \nu^{-1})U, \\ h^{\pm}(\widetilde{\nu}\widetilde{\sigma}\widetilde{\nu}^{-1}) &= V_1 + \epsilon_1 \nu S_1 + \epsilon_1 \nu^{-1}(1 - \nu^4 \sigma)U, \\ h^{\pm}(\widetilde{\nu}\widetilde{\tau}\widetilde{\nu}^{-1}) &= W_1 + \epsilon_1 \nu T_1 + \epsilon_1(\nu^{-1} - \nu\tau\nu^{-2})U. \end{aligned}$$

Fix $\epsilon_1 = \pm 1$ and put

$$h^{\pm}(\widetilde{\sigma}) = S, \quad h^{\pm}(\widetilde{\tau}) = T. \tag{6.34}$$

Then V_1 and W_1 are eliminated and we obtain

$$h^{\pm}(\widetilde{\nu}^2) = (1 + \epsilon_1 \nu^{-1})U, \tag{6.35}$$

$$h^{\pm}(\widetilde{\nu}\widetilde{\sigma}\widetilde{\nu}^{-1}) = \epsilon_1 \nu S + \epsilon_1 \nu^{-1}(1 - \nu^4 \sigma)U, \tag{6.36}$$

$$h^{\pm}(\widetilde{\nu}\widetilde{\tau}\widetilde{\nu}^{-1}) = \epsilon_1 \nu T + \epsilon_1(\nu^{-1} - \nu\tau\nu^{-2})U. \tag{6.37}$$

Clearly S, T and U can take arbirary three vectors of V. The formulas (6.34) $\sim$ (6.37) describe a general element of $Z^1(\mathcal{F}R^*, V)^{\pm}$. The conditions for $h^{\pm}$ to vanish on the elements (iv*) and (v*) are

$$\{\nu\tau\nu^{-1} - 1 + \epsilon_1(1 - \tau)\nu\}T + \epsilon_1(1 - \tau)(\nu^{-1} - \nu^{-1}\tau\nu^{-2})U = 0, \tag{6.38}$$

$$(6.39)\quad (1+\epsilon_1\tau\nu-\nu^2)T+\{\epsilon_1\tau(\nu^{-1}-\nu\tau\nu^{-2})-(1-\nu^2\tau\nu^{-2})(1+\epsilon_1\nu^{-1})\}U=0$$

respectively. For $h^{\pm} \in Z^1(\mathcal{F}R^*, V)^{\pm}$ as above, we have

$$(6.40)\qquad h^{\pm}((\widetilde{\sigma}\widetilde{\nu})^2) = (1+\epsilon_1\sigma\nu)S + (\nu^{-2}+\epsilon_1\sigma\nu^{-1})U.$$

Now we consider the case $\epsilon_1 = -1$. Let B_A^- be the subspace of V generated by $(1-\sigma\nu)S+(\nu^{-2}-\sigma\nu^{-1})U$ when S, T, U extend over vectors of V satisfying the relations (6.21), (6.22), (6.38), (6.39) and

$$(6.41)\qquad (\delta-1)\{(1-\sigma\nu)S+(\nu^{-2}-\sigma\nu^{-1})U\}=0.$$

We have $B_A^- \subset Z_A^-$. As shown in §4.1, we have

$$\zeta^-(B_A^-)=\{0\}\quad \text{if } l_1\neq l_2, \qquad \dim\zeta^-(B_A^-)\le 1 \quad \text{if } l_1=l_2.$$

Using Proposition 5.2, (2), we can show that

$$\dim Z_A^-/B_A^- \ge \dim S_{l_1+2.l_2+2}(\Gamma) \qquad \text{if } l_2\ge 2,\ l_1\neq l_2 \text{ or if } \ l_1=l_2,\ l_2\ge 4.$$

Now by numerical computations, we have verified:

Fact 4. Suppose $0 \le l_2 \le l_1 \le 20$. Then $\dim S_{l_1+2,l_2+1}(\Gamma) = \dim Z_A^-/B_A^-$.

The formula (6.11) can be generalized in the following way. We put $g^- = T(\varpi)f^-$ where g^- is defined by (5.4). Let $\varphi \in H^1(R^*, V)^{\Gamma,-}$ be a corresponding element to f^-. We may assume that (6.31) holds. There exists a 1-cochain $b \in C^1(\mathcal{F}R^*, V)$ such that

$$(6.42)\quad f^-(\pi^*(x_1),\pi^*(x_2)) = x_1b(x_2)+b(x_1)-b(x_1x_2), \qquad x_1,x_2\in\mathcal{F}R^*.$$

As the intial conditions, we may assume that

$$b(\widetilde{\sigma})=0,\quad b(\widetilde{\nu}^2)=0,\quad b(\widetilde{\tau})=0,\quad b(\widetilde{\nu}\widetilde{\sigma}\widetilde{\nu}^{-1})=0,\quad b(\widetilde{\nu}\widetilde{\tau}\widetilde{\nu}^{-1})=0$$

for the free generators of $\mathcal{F}R^*$. Then the formula (5.9) holds when $b(\widetilde{\gamma}_j)=0$, $1\le j\le m$. This condition holds if $\widetilde{\gamma}_j$ is equal to one of the five free generators as above or their inverses. In particular, $\psi = b|R^*$ is given by

$$\psi(\widetilde{\gamma}_1\widetilde{\gamma}_2\cdots\widetilde{\gamma}_m)$$
$$=c\sum_{i=1}^{d}\beta_i^{-1}\varphi(\widetilde{\beta_i\gamma_1\beta_{q_1(i)}^{-1}}\widetilde{\beta_{q_1(i)}\gamma_2\beta_{q_2(i)}^{-1}}\cdots\widetilde{\beta_{q_{m-1}(i)}\gamma_m\beta_{q_m(i)}^{-1}}(\widetilde{\beta_i\gamma_1\gamma_2\cdots\gamma_m\beta_{q_m(i)}^{-1}})^{-1})$$

provided $\widetilde{\gamma}_j$ is equal to one of the five free generators of $\mathcal{F}R^*$ or their inverses and $\gamma_1\gamma_2\cdots\gamma_m=1$. The above formula is the same as (6.11) but there is one important point about which we must be careful. This ψ belongs to $H^1(R^*, V)^\Gamma$ and gives a corresponding element to g^- but it does not necessarily belong to $H^1(R^*, V)^{\Gamma,-}$. We obtain $\psi^- \in H^1(R^*, V)^{\Gamma,-}$ corresponding to g^- by $\psi^- = \frac{1}{2}(1-e)\psi$ (cf. §6.8).

Example 6.10. We take $l_1 = 12$, $l_2 = 8$. We have $\dim S_{14,10}(\Gamma) = 2$. Moreover we have $\zeta^-(Z_A^-) = 2$ in this case. Hence ζ^- gives an isomorphism of Z_A^-/B_A^- into $\mathbf{C}^{l_2+1}$. Take an eigenvector of $T(2)$ in Z_A^-/B_A^- and map it by ζ^-. Then we find

$$R(10,\Omega)/R(8,\Omega) = 50 - \sqrt{106},$$

if $0 \neq \Omega \in S_{14,10}(\Gamma)$ satisfies $\Omega|T(2) = (-2560 + 960\sqrt{106})\Omega$. If $0 \neq \Omega \in S_{14,10}(\Gamma)$ satisfies $\Omega|T(2) = (-2560 - 960\sqrt{106})\Omega$, then we have

$$R(10,\Omega)/R(8,\Omega) = 50 + \sqrt{106}.$$

Let Ω be a Hecke eigenform of $S_{14,10}(\Gamma)$. Then $L(m,\Omega)$ is a critical value for integers in the range $3 \leq m \leq 11$ (cf. (5.14)). We have $L(s,\Omega) = L(14-s,\Omega)$ (cf. (2.7)). By Examples 6.6 and 6.10, we have treated all critical values on the right of the critical line.

Example 6.11. We take $l_1 = l_2 = 18$. We have $\dim S_{20,20}(\Gamma) = 7$. Calculating the action of $T(2)$ on Z_A^+/B_A^+ using (6.11), we find that the characteristic polynomial of $T(2)$ is (we can use Z_A^-/B_A^- which gives the same result)

$$\begin{aligned}&(X-97280)^2(X+840640)(X^4 - 1286780X^3 + 19006483200X^2\\&+ 27181090390835200X - 22979876427231395840000).\end{aligned}$$

The irreducible factor of degree four corresponds to the base change part from $S_{20}(\Gamma_0(5), (\frac{\ }{5}))$; $X + 840640$ corresponds to the base change part from $S_{20}(\mathrm{SL}_2(\mathbf{Z}))$; the factor $(X - 97280)^2$ corresponds to the non base change part. Let $\Omega \in \dim S_{20,20}(\Gamma)$ be a Hecke eigenform in the non base change part. A calculation for the plus part yields the result

$$R(18,\Omega)/R(10,\Omega) = 39355680000, \qquad R(16,\Omega)/R(10,\Omega) = 33163650,$$

$$R(14,\Omega)/R(10,\Omega) = \frac{1266460}{27}, \qquad R(12,\Omega)/R(10,\Omega) = \frac{26075}{216}.$$

A calculation for the minus part yields the result

$$R(17,\Omega)/R(11,\Omega) = \frac{111006792000}{803}, \qquad R(15,\Omega)/R(11,\Omega) = \frac{54618434}{365},$$

$$R(13,\Omega)/R(11,\Omega)=\frac{453159}{1606}.$$

We note that though there are two Hecke eigenforms in the non base change part, these ratios are the same for them. [5]

§7. Numerical examples II

7.1. In this section, we treat the case $F=\mathbf{Q}(\sqrt{13})$. We use the same notation as in the previous section. Many results there remain valid in the present case so we will be brief.

The fundamental unit of F is $\epsilon=\frac{3+\sqrt{13}}{2}$. The elements σ, ν and τ of Γ^* satisfy the relations (i′) $\sim$ (iv′) in §6.1 and

$$\text{(v}'\text{)} \qquad \nu^2\tau\nu^{-2}=\tau(\nu\tau\nu^{-1})^3.$$

Though we do not know that (i′) $\sim$ (v′) are the fundamental relations, we will show that it is possible to calculate ratios of critical values of L-functions rigorously.

Let $\mathcal{F}^*$ be the free group on three letters $\widetilde{\sigma}$, $\widetilde{\nu}$, $\widetilde{\tau}$. We define a surjective homomorphism $\pi^*:\mathcal{F}^*\longrightarrow\Gamma^*$ by $\pi^*(\widetilde{\sigma})=\sigma$, $\pi^*(\widetilde{\nu})=\nu$, $\pi^*(\widetilde{\tau})=\tau$. Let R^* be the kernel of π^*. Then R^* contains the elements (i*) $\sim$ (iv*) in §6.1 and

$$\text{(v}^*\text{)} \qquad \widetilde{\nu}^2\widetilde{\tau}\widetilde{\nu}^{-2}\{\widetilde{\tau}(\widetilde{\nu}\widetilde{\tau}\widetilde{\nu}^{-1})^3\}^{-1}.$$

For every $\gamma\in\Gamma^*$, we choose $\widetilde{\gamma}\in\mathcal{F}^*$ so that $\pi^*(\widetilde{\gamma})=\gamma$. We use the same algorithm as in the previous section.

We consider f^+ (cf. §5.6). We put $f^*=\widetilde{T}(f)$. Then $f^*\in Z^2(\Gamma^*,V)$ and $f^*|\Gamma=f^+$ (cf. §5.3). Let $\varphi\in H^1(R^*,V)^{\Gamma^*}$ be a corresponding element to f^*. We may assume that (6.4) and (6.5) hold. We may also assume that $\varphi(\widetilde{\sigma}^2)=0$. We need to analyze the process of adding $h|R^*$ to φ. For S, T, $U\in V$, there exists $h\in H^1(\mathcal{F}^*,V)$ such that

$$h(\widetilde{\sigma})=S,\qquad h(\widetilde{\tau})=T,\qquad h(\widetilde{\nu})=U.$$

We find that the conditions for h to vanish on the elements (iv*) and (v*) are (6.7) and

(7.1)
$$\begin{aligned}&[\nu^2-\tau\{1+\nu\tau\nu^{-1}+(\nu\tau\nu^{-1})^2\}\nu-1]T\\ &+[(1-\nu^2\tau\nu^{-2})(1+\nu)-\tau\{1+\nu\tau\nu^{-1}+(\nu\tau\nu^{-1})^2\}(1-\nu\tau\nu^{-1})]U=0\end{aligned}$$

[5] We can show that the L-functions (2.47) are the same for two Hecke eigenforms in the non base change part. In fact, let $\Omega\neq0$ be a Hecke eigenform in the non base change part and let $\lambda(\mathfrak{m})$ be the eigenvalue of $T(\mathfrak{m})$ for Ω. For the nontrivial automorphism σ of F, there exists a Hecke eigenform $\Omega_\sigma\neq0$ such that $\Omega_\sigma|T(\mathfrak{m})=\lambda(\mathfrak{m}^\sigma)\Omega_\sigma$ (cf. [Y2], p. 1035, Remark). Since Ω is not a base change, we have $\lambda(\mathfrak{m})\neq\lambda(\mathfrak{m}^\sigma)$ for some $\mathfrak{m}$. Hence Ω_σ is not a constant multiple of Ω. On the other hand, $L(s,\Omega_\sigma)$ is equal to $L(s,\Omega)$.

respectively. We put

$$A = \varphi((\widetilde{\sigma}\widetilde{\nu})^2), \qquad B = \varphi((\widetilde{\sigma}\widetilde{\tau})^3).$$

Then (6.10) holds. As in the previous section, our objective is to determine A explicitly.

7.2. Let us consider the Hecke operators. We put $g^* = T(\varpi)f^*$ where g^* is defined by (5.4) with Γ^* in place of Γ. Let $\psi \in H^1(R^*, V)^{\Gamma^*}$ be a corresponding element to g^*. We may assume that ψ is given by (6.11).

We have $3 = (4+\sqrt{13})(4-\sqrt{13})$ in F. Put $\varpi = 4-\sqrt{13} = -2\epsilon + 7$, $\mathfrak{p} = (\varpi)$ and we consider the Hecke operator $T(\mathfrak{p}) = T(\varpi)$. We may take

$$\beta_1 = \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 1 & 1 \\ 0 & \varpi \end{pmatrix}, \quad \beta_3 = \begin{pmatrix} 1 & \epsilon \\ 0 & \varpi \end{pmatrix}, \quad \beta_4 = \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}.$$

Using (6.11), we can compute $\psi(\widetilde{\sigma}^2)$, $\psi((\widetilde{\sigma}\widetilde{\nu})^2)$ and $\psi((\widetilde{\sigma}\widetilde{\tau})^3)$. Remarkably it turns out that these quantities can be expressed by A and B. Since this is technically the essential part of calculation, we are going to explain the computation of $\psi((\widetilde{\sigma}\widetilde{\tau})^3)$ in some detail. By (6.11), we have

$$\psi((\widetilde{\sigma}\widetilde{\tau})^3) = c\beta_3^{-1}Z_3,$$

where

$$Z_3 = \varphi((\widetilde{\sigma}\widetilde{\begin{pmatrix} \epsilon^{-1} & 2\epsilon-7 \\ 0 & \epsilon \end{pmatrix}}\widetilde{\sigma}\widetilde{\begin{pmatrix} 1 & -2\epsilon \\ 0 & 1 \end{pmatrix}})^3). \tag{7.2}$$

For $x \in \mathcal{O}_F$ and $u \in E_F$ such that x divides $u-1$, we put

$$\{x,u\}_4 = \widetilde{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}}\widetilde{\sigma}\widetilde{\begin{pmatrix} 1 & (1-u)/x \\ 0 & 1 \end{pmatrix}}\widetilde{\sigma}\widetilde{\begin{pmatrix} 1 & -x/u \\ 0 & 1 \end{pmatrix}}\widetilde{\sigma}\widetilde{\begin{pmatrix} 1 & -u(1-u)/x \\ 0 & 1 \end{pmatrix}}\widetilde{\sigma}\widetilde{\begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}}.$$

Then $\{x,u\}_4 \in R^*$. As a quantitative version of Lemma A.6, (3) of Appendix, we can show that

$$\begin{aligned} \varphi(\{x,u^e\}_4) &= \varphi(\{x,u\}_4) \\ &+ \sigma\begin{pmatrix} u^{-1} & u^{-e+1}(1-u^e)/x \\ 0 & u \end{pmatrix}\sigma\begin{pmatrix} 1 & u^{e-2}x \\ 0 & 1 \end{pmatrix}\varphi(\{-u^{e-2}x, u^{e-1}\}_4) \\ &- \sigma\begin{pmatrix} u^{-1} & u^{-e+1}(1-u^e)/x \\ 0 & u \end{pmatrix}\sigma\varphi((\widetilde{\sigma}\widetilde{\begin{pmatrix} u^{1-e} & 0 \\ 0 & u^{e-1} \end{pmatrix}})^2) \\ &+ \sigma\begin{pmatrix} u^{-e} & 0 \\ 0 & u^e \end{pmatrix}\varphi((\widetilde{\sigma}\widetilde{\begin{pmatrix} u^{-e} & 0 \\ 0 & u^e \end{pmatrix}})^2) \\ &- \sigma\begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}\varphi((\widetilde{\sigma}\widetilde{\begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}})^2) \end{aligned} \tag{7.3}$$

for $e \in \mathbf{Z}$. (This formula holds for any real quadratic field F.) By (7.3) and using the formulas given in §6.5, we can express $\psi((\widetilde{\sigma}\widetilde{\tau})^3)$ in terms of A and B.

7.3. We assume $\varphi \in H^1(R^*, V)^{\Gamma^*,+}$ (cf. §5.5). Then, as in §6.6, we have

$$(\delta\tau + 1)B = 0, \qquad (\delta - 1)A = 0.$$

Fact 1. Suppose $0 \le l_2 \le l_1 \le 20$. Then adding $h|R^*$, $h \in H^1(\mathcal{F}^*, V)$ to φ (keeping φ in the plus space under the action of δ), we may assume $B = 0$.

Therefore our task is to find constraints on $A = \varphi((\widetilde{\sigma}\widetilde{\nu})^2)$. We put $x = \sigma \begin{pmatrix} \epsilon^{-1} & 2\epsilon - 7 \\ 0 & \epsilon \end{pmatrix} \sigma \begin{pmatrix} 1 & -2\epsilon \\ 0 & 1 \end{pmatrix}$ and let

$$Z_A^+ = \{\mathbf{v} \in V \mid (\sigma\nu - 1)\mathbf{v} = 0,\ (\delta - 1)\mathbf{v} = 0,\ xZ_3 = Z_3\}. \tag{7.4}$$

Here Z_3 is defined by (7.2) and the meaning of $xZ_3 = Z_3$ is the same as in §6.6. Namely, $xZ_3 = Z_3$ must hold because $x^3 = 1$; since Z_3 can be expressed by A, $xZ_3 = Z_3$ gives a constraint on A.

We consider the contribution of $H^1(\mathcal{F}^*, V)$ to Z_A^+. Take $h \in H^1(\mathcal{F}^*, V)$ and put

$$h(\widetilde{\sigma}) = S, \qquad h(\widetilde{\nu}) = U, \qquad h(\widetilde{\tau}) = T.$$

We require that $h|R^*$ vanishes on the elements (i*), (ii*), (iv*), (v*). These conditions are equivalent to (6.21), (6.22), (6.7) and and (7.1). We have

$$h((\widetilde{\sigma}\widetilde{\nu})^2) = (\sigma\nu + 1)(\sigma U + S).$$

We also require that (6.23) holds. Let B_A^+ be the subspace of V generated by $(\sigma\nu + 1)(\sigma U + S)$ when S, T, U extend over vectors of V satifying the relations (6.7), (6.21), (6.22), (6.23) and (7.1). We have $B_A^+ \subset Z_A^+$. As shown in §4.1, (6.24) holds. By Proposition 5.5, we have

$$\dim Z_A^+/B_A^+ \ge \dim S_{l_1+2.l_2+2}(\Gamma) \qquad \text{if } l_2 \ge 4,\ l_1 \ne l_2 \text{ or if } \ l_1 = l_2,\ l_2 \ge 6.$$

Now by numerical computations, we have verified:

Fact 2. Suppose $0 \le l_2 \le l_1 \le 20$. Then $\dim S_{l_1+2,l_2+1}(\Gamma) = \dim Z_A^+/B_A^+$.

This fact means that the constraints posed on $A = \varphi((\widetilde{\sigma}\widetilde{\nu})^2)$ is enough.

Example 7.1. We take $l_1 = l_2 = 6$. We have $\dim S_{8,8}(\Gamma) = 5$. Calculating the action of $T(\mathfrak{p})$ on Z_A^+/B_A^+ using (6.11), we find that the characteristic polynomial of $T(\mathfrak{p})$ is

$$(X^2 - 40X - 3957)(X^3 + 28X^2 - 2601X - 71748).$$

The quadratic factor corresponds to the non base change part; the irreducible factor of degree three corresponds to the base change part from $S_8(\Gamma_0(13), (\frac{}{13}))$. Let $\Omega \in S_{8,8}(\Gamma)$ be the Hecke eigenform such that $\Omega|T(\mathfrak{p}) = (20 + \sqrt{4357})\Omega$. Then we find

$$R(6, \Omega)/R(4, \Omega) = 70/3.$$

Example 7.2. We take $l_1 = l_2 = 8$. We have $\dim S_{10,10}(\Gamma) = 7$. We find that the characteristic polynomial of $T(\mathfrak{p})$ is

$$(X^2-16X-42789)(X^5+X^4-66033X^3+1260423X^2+530326440X+14266185264).$$

The quadratic factor corresponds to the non base change part. Let $\Omega \in S_{10,10}(\Gamma)$ be the Hecke eigenform such that $\Omega|T(\mathfrak{p}) = (8 + \sqrt{42853})\Omega$. Then we find

$$R(7, \Omega)/R(5, \Omega) = 50.$$

Example 7.3. We take $l_1 = l_2 = 10$. We have $\dim S_{12,12}(\Gamma) = 11$. We find that the characteristic polynomial of $T(\mathfrak{p})$ is

$$\begin{aligned}&(X - 252)(X^4 + 252X^3 - 496198X^2 - 116604684X + 25202349477)\\&(X^6 + 244X^5 - 665334X^4 - 129598956X^3 + 109163403621X^2\\&+ 14522233287672X - 255121008509808).\end{aligned}$$

The irreducible factor of degree four corresponds to the non base change part; $X - 252$ corresponds to the base change part from $S_{12}(\mathrm{SL}_2(\mathbf{Z}))$ and the irreducible factor of degree six corresponds to the base change part from $S_{12}(\Gamma_0(13), (\frac{}{13}))$. Put

$$f(X) = X^4 + 252X^3 - 496198X^2 - 116604684X + 25202349477.$$

Let θ be a root of $f(X)$ and put $K = \mathbf{Q}(\theta)$. We find that K contains a quadratic subfield $F = \mathbf{Q}(\sqrt{7 \cdot 5167})$. Put $d = 7 \cdot 5167$. Then a root of $f(X)$ is given by

$$\psi = -(63 + \sqrt{d}) + \sqrt{223837 - 360\sqrt{d}}.$$

We have

$$N(223837 - 360\sqrt{d}) = 13 \cdot 563 \cdot 6205151.$$

This number and the quadratic fields in Examples 7.1 and 7.2 are consistent with the table given in Doi-Hida-Ishii [DHI].

For the Hecke eigenform $\Omega \in S_{12,12}(\Gamma)$ such that $\Omega | T(\varpi) = \psi\Omega$, we find

$$R(10, \Omega)/R(6, \Omega) = \frac{3732099 + 18663\sqrt{d}}{5},$$

$$R(8, \Omega)/R(6, \Omega) = \frac{24367 + 121\sqrt{d}}{20}.$$

§8. A comparison of two methods

In [Sh3], Shimura gave a method to calculate critical values of $D(s, f, g)$ for two elliptic modular forms f and g. Here $D(s, f, g)$ is the Rankin-Selberg convolution of f and g. Shortly later he gave a generalization to the case of Hilbert modular forms ([Sh4]). Taking one argument in the convoluted L-function as a suitable Eisenstein series, this method enables us to calculate the ratios of critical values of $L(s, \Omega)$ for a Hilbert modular form Ω. We call this technique method A. We call the cohomological technique method B, which was initiated in [Sh1] and studied in this paper when $[F : \mathbf{Q}] = 2$. It is interesting to compare A and B.

(0) Method A is more general and conceptually simpler. It has the advantage to give the relation of the product of the plus and minus periods to the Petersson norm. It is applicable also to modular forms of half integral weights.

(1) If $n = [F : \mathbf{Q}] > 2$, the method B has to calculate $H^n(\Gamma, V)$, which is beyond the reach at present. Therefore when $[F : \mathbf{Q}] > 2$, A is definitively superior than B.

(2) Suppose that $[F : \mathbf{Q}] = 2$. The method B is still incomplete. But in the cases well worked out, $F = \mathbf{Q}(\sqrt{5})$ for example, B has the advantage that we can write a program which calculates everything by machine. It can also be used to calculate the characteristic polynomials of Hecke operators. (In this respect, it is desirable to solve the problem mentioned at the end of subsection 6.5.) We employed essentially a single program to obtain examples in section 7. Therefore in some cases at least, B will have the advantage over A. But in general the method A is conceptually simpler.

In Doi-Goto [DG] and Doi-Ishii [DI], the authors gave interesting examples of critical values of $D(s, f, g)$ for Hilbert modular forms f and g. Their interests was the relation of this value to the congruences between Hilbert modular forms. However they did not give examples of critical values of $L(s, \Omega)$. Recently Dr. K. Okada calculated the ratios of critical values of $L(s, \Omega)$ and confirmed the numerical value of Example 7.1 by method A. He obtained one more example for $F = \mathbf{Q}(\sqrt{17})$.

(3) Suppose that $F = \mathbf{Q}$. The method B is developed into the theory of modular symbols which is presently used to calculate characteristic polynomials of Hecke operators. For the L-values, the author doesn't know which is faster. But the calculation of [Sh1] reviewed in the introduction suggests that B would not be more complex than A.

Appendix. Generators and relations

Let F be a real quadratic field and ϵ be the fundamental unit of F. Let $\{1, \omega\}$ be an integral basis of $\mathcal{O}_F$, i.e., $\mathcal{O}_F = \mathbf{Z} \oplus \mathbf{Z}\omega$. We write

$$\epsilon^2 = A + B\omega, \qquad \epsilon^2\omega = C + D\omega. \tag{A.1}$$

We put $\Gamma = \mathrm{PSL}(2, \mathcal{O}_F)$, $\widetilde{\Gamma} = \mathrm{SL}(2, \mathcal{O}_F)$,

$$\widetilde{P} = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \,\middle|\, a \in E_F, b \in \mathcal{O}_F \right\}, \qquad P = \widetilde{P}/\{\pm 1_2\}.$$

We define elements of $\widetilde{\Gamma}$ by

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mu = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \eta = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}.$$

Then it is known that σ, μ, τ and η generate $\widetilde{\Gamma}$ (cf. Vaserštein [V]). This fact can be proved in elementary way if $\mathcal{O}_F$ is a Euclidean ring, $F = \mathbf{Q}(\sqrt{5})$ for example. We use same letters σ, μ, τ and η for their classes in Γ, since this will cause no confusion. Now we have relations among them:

(i) $$\sigma^2 = 1.$$

(ii) $$(\sigma\tau)^3 = 1.$$

(iii) $$(\sigma\mu)^2 = 1.$$

(iv) $$\tau\eta = \eta\tau.$$

(v) $$\mu\tau\mu^{-1} = \tau^A\eta^B.$$

(vi) $$\mu\eta\mu^{-1} = \tau^C\eta^D.$$

If we can take $\omega = \epsilon$ and $-\epsilon^{-1} = A' + B'\epsilon$, then we have

$$\text{(vii)} \qquad \sigma\eta\sigma = \tau^{A'}\eta^{B'}\sigma\eta^{-1}\mu.$$

The relations (ii) and (vii) follow from

$$\text{(A.2)} \qquad \sigma \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \sigma = \begin{pmatrix} 1 & -t^{-1} \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} -t & 1 \\ 0 & -t^{-1} \end{pmatrix}, \quad t \in E_F.$$

It is easy to see that μ, τ and η generate P and (iv) $\sim$ (vi) are their fundamental relations.

The purpose of this appendix is to prove the following theorem.

Theorem A.1. *Let $F = \mathbf{Q}(\sqrt{5})$ and $\Gamma = \mathrm{PSL}(2, \mathcal{O}_F)$. We take $\omega = \epsilon$. The fundamental relations satisfied by the generators σ, μ, τ and η are (i) $\sim$ (vii).*

We note that if $F = \mathbf{Q}(\sqrt{5})$ then $A = 1$, $B = 1$, $C = 1$, $D = 2$, $A' = 1$, $B' = -1$. The relations (i) to (vi) and (A.2) hold for any real quadratic field. Our theorem states that the minimal relations are enough when $F = \mathbf{Q}(\sqrt{5})$. This minimality will be satisfied by some more real quadratic fields with small discriminants but will not hold in general.

We begin by preliminary considerations on generators and relations of Γ. [6] Since Γ is generated by P and σ, every relation among elements of P and σ takes the form

$$p_1\sigma p_2\sigma \cdots p_m\sigma = 1, \qquad p_i \in P,\ 1 \le i \le m.$$

Using (i) and (iii) $\sim$ (vi), this relation can be written as

$$\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix} \sigma \cdots \begin{pmatrix} 1 & x_m \\ 0 & 1 \end{pmatrix} \sigma = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \quad x_i \in \mathcal{O}_F,\ u \in E_F.$$

We call a relation of this type an m *terms relation* counting the number of σ involved.

Lemma A.2. *Using relations (i) and (iii) $\sim$ (vi), every three terms relation can be reduced to (A.2).*

Proof. If we have a two terms relation

$$\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix} \sigma = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix},$$

[6] For this part, we do not assume $F = \mathbf{Q}(\sqrt{5})$.

we have $x_1 = x_2 = 0$, $u = \pm 1$. Hence the two terms relation reduces to (i). Let

$$\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & x_3 \\ 0 & 1 \end{pmatrix} \sigma = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$$

be a three terms relation. Then we see that $x_2 = \pm u \in E_F$. Using (A.2), we have $\sigma \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix} \sigma = p_1 \sigma p_2$ with some p_1, $p_2 \in P$ and the three terms relation in question reduces to a two terms relation. This completes the proof.

Lemma A.3. *Assume that we can take $\omega = \epsilon$. The relation (A.2) can be reduced to the relations (i) $\sim$ (vii). In other words, the relation (A.2) for $t \in E_F$ can be reduced to the relations (A.2) for $t = 1$, ϵ using relations (i) and (iii) $\sim$ (vi).*

Proof. We write the relation (A.2) as $\{t\}$. Using (i), the relation (iii) implies the relation $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \sigma = \sigma \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}$ for $u \in E_F$. Then we obtain the relation $\{-t\}$ taking the inverse of the both sides of (A.2), using (i), (iv) $\sim$ (vi). Taking the conjugate by μ of both sides of (A.2), we obtain the relation $\{\epsilon^{-2} t\}$ using (i), (iii) $\sim$ (vi). Since E_F is generated by ϵ and ± 1, this completes the proof.

Next we consider the four terms relation.

$$\text{(A.3)} \qquad \begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & x_3 \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & x_4 \\ 0 & 1 \end{pmatrix} \sigma = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$$

We write the relation (A.3) as $\{x_1, x_2, x_3, x_4; u\}$.

Lemma A.4. *The four terms relation (A.3) reduces to (i) $\sim$ (vi) and (A.2) if $x_i \in E_F$ for some i, $1 \leq i \leq 4$.*

Proof. Suppose that $x_2 \in E_F$. By (A.2), we have $\sigma \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix} \sigma = p_1 \sigma p_2$ with some p_1, $p_2 \in P$. Using this expression, we find that (A.3) reduces to a three terms relation. We write (A.3) as

$$\begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & x_3 \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & x_4 \\ 0 & 1 \end{pmatrix} \sigma = \sigma \begin{pmatrix} 1 & -x_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}.$$

Using (i) $\sim$ (vi), the right-hand side can be written as $\begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} \sigma \begin{pmatrix} 1 & -u^{-2} x_1 \\ 0 & 1 \end{pmatrix}$. Hence $\{x_1, x_2, x_3, x_4; u\}$ is equivalent to $\{x_2, x_3, x_4, u^{-2} x_1; u^{-1}\}$ under (i) $\sim$ (vi). By this cyclic rotation, any x_i can be brought to the second position at the cost of multiplying by a unit. Hence the assertion follows.

For $u \in E_F$, $x \in \mathcal{O}_F$, we have the relation

$$\text{(A.4)}\qquad \begin{aligned} &\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & (1-u)/x \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & -x/u \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & -u(1-u)/x \\ 0 & 1 \end{pmatrix} \sigma \\ &= \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \end{aligned}$$

if x divides $u-1$.

Lemma A.5. *Under (i) $\sim$ (vi) and (A.2), the four terms relation (A.3) can be reduced to (A.4) with some x and u.*

Proof. We see easily that the four terms relation (A.3) is equivalent to a relation of the form

$$\text{(A.3}'\text{)}\qquad \sigma \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \sigma = \begin{pmatrix} 1 & y_1 \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & y_2 \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & y_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix}.$$

Here x, $y_i \in \mathcal{O}_F$, $1 \leq i \leq 3$ and $h \in E_F$. By a direct computation, we get

$$h(y_1 y_2 - 1) = -\omega, \qquad h y_2 = \omega x, \qquad h^{-1}(y_2 y_3 - 1) = -\omega,$$

where $\omega = \pm 1$. Putting $u = \omega h^{-1}$, we have

$$y_2 = ux, \qquad y_1 = \frac{1-u}{ux}, \qquad y_3 = \frac{1-u^{-1}}{ux}.$$

Hence we see that x divides $u-1$ and that (A.3$'$) is equivalent to

$$\text{(A.3}''\text{)}\qquad \begin{aligned} &\sigma \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \sigma \\ &= \begin{pmatrix} 1 & (1-u)/ux \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & ux \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & (1-u^{-1})/ux \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}. \end{aligned}$$

On the other hand, under (i) $\sim$ (vi), (A.4) is equivalent to

$$\text{(A.4}'\text{)}\qquad \begin{aligned} &\sigma \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \sigma \\ &= \begin{pmatrix} 1 & (1-u)/x \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & -x/u \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & -u(1-u)/x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}. \end{aligned}$$

We obtain (A.3$''$) from (A.4$'$) by substituting x by $-x$ and u by u^{-1}. This completes the proof.

We denote the four terms relation (A.4) by $\{x, u\}$. We have $\{x, u\} = \{x, (1-u)/x, -x/u, -u(1-u)/x; u\}$. Under (i) $\sim$ (vi), the relation of the form (A.3$'$) is equivalent to $\{x, u\}$ and the relation $\{x_1, x_2, x_3, x_4; u\}$ is equivalent to $\{x_2, x_3, x_4, u^{-2}x_1; u^{-1}\}$ (cf. the proofs of Lemmas A.4 and A.5). Therefore $\{x, u\}$ is equivalent to $\{(1-u)/x, u^{-1}\}$ under (i) $\sim$ (vi). By Lemma A.4, $\{x, u\}$ is reducible to (i) $\sim$ (vi) and (A.2) if $x \in E_F$ or $(1-u)/x \in E_F$.

Lemma A.6. *Assuming (i) $\sim$ (vi) and (A.2), the following assertions hold.*

(1) $\{x, u\}$ is equivalent to $\{-x, u^{-1}\}$.

(2) $\{x, u\}$ is equivalent to $\{t^2 x, u\}$ for every $t \in E_F$.

(3) We assume the four terms relation $\{x, u\}$. Then $\{x, u^e\}$ is equivalent to $\{u^e x, u^{1-e}\}$ for $e \in \mathbf{Z}$.

(4) $\{x, u\}$ is equivalent to $\{(1-u)/x, u^{-1}\}$.

(5) Suppose that $(x) = (2)$. Then $\{x, u\}$ is equivalent to $\{x, -u\}$.

Proof. We write $\{-x, u^{-1}\}$ in the form of (A.3″). Taking the inverses of both sides, we obtain (1). We obtain (2) taking the conjugates of both sides by $\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$. To prove (3), we set the right-hand side of (A.3″) is equal for $\{x, u\}$ and for $\{x, u^e\}$. By a simple computation, we find that the resulting equality is

$$\begin{aligned} &\sigma \begin{pmatrix} 1 & u^e x \\ 0 & 1 \end{pmatrix} \sigma \\ &= \begin{pmatrix} 1 & (u^{e-1}-1)/u^e x \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & ux \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & (u^{-1}-u^{e-2})/x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u^{e-1} & 0 \\ 0 & u^{1-e} \end{pmatrix}, \end{aligned}$$

which is $\{u^e x, u^{1-e}\}$. Hence we obtain (3). We noted (4) already in the discussion before Lemma A.6. To prove (5), we set the right-hand side of (A.3″) is equal for $\{x, u\}$ and for $\{x, -u\}$. The resulting equality is

$$\begin{aligned} &\sigma \begin{pmatrix} 1 & ux \\ 0 & 1 \end{pmatrix} \sigma \\ &= \begin{pmatrix} 1 & -2/ux \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & -ux \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & -2/ux \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned}$$

Since $-2/ux \in E_F$, this relation reduces to a three terms relation by Lemma A.4. In view of Lemma A.2, this completes the proof.

Remark A.7. Suppose that $(1-u)/x \in E_F$. Then, by Lemma A.4, $\{tx, u\}$ can be reduced to (i) $\sim$ (vi) and (A.2) for every $t \in E_F$. By (1) and (3) of Lemma A.6, we see that $\{x, u^e\}$ can be reduced to (i) $\sim$ (vi) and (A.2) for all $e \in \mathbf{Z}$.

The following Lemma is of some interest though it will not be used in this paper.

Lemma A.8. *Suppose that there exist sequences of integers* $x_0, x_1, \ldots, x_k \in \mathcal{O}_F$ *and units* $u_0, u_1, \ldots, u_k \in E_F$ *such that*

$$x_{i-1}x_i = 1 - u_i, \quad 1 \leq i \leq k.$$

We assume that $u_i = u_{i-1}^{m_i}$, $1 \leq i \leq k$ *with a nonzero interger* m_i. *If* $(1-u_0)/x_0 \in E_F$, *then the four terms relation* $\{x_k, u_k\}$ *reduces to (i)* $\sim$ *(vi) and (A.2).*

Proof. Using Lemma A.6, the reducibility of $\{tx_i, u_i^e\}$, $t \in E_F$, $e \in \mathbf{Z}$ can be shown easily by induction on i.

Let G be a group with generators $\sigma_1, \ldots, \sigma_m$. Let $\mathcal{F}$ be a free group on the free generators $\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_m$. Then we can define a surjective homomorphism $\pi : \mathcal{F} \longrightarrow G$ by $\pi(\widetilde{\sigma}_i) = \sigma_i$, $1 \leq i \leq m$. Let R be the kernel of π. Next let S be a finite subset of G which generates G. For $\gamma \in S$, we prepare a symbol $[\gamma]$ and let $\mathcal{F}'$ be the free group on the free generators $[\gamma]$, $\gamma \in S$. We can define a surjective homomorphism $\pi' : \mathcal{F}' \longrightarrow G$ by $\pi'([\gamma]) = \gamma$, $\gamma \in S$. Let R' be the kernel of π'. Clearly $([\gamma_1][\gamma_2])^{-1}[\gamma_1\gamma_2] \in R'$ if γ_1, γ_2, $\gamma_1\gamma_2 \in S$. We assume that R' is generated by the elements of this form and their conjugates.

Now for every $\gamma \in S$, we take and fix an expression

$$\gamma = \sigma_{i_1}^{\epsilon_1} \cdots \sigma_{i_k}^{\epsilon_k}, \qquad i_j \in [1, m], \quad \epsilon_j = \pm 1$$

and put $\widetilde{\gamma} = \widetilde{\sigma}_{i_1}^{\epsilon_1} \cdots \widetilde{\sigma}_{i_k}^{\epsilon_k}$. (If $\gamma = \sigma_i \in S$, we put $\widetilde{\gamma} = \widetilde{\sigma}_i$.) By the universality of the free group, there exists a homomorphism $\varphi : \mathcal{F}' \longrightarrow \mathcal{F}$ which satisfies $\varphi([\gamma]) = \widetilde{\gamma}$, $\gamma \in S$. Then we have $\pi' = \pi \circ \varphi$. Let R_0 be the normal subgroup of $\mathcal{F}$ generated by $(\widetilde{\gamma}_1\widetilde{\gamma}_2)^{-1}\widetilde{\gamma_1\gamma_2}$, γ_1, γ_2, $\gamma_1\gamma_2 \in S$ and their conjugates. We have $R_0 \subset R$. Since $\varphi(R') \subset R_0$ by the assumption, φ induces the homomomorphism $\bar{\varphi} : \mathcal{F}'/R' \longrightarrow \mathcal{F}/R_0$ which satisfies $\bar{\varphi}(g \mod R') = \varphi(g) \mod R_0$, $g \in \mathcal{F}'$.

Lemma A.9. *Let the notation be the same as above. If* $\sigma_i \in S$, $1 \leq i \leq m$, *then we have* $R_0 = R$.

Proof. Define a homomorphism $\pi_0 : \mathcal{F}/R_0 \longrightarrow G$ by $\pi_0(h \mod R_0) = \pi(h)$, $h \in \mathcal{F}$. Since $(\pi_0 \circ \bar{\varphi})(g \mod R') = (\pi \circ \varphi)(g) = \pi'(g)$, $g \in \mathcal{F}'$, $\pi_0 \circ \bar{\varphi}$ is injective. Hence $\pi_0|\bar{\varphi}(\mathcal{F}'/R')$ is injective. We can write $\bar{\varphi}(\mathcal{F}'/R') = H/R_0$ with a subgroup H of $\mathcal{F}$. Now the assumption of the Lemma implies $H = \mathcal{F}$. Therefore π_0 is injective and we obtain $R_0 = R$.

For the proof of Theorem A.1, we use the following theorem of Macbeath (cf. Theorem 1 of [Mac] and also Theorem 1.1 of [Sw]).

Theorem M. *Let X be a path connected Hausdorff topological space and Γ be a group which acts on X as homeomorphisms. We assume that the fundamental group $\pi_1(X)$ of X is trivial. Let V be a path connected open subset of X such that $X = \Gamma V$. Define a subset S of Γ by*

$$S = \{\gamma \in \Gamma \mid V \cap \gamma V \neq \emptyset\}.$$

Then S generates Γ. [7] *Let $\mathcal{F}$ be the free group which has the symbols $[\sigma]$, $\sigma \in S$ as free generators. Define a homomorphism $\pi : \mathcal{F} \longrightarrow \Gamma$ by $\pi([\sigma]) = \sigma$. Let R be the kernel of π. Then R is generated by $([\sigma][\tau])^{-1}[\sigma\tau]$ and their conjugates, where σ and τ are elements of S which satisfy*

$$(*) \qquad V \cap \sigma V \cap \sigma\tau V \neq \emptyset.$$

In other words, Γ has a presentation $\Gamma = \mathcal{F}/R$.

Swan ([Sw]) generalized this theorem to the case where $\pi_1(X) \neq 1$ and obtained generators and relations for $\mathrm{SL}(2, \mathcal{O}_K)$, for several imaginary quadratic fields K with small discriminants.

Let the notation be the same as in Theorem M. For a subset T of X, we put

$$S(T) = \{\gamma \in \Gamma \mid T \cap \gamma T \neq \emptyset\}.$$

Let D be a closed subset of X such that $\Gamma D = X$.

Lemma A.10. *Suppose in addition that the topological space X is normal. Then we have*

$$\cap_{U \supset D,\ U \text{ is open}} S(U) = S(D).$$

Proof. Clearly the left-hand side contains the right-hand side. Pick an element γ of the left-hand side. Assume that $D \cap \gamma D = \emptyset$. Since X is normal, we can find open subsets U and U' of X so that

$$U \supset D, \qquad U' \supset \gamma D, \qquad U \cap U' = \emptyset.$$

Put $U'' = U \cap \gamma^{-1} U'$. Then we have $U'' \supset D$, $U'' \cap \gamma U'' \subset U \cap U' = \emptyset$. This is a contradiction and we complete the proof.

Next we assume that $S(D)$ is finite and that $S(U)$ is finite for an open set U which contains D. We put

$$S(D) = \{\gamma_1, \ldots, \gamma_m\}, \qquad S(U) = \{\gamma_1, \ldots, \gamma_m, \gamma_{m+1}, \ldots, \gamma_n\}$$

[7] This fact is an old result of Siegel, cf. [Si1].

assuming $S(U) \supsetneqq S(D)$. By Lemma A.10, for every γ_i, $i > m$, there exists an open set $U_i \supset D$ such that $\gamma_i \notin S(U_i)$. Put $V = U \cap (\cap_{i=m+1}^{n} U_i)$. Then we have $\gamma_i \notin S(V)$. Therefore we conclude that $S(D) = S(V)$ for an open set V which contains D. This means that we may replace S to $S(D)$ in Theorem M if such a V is path connected. (Note that in Theorem M, $([\sigma][\tau])^{-1}[\sigma\tau] \in R$ for σ, $\tau \in S$ such that $\sigma\tau \in S$. Thus the condition $(*)$ may be dropped. However $(*)$ reduces the number of relations and can be essential for the practical purpose.)

Now let F be a totally real field of degree n. Let us review the fundamental domain of $\Gamma = \mathrm{PSL}(2, \mathcal{O}_F)$ acting on $\mathfrak{H}^n$ (cf. [Si2]). Let $\sigma_1, \ldots, \sigma_n$ be all the isomorphisms of F into $\mathbf{R}$. For $a \in F$, we put $a^{(i)} = a^{\sigma_i}$. Take an integral basis of $\mathcal{O}_F$ so that

$$\mathcal{O}_F = \mathbf{Z}\omega_2 + \mathbf{Z}\omega_2 + \cdots + \mathbf{Z}\omega_n$$

and let $\epsilon_1, \ldots, \epsilon_{n-1}$ be generators of a free part of E_F. For $x = (x_1, \ldots, x_n) \in \mathbf{C}^n$, we put $N(x) = x_1 \cdots x_n$. For simplicity, we assume that the class number of F is one. Take $z = (z_1, \ldots, z_n) \in \mathfrak{H}^n$. Put $z_j = x_j + iy_j$, x_j, $y_j \in \mathbf{R}$. We define the local coordinates of z relative to the cusp ∞ by the formulas (cf. [Si2], p. 249)

$$\text{(A.5)} \quad Y_1 \log|\epsilon_1^{(k)}| + \cdots + Y_{n-1} \log|\epsilon_{n-1}^{(k)}| = \frac{1}{2} \log \frac{y_k}{\sqrt[n]{N(y)}}, \qquad 1 \le k \le n-1.$$

$$\text{(A.6)} \qquad X_1 \omega_1^{(l)} + \cdots + X_n \omega_n^{(l)} = x_l, \qquad 1 \le l \le n,$$

Here $y = (y_1, \ldots, y_n)$. We put

$$D_\infty = \{z \in \mathfrak{H}^n \mid -\frac{1}{2} \le Y_i < \frac{1}{2},\ 1 \le i \le n-1, \quad -\frac{1}{2} \le X_j < \frac{1}{2},\ 1 \le j \le n\}.$$

Then D_∞ is a fundamental domain of P. (P is the subgroup of Γ consisting of all elements which are represented by upper triangular matrices.) We define

$$\text{(A.7)} \quad \begin{aligned} D = \{z \in \overline{D_\infty} \mid N(|cz+d|) \ge 1 \ \text{ whenever } \\ c \text{ and } d \text{ are relatively prime integers of } \mathcal{O}_F\}. \end{aligned}$$

Here $\overline{D_\infty}$ denote the closure of D_∞ and $|cz+d| = (|c^{(1)} z_1 + d^{(1)}|, \ldots, |c^{(n)} z_n + d^{(n)}|)$. Then D satisfies that (cf. [Si2], p. 266–268):

1. D is a closed subset of $\mathfrak{H}^n$ such that $\Gamma D = \mathfrak{H}^n$.

2. Two distinct interior points of D cannot be transformed each other by an element of Γ.

3. There are only finitely many $\gamma \in \Gamma$ such that $D \cap \gamma D \neq \emptyset$. Furthermore D and γD, $\gamma \neq 1$ can intersect only on the boundary of D.

Now we assume that $[F : \mathbf{Q}] = 2$. We may assume that $\omega_1 = 1$, $\omega_2 = \omega$, $\epsilon^{(1)} = \epsilon$. Then we have

(A.8)

$$D = \Bigg\{ z \in \mathfrak{H}^2 \mid \epsilon^{-2} \leq \frac{y_2}{y_1} \leq \epsilon^2,$$
$$-\frac{1}{2} \leq \frac{1}{\omega - \omega'}(\omega' x_1 - \omega x_2) \leq \frac{1}{2}, \quad -\frac{1}{2} \leq \frac{1}{\omega - \omega'}(x_1 - x_2) \leq \frac{1}{2},$$
$$N(|cz + d|) \geq 1 \text{ whenever } c \text{ and } d \text{ are relatively prime integers of } \mathcal{O}_F \Bigg\}.$$

Here ω' denotes the conjugate of ω.

Hereafter in this section, we assume that $F = \mathbf{Q}(\sqrt{5})$. We take $\omega = \epsilon$. The next lemma is the essential ingredient of the proof of Theorem A.1.

Lemma A.11. *Let $F = \mathbf{Q}(\sqrt{5})$ and take $\omega = \epsilon$. Put $S = \{\gamma \in \Gamma \mid D \cap \gamma D \neq \emptyset\}$. Then S is a finite set and we have $S \subset S_0 \sqcup S_1 \sqcup S_2$, where*

$$S_0 = P, \qquad S_1 = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ c \in E_F\},$$

$$S_2 = \left\{ \gamma = \begin{pmatrix} \pm\epsilon^3 & b \\ 2\epsilon & \pm\epsilon^3 \end{pmatrix}, \qquad \begin{pmatrix} \pm 1 & b \\ 2\epsilon^{-2} & \pm 1 \end{pmatrix} \right\}.$$

Here $\pm$ can be taken arbirarily and $b \in \mathcal{O}_F$ is chosen so that $\det \gamma = 1$. *(S_2 consists of eight elements.)*

We give a proof of Theorem A.1 assuming Lemma A.11.

Proof of Theorem A.1. We consider $\mathfrak{H}^2 \subset \mathbf{C}^2$ and let d denote the Euclidean metric induced by this embedding. For $\delta > 0$, we put

$$D_\delta = \{z \in \mathfrak{H}^2 \mid d(z, D) < \delta\}.$$

We see easily that D is path connected. Let $z \in D_\delta$. Then there exists $z_1 \in D$ such that $d(z, z_1) < \delta$. Hence z is connected by a path to z_1. Therefore D_δ is path connected. By using the argument of Lemma A.10, we see that $\cap_{\delta > 0} S(D_\delta) = S$. Moreover we can show without difficulty that $S(D_\delta)$ is finite when δ is sufficiently small. Therefore $S(D_\delta) = S$ when δ

is sufficiently small and Theorem M can be applied with S given in Lemma A.11.

For $\gamma \in S$, we prepare a symbol $[\gamma]$ and consider the free group $\mathcal{F}'$ on the free generators $[\gamma]$. By Theorem M, it is sufficient to show that $[\gamma_2]^{-1}[\gamma_1]^{-1}[\gamma_1\gamma_2]$, γ_1, γ_2, $\gamma_1\gamma_2 \in S$ can be reduced to a three term relation. We put $S'_i = S \cap S_i$, $0 \leq i \leq 2$. We can check easily that σ, μ, τ, $\eta \in S$. Hence Lemma A.9 is applicable. Let $\mathcal{F}$ be the free group on the free generators $\widetilde{\sigma}$, $\widetilde{\mu}$, $\widetilde{\tau}$ and $\widetilde{\eta}$. We define a homomorphism $\pi : \mathcal{F} \longrightarrow \Gamma$ by $\pi(\widetilde{\sigma}) = \sigma$, $\pi(\widetilde{\mu}) = \mu$, $\pi(\widetilde{\tau}) = \tau$, $\pi(\widetilde{\eta}) = \eta$. For $\gamma \in S$, we define $\widetilde{\gamma} \in \mathcal{F}$ such that $\pi(\widetilde{\gamma}) = \gamma$ as follows.

If $\gamma \in P$, we write $\gamma = \mu^a\tau^b\eta^c$. Then we define $\widetilde{\gamma} = \widetilde{\mu}^a\widetilde{\tau}^b\widetilde{\eta}^c$. In particular, this rule applies to an element $\gamma \in S'_0$. We have

$$\text{(A.9)} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & c^{-1}a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -c & -d \\ 0 & -c^{-1} \end{pmatrix}, \qquad c \in E_F.$$

Hence $\gamma \in S'_1$ can be written as $\gamma = p_1\sigma p_2$, p_1, $p_2 \in P$. We fix such an expression and define $\widetilde{\gamma} = \widetilde{p}_1\widetilde{\sigma}\widetilde{p}_2$. Suppose $\gamma \in S'_2$. We write γ in the form $\gamma = \begin{pmatrix} u & \beta \\ 2\epsilon^m & u^* \end{pmatrix}$, u, $u^* \in E_F$, $\beta \in \mathcal{O}_F$, $m \in \mathbf{Z}$. We have

$$\text{(A.10)} \qquad \begin{pmatrix} u & \beta \\ 2\epsilon^m & u^* \end{pmatrix} = \sigma \begin{pmatrix} 1 & -2u^{-1}\epsilon^m \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} -u & -\beta \\ 0 & -u^{-1} \end{pmatrix}.$$

We fix this expression $\gamma = \sigma p_1 \sigma p_2$, p_1, $p_2 \in P$ and define $\widetilde{\gamma} = \widetilde{\sigma}\widetilde{p}_1\widetilde{\sigma}\widetilde{p}_2$.

By Lemma A.9, it is sufficient to show that $\widetilde{\gamma}_2^{-1}\widetilde{\gamma}_1^{-1}\widetilde{\gamma_1\gamma_2}$ reduces to a three terms relation (under (i) $\sim$ (vi) and (A.2)) when γ_1, γ_2, $\gamma_1\gamma_2 \in S$. We see that there cannot arise the case where all of γ_1, γ_2, $\gamma_1\gamma_2$ belong to S'_2, by inspecting the $(2,1)$-component of $\gamma_1\gamma_2$. This implies that if two of γ_1, γ_2, $\gamma_1\gamma_2$ belong to S'_2, then the other one must belong to S'_0. Therefore $\widetilde{\gamma}_2^{-1}\widetilde{\gamma}_1^{-1}\widetilde{\gamma_1\gamma_2}$ defines at most a four terms relation. We may assume that $\widetilde{\gamma}_2^{-1}\widetilde{\gamma}_1^{-1}\widetilde{\gamma_1\gamma_2}$ defines a four terms relation. Then one of γ_1, γ_2, $\gamma_1\gamma_2$ belongs to S'_2. By (A.10), this relation takes the form (A.3′) with $x \in \mathcal{O}_F$ such that $(x) = (2)$. As shown in the proof of Lemma A.5, it suffices to consider the four terms relation $\{x, u\}$ for $u \in E_F$ such that x divides $u - 1$. Now the group $E_{(2)} = \{u \in E_F \mid u \equiv 1 \bmod 2\}$ is generated by -1 and ϵ^3. By $\epsilon^3 - 1 = 2\epsilon$ and Remark A.7, we see that $\{x, \epsilon^{3e}\}$ is reducible to (i) $\sim$ (vi) and (A.2) for $e \in \mathbf{Z}$. By Lemma A.6, (5), $\{x, -\epsilon^{3e}\}$ is reducible to (i) $\sim$ (vi) and (A.2). This completes the proof.

Now we are going to prove Lemma A.11. We consider an element $\gamma \in \Gamma$ such that for a point $z \in D$, $\gamma z \in D$ holds, i.e., $D \cap \gamma^{-1}D \neq \emptyset$. [8]. We put

[8] Since S_i, $i = 0, 1, 2$ is stable under $\gamma \mapsto \gamma^{-1}$, it suffices to determine γ which satisfies $D \cap \gamma^{-1}D \neq \emptyset$.

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $z' = \gamma z$, $z' = (z'_1, z'_2)$, $z'_j = x'_j + iy'_j$, $j = 1, 2$, $y' = (y'_1, y'_2)$. We have

$$N(y') = \frac{N(y)}{N(|cz+d|)^2}.$$

Hence $N(y') \leq N(y)$. Changing the roles of z and z', we have $N(y) \leq N(y')$. Hence we see that $N(y') = N(y)$ and

$$\text{(A.11)} \qquad N(|cz+d|) = 1.$$

Since we are assuming that $F = \mathbf{Q}(\sqrt{5})$, $\omega = \epsilon$, we have

$$x_1 = X_1 + \frac{1+\sqrt{5}}{2} X_2, \quad x_2 = X_1 + \frac{1-\sqrt{5}}{2} X_2, \quad -\frac{1}{2} \leq X_1 \leq \frac{1}{2}, \ -\frac{1}{2} \leq X_2 \leq \frac{1}{2}.$$

Then $x_1 x_2 = X_1^2 - X_2^2 + X_1 X_2$ and we see that

$$\text{(A.12)} \qquad |x_1 x_2| \leq \frac{5}{16}, \qquad |x_1| \leq \frac{3+\sqrt{5}}{4}, \qquad |x_2| \leq \frac{1+\sqrt{5}}{4}.$$

Since $z \in D$, we have

$$\text{(A.13)} \qquad N(|z|)^2 = (x_1^2 + y_1^2)(x_2^2 + y_2^2) \geq 1.$$

Put $k = y_1 y_2$. Since $\epsilon^{-2} \leq y_1/y_2 \leq \epsilon^2$, we have $\epsilon^{-1}\sqrt{k} \leq y_1$, $y_2 \leq \epsilon\sqrt{k}$. Then by (A.13), we have

$$k^2 + (x_1^2 + x_2^2)\epsilon^2 k + x_1^2 x_2^2 - 1 \geq 0.$$

We consider the equation with respect to t:

$$\text{(A.14)} \qquad t^2 + (x_1^2 + x_2^2)\epsilon^2 t + x_1^2 x_2^2 - 1 = 0.$$

Let ξ be the positive root of (A.14) and let $\kappa^* = \min \xi$. Here the minimum is taken with respect to X_1 and X_2, regarding x_1 and x_2 as the functions of X_1 and X_2; X_1 and X_2 extend over the domain $-1/2 \leq X_1, X_2 \leq 1/2$. Let κ be the positive root of the equation

$$t^2 + \frac{7(3+\sqrt{5})}{8} t - \frac{15}{16} = 0.$$

This is the positive root of (A.14) when $X_1 = X_2 = 1/2$, $x_1 = (3+\sqrt{5})/4$, $x_2 = (3-\sqrt{5})/4$. We have $\kappa = 0.19622\cdots$. By elementary but somewhat tedious calculation, which we omit the details, we can show that $\kappa^* = \kappa$. Hence we have

$$y_1 y_2 \geq \kappa = 0.19622\cdots. \tag{A.15}$$

If $c = 0$, then $\gamma \in S_0$. It suffices to show that $\gamma \in S_1 \sqcup S_2$ assuming $c \neq 0$. By (A.11), we have

$$|N(c)|y_1 y_2 \leq 1. \tag{A.16}$$

By (A.15), we have $|N(c)| \leq 1/\kappa$. Therefore $|N(c)| = 1$ or 4 or 5. If $|N(c)| = 1$, then $c \in E_F$ and $\gamma \in S_1$. Hereafter we assume $|N(c)| = 4$ or 5. By (A.15) and (A.16), noting $\epsilon^{-2} \leq y_1/y_2 \leq \epsilon^2$, , we obtain

$$\epsilon^{-1}\sqrt{\kappa} \leq y_1, y_2 \leq \frac{\epsilon}{\sqrt{|N(c)|}}. \tag{A.17}$$

Since $N(|z|) \geq 1$, we have $(x_1^2 + y_1^2)(x_2^2 + y_2^2) \geq 1$. Using $y_1 y_2 \leq 1/|N(c)|$, we have

$$x_1^2 y_2^4 - (1 - x_1^2 x_2^2 - \frac{1}{N(c)^2})y_2^2 + \frac{x_2^2}{N(c)^2} \geq 0. \tag{A.18}$$

If $x_1 = 0$, we obtain

$$y_1^2 x_2^2 \geq 1 - \frac{1}{N(c)^2} \geq 1 - \frac{1}{16}$$

from $N(|z|) \geq 1$ and (A.16). By (A.12), we have

$$y_1 \geq \sqrt{1 - \frac{1}{16}} \cdot \frac{2}{\epsilon} = 1.19681\cdots.$$

This contradics (A.17). Hence we have $x_1 \neq 0$.

First we exclude the case $|N(c)| = 5$. To this end, we assume $|N(c)| = 5$ and consider the equation (cf. (A.18))

$$x_1^2 t^2 - (1 - x_1^2 x_2^2 - \frac{1}{25})t + \frac{x_2^2}{25} = 0. \tag{A.19}$$

Let $f(t)$ be the polynomial of t on the left-hand side. For $t_0 = \epsilon^{-2}\kappa$, we have

$$f(t_0) \leq (\frac{\epsilon + 1}{2})^2 t_0^2 - (1 - \frac{25}{256} - \frac{1}{25})t_0 + \frac{1}{25}(\frac{\epsilon}{2})^2 = -0.02882\cdots < 0$$

using (A.12). Let $\eta_1 > \epsilon^{-2}\kappa > \eta_2$ be the roots of the equation (A.19). By (A.17) and (A.18), we must have $y_2 \geq \sqrt{\eta_1}$. We note that (cf. (A.17))

$$y_1, y_2 \leq \frac{\epsilon}{\sqrt{5}} = 0.72360\cdots. \tag{A.20}$$

We consider η_1 as a function of X_1 and X_2 defined in the domain $-1/2 \leq X_1, X_2 \leq 1/2$. First we consider η_1 on the subdomain defined by the condition $x_1 > 0$. It is not difficult to check that η_1 is monotone decreasing with respect to the both arguments X_1 and X_2. For $X_1 = 1/2$, $X_2 = 0.4985$, we have $\sqrt{\eta_1} = 0.72377\cdots$. For $X_1 = 0.4985$, $X_2 = 1/2$, we have $\sqrt{\eta_1} = 0.72389\cdots$. In view of (A.20), we must have X_1, $X_2 > 0.4985$. Similarly, in the subdomain $x_1 < 0$, we must have X_1, $X_2 < -0.4985$.

First we consider the case X_1, $X_2 > 0.4985$. For relatively prime integers α, $\beta \in \mathcal{O}_F$, we have (cf. (A.8)) $N(|\alpha z + \beta|) \geq 1$. Take $\alpha = 2$, $\beta = -\epsilon^2$. We have

$$|2x_1 - \epsilon^2| \leq 0.03(1+\epsilon), \qquad |2x_2 - \epsilon^{-2}| \leq 0.03(1+|\epsilon'|).$$

Here $\epsilon' = (1-\sqrt{5})/2$ is the conjugate of ϵ. Then we find

$$\begin{aligned} &N(|2z-\epsilon^2|)^2 = \{(2x_1-\epsilon^2)^2 + 4y_1^2\}\{(2x_2-\epsilon^{-2})^2 + 4y_2^2\} \\ &= 16y_1^2y_2^2 + 4y_1^2(2x_2-\epsilon^{-2})^2 + 4y_2^2(2x_1-\epsilon^2)^2 + (2x_1-\epsilon^2)^2(2x_2-\epsilon^{-2})^2 \\ &\leq \frac{16}{25} + 4y_1^2\{0.03(1+\epsilon)\}^2 + 4y_2^2\{0.03(1+|\epsilon'|)\}^2 \\ &\qquad\qquad + \{0.03(1+\epsilon)\}^2\{0.03(1+|\epsilon'|)\}^2. \end{aligned}$$

Since y_1, $y_2 \leq 0.72360\cdots$, this contradicts $N(|2z-\epsilon^2|) \geq 1$. When X_1, $X_2 < -0.4985$, we obtain a contradiction similarly by taking $\alpha = 2$, $\beta = \epsilon^2$. Thus we have shown that the case $|N(c)| = 5$ cannot occur.

It remains to show that $\gamma \in S_2$ assuming $|N(c)| = 4$. We can write $c = \pm 2\epsilon^m$ with $m \in \mathbf{Z}$. Changing γ to $-\gamma$ if necessary, we may assume that $c = 2\epsilon^m$. We put $z' = (z_1', z_2') = \gamma z$, $z_j' = x_j' + iy_j'$, $j = 1, 2$. Since $z = \gamma^{-1}z'$, $\gamma^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, the estimate (A.17) holds also for y_1' and y_2'. We have

$$\epsilon^{-1}\sqrt{\kappa} = 0.27376\cdots \leq y_1, y_2, y_1', y_2' \leq \frac{\epsilon}{\sqrt{|N(c)|}} = 0.80901\cdots. \tag{A.21}$$

We have

$$|c^{(j)}z_j + d^{(j)}|^2 = \frac{y_j}{y_j'}, \qquad j = 1, 2.$$

Hence we obtain

$$\epsilon^{-2}\sqrt{\kappa}\sqrt{|N(c)|} \leq |c^{(j)}z_j + d^{(j)}|^2 \leq \frac{\epsilon^2}{\sqrt{\kappa}\sqrt{|N(c)|}}, \qquad j = 1, 2. \tag{A.22}$$

In particular, we have

$$(c^{(j)})^2 y_j^2 \leq \frac{\epsilon^2}{\sqrt{\kappa}\sqrt{|N(c)|}}, \qquad j = 1, 2.$$

Using (A.21), we obtain

$$|c^{(j)}| \leq \epsilon^2 \kappa^{-3/4} |N(c)|^{-1/4} = 6.27915\cdots, \qquad j = 1, 2. \tag{A.23}$$

From (A.23), we obtain $m = 0, \pm 1, \pm 2$.

Next we are going to restrict possibilities of d. A preliminary table of listing all possible d can be obtained by (A.22) and (A.23). By (A.11) and (A.21), we have

$$\{(2\epsilon^m x_1 + d^{(1)})^2 + 4\epsilon^{2m} \cdot \epsilon^{-2}\kappa\}\{(2(\epsilon')^m x_2 + d^{(2)})^2 + 4\epsilon^{-2m} \cdot \epsilon^{-2}\kappa\} \leq 1. \tag{A.24}$$

We consider the equation (cf. (A.18))

$$x_1^2 t^2 - (1 - x_1^2 x_2^2 - \frac{1}{16})t + \frac{x_2^2}{16} = 0. \tag{A.25}$$

Let $g(t)$ be the polynomial of t on the left-hand side. For $t_0 = \epsilon^{-2}\kappa$, we can check $g(t_0) < 0$. Let $\eta_1 > t_0 > \eta_2$ be the roots of $g(t)$. By (A.18) and (A.21), we have $y_2 \geq \sqrt{\eta_1}$. As in the case where $|N(c)| = 5$, we consider η_1 as a function of X_1 and X_2 defined in the domain $-1/2 \leq X_1, X_2 \leq 1/2$. On the subdomain defined by the condition $x_1 > 0$, we check that η_1 is monotone decreasing with respect to the both arguments X_1 and X_2. For $X_1 = 1/2$, $X_2 = 0.39$, we have $\sqrt{\eta_1} = 0.81291\cdots$. For $X_1 = 0.38$, $X_2 = 1/2$, we have $\sqrt{\eta_1} = 0.81101\cdots$. In view of (A.21), we must have $X_1 > 0.38$, $X_2 > 0.39$. Similarly, in the subdomain $x_1 < 0$, we must have $X_1 < -0.38$, $X_2 < -0.39$. Let V be the closed domain

$$V = \{(X_1, X_2) \mid 0.38 \leq |X_1| \leq 1/2,\ 0.39 \leq |X_2| \leq 1/2\}$$

and consider the function

$$f(X_1, X_2) = \{(2\epsilon^m x_1 + d^{(1)})^2 + 4\epsilon^{2m-2}\kappa\}\{(2(\epsilon')^m x_2 + d^{(2)})^2 + 4\epsilon^{-2m-2}\kappa\}$$

on V. By (A.24), we see that:

(C1) The minimum of $f(X_1, X_2)$ on V does not exceed 1.

Next let ξ be the positive root of (A.14). Since $y_1 y_2 \geq \xi$, we have y_1, $y_2 \geq \epsilon^{-1}\sqrt{\xi}$. By (A.11), we obtain another inequality:

$$\begin{aligned}(2\epsilon^m x_1 + d^{(1)})^2 (2(\epsilon')^m x_2 + d^{(2)})^2 + 4\epsilon^{-2m-2}\xi(2\epsilon^m x_1 + d^{(1)})^2 \\ + 4\epsilon^{2m-2}\xi(2(\epsilon')^m x_2 + d^{(2)})^2 + 16\xi^2 \leq 1.\end{aligned} \tag{A.26}$$

We regard x_1, x_2 and ξ as the functions of X_1 and X_2 and let $g(X_1, X_2)$ be the function on the left-hand side of (A.26). Then (A.26) implies:

(C2) The minimum of $g(X_1, X_2)$ on V does not exceed 1.

By numerical computations using a computer, we find the following:

For $m = 0$, (C1) leaves possibilities $d = \pm 1, \pm\epsilon, \pm\epsilon^2, \pm\epsilon^{-1}$. If combined with (C2), the only possibility is $d = \pm\epsilon^2$. For $m = 1$, (C1) leaves possibilities $d = \pm 1, \pm\epsilon, \pm\epsilon^2, \pm\epsilon^3$. If combined with (C2), the only possibility is $d = \pm\epsilon^3$. For $m = 2$, (C1) leaves possibilities $d = \pm\epsilon, \pm\epsilon^2, \pm\epsilon^3, \pm\epsilon^4$. If combined with (C2), the only possibility is $d = \pm\epsilon^4$. For $m = -1$, (C1) leaves possibilities $d = \pm 1, \pm\epsilon, \pm\epsilon^{-1}, \pm\epsilon^{-2}$. If combined with (C2), the only possibility is $d = \pm\epsilon$. For $m = -2$, (C1) leaves possibilities $d = \pm 1, \pm\epsilon^{-1}, \pm\epsilon^{-2}, \pm\epsilon^{-3}$. If combined with (C2), the only possibility is $d = \pm 1$.

Thus, in every case where $c = 2\epsilon^m$, we have $d = \pm\epsilon^n$ with n depending only on m. Changing the roles of z and z' and noting that $-\gamma^{-1} = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}$, we see that a must have the same form $a = \pm\epsilon^n$. (Here the $\pm$ sign is arbitrary but n is the same for d and a.) By $\det\gamma = 1$, we have $ad \equiv 1 \mod 2$, which implies $n \equiv 0 \mod 3$. Therefore only the cases $m = 1, -2$ can survive and we see that $\gamma \in S_2$. This completes the proof of Lemma A.11.

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