

On a Fourth Order Lichnerowicz Type Equation Involving The Paneitz-Branson Operator.

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Abstract In this paper, we study some fourth order singular critical equations of Lichnerowicz type involving the Paneitz-Branson operator, and we prove existence and non existence results under given assumptions.

1 Introduction

During the last years there have been effective studies of conformal operators and their relative invariants due to their application in geometry or mathematical physics. For instance the Yamabe problem played an essential role in the evolution of the analytical and geometrical tools also it was with crucial importance for the study of the Einstein-Hilbert functional without forgetting the input in relativity for the study of the conformal Einstein constraint equations (see [9],[8]). And In 1983, Paneitz [20] introduced a conformally fourth order operator defined on 4-dimensional Riemannian manifolds Bran-

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son [3] generalized the definition to n -dimensional Riemannian manifolds, $n \geq 5$. He introduced another geometric quantity that defines another conformal invariant for $n > 3$ that is the Q -curvature that behaves in a very similar way to the scalar curvature. And its variation after a conformal change involves a fourth order operator. That is the Paneitz-Branson Operator. One can think about the Q -curvature and the Branson-Paneitz operator like the scalar curvature and the conformal laplacian. There was a lot of published work concerning prescribing the Q -curvature where one can notice that the conditions that we get are similar to the scalar curvature one modulo some technical assumptions (see [13],[14],[2]). One of the issues that we meet while dealing with this operator, is the fact that there is no maximum principle, thus getting good and effective estimate is not as easy as for the conformal Laplacian. Many authors have studied the positivity and coercivity of the Paneitz, one can consult [24] or [16] for example. As interaction with mathematical physics we can see the work of Choquet-Bruhat in [9] with the Conformal Laplacian, where they study the scalar field equation that leads to a Lichnerowicz type semi-linear PDE. In this work we attempt to study another action functional as a proposal for a relativistic model since it is conformally invariant and we will see a scalar-field perturbation of it. The study of such functional leads to the resolution of a Lichnerowicz type equation but it is a fourth order one, with the Paneitz operator as a differential part. So in this paper we will investigate the existence of positive solutions under some assumption that we will mention later to that equation and also we will give a non-existence result.

2 Preliminaries and Motivations

Let (M, g) be a n -dimensional closed compact manifold with $n \geq 3$, recall that if R_g is the scalar curvature then under the conformal change $\tilde{g} = u^{\frac{4}{n-2}}g$, one gets the following relation relation the new curvature with the old one :

$$-\Delta_g u + \frac{(n-2)}{4(n-1)}R_g u = R_{\tilde{g}} u^{\frac{n+2}{n-2}}, \quad (1)$$

Let $-L_g u = -\Delta_g u + \frac{(n-2)}{4(n-1)}R_g u$ this operator is called the conformal Laplacian to see more of its property one could check [18]. Similarly if we consider the following quantity which is the Branson Q - curvature introduced in [3], defined by

$$\begin{aligned} Q & : = \frac{n^2 - 4}{8n(n-1)^2}R^2 - \frac{2}{(n-1)^2} \left| Ric - \frac{R}{n}g \right|^2 + \frac{1}{2(n-1)}\Delta R \\ & = c_n R^2 + d_n |Ric|^2 + \frac{1}{2(n-1)}\Delta R. \end{aligned}$$

Then after a conformal change $\tilde{g} = u^{\frac{4}{n-4}}g$ of the metric, one gets

$$Q_{\tilde{g}} u^{\frac{n+4}{n-4}} = Pu \quad (2)$$

where

$$Pu := \Delta_g^2 u - \operatorname{div} \left(\left(\frac{(n-2)^2 + 4}{2(n-2)(n-1)}Rg - \frac{4}{n-2}Ric \right) du \right) + \frac{n-4}{2}Qu.$$

We will set P_0 its differential part, that is

$$P_0 u = \Delta_g^2 u - \operatorname{div} \left(\left(\frac{(n-2)^2 + 4}{2(n-2)(n-1)}Rg - \frac{4}{n-2}Ric \right) du \right),$$

One can see that if g is an Einstein metric then P is with constant coefficient,

$$Pu = \Delta_g^2 u + \frac{n^2 - 2n - 4}{2n(n-1)}R\Delta_g u + \frac{(n-4)(n^2-4)}{16n(n-1)^2}R^2 u$$

and satisfies the maximum principle since it can be written as a product of two second order operator satisfying the maximum principle.

Remark that the natural space to work on for the prescribed scalar curvature problem is $H^1(M)$ and thus (1) is a critical semi-linear problem. Also the natural space for prescribing the Branson Q -curvature is $H^2(M)$ and again (2) is a critical problem since we are in the borderline of the Sobolev embeddings.

There was an extensive work concerning (1) to find a metric with constant scalar curvature, which is a critical point of the Einstein-Hilbert functional

$$F_R : g \mapsto \frac{\int_M R_g}{V_g^{\frac{n-4}{n}}}$$

restricted to the conformal Class of a given metric. Same thing can be applied to the functional

$$F_Q : g \mapsto \frac{\int_M Q_g}{V_g^{\frac{n-4}{n}}},$$

For instance one could check that the functional is Riemannian and Einstein metrics are critical points of this functional.

There have been many proposal in relativity to replace the Hilbert-Einstein total curvature functional with a conformally invariant functional, like for instance the case of Bach relativity where the functional is replaced by $\int_M |C_g|^2 dv_g$ and C is the Weyl tensor (see [4]). In this case we will consider another proposal consisting of the total Paneitz-Branson curvature F_Q . Therefore one can think about a scalar field perturbation of the previous one, that is

$$F_\psi(g) = \int_M Q_g - |\nabla_g \psi|^2 - V(\psi) d\mu_g,$$

this functional was studied in [10] for the case Hilbert-Einstein functional under conformal change where the authors try to solve a conformal constraint for the Einstein scalar field equation also in [15] where the author studies the problem from a variational point of view.

Now if we take a closer look to this functional, one can see that if we restrict it to the conformal class of g one have

$$F_\psi \left(u^{\frac{4}{n-4}} g \right) = \frac{1}{a_n} \int_M u P u - a_n |\nabla_g \Psi|^2 u^2 d\mu_g.$$

Where $a_n = \frac{n-4}{4}$, therefore, the associated Euler-Lagrange equation to this problem is

$$P_{g,\psi} u = P u - a_n |\nabla_g \Psi|^2 u = \left(\tilde{Q} - |\nabla_{\tilde{g}} \Psi|^2 \right) u^{\frac{n+4}{n-4}},$$

that is

$$P_0 u + a_n \left(Q - |\nabla_g \Psi|^2 \right) u = \left(\tilde{Q} - |\nabla_{\tilde{g}} \Psi|^2 \right) u^{\frac{n+4}{n-4}}$$

and the constant

$$\mathcal{P}[g, \Psi] = \inf_{u>0, u \in C^\infty(M)} \frac{1}{a_n} \frac{\int_M u P u - a_n |\nabla_g \Psi|^2 u^2}{\left(\int_M u^{\frac{2n}{n-4}} \right)^{\frac{n-4}{n}}},$$

is a conformal invariant.

Let us recall the following result about the coercivity of the Paneitz-Branson operator.

Theorem 2.1 ([24]). *Let (M, g) be a closed Riemannian manifold of dimension at least 6. If the Yamabe invariant of g is non-negative, then with respect to any conformal metric of positive scalar curvature P_0 has a non-negative first eigenvalue and $\ker P_0 = \{\text{constant}\}$. The last statement also holds in dimension five, provided we assume the Yamabe metric has positive Q -curvature.*

Proposition 2.2. *Under the assumptions of the previous theorem, then the sign of $\mathcal{P}[g, \Psi]$ is the sign of the first eigenvalue of the Operator $P_{\tilde{g}, \Psi}$ for every \tilde{g} in the same conformal class.*

Proof. Assume $\mathcal{P}[g, \Psi] > 0$, then if we take φ_1 the first eigenvalue as a test function, one gets

$$\lambda_1 \|\varphi_1\|_2^2 = E(\varphi_1) \geq \mathcal{P}[g, \Psi] \|\varphi_1\|_{2^\sharp}^2$$

thus $\lambda_1 > 0$. Now if we assume that $\mathcal{P}[g, \Psi] = 0$ then using the same argument we get that $\lambda_1 \geq 0$, but if $\lambda_1 > 0$ then using Sobolev inequalities we get $\mathcal{P}[g, \Psi] > 0$ which is not the case, thus $\lambda_1 = 0$.

Now if $\mathcal{P}[g, \Psi] < 0$ then there exist a function v , such that $E(v) < 0$ thus we get $\lambda_1 < 0$. □

From now on we will assume that the Yamabe and the Paneitz invariants are positive and P is positive, therefore we guaranty that the operator $P_{g, \Psi}$ is coercive and satisfies the maximum principle and up to a conformal change we can assume that $Q_\psi = Q - |\nabla\psi|_g^2$ id positive on M .

And if we follow the procedure of the Authors in [10] to find the Einstein-scalar field conformal constraint equation one gets a a Lichnerowicz type problem but of fourth order of the following form :

$$\begin{cases} P_{g, \Psi} u = \frac{A(x)}{u^{2^\sharp+1}} - B(x)u^{2^\sharp-1} \\ u > 0 \end{cases} .$$

where $2^\sharp = \frac{2n}{n-4}$, A and B two smooth functions. Therefore, the object of the rest of this paper is to investigate the existence of positive solutions to problem of the following form :

$$\begin{cases} P_{g, \Psi} u = \frac{A(x)}{u^p} - B(x)u^q \\ u > 0 \end{cases} ,$$

where $p > 1$ and $1 < q \leq 2^\# - 1$.

3 Existence Via heat flow

Let E be a Banach space with norm $\|-\|_E$. E is partially ordered by a closed cone $P \subset X$, and we assume that it has non-empty interior $\overset{\circ}{P}$. We define also $\overset{\bullet}{P} = P - \{0\}$. The element of $\overset{\bullet}{P}$ are called positive and element of $-\overset{\bullet}{P}$ are called negative.

Now, if $u, v \in E$ we will use the following notations to distinguish how they are comparable :

$$\left\{ \begin{array}{l} u \leq v \text{ if } v - u \in P \\ u < v \text{ if } v - u \in \overset{\bullet}{P} \\ u \ll v \text{ if } v - u \in \overset{\circ}{P} \end{array} \right. .$$

A map f , we set $D(f) \subset E$ its domain. Now a map $f : D(f) \rightarrow E$ is said order preserving, if for every $u, v \in D(f)$ such that $u \leq v$ then $f(u) \leq f(v)$.

And we say that

$$\lim_{x \rightarrow +\infty} f(x) = +\infty$$

if for every $u \in P$ there exist $x \in P$ such that $f(v) \geq u$, for every $v \geq x$.

And finally we define the set

$$[u, v] = \{w \in E; u \leq w \leq v\}$$

and sometimes if needed for a set $D \subset E$,

$$[u, v]_D = \{w \in D; u \leq w \leq v\} .$$

An OBS is said normal if there exist $\delta > 0$ such that for every $u \leq v$ in E ,

$$\|u\|_E \leq \delta \|v\|_E .$$

Theorem 3.1 (Krein-Rutman). *Let E be a total ordered Banach space and $T : E \rightarrow E$ a compact order preserving linear operator, then $r(T)$ is an eigenvalue with eigenvector $u \in \overset{\bullet}{P}$ and if in addition we assume that T is strongly order preserving (That is $Tu \gg 0$ if $u > 0$) then $r(T) > 0$ and is a simple eigenvalue with positive eigen vector.*

Let us consider the following problem

$$\begin{cases} \frac{d}{dt}u + Au = F(u) \\ u(0) = u_0 \end{cases} \quad (3)$$

where $F : \overset{\circ}{P} \rightarrow X$ is a C^1 map, A is a densely defined compact resolvent positive operator and $u_0 \in \overset{\circ}{P} \cap D(A)$.

Theorem 3.2. *Assume that*

$$\lim_{x \rightarrow +\infty} F(x) = -\infty$$

and

$$\lim_{x \rightarrow 0^+} F(x) = +\infty$$

and for every bounded set K there exist a constant λ such that $F + \lambda I$ is order preserving in K , then the problem admits a positive solution.

In the applications we can know more about the solution and we will deal with that further in this paper.

Proof. First remark that there exist u_1 and u_2 such that

$$\begin{aligned} \frac{d}{dt}u_1 + Au_1 &\leq F(u_1) \\ u_1(0) &\leq u_0 \end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}u_2 + Au_2 &\geq F(u_2) \\ u_2(0) &\geq u_0\end{aligned}$$

In fact u_1 and u_2 could be chosen of the form se where $e \in \overset{\circ}{P}$ and $s > 0$.
set

$$K = \{u_1 \leq u \leq u_2\}$$

then K is a bounded set, so there exist $\lambda > 0$ so that $F + \lambda I$ is order preserving on K so let \tilde{A} and \tilde{F} denote respectively $A + \lambda I$ and $F + \lambda I$.

Now let us construct the following sequence : u^1 being the unique solution of

$$\frac{d}{dt}u + \tilde{A}u = \tilde{F}(u_1)$$

and u^{k+1} is the unique solution of

$$\frac{d}{dt}u + \tilde{A}u = \tilde{F}(u^k)$$

By induction one can easily show that the sequence (u^k) is monotone non-decreasing and $u^k \in K, \forall k \geq 1$. Let us show the first step, that is $u^1 \geq u_1$.

First using the assumptions on A we have the existence of a compact positive semi-group $S(t)$, generated by \tilde{A} (see [21]). So we have

$$\begin{aligned}u_1 &= S(t)u_1(0) + \int_0^t S(t-s) \left(\frac{d}{dt}u_1 + \tilde{A}u_1 \right) \\ &\leq S(t)u_0 + \int_0^t S(t-s) \tilde{F}(u_1(s)) ds \\ &\leq u^1\end{aligned}$$

Now since A has compact resolvent and K is bounded we can extract for fixed time a subsequence that we will call also (u^k) such that $S(t)u^k(s)$

converges to $S(t)u(s)$, thus by writing

$$u^{k+1} = S(t)u_0 + \int_0^t S(t-s)\tilde{F}(u^k),$$

one can see that u satisfies

$$u = S(t)u_0 + \int_0^t S(t-s)\tilde{F}(u)$$

And this gives a positive solution to (3).

Now notice that $u(t)$ is bounded in $D(A)$ thus there exists a sequence $(t_k)_k$ going to infinity such that $u(t_k)$ converges to some \tilde{u} , and in fact the convergence occurs in $D(A)$. Thus knowing that $\lim \int_0^t S(t-s)xd s = (-A)^{-1}x$ we get by passing to the limit that \tilde{u} is a solution of the steady-state problem. \square

Now we will consider a problem of the form

$$\begin{cases} u_t + P_{g,\Psi}u = f(x, u) \\ u(0) = u_0 \end{cases} \quad (4)$$

where P is the Paneitz-Branson operator and $f : M \times \mathbb{R}_+^* \rightarrow \mathbb{R}$ is C^1 such that $\lim_{x \rightarrow +\infty} f(x, t) = -\infty$ uniformly on x and $\lim_{x \rightarrow 0} f(x, t) = +\infty$ uniformly on x , Then for every u_0 smooth and positive, there exist a positive solution to (4) moreover there exist a sequence t_k going to infinity such that $u(t_k) \rightarrow \tilde{u}$ a solution of the steady-state problem.

One also can write the problem as an integral equation using the positivity preserving flow like in [22].

Corollary 3.3. *Take A and B two positive functions defined on M and consider the singular problem :*

$$\begin{cases} Pu = \frac{A(x)}{u^p} - B(x)u^q \\ u > 0 \end{cases} \quad (5)$$

then using Theorem (3.2) we have the existence of a solution more than that, it is the unique solution.

Remark that in this case we can take $q \geq 2^\sharp - 1$ since we do not need the compact or continuous embedding in L^p spaces.

Now, if we suppose that B is just non-negative. we can show indeed that even in that case we have a solution.

Corollary 3.4. *Take $A > 0$ and $B \geq 0$, two smooth functions defined on M and consider the singular problem :*

$$\begin{cases} P_{g,\psi}u = \frac{A(x)}{u^p} - B(x)u^q \\ u > 0 \end{cases}$$

where $q \leq 2^\sharp - 1$, then it has a unique solution.

Let u_ε be the solution obtained by Corollary (3.3), of

$$\begin{cases} P_{g,\psi}u = \frac{A(x)}{u^p} - B_\varepsilon(x)u^q \\ u > 0 \end{cases}$$

where $B_\varepsilon = B + \varepsilon$. First remark that u_ε is uniformly bounded from below (it is by construction of the sub and super solution in the proof of Theorem (3.2)).

So

$$\begin{aligned} \int_M u P_{g,\psi}u &= \int_M \frac{A(x)}{u_\varepsilon^{p-1}} - \int_M (B(x) + \varepsilon) u_\varepsilon^{q+1} \\ &\leq \int_M \frac{A(x)}{\delta} = C \end{aligned}$$

where δ is the uniform lower bound of u_ε . Therefore $(u_\varepsilon)_\varepsilon$ is bounded in $H^2(M)$ and if $q + 1 \leq 2^\sharp$, we have $u_\varepsilon \rightarrow u$ in L^2 and weakly in $H^2(M)$ and $L^{2^\sharp}(M)$.

So if we take $\varphi \in C^\infty(M)$, we have a weak solution which we can show using the regularity theory that is indeed smooth.

$$\int_M \varphi P_{g,\psi} u_\varepsilon = \int_M \frac{A(x)}{u_\varepsilon^{p-1}} \varphi - \int_M (B(x) + \varepsilon) u_\varepsilon^q \varphi$$

so by letting $\varepsilon \rightarrow 0$ we get that

$$\int_M \varphi P_{g,\psi} u = \int_M \frac{A(x)}{u^{p-1}} \varphi - \int_M B(x) u^q \varphi$$

so u is a weak solution and using elliptic regularity we get the fact that it is indeed a smooth one.

For the uniqueness, if we consider two smooth positive solutions u and v of then $w = u - v$ satisfies :

$$\begin{aligned} P_{g,\psi} w &= \frac{A(x)}{u^p} - \frac{A(x)}{v^p} + B_\varepsilon(x) v^q - B_\varepsilon(x) u^q \\ &= -C(x)(u - v) = -C(x)w \end{aligned}$$

where $C(x)$ is a non-negative function that we get from the mean value theorem, therefore using the maximum principle we get the desired result.

As an improvement of the previous result we have :

Theorem 3.5. *Assume that B^+ is non-zero then problem (5) has at least one positive solution if the following inequality is satisfied*

$$\max_M \left(A^{\frac{q-1}{p+q}} B_-^{\frac{p+1}{p+q}} \varphi_1^{q \frac{p+1}{p+q} - p \frac{q-1}{p+q} - 1} \right) \leq \frac{\lambda_1}{\left(\frac{q-1}{p+1} \right)^{\frac{p+1}{p+q}} + \left(\frac{p+1}{q-1} \right)^{\frac{q-1}{p+q}}}, \quad (6)$$

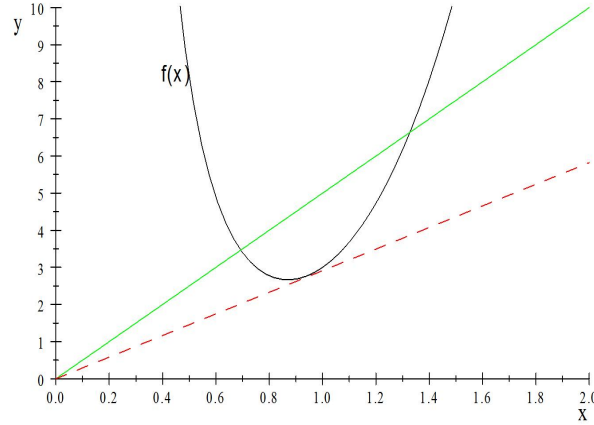
where λ_1 and φ_1 are the first eigenvalue and eigenfunction of $P_{g,\psi}$, respectively.

first let \underline{u} be a solution of

$$\begin{cases} P_{g,\psi} u = \frac{A(x)}{u^p} - B^+(x) u^q \\ u > 0 \end{cases}$$

In fact since we are going to use this process another time let us give the picture and the idea behind :

Consider a convex function positive $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ and, so for it to intersect a line L passing through the origin its slope should be greater than the one of the unique tangent to the graph of passing through the origin as shown in the following figure :



and the condition to find the slope at zero is by solving

$$\frac{f(t)}{t} = f'(t). \quad (7)$$

So if we take φ_1 the first eigenfunction of P , we get

$$\begin{aligned} tP\varphi_1 - \frac{A(x)}{t^p\varphi_1^p} + B(x)t^q\varphi_1^q &= t\lambda_1\varphi_1 - \frac{A(x)}{t^p\varphi_1^p} + B(x)t^q\varphi_1^q \\ &\geq t\lambda_1\varphi_1 - \frac{A(x)}{t^p\varphi_1^p} - B^-(x)t^q\varphi_1^q. \end{aligned}$$

And here we can see that in fact we are comparing $t \rightarrow t\lambda_1\varphi_1$ and $t \rightarrow \frac{A(x)}{t^p\varphi_1^p} + B^-(x)t^q\varphi_1^q$ which is convex, thus using the previous remark we can see that the inequality (6) insures that we are in the same situation as ?? and

thus there exist $t_0 > 0$ such that $t_0\varphi_1$ is a super-solution to (5) therefore, using the classical monotone iteration method we get a positive solution.

3.1 Further investigations and existence results

Here we investigate the case where $B < 0$. The coercivity assumption implies that

$$\|u\|_\psi = \left(\int_M u P_{g,\psi} u \right)^{\frac{1}{2}},$$

defines a norm equivalent to the $H^2(M)$ norm. So we will use that norm instead of the usual one. Also we take S_ψ the best Sobolev constant with respect to this norm, that is S_ψ is the est constant satisfying

$$\|u\|_{L^{2^\sharp}}^2 S_\psi \leq \|u\|_\psi^2$$

Remark that for $B < 0$ this condition still work, but let us try to find another condition that works in a weaker setting. We will rewrite the problem as

$$\begin{cases} P_{g,\psi} u = \frac{A(x)}{u^p} + B(x)u^q \\ u > 0 \end{cases}, \quad (8)$$

and B here is taken to be positive (in fact one get a similar result if B has a negative part up to a small modification to the assumption in the following theorem).

For the regularity issues we refer to [5] and [5], there one can find the necessary regularity and bootstrapping argument to deal with it.

Theorem 3.6. *Assume that P is strongly positive (that is it satisfies the strong maximum principle). If there exist a function $\varphi > 0$ in $H^2(M)$ such that*

$$\|\varphi\|_\psi^{(p-1)} \|B\|_{L^{\frac{2^\sharp}{2^\sharp - q - 1}}}^{\frac{p+1}{q-1}} \left(\int_M \frac{A}{\varphi^{p-1}} \right) < C \quad (9)$$

then problem (8) has at least one positive smooth solution.

In fact we will compute an exact value of C , That is

$$C = S_\psi^{-\frac{(q+1)(p+2q+1)}{2(q-1)}} \left(\frac{(q-1)(p-1)}{2} \right) \quad (10)$$

Before Starting the proof let us state the following lemma which appears to be helpful in our situation.

Lemma 3.7. *Let E, E_1, E_2 be three C^1 functional on a Banach space X . Assume that $E_1(0) = 0$ and $\lim E(t\varphi) = -\infty$. and $E_2 \geq 0$. Then If E_1 has the mountainpass geometry around zero, (that is there exist $r > 0$ such that $\delta = \inf_{\partial B(0,r)} E_1 > 0$) and there exist $u \in B(0,r)$ such that $E_2(u) < \delta$, the functional E has a Palais-Smale sequence.*

Proof of Lemma. Here is is easy to see that if we consider the set

$$\Gamma = \{\gamma : [0, 1] \longrightarrow X \text{ such that } \gamma(0) = u \text{ and } \gamma(1) = t\varphi\}$$

then we get a Palais-Smale sequence at the level

$$c = \inf_{\gamma \in \Gamma} \max E(\gamma([0, 1]))$$

Since each curve crosses $\partial B(0, r)$, then $c > \max(E(u), E(t\varphi))$, and thus we have a mountain pass geometry. \square

Proof. In fact let φ be a positive function such that $\|\varphi\|_\psi = 1$ and the energy functional

$$E(u) = \frac{1}{2} \|u\|_\psi^2 + \frac{1}{p-1} \int_M \frac{A}{(\varepsilon + u^2)^{\frac{p-1}{2}}} - \frac{1}{q+1} \int_M Bu^{q+1}.$$

Clearly the functional E_1 defined by

$$E_1(u) = \frac{1}{2} \|u\|_\psi^2 - \frac{1}{q+1} \int_M Bu^{q+1}$$

has the mountain pass geometry and in fact if $r_0 = \|B\|_{L^s}^{-\frac{1}{q-1}} S_\psi^{-\frac{q+1}{2(q-1)}}$ then

$$\inf_{u \in \partial B(0, r_0)} E_1(u) = \|B\|_{L^s}^{-\frac{2}{q-1}} S^{-\frac{q+1}{q-1}} \left(\frac{q-1}{2} \right)$$

And therefore the inequality (9) is exactly saying that that $t_0\varphi$ satisfies the assumption of Lemma (3.7) for $t_0 < r_0$ and thus we have the existence of $t_0 < r_0 < t_2$ such that,

$$\max(E(t_0\varphi), E(t_2\varphi)) < E(r_0\varphi)$$

And in fact we can apply the lemma for the following approximated energy functional

$$E_\varepsilon(u) = \frac{1}{2} \|u\|_\psi^2 + \frac{1}{p-1} \int_M \frac{A}{\left(\varepsilon + (u^+)^2\right)^{\frac{p-1}{2}}} - \frac{1}{q+1} \int_M B (u^+)^{q+1}$$

for $\varepsilon > 0$ and small. Remark that we have uniform convergence of $t \rightarrow E_\varepsilon(t\varphi)$ to $t \rightarrow E(t\varphi)$, on every compact of \mathbb{R} . Therefore there exist $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \leq \varepsilon_0$ one have

$$\max(E_{\varepsilon_0}(t_0\varphi), E_{\varepsilon_0}(t_2\varphi)) \leq \max(E_\varepsilon(t_0\varphi), E_\varepsilon(t_2\varphi)) < E_\varepsilon(r_0\varphi) \leq E(r_0\varphi). \quad (11)$$

Therefore if we take

$$\Gamma = \{\gamma : [0, 1] \rightarrow H^2(M) \text{ such that } \gamma(0) = t_0\varphi; \gamma(1) = t_2\varphi\}$$

we have a Palais-smale sequence at the level

$$c_\varepsilon = \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0, 1])} E_\varepsilon(u)$$

notice that from (11)

$$\inf_{u \in \partial B(0, r_0)} E_1(u) < c_\varepsilon < E(r_0\varphi)$$

and thus c_ε is uniformly bounded. Let us call that Palais-Smale sequence $(u_k^\varepsilon)_k$, it satisfies then

$$E_\varepsilon(u_k^\varepsilon) \longrightarrow c_\varepsilon \text{ and } E'_\varepsilon(u_k^\varepsilon) \longrightarrow 0 \text{ in } H^{-2} \text{ as } k \longrightarrow \infty.$$

Thus The following holds

$$\begin{aligned} O(\|u_k^\varepsilon\|_\psi) &= (q+1)E_\varepsilon(u_k^\varepsilon) - \langle E'_\varepsilon(u_k^\varepsilon), u_k^\varepsilon \rangle \\ &= \frac{(q-1)}{2} \|u_k^\varepsilon\|_\psi^2 + \left(\frac{q+1}{p-1} - 1\right) \int \frac{A}{\left(\varepsilon + (u_k^{\varepsilon+})^2\right)^{\frac{p-1}{2}}} + \varepsilon \int \frac{A}{\left(\varepsilon + (u_k^{\varepsilon+})^2\right)^{\frac{p+1}{2}}} \end{aligned} \quad (12)$$

Therefor

$$\|u_k^\varepsilon\|_\psi = O(1),$$

which implies the boundedness of $(u_k^\varepsilon)_k$ in $H^2(M)$ and thus the existwence of $u_\varepsilon \in H^2(M)$ such that

$$\begin{cases} u_k^\varepsilon \rightharpoonup u_\varepsilon \text{ weakly in } H^2 \\ u_k^\varepsilon \longrightarrow u_\varepsilon \text{ strongly in } L^2, \\ u_k^\varepsilon \rightharpoonup u_\varepsilon \text{ weakly in } L^{2^\sharp} \end{cases}$$

so take $\eta \in C^\infty(M)$, the previous assertion gives that

$$\int_M \eta P_{g,\psi} u_\varepsilon = \int_M \frac{A u_\varepsilon^+ \eta}{\left(\varepsilon + (u_\varepsilon^+)^2\right)^{\frac{p-1}{2}}} + \int_M B u_\varepsilon^q \eta$$

thus u_ε is a weak solution to the problem

$$P_{g,\psi} u_\varepsilon = \frac{A u_\varepsilon^+}{\left(\varepsilon + (u_\varepsilon^+)^2\right)^{\frac{p+1}{2}}} + B u_\varepsilon^q \quad (14)$$

hence u_ε is smooth and positive.

First, assume that $\left(\frac{q+1}{p-1} - 1\right) > 0$, then

$$(q+1)E_\varepsilon(u_\varepsilon) - \langle E'_\varepsilon(u_\varepsilon), u_\varepsilon \rangle = (q+1)c_\varepsilon$$

therefore from (12) we get

$$\begin{aligned} \int_M \frac{A}{\left(\varepsilon + (u_\varepsilon)^2\right)^{\frac{p+1}{2}}} &< C_1 \\ \|u_\varepsilon\|_{H^2} &< C_2 \end{aligned} \quad (15)$$

where C_1, C_2 are constants independent of ε . Thus we can extract a subsequence of $(u_\varepsilon)_\varepsilon$ that we will call $(u_{\varepsilon_k})_{\varepsilon_k}$ so that

$$\begin{cases} u_{\varepsilon_k} \rightharpoonup u \text{ weakly in } H^2(M) \\ u_{\varepsilon_k} \longrightarrow u \text{ strongly in } L^s(M) \text{ for } 1 < s < 2^\sharp \\ u_{\varepsilon_k} \longrightarrow u \text{ a.e on } M. \end{cases}$$

Thus using Fatou's lemma in (15) we get

$$\int_M \frac{1}{u^{p+1}} < C_1. \quad (16)$$

Assume now that there exist $x_k \longrightarrow \bar{x}$ such that $u_{\varepsilon_k}(x_k) \longrightarrow 0$. Then using the integral representation we get

$$u_{\varepsilon_k}(x_k) \geq \int_M G(x_k, y) B(y) u_{\varepsilon_k}^q(y) dy$$

where G is the Green's function of the operator $P_{g,\psi}$. Taking $k \longrightarrow \infty$ we get

$$\int_M G(\bar{x}, y) B(y) u^q(y) dy = 0,$$

thus $u = 0$ which is impossible because of (16), therefore u_ε is uniformly bounded from below.

So now we can pass to the weak limit in (14) to get.

$$\int_M \eta P_{g,\psi} u = \int_M \frac{\eta A}{u^p} + \int_M B u^q \eta, \text{ for every } \eta \in C^\infty(M),$$

hence, since u is positively bounded from below, we get a smooth positive solution to

$$P_{g,\psi} u = \frac{A}{u^p} + B u^q.$$

If $p-1 = q+1$ (and that is the case of the Lichnerowicz Equation), to find a uniform bound on $\int_M \frac{A}{(\varepsilon+(u_\varepsilon)^2)^{\frac{p+1}{2}}}$, we use the fact that $\|u_\varepsilon\|_\psi$ is uniformly bounded, and Sobolev embedding to get a uniform bound on $\int_M B u_\varepsilon^{q+1}$ and thus we get the desired bound. \square

Corollary 3.8. *Under the assumption of the previous theorem, we have the existence of another positive solution*

Proof. If we take a look at the inequality (9) we notice that it is open, that is if we perturb B a small perturbation, we still get the same existence result. So let us call u_B the solution corresponding to B . Then using a comparison principle, we get $u_{B-\varepsilon} < u_{B+\varepsilon}$ and they are a pair of sub and super-solution to the problem (8), therefore we have the existence of a solution \tilde{u} to the problem, and to guaranty that $u_B \neq \tilde{u}$ we use a degree theory argument since every positive smooth solution is in the set $A = \{u \in C^{4,\alpha}(M); \frac{1}{C} < u < C\}$ for $C > 0$ large enough and uniform. \square

Corollary 3.9. *There exist a constant $C = C(n, M, Q_\psi) > 0$ such that, if $P_{g,\psi}$ is strongly positive and*

$$(\max B)^{\frac{3n-4}{8}} \int_M A < C$$

the Paneitz-Lichnerowicz Problem admits at least one positive solution.

For the proof of this corollary, we just take $\varphi = 1$ in (9), and the Sobolev embedding

$$H^2(M) \hookrightarrow L^{2^*}(M).$$

4 Non existence Result

Theorem 4.1. *Assume that $A, B \geq 0$, then if*

$$\left(\int_M A^{\frac{q}{p+q}} B^{\frac{p}{q+p}} \right)^{\frac{(p+q)(q-3)}{q(p+q-2)}} \left(\left(\frac{q-1}{p+1} \right)^{\frac{1-q}{p+q-2}} \left(\int_M A^{\frac{q}{p+q}} B^{\frac{p}{q+p}} \right)^{\frac{(p+q)2}{q(p+q-2)}} + \left(\frac{q-1}{p+1} \right)^{\frac{p+1}{p+q-2}} \right) > \left(\int_M (Q_\psi^+)^{\frac{q}{q-1}} B^{-\frac{1}{q-1}} \right)^{\frac{q-1}{q}},$$

then the problem does not posses any positive smooth solution, where $Q_\psi = Q - |\nabla\psi|_g^2$

Proof. Let u be a positive solution, then the follwong holds :

$$\int_M \frac{A}{u^p} + \int_M B u^q = \int_M Q_\psi u$$

using the fact that

$$\int_M Q_\psi u \leq \left(\int_M (Q_\psi^+)^{\frac{q}{q-1}} B^{-\frac{1}{q-1}} \right)^{\frac{q-1}{q}} \left(\int_M B u^q \right)^{\frac{1}{q}}$$

Also

$$\int_M A^{\frac{q}{p+q}} B^{\frac{p}{q+p}} \leq \left(\int_M \frac{A}{u^p} \right)^{\frac{q}{p+q}} \left(\int_M B u^q \right)^{\frac{p}{q+p}}$$

therefore if we set $X = \int B u^q$, one gets

$$X + \left(\left(\int_M A^{\frac{q}{p+q}} B^{\frac{p}{q+p}} \right) \right)^{\frac{p+q}{q}} X^{-\frac{p}{q}} \leq \left(\int_M (Q_\psi^+)^{\frac{q}{q-1}} B^{-\frac{1}{q-1}} \right)^{\frac{q-1}{q}} X^{\frac{1}{q}}$$

Which is equivalent to say that

$$X^{1-\frac{1}{q}} + \left(\int_M A^{\frac{q}{p+q}} B^{\frac{p}{q+p}} \right)^{\frac{p+q}{q}} X^{-\frac{p+1}{q}} \leq \left(\int_M (Q_\psi^+)^{\frac{q}{q-1}} B^{-\frac{1}{q-1}} \right)^{\frac{q-1}{q}}$$

Therefore if

$$\left(\int_M A^{\frac{q}{p+q}} B^{\frac{p}{q+p}} \right)^{\frac{(p+q)(q-3)}{q(p+q-2)}} \left(\left(\frac{q-1}{p+1} \right)^{\frac{1-q}{p+q-2}} \left(\int_M A^{\frac{q}{p+q}} B^{\frac{p}{q+p}} \right)^{\frac{(p+q)2}{q(p+q-2)}} + \left(\frac{q-1}{p+1} \right)^{\frac{p+1}{p+q-2}} \right) > \left(\int_M (Q_\psi^+)^{\frac{q}{q-1}} B^{-\frac{1}{q-1}} \right)^{\frac{q-1}{q}}$$

then there is no smooth positive solution to the problem. \square

5 Conclusion

As a conclusion of the previous existence and non-existence result, we can set for the sake of simplicity, $A = 1$, $B = \lambda \in \mathbb{R}$, and we get the following corollary if we consider the following problem

$$P_{g,\psi}u = \frac{1}{u^p} + \lambda u^q. \quad (17)$$

Corollary 5.1. *If $P_{g,\psi}$ is strongly positive, then there exist a constant $\lambda^* > 0$ such that*

i) Problem has no positive smooth solution if $\lambda > \lambda^$.*

ii) Problem has at least one positive solution if $\lambda < \lambda^$.*

Moreover we have the following estimate

$$\left(Vol(M)^{-\frac{2^\sharp - q - 1}{2^\sharp}} C \left(\frac{n-4}{2} \int_M Q_\psi \right)^{-(p-1) \frac{q-1}{p+1}} \right) < \lambda^* < Vol(M)^{-\frac{(p+q)(q-1)}{pq+q-2}} \left(\frac{q-1}{p+1} \right)^{\frac{q(q-1)}{pp+q-2}} \|Q_\psi\|^{\frac{q(p+q-2)}{pq+q-2} \frac{q}{q-1}}$$

where C is the constant (10).

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