FOLIATED STOCHASTIC CALCULUS: HARMONIC MEASURES

PEDRO J. CATUOGNO, DIEGO S. LEDESMA AND PAULO R. RUFFINO

Abstract. In this article we present an intrinsec construction of foliated Brownian motion via stochastic calculus adapted to foliation. The stochastic approach together with a proposed foliated vector calculus provide a natural method to work on harmonic measures. Other results include a decomposition of the Laplacian in terms of the foliated and basic Laplacians, a characterization of totally invariant measures and a differential equation for the density of harmonic measures.

1. Introduction

Harmonic measures in a foliated Riemannian manifold are invariant measures for foliated Brownian motion (FoBM), which is a diffusion associated to the foliated Laplacian Δ_E. A Borel measure μ is harmonic if for any leafwise C^2-function f,

$$\int_M \Delta_E f \; d\mu = 0.$$

Harmonic measures have a central place in the ergodic theory of foliations and in the study of asymptotic properties of the leaves. It has been introduced by Garnett [6] and developed in many articles by Candel [2], Kaimanovich [11], Yue [19], Adams [1], Ghys [8], Ledrappier [13] and others.

In this article we apply stochastic calculus in foliated Riemannian manifold focusing mainly the theory of harmonic measures. Precisely, we present an intrinsec construction of foliated Brownian motion via stochastic calculus adapted to foliation (Theorem 3.4). The stochastic approach together with a proposed foliated vector calculus provide a natural method to work on harmonic measures. Other results include a decomposition of the Laplacian in terms of the foliated and

Key words and phrases. Foliation, diffusion process, stochastic calculus.

2010 *Mathematics Subject Classification. Primary: 58J65, 53C12 ; Secondary: 60H30, 60J60.*

Research partially supported by FAPESP 04/13758 and 08/13758 and CNPQ 302704/2008-6.

basic Laplacians (Theorem 2.1), a characterization of totally invariant measures (Theorem 4.3) and a differential equation for the density of harmonic measures (Theorem 4.6).

The article is organized in the following way. In section 2, we introduce the foliated operators and study properties of foliated Laplacian. In section 3, we develop the foliated stochastic calculus, where we construct a FoBM in a similar way of the construction of the Brownian motion of Eells-Elworthy-Malliavin (e.g. [4], [10], among others). In section 5, we study the harmonic measures. Some examples are given along the text.

2. The foliated Laplacian

In this section we introduce the fundamental operators which are structural in the theory of foliated spaces. Our framework here is a Riemannian manifold (M, g) which is foliated by a family of submanifolds $\mathcal{F}$ which is characterized by the integrable distribution $E \subseteq TM$ given by the tangent bundle of the leaves. Let $\pi : TM \to E$ be the orthogonal projection on E, naturally the metric g induces metrics g_E and $g^{\perp}$ in E and $E^{\perp}$ respectively.

Let ∇ be the Levi-Civita connection in M with respect to g. Denote by ∇^E the connection on E induced by ∇, i.e.

$$\nabla^E_X Y = \pi \nabla_X Y$$

for all $X \in TM$ and Y in $\Gamma(E)$, the space of smooth sections of E over M. The connection ∇^E is the Levi-Civita connection on the leaves with respect to g_E.

Definition 2.1. *Let $f : M \to \mathbf{R}$ be a smooth function and X, Y sections of E. We define the foliated operators:*

a) $\operatorname{grad}_E f = \pi(\operatorname{grad} f)$;

b) $\operatorname{div}_E Y = \operatorname{Tr}_E g(\nabla^E_{\cdot} Y, \cdot)$, *where* Tr_E *is the trace on* E;

c) $\operatorname{Hess}_E(f)(X, Y) = XY(f) - \nabla^E_X Y f$;

d) $\Delta_E f = \operatorname{div}_E(\operatorname{grad}_E f) = \operatorname{Tr}_E \operatorname{Hess}_E f$.

Extending vector fields and smooth functions from a leaf to the manifold M, one sees that the operators above are the natural extension of the corresponding operator on the leaf.

Given $\{X_1, \ldots, X_p\}$ a local orthonormal basis of E, the following classical formulae hold:

a) $\operatorname{grad}_E f = \sum_{i=1}^p (X_i f)\, X_i$,

b) $\operatorname{div}_E Y = \sum_{i=1}^{p} \left(g(\nabla^E_{X_i} Y, X_i) \right)$,

c) $\Delta_E f = \sum_{i=1}^{p} \operatorname{Hess}_E f(X_i, X_i)$.

Comparing the Hessian on M with the foliated operator Hess_E, we have that

$$\operatorname{Hess} f(X,Y) = \operatorname{Hess}_E f(X,Y) - W(X,Y) f \tag{1}$$

where $W(X,Y)$ is the second fundamental form of the foliation. Also, by this formula one finds that

$$\Delta_E f = \operatorname{Tr}_E (\operatorname{Hess} f) + K f, \tag{2}$$

where K is the mean curvature of the foliation, defined as $K = \operatorname{Tr}_E W$.

Given smooth functions f and h, the classical formulae below hold:

$$\operatorname{Hess}_E(fh) = h \operatorname{Hess}_E f + f \operatorname{Hess}_E h + dh|_E \otimes df|_E + df|_E \otimes dh|_E \tag{3}$$

and

$$\Delta_E(fh) = f \Delta_E h + h \Delta_E f + 2\, g(\operatorname{grad}_E f, \operatorname{grad}_E h). \tag{4}$$

Following the definition presented by Rumler [16] and [17] we introduce the characteristic form of $\mathcal{F}$.

Definition 2.2. *Let E be an orientable bundle. The characteristic form of $\mathcal{F}$, denoted by χ_E, is the differential p-form on M defined by*

$$\chi_E(Y_1, \ldots, Y_p) = \det[g(Y_i, E_j)]$$

where $Y_1, \ldots, Y_p \in TM$ and $\{E_1, \ldots, E_p\}$ is a local positively oriented orthonormal basis of sections of E.

The restriction of χ_E to tangent vectors to a leaf $L \in \mathcal{F}$ is an induced volume form in L. The characteristic form fits well in the approach proposed here in the sense that we recover the classical formula:

Lemma 2.1. *Let $Y \in \Gamma(E)$. Then*

$$\operatorname{div}_E (Y) \chi_E = L_Y \chi_E$$

where L_Y is the Lie derivative.

Proof. Let $\{E_1, \ldots, E_p\}$ be a local positively oriented orthonormal basis of sections of E. Symmetry of the connection implies that

$$g_E(L_Y E_j, E_j) = -g_E(\nabla^E_{E_j} Y, E_j),$$

hence

$$\begin{aligned} L_Y\chi_E(E_1,\dots,E_p) &= -\sum_{j=1}^{p}\chi_E(E_1,\dots,L_YE_j,\dots,E_p) \\ &= -\sum_{j=1}^{p} g_E(L_YE_j,E_j)\chi_E(E_1,\dots,E_p) \\ &= \sum_{j=1}^{p} g_E(\nabla^E_{E_j}Y,E_j)\chi_E(E_1,\dots,E_p) \\ &= \operatorname{div}_E(Y)\chi_E(E_1,\dots,E_p). \end{aligned}$$

□

Again, using the terminology of Rumler [16], we introduce the following

Definition 2.3. *Let $E^\perp$ be a orientable bundle and $q = n - p$. The characteristic transverse form of $\mathcal{F}$, denoted by $\chi_{E^\perp}$, is the differential q-form on M defined by*

$$\chi_{E^\perp}(Y_1,\dots,Y_q) = \det[g(Y_i,E_j)]$$

where $Y_1,\dots,Y_q \in TM$ and $\{E_{p+1},\dots,E_n\}$ is a local positively oriented orthonormal basis of sections of $E^\perp$.

The following identity holds:

Lemma 2.2.

$$d\chi_{E^\perp} = -\kappa^\flat \wedge \chi_{E^\perp}, \tag{5}$$

where $\kappa = \pi_E\left(\sum_{j=p+1}^{n}\nabla_{E_j}E^j\right)$.

Proof. Let $\{E_{p+1},\dots,E_n\}$ be a local positively oriented orthonormal basis of sections of $E^\perp$.

Initially, note that if $Y \in E^\perp$, both sides of the Equation (5) vanish in the $(q+1)$-uple $(Y,E_{p+1},\dots,E_n)$. For $Y \in E$ we have that

$$g(L_YE_j,E_j) = g(Y,\nabla_{E_j}E_j).$$

Since $i_Y\chi_{E^\perp}=0$, by Cartan formula:

$$\begin{aligned} d\chi_{E^\perp}(Y,E_{p+1},\ldots,E_n) &= L_Y\chi_{E^\perp}(E_{p+1},\ldots,E_n)\\ &= Y(\chi_{E^\perp}(E_{p+1},\ldots,E_n))\\ &\quad -\sum_{j=p+1}^{n}\chi_{E^\perp}(E_{p+1},\ldots,L_YE_j,\ldots,E_n)\\ &= -g\left(Y,\sum_{j=p+1}^{n}\nabla_{E_j}E_j\right)\\ &= -g(Y,\kappa). \end{aligned}$$

Thus

$$d\chi_{E^\perp}(Y,E_{p+1},\ldots,E_n)=-\sum_{j=p+1}^{n}g(Y,\pi_E\nabla_{E_j}E_j)=-\kappa^\flat(Y),$$

for all $Y\in TM$. □

The vector field κ defined in the lemma above can also be characterized as the unique section on E such that

$$\kappa^\flat(X)=\operatorname{div}_E(X)-\operatorname{div}(X), \tag{6}$$

for all $X\in E$. Yet, κ is the trace on $E^\perp$ of the bilinear form b, defined by

$$b(V,W)=\pi\nabla_VW.$$

The Laplacian Δ_M can be written in terms of the foliated Laplacian Δ_E, the section κ and the basic Laplacian Δ_b. Let δ_b is the formal adjoint of the exterior derivative d restricted to basic forms and denote by $*$ the Hodge star operator (see e.g. Tondeur [18, p. 134]). The basic Laplacian $\Delta_b f=\delta_b df$ is the second order operator given by

$$\Delta_b f=(-1)^{p(q-1)}*\left[K^\flat\wedge(*(df\wedge\chi_E))\wedge\chi_E)-d(*(df\wedge\chi_E))\wedge\chi_E)\right].$$

Theorem 2.1. *If M is oriented, then*

$$\Delta f=(\Delta_E f-\kappa f)+(-1)^{pq+1}\Delta_b f.$$

Proof. Orientability of M implies that $\chi^\perp=*\chi$. Iniciatially, consider a vector field Y in M. We claim that

$$i_Y\chi^\perp=(-1)^p*(Y^\flat\wedge\chi). \tag{7}$$

In fact, let $\{V_1,\ldots,V_q,E_1,\ldots,E_p\}$ be an adapted orthonormal basis of TM and consider its dual basis $\{V_1^\flat,\ldots,V_q^\flat,E_1^\flat,\ldots,E_p^\flat\}$. We have

$$\chi=E_1^\flat\wedge\ldots\wedge E_p^\flat \qquad\text{and}\qquad \chi^\perp=V_1^\flat\wedge\ldots\wedge V_q^\flat.$$

It is enough to prove the equality for $Y = V_j$, for $j = 1, \ldots, q$. But, for a fixed j, Equation (7) follows from

$$i_{V_j}\chi^\perp = (-1)^{j+1}(V_1^\flat \wedge \ldots \wedge \widehat{V_j^\flat} \wedge \ldots \wedge V_q^\flat)$$

and

$$*(V_j^\flat \wedge \chi) = (-1)^{p+j-1}(V_1^\flat \wedge \ldots \wedge \widehat{V_j^\flat} \wedge \ldots \wedge V_q^\flat).$$

Secondly, let f be a smooth function in M, by Cartan formula,

$$\begin{aligned} di_{(\mathrm{grad}\, f)}\ \chi^\perp &= di_{(\pi^\perp \mathrm{grad}\, f)}\ \chi^\perp \\ &= L_{(\pi^\perp \mathrm{grad}\, f)}\ \chi^\perp - i_{(\pi^\perp \mathrm{grad}\, f)}\ d\chi^\perp, \end{aligned} \tag{8}$$

where $\pi^\perp : TM \to E^\perp$ is the orthogonal projection. Equation (5) implies that,

$$\begin{aligned} i_{\pi^\perp \mathrm{grad}\, f} d\chi^\perp &= k^\flat \wedge i_{\pi^\perp \mathrm{grad}\, f}\chi^\perp \\ &= k^\flat \wedge i_{\mathrm{grad}\, f}\chi^\perp. \end{aligned} \tag{9}$$

Rumler formula (Tondeur [18, pg. 66]) says that

$$L_Z\chi|_E + K^\flat(Z)\chi|_E = 0, \qquad \forall Z \in E^\perp. \tag{10}$$

Hence, combining the above equations, we have that

$$\begin{aligned} (-1)^p d * (df \wedge \chi) \wedge \chi &= (di_{\mathrm{grad}\, f}\chi^\perp) \wedge \chi && \text{by (7)} \\ &= (L_{\pi^\perp \mathrm{grad}\, f}\chi^\perp) \wedge \chi \\ &\quad -k^\flat \wedge (i_{\mathrm{grad}\, f}\chi^\perp) \wedge \chi && \text{by (8) and (9)} \\ &= L_{\pi^\perp \mathrm{grad}\, f}\mu_g \\ &\quad -\chi^\perp \wedge (L_{\pi^\perp \mathrm{grad}\, f}\chi) \\ &= L_{\pi^\perp \mathrm{grad}\, f}\mu_g + K(f)\mu_g, \end{aligned}$$

by Equation (10). Again, by Equation (7) and the fact that $K^\flat \wedge \chi^\perp = 0$ one can show that

$$\begin{aligned} (-1)^p K^\flat \wedge *(df \wedge \chi) \wedge \chi &= K^\flat \wedge i_{\mathrm{grad}\, f}\chi^\perp \wedge \chi \\ &= -i_{\mathrm{grad}\, f}(K^\flat \wedge \chi^\perp) \wedge \chi + (i_{\mathrm{grad}\, f}K^\flat)\ \chi^\perp \wedge \chi \\ &= K(f)\mu_g. \end{aligned}$$

Now, replacing this formula in the definition of Δ_b, we obtain

$$\Delta_b f = -(-1)^{pq}\mathrm{div}(\pi^\perp \mathrm{grad}\, f).$$

Thus, decomposing $\mathrm{grad}\, f$ into tangential and normal components:

$$\Delta f = (\Delta_E f - \kappa f) + (-1)^{pq+1}\Delta_b f.$$

□

In order to construct foliated Brownian motion in the next section, we have to study the horizontal lift of the foliated Laplacian. We consider the principal bundle $O(E)$ of orthonormal frames in E, with projection $r : O(E) \to M$ and structural group $O(p)$.

The induced connection ∇^E gives a partition of the tangent bundle of $O(E)$ into a vertical space $VO(E)$ and a horizontal space $HO(E)$ such that $TO(E) = VO(E) \oplus HO(E)$ (see e.g. Kobayashi and Nomizu [12]).

For each v in $\mathbf{R}^p$ the standard vector field H_v in $O(E)$, is given by the unique $H_v(u) \in HO(E)_u$ such that $r_*(H_v(u)) = uv$. For an orthonormal frame $\{e_1, \ldots, e_p\}$ of $\mathbf{R}^p$, we define the horizontal foliated Laplacian in $O(E)$ as

$$\Delta_E^H = \sum_{i=1}^{p} (H_{e_i})^2.$$

One checks that it is independent of the basis.

The following lemma shows that Δ_E^H is the horizontal lift of Δ_E:

Lemma 2.3. *For $f \in C^\infty(M)$, the following identity holds*

$$\Delta_E^H(f \circ r) = (\Delta_E f) \circ r.$$

Proof. We first observe that

$$H_{e_j}(f \circ r)(u) \quad = \quad g\left(\mathrm{grad}_E f(r(u)), ue_j\right).$$

For the second derivative, consider a horizontal curve u_t in $O(E)$ such that $u_0 = u$ and $\dot{u}_t = H_{e_i}(u_t)$. Then, for each e_i, the vector field $u_t e_i$ is the parallel transport of ue_i along $\gamma_t = r(u_t)$ with respect to the connection ∇^E. Hence,

$$\begin{aligned} H_{e_i} H_{e_j}(f \circ r)(u) &= \left.\frac{d}{dt}\right|_{t=0} g\Big(\mathrm{grad}_E f(r(u_t)), u_t e_j\Big) \\ &= g\left(\nabla^E_{ue_i} \mathrm{grad}_E f(r(u)), ue_j\right) \\ &= \mathrm{Hess}_E f(ue_i, ue_j) \circ r(u). \end{aligned}$$

So,

$$\Delta_E^H(f \circ r)(u) \quad = \quad \left(\sum_{i=1}^{p} \mathrm{Hess}_E f(ue_i, ue_i)\right) \circ r(u).$$

□

3. The foliated Brownian motion

In this section we introduce the probabilistic aspects which are the key points of our approach. We shall denote by $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}, \mathbf{P})$ a filtered probability space satisfying the usual completeness conditions.

A semimartingale X in M will be called *foliated* if each trajectory stays in a single leaf. Furthermore, a foliated semimartingale X will be called a *foliated martingale* if for any smooth function f,

$$f(X) - f(X_0) - \frac{1}{2}\int_0 \mathrm{Hess}_E f(dX, dX)$$

is a local martingale. Foliated martingales may not be martingales in M; precisely, this fact depends on the geometry of the foliation:

Proposition 3.1. *Foliated martingales are martingales in M if and only if the foliation is totally geodesic.*

Proof. A foliation is totally geodesic when $\mathrm{Hess}_E f(X,Y) = \mathrm{Hess} f(X,Y)$ for all $f \in C^\infty(M)$, hence the result follows.

Conversely, consider γ a geodesic of a leaf. We have to proof that γ is a geodesic in M. For a linear Brownian motion B, the process $X = \gamma(B)$ is a foliated martingale, hence, by hypothesis, it is a martingale in M. By standard calculation

$$\begin{aligned}
\int \frac{d^2}{dt^2} f(\gamma)(B)dt &= \int \mathrm{Hess}_M f(dX, dX) \\
&= \int (\gamma_* \otimes \gamma_*)^* \mathrm{Hess}_M f(dB, dB) \\
&= \int \mathrm{Tr}(\gamma_* \otimes \gamma_*)^* \mathrm{Hess}_M f(B)dt \\
&= \int \mathrm{Hess}_M f(\dot\gamma(B), \dot\gamma(B))dt.
\end{aligned}$$

It follows that

$$\frac{d^2}{dt^2} f(\gamma)(t) = \mathrm{Hess}_M f(\dot\gamma(t), \dot\gamma(t)),$$

i.e. γ is a geodesic in M. □

Next result says that the foliated martingales also satisfy the non-confluence property:

Proposition 3.2. *For each $x \in M$ there is an open neighborhood $U_x \subset M$ such that, if X and Y are foliated martingales in U_x such that $X_T = Y_T$ for a stopping time T then $X_t = Y_t$ a.e. for $0 \le t \le T$.*

Proof. The proof follows similar ideas for nonconfluence of martingales in a manifold as in Emery [5, p.52-53]. For a fixed point $p \in M$ we consider a convex function f defined on a neighborhood $U \subset L_p \times L_p$ where L_p is the leaf through of p. By continuity, we extend this function to $\widetilde{f} : \widetilde{U} \subset M \times M \to \mathbf{R}$ such that $\text{Hess}_{E\times E}\widetilde{f}(A, A) \geq 0$ for all $A \in E \times E$ and $\widetilde{f}|\{(x, x) \in \widetilde{U}\} = 0$. Clearly, there exists U_x neighborhood of x such that $U_x \times U_x \subset \widetilde{U}$. Let X and Y be foliated martingales in U_x such that $X_T = Y_T$ a.e.. Using that $\widetilde{f}(X, Y)$ is a positive bounded submartingale null at time T, we conclude that $X_t = Y_t$ a.e. for $0 \leq t \leq T$. □

Let X be a foliated semimartingale. We says that X is a *foliated Brownian motion* (FoBM) if for any smooth function f,

$$f(X) - f(X_0) - \frac{1}{2}\int_0 \Delta_E f(X)\, dt$$

is a local martingale. Note that a process X is a FoBM if and only if it is a foliated martingale and for any smooth function f,

$$[f(X), f(X)] = \int_0 |\text{grad}_E f(X)|^2 dt. \tag{11}$$

The geometry of the foliation determines probabilistic properties of FoBM:

Proposition 3.3. *FoBM are martingales in M if and only if the leaves are minimal submanifolds, i.e. the foliation is harmonic.*

Proof. Let X be a FoBM. By Equation (2) and the definition, we have that for all smooth function f,

$$f(X) - f(X_0) - \frac{1}{2}\int_0 \text{Hess} f\ (dX, dX) - \frac{1}{2}\int_0 K f(X)\ dt \tag{12}$$

is a local martingale. Since X is a martingale in M,

$$\int_0 K f(X)\ dt = 0,$$

hence $K = 0$.

Conversely, from Equation (12) and $K = 0$ we have that

$$f(X) - f(X_0) - \frac{1}{2}\int_0 \text{Hess} f\ (dX, dX)$$

is a local martingale. Thus X is a martingale in M. □

Garnett [6] introduced foliated heat kernels via foliated semigroups of operators which depends strongly on the geometry of the foliation, (see also Candel [2]). We can recovery the same semigroup considering the semigroup associated to a FoBM, provided one can guarantees the existence of this stochastic process.

Focusing in this direction, we present an intrinsec construction of FoBM. Our argument corresponds to an adaptation to foliated spaces of the techniques of Eells-Elworthy-Malliavin, classically used to construct Brownian motions in a Riemaniann manifold (see e.g. [4], [10] and references therein) .

Theorem 3.4. *Let u_t be the solution of the Stratonovich equation*

$$du_t = \sum_{i=1}^{p} H_{e_i}(u_t) \circ dB^i,$$

where $(B^1, \dots, B^p)$ is the Brownian motion on $\mathbf{R}^p$ with u_0 as initial condition in $O(E)$. Then $r(u_t)$ is a FoBM in M starting at $r(u_0)$.

Proof. For any smooth function f in M, applying Lemma 2.3, we have that

$$\begin{aligned} f(r(u)) - f(r(u_0)) &= \sum_{i=1}^{p} \int H_{e_i}(f \circ r)(u)\, dB^i + \\ &\quad \frac{1}{2} \int \sum_{i=1}^{p} H_{e_i}^2 (f \circ r)(u) dt \\ &= \sum_{i=1}^{p} \int H_{e_i}(f \circ r)(u)\; dB_t^i \\ &\quad + \frac{1}{2} \int \Delta_E f(r(u)) dt. \end{aligned}$$

□

Example 1: Let N and L be two Riemannian manifolds. Consider the product $M = N \times L$ with the canonical foliation given by $E = TL \subset TM = TN \oplus TL$. The foliated Laplacian $\Delta_E = \Delta_L$, hence if W is a Brownian motion in L then $B = (x_0, W)$ is a FoBM.

□

Example 2: (Kronecker foliation) Consider the totally geodesic foliation of the plane $\mathbf{R}^2$ along lines parallel to the vector $(a, 1)$. The process

$$B = \frac{(a,1)}{\sqrt{a^2+1}}\, W,$$

where W is a linear Brownian motion is a foliated Brownian motion. Now, denoting by $\mathbf{T}^2$ the 2-torus $S^1 \times S^1 \subset \mathbf{R}^4$, let $\phi : \mathbf{R}^2 \to \mathbf{T}^2$ given by

$$\phi(x, y) = (\cos(x), \sin(x), \cos(y), \sin(y)).$$

The induced foliation by ϕ in $\mathbf{T}^2$ is called the Kronecker foliation, Candel and Conlon [3]. We claim that $\phi(B)$ is a FoBM. In fact, $E = \{\lambda(a\partial_x + \partial_y),\ \lambda \in \mathbf{R}\}$ and

$$\Delta_E = \frac{1}{a^2+1}(a^2\partial_x^2 + 2a\partial_{xy}^2 + \partial_y^2),$$

where $\partial_x = \phi_*(e_1)$ and $\partial_y = \phi_*(e_2)$.

For all smooth function f in $\mathbf{T}^2$,

$$\begin{aligned} f(\phi(B_t)) &= f(\phi(x_0)) + \int_0^t a(\partial_x f)(\phi(B_s)) + (\partial_y f)(\phi(B_s))dW_s \\ &\quad + \frac{1}{2}\int_0^t (\Delta_E f)(\phi(B_s))ds. \end{aligned}$$

Hence $\phi(B)$ is a foliated Brownian motion in the Kronecker foliation of the torus.

□

Next proposition shows that 1-dimensional foliations generated by unitary vector fields have an easy construction of FoBM:

Proposition 3.5. *Let M be a foliated Riemannian manifold where the distribution E is generated by a smooth unitary vector field Y. If ϕ_t is the flow of diffeomorphisms associated to Y and B is a linear Brownian motion, then $\phi_B(x_0)$ is a FoBM starting at $x_0 \in M$.*

Proof. We have that $\Delta_E = Y^2$ and $\phi_t(x_0)$ is a geodesic in the leaf. Hence $\phi_B(x_0)$ is a martingale, so that

$$f(\phi_B(x_0)) - f(\phi_0(x_0)) - \frac{1}{2}\int \Delta_E f(\phi_B(x_0))\ dt$$

is a local martingale for any smooth function f. So, $\phi_B(x_0)$ is a FoBM starting at x_0. □

4. The Harmonic Measures

In this section we focus on the theory of harmonic measures, according to Garnett [6], Candel [2] and others. The Markov semigroup $P(t, x, A)$ associated to the FoBM generates a Feller semigroup in

$C^2(M)$ whose infinitesimal generator is $1/2\Delta_E$. The adjoint semigroup, denoted by $\{T_t^*\}$, acts on the measure space of M in the following way

$$T_t^*\mu(A) = \int_M P(t,x,A)\,\mu(dx).$$

A measure μ is called invariant if $T_t^*\mu = \mu$ for all $t \geq 0$. We recall that a measure μ on a foliated manifold M is called *harmonic* if for all smooth function f,

$$\int_M \Delta_E f d\mu = 0.$$

Lemma 4.1. *A Borel measure μ on M is harmonic if and only if it is an invariant measure for FoBM.*

Proof. The proof follows directly from the equality

$$\frac{d}{dt}\int_M T_t f\ d\mu = \frac{1}{2}\int_M \Delta_E T_t f\ d\mu,$$

for all smooth function f.

□

As a consequence of the support theorem (see e.g. Ikeda-Watanabe [10]), it follows that the support of a harmonic measure is a satured set, i.e. it consists in a union of leaves. Next theorem extends results of Garnett [6] on existence of harmonic measures.

Theorem 4.1. *Let M be a foliated Riemannian manifold*

1) *If M is compact then there exist harmonic probability measure;*
2) *If the leaves are stochastically complete and there exists a smooth function $\varphi \geq 0$ on M such that*

$$\lim_{d(x,x_0)\to\infty} \Delta_E \varphi(x) = -\infty,$$

then there exist harmonic probability measure.

3) *If $\Delta_E f \equiv 0$ then f is constant in the leaves of the support of any harmonic measure.*

Proof. Item (1) follows directly from Lemma 4.1 and the existence of invariant measure of diffusions in compact manifolds.

Item (2) is a consequense of Khas'minskii criterium for existence of invariant measures, see e.g. Lorenzi and Bertoldi [14, p.172].

For (3), let $f : M \to \mathbf{R}$ be a leafwise harmonic function ($\Delta_E f = 0$) and μ a harmonic measure. Using Equation (4) one finds

$$\int_M |\mathrm{grad}_E f|^2(x)\ \mu(dx) = \frac{1}{2}\int_M \Delta_E f^2(x)\ \mu(dx) = 0.$$

Then, by continuity of $|\mathrm{grad}_E f|$, f is constant in the leaves of the support of μ.

□

Remark: Stochastic completeness in item (2) of the above Theorem can be guaranteed by well known geometrical conditions on the leaves, say, for example if the leaves are complete and

$$\int_{c>0}^{\infty} \frac{r\ dr}{\ln |B(r)|} = \infty,$$

where $|B(r)|$ is the volume of the geodesic ball of radius r, see Grigorian [9, Thm. 9.1, p.184] or yet, in terms of curvature, if we have lower bounds in the Ricci curvature, Elworthy [4].

Given M and M' two foliated Riemannian manifolds, we say that a smooth map $\phi : M \to M'$ is a foliated map if it preserves leaves. If $\phi_* \Delta_E = \Delta'_E$, and μ is a harmonic measure in M, note that the induced measure $\phi_*\mu$ is a harmonic measure in M'.

Our formalism allows a direct proof of the result on superharmonic functions on foliations in Adams [1]:

Theorem 4.2 (Adams)**.** *Let μ be a harmonic probability measure. Consider $f : M \to (0,\infty)$ a measurable function such that, for μ-a.s. $\Delta_E f \leq 0$. Then f is constant in each leaf μ-a.s.*

Proof. Using the fact that for any smooth function ϕ we have that

$$\Delta_E(\phi \circ f) = \phi'(f(x))\Delta_E f(x) + \phi''(f(x))|\mathrm{grad}_E f(x)|^2$$

then for $u = \ln(f+1)$, u is positive and satisfies

$$|\mathrm{grad}_E u|^2 + \Delta_E u \ = \ \frac{\Delta_E f}{f+1} \ \leq \ 0, \tag{13}$$

on μ-a.s., thus

$$\int_M |\mathrm{grad}_E u|^2 \ d\mu \leq 0,$$

hence $\mathrm{grad}_E f = (1+f)\ \mathrm{grad}_E u$ vanishes.

□

Example 3: Consider a torus $\mathbf{T}^2 \subset \mathbf{R}^3$ imersed isometrically with the covering coordinate system $\phi : \mathbf{R}^2 \to \mathbf{T}^2$ given by

$$\phi(x,y) = \big((b+\cos(x))\cos(y), (b+\cos(x))\sin(y), \sin(x)\big),$$

with $b > 1$. In these coordinates the induced metric is

$$g = dx^2 + (b+\cos(x))^2 dy^2,$$

and the associated Riemannian connection ∇ is characterised by

$$\nabla_{\partial_x}\partial_x = 0, \quad \nabla_{\partial_x}\partial_y = \frac{-\sin(x)}{b+\cos(x)}\partial_y, \quad \nabla_{\partial_y}\partial_y = (b+\cos(x))\sin(x)\partial_x.$$

Consider the foliation E on $\mathbf{T}^2$ generated by

$$Y = \frac{1}{\sqrt{\alpha^2+1}}\left(\alpha\partial_x + \frac{1}{(b+\cos(x))}\partial_y\right).$$

The leaf through (x_0, y_0) is the flow line of Y through (x_0, y_0), where the flow ψ of Y can be represented in local coordinates as $\psi_t(x_0, y_0) = (x_t\,, y_t)$ with

$$\begin{aligned}
x_t &= x_0 + \frac{\alpha\, t}{\sqrt{1+\alpha^2}} && (2\pi \mod),\\
y_t &= y_0 + A + \frac{2}{\alpha\sqrt{b^2-1}}\arctan\left(\sqrt{\frac{b-1}{b+1}}\tan\left(\frac{x_t}{2}\right)\right) && (2\pi \mod),\\
A &= \frac{-2}{\alpha\sqrt{b^2-1}}\arctan\left(\sqrt{\frac{b-1}{b+1}}\tan\left(\frac{x_0}{2}\right)\right) && (2\pi \mod).
\end{aligned}$$

We observe that

$$\nabla_Y Y = \frac{\sin(x)}{(1+\alpha^2)(b+\cos(x))}\,\partial_x - \frac{\alpha\sin(x)}{(1+\alpha^2)(b+\cos(x))^2}\,\partial_y.$$

Then $\nabla^E_Y Y = 0$ and $\Delta_E = Y^2$. So, a FoBM in $\mathbf{T}^2$ is a solution of the stochastic differential equation

$$\begin{aligned}
dW &= Y(W)\circ dB\\
W_0 &= p_0 \in M
\end{aligned}$$

with B the Brownian motion in $\mathbf{R}$. Therefore the FoBM can be written in coordinates as the solution of

$$\begin{aligned}
dX_t &= \frac{\alpha}{\sqrt{1+\alpha^2}}\,dB_t,\\
dY_t &= \frac{1}{(b+\cos(X_t))\sqrt{1+\alpha^2}}\,dB_t + \frac{1}{2}\frac{\alpha\sin(X_t)}{(1+\alpha^2)(b+\cos(X_t))^2}\,dt.
\end{aligned}$$

So,

$$\begin{aligned}
X_t &= x_0 + \frac{\alpha}{\sqrt{1+\alpha^2}}\,B_t \qquad (2\pi \mod),\\
Y_t &= y_0 + A + \frac{2}{\alpha\sqrt{b^2-1}}\arctan\left(\sqrt{\frac{b-1}{b+1}}\tan\left(\frac{X_t}{2}\right)\right)\\
&\qquad\qquad (2\pi \mod).
\end{aligned}$$

Therefore, cf. Proposition 3.5, the FoBM starting at (x_0, y_0) is given by $W_t = \psi_{B_t}(x_0, y_0)$. A measure $\mu = h\,\mu_g$ is harmonic for a smooth function h if and only if h satisfies

$$Y^2(h) + 2\ \mathrm{div}(Y)Y(h) + \big(Y(\mathrm{div}(Y)) + \mathrm{div}(Y)^2\big)\,h = 0.$$

Considering the case of h depending only on x, equation above reduces to

$$(b + \cos(x))h''(x) - 2\sin(x)h'(x) - \cos(x)h(x) = 0.$$

Whose unique non-trivial normalized periodic solution is

$$h(x) = \frac{1}{4\pi^2}\frac{1}{(b + \cos(x))}.$$

□

In a foliated space M with orientable leaves which admit a holonomy invariant measure ν, a harmonic measure can be constructed in terms of ν. The p-current φ_ν associated to ν is the funcional in $\Lambda^p(M)$ given by

$$\varphi_\nu(\omega) = \sum_{\alpha\in\mathcal{U}} \int_{S_\alpha} \left(\int_P \lambda_\alpha\omega\right)\ d\nu(P)$$

where λ_α is a partition of unity subordinated to a foliated atlas $\mathcal{U}$, P are plaques in $U_\alpha \in \mathcal{U}$ and S_α is transversal in U_α (see Plante [15, p.330] and Candel [2, p.235]).

The measure μ_ν associated to the positive functional $f \mapsto \varphi_\nu(f\chi_E)$ is called in the literature a *totally invariant measure* (e.g. [15]). Such associated measures have a further characterization which generalizes similar result in [6]:

Theorem 4.3. *Let M be a compact foliated Riemannian manifold leafwise orientable. A measure μ is totally invariant if and only if*

$$\int_M \mathrm{div}_E\, X\ d\mu = 0$$

for any $X \in \Gamma(E)$.

Proof. Let μ be a totally invariant measure associated to ν. We have to prove that

$$\varphi_\nu(\mathrm{div}_E\, X\chi_E) = 0.$$

But

$$\mathrm{div}_E\,(X)\chi_E = di_X\chi_E + i_Xd\chi_E,$$

$\varphi_\nu(d\alpha) = 0$ and $i_X\,(d\chi_E) = 0$ restricted to the leaves (cf. [18, p.69]).

For the converse, note that μ is harmonic. There exists a foliated atlas $\{U_i \simeq T_i \times P\}$ and an associated family of leafwise positive harmonic functions $h_i : U_i \to \mathbf{R}$ with corresponding transverse measures ν_i such that for any measurable function f,

$$\int_{U_i} f \; d\mu = \int_{T_i} \int_{\{t\}\times D_t} f \chi_E \; d\nu_i$$

see Garnett [6, Theorem 1-c] or Candel and Conlon [3, Vol II, Prop.2.4.10]. We take a partition of unity $\{\lambda_i\}$ subordinated to the foliated atlas $\{U_i \simeq T_i \times P\}$. Hence

$$\begin{aligned}
\int_M \mathrm{div}_E\,(X)\; d\mu &= \sum_i \int_M \mathrm{div}_E\,(\lambda_i X)\mu \\
&= \sum_i \int_{T_i} \left(\int_{\{t\}\times D_t} \mathrm{div}_E\,(\lambda_i X) h_i \; \chi_E(t) \right) d\nu_i(t) \\
&= \sum_i \int_{T_i} \left(\int_{\{t\}\times D_t} \mathrm{div}_E\,(h_i \lambda_i X) \; \chi_E(t) \right) d\nu_i(t) \\
&\quad - \sum_i \int_{T_i} \left(\int_{\{t\}\times D_t} \lambda_i X(h_i) \; \chi_E(t) \right) d\nu_i(t) \\
&= -\sum_i \int_{T_i} \left(\int_{\{t\}\times D_t} \lambda_i X(h_i) \; \chi_E(t) \right) d\nu_i(t)
\end{aligned}$$

where each $\nu_i = p_*(\mu|_{U_i})$ is a measure over T_i induced by the projection $p : U_i \to T_i$. Thus, $\int_M \mathrm{div}_E\,(X)\mu = 0$ for any X if and only if h_i is constant along the leaf, that is μ is associated to the transverse measure $\nu = \sum_i h_i \nu_i$ which is invariant under holonomy transformations.

□

Corollary 4.4. *Let M be a compact foliated Riemannian manifold leafwise orientable. A measure μ is totally invariant if and only if*

$$\int_M div(X) \; \mu = -\int_M \kappa^\flat(X) \; \mu$$

for all $X \in \Gamma(E)$.

Proof. It follows directly from Equation (6). □

Proposition 4.5. *Let M be a compact foliated Riemannian manifold leafwise orientable with ν a holonomy invariant measure. If μ and $\widetilde{\mu}$ are two harmonic measure such that $\tilde{\mu}$ has a leafwise C^2 Radon-Nikodym derivative h with respect to μ, then h is constant in the leaf (a.e).*

Proof. We have by Equation (4),

$$\int_M |\mathrm{grad}_E h|^2 \mu = \frac{1}{2}\int \Delta_E h^2 \ \mu - \int_M (\Delta_E h)\ \tilde{\mu} = 0.$$

□

Harmonic measures which are absolutely continuous with respect to the Riemanian volume μ_g are characterized in the following:

Theorem 4.6. *Let M be a compact foliated Riemannian manifold without boundary and h be a non negative function which is C^2 leafwise. Then $h\mu_g$ is harmonic if and only if h satisfies*

$$\mathrm{div}(\mathrm{grad}_E h - h\kappa) = 0 \qquad \mu_g - a.s. \tag{14}$$

Proof. Firstly, we claim that the operator $\Delta_E - \kappa$ is self-adjoint. In fact, by Equation (6) we have that

$$(\Delta_E f - \kappa f)\mu_g = \mathrm{div}(\mathrm{grad}_E f)\mu_g.$$

Using that

$$\mathrm{div}\,(h\ \mathrm{grad}_E f) = g(\mathrm{grad}_E f, \mathrm{grad}_E h) + h\ \mathrm{div}(\mathrm{grad}_E f),$$

one finds that

$$\begin{aligned}
\int_M h(\Delta_E - \kappa) f \mu_g &= \int_M h\ \mathrm{div}(\mathrm{grad}_E f)\mu_g \\
&= -\int_M g(\mathrm{grad}_E f, \mathrm{grad}_E h)\mu_g \\
&= \int_M f\ \mathrm{div}(\mathrm{grad}_E h)\mu_g \\
&= \int_M f(\Delta_E - \kappa) h \mu_g.
\end{aligned}$$

For any smooth function f we have that

$$\mathrm{div}(fh\kappa) = h\kappa(f) + f\kappa(h) + fh\mathrm{div}(\kappa).$$

Hence,

$$\begin{aligned}
\int_M f\mathrm{div}(\mathrm{grad}_E h - h\kappa)\ \mu_g &= \int_M f(\Delta_E h - 2\kappa h - \mathrm{div}(\kappa)h)\mu_g \\
&= \int_M f(\Delta_E h - \kappa h)\mu_g + \int_M (\kappa f) h \mu_g \\
&= \int_M (\Delta_E f) h \mu_g
\end{aligned}$$

which vanishes for any f if and only if $h\mu_g$ is harmonic.

□

Corollary 4.7. *Let M be a compact foliated Riemannian manifold without boundary. Then* $\mathrm{div}(\kappa) = 0$ *if and only if for every non-negative leafwise constant function h the measure $\mu = h\mu_g$ is harmonic.*

Corollary 4.8. *Let M be a compact foliated Riemannian manifold without boundary. Then*

$$\int_M |\mathrm{grad}_E h|^2 \; d\mu_g = -\frac{1}{2} \int_M h^2 \, div(\kappa) \; \mu_g,$$

for any leafwise harmonic function h.

Proof. One uses that $\Delta_E - \kappa$ is selfadjoint, Equation (4) and the Gauss theorem to find that

$$0 = \int_M (\Delta_E - \kappa) h^2 \mu_g = 2 \int_M |\mathrm{grad}_E h|^2 \mu_g - \int_M \kappa(h^2) \mu_g.$$

□

Example 4: Let M be a quotient of the universal covering of $Sl(2, \mathbf{R})$ by a cocompact lattice. Denote by $\{X, Y, H\}$ an orthonormal basis of TM satisfying

$$[X, H] = X \qquad [X, Y] = -H \qquad [H, Y] = Y.$$

Consider the foliation induced by $E = \mathrm{span}\{X, H\}$. We have that

$$\Delta_E = X^2 + H^2 + H.$$

Hence FoBM satisfies the following stochastic differential equation

$$dB = H dt + H \circ dB^1 + X \circ dB^2$$

where (B^1, B^2) is the Brownian motion in $\mathbf{R}^2$. In this case we also have that $\kappa = H$ and $\mathrm{div}(\kappa) = 0$ which implies that the volume measure μ_g and any $h\mu_g$ with h constant in the leaves are harmonic (cf. Corollary 4.7). But any smooth h which is leafwise constant is constant in M: In fact, note that for any smooth function f,

$$\int_M Hf \; \mu_g = \int_M L_H(f\mu_g) - \int_M f \mathrm{div} H \mu_g \tag{15}$$

and both terms on the right hand side vanishes by Cartan formula and Stokes theorem. Moreover, we observe that

$$\begin{aligned} H(Yh) &= Yh + YHh \\ &= Yh. \end{aligned}$$

Hence, applying Equation (15) to $(Yh)^2$,

$$
\begin{aligned}
0 = \int_M H(Yh)^2 \mu_g &= 2 \int_M (Yh) H(Yh) \mu_g \\
&= 2 \int_M (Yh)^2 \mu_g .
\end{aligned}
$$

Then h is constant.

$\square$

Example 5: (Lie foliations) Let M be a manifold and $\mathfrak{g}$ a Lie algebra of dimension q. Assume that there exists a non singular surjective $\mathfrak{g}$-valued 1-form θ which satisfies the Maurer-Cartan formula

$$
d\theta + \frac{1}{2}[\theta, \theta] = 0.
$$

Consider the Lie foliation $E = \ker \theta_x$.

Let $Y_1, \ldots, Y_q$ be vector fields in TM such that $\theta(Y_i)$, $i = 1. \ldots, q$ is an orthonormal basis in $\mathfrak{g}$. We introduce an adapted metric on M, i.e., a metric g such that $g(Y_k, Y_j) = \delta_{kj}$ and $Y_1, \ldots, Y_q$ is an orthonormal basis of $E^{\perp}$. Hence, for all $X \in E$,

$$
g(\kappa, X) = \sum_{k=1}^{q} g(\nabla_{Y_k} Y_k, X) = g([Y_k, X], Y_k) = 0.
$$

Then $\kappa = 0$ therefore the volume measure μ_g and any $h\mu_g$ with h constant in the leaves are harmonic (cf. Corollary 4.7).

$\square$

References

[1] S. R. Adams, *Superharmonic functions on foliations.* Trans. Amer. Math. Soc. 330 (1992), 2, 625-635.

[2] A. Candel, *The harmonic measures of Lucy Garnett.* Adv. Math. 176 (2003), 2, 187-247.

[3] A. Candel, L. Conlon. *Foliations I.* Graduate Studies in Mathematics, 23. American Mathematical Society. *Foliations II.* Graduate Studies in Mathematics, 60. American Mathematical Society.

[4] K. D. Elworthy, *Stochastic Differential Equations on Manifolds.* London Math. Society (Lecture Notes Series 70) Cambridge University Press 1982.

[5] M. Emery, *Stochastic Calculus in Manifolds.* Universitext. Springer-Verlag,. Berlin, 1989.

[6] L. Garnett, *Foliation, The ergodic theorem and Brownian motion.* Journal of Functional Analysis 51, (1983)pp. 285-311

[7] E. Ghys, *Gauss-Bonnet theorem for 2-dimensional foliations*, J. Functional Analysis 77 (1988) pp. 51-59.

[8] E. Ghys, *Topologie des feuilles génériques*, Ann. of Math. 141 (1995) pp. 387-422.

[9] A. A. Grigorian, *Analytic and geometric background of recurrence and non-explosion of Brownian motion on Riemannian manifolds* Bull. AMS, **36**, no. 2 (1999), 135-249.

[10] N. Ikeda and S. Watanabe *Stochastic differential equations and diffusion processes*. Second edition. North-Holland Mathematical Library, 24. North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, 1989.

[11] V. A. Kaimanovich,*Brownian motion on foliations: Entropy, invariant measures, mixing* Functional Analysis and Its Applications, Vol. 22, N^0 4, (1988) pp. 326-328

[12] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vol I, Interscience Publishers, New York 1963.

[13] F. Ledrappier, *Ergodic properties of the stable foliations*, Lecture Notes in Math., Vol. 1514, Springer, Berlin, (1993) pp. 131-145.

[14] L. Lorenzi and M. Bertoldi, *Analytical methods for Markov semigroups.* Chapman and Hall/CRC, Boca Raton, 2007.

[15] J. F. Plante, *Foliations with measure preserving holonomy.* Annals of Mathematics, 102 (1975), 327-361.

[16] H. Rumler, *Quelques notions simples en géométrie riemannienne et leurs applications aux feuilletages compacts.* Comment. Math. Helvetici, Vol 54 (1979), pp. 224-239.

[17] H. Rumler, *Differential forms, Weitzembock formulae and foliations.* Publicacions Matematiques, Vol 33 (1989), pp. 543-554.

[18] P. Tondeur *Foliations on Riemannian manifolds.* Universitext, Springer Verlag, Berlin-Heidelberg-New York, 1988.

[19] Ch. Yue. *Brownian motion on Anosov foliations and manifolds of negative curvature* J. Differential Geom. Vol. 41, N^0 1 (1995), pp.159-183.

Departamento de Matemática, Universidade Estadual de Campinas,, 13.081-970 - Campinas - SP, Brazil.

E-mail address: `pedrojc@ime.unicamp.br ; dledesma@ime.unicamp.br and ruffino@ime.unicamp.br`