A PHYSICS PATHWAY TO THE RIEMANN HYPOTHESIS

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Abstract. We present a brief review of the spectral approach to the Riemann hypothesis, according to which the imaginary part of the non trivial zeros of the zeta function are the the eigenvalues of the Hamiltonian of a quantum mechanical system.

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1. Introduction

The Riemann hypothesis (RH) is the statement that the complex zeros of the classical zeta function all have imaginary part equal to 1/2. It was first suggested by Riemann in his famous memoir in 1859 [1]. The RH is important for its connection with the distribution of prime numbers [2, 3]. The average number of primes less than a given number *x*, which is denoted as $\pi(x)$, behaves asymptotically as $x/\log x$. This statement is called the Prime Number Theorem (PNT) and was proved independently by Hadamard and de La Vallée-Poussin in 1896 [4, 5]. The truth of the RH implies that the fluctuations of $\pi(x)$, around its average value, behaves asymptotically as $x^{1/2} \log x$, which also gives the best possible bound for the error of the PNT. The RH is not an isolated property of a particular function, but it holds for the Dirichlet L-functions, for curves over finite fields, etc. It is expected that a proof of the RH for the zeta function, will be generalizable to other L-functions as well. However the consensus is that some key idea is required for this goal.

One of the most promising pathways for a proof of the RH was suggested by Polya and Hilbert around 1910, but never published apparently. The suggestion is that there exist a selfadjoint operator H, whose spectrum contains the imaginary part of the Riemann zeros. The selfadjointness of such an operator would inmediately prove the RH:

If
$$\zeta\left(\frac{1}{2}+iE_n\right)=0, \quad \forall n \Longrightarrow H |\psi_n\rangle = E_n |\psi_n\rangle \Longrightarrow E_n = E_n^*, \quad \forall n$$
 (1)

In the latter equation one must exclude the trivial Riemann zeros, so a most appropriate formulation fo the problem is to use the $\xi(s)$ function, defined as [4]

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{2}$$

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whose zeros are those of $\zeta(s)$ except the trivial ones. Polya and Hilbert conjecture is known as the spectral approach to the RH, and, as we shall see later on, it is supported by several "phenomenological" results and heuristic arguments, which suggest that the operator *H* is the quantum Hamiltonian of a physical system. This is one of the reasons why the RH has attracted the interest of physicist working in disciplines apparently unrelated to Number Theory (see [6] for an extensive list of references concerning several approaches to the RH).

Assuming the RH (i.e. $E_n^* = E_n, \forall n$), one can define a diagonal operator H whose entries are E_n , but nothing is learned from this construction. Eq.(1), implies that H must encode in itself, the zeta function $\zeta(1/2 + iE)$, without assuming the truth of the RH, which will be a consequence of its selfadjointness. If $\zeta(1/2 + iE)$, or $\xi(1/2 + iE)$, where a polynomial in E, this encoding could be realized by a finite dimensional matrix H whose characteristic polynomial were proportional to $\zeta(1/2 + iE)$. The Euler product formula of the zeta function, in terms of the prime numbers, implies that H must also know about these numbers. Hence, the relation found by Riemann, between prime numbers and zeros of the zeta function, will be justified from the common dynamical origin of these quantities. The precise mathematical formulation of these relations is given by the, so called, trace formulas in Number Theory and Quantum Chaos [7]. The simplicity of the definition of the zeta function, as the series $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ (Re s > 1), has lead some researches, as Berry, to suggest that the Hamiltonian H may have a subtle but simple definition, allowing the observation of the Riemann zeros as spectral lines in an experimental set up [8]. The existence of such a "Riemann calculator" would place Number Theory in the realm of Quantum Mechanics with far reaching consequences. We shall briefly summarize below some phenomenologial and heuristic hints that support the spectral approach to the RH.

2. Selberg's trace formula (1956)

Consider a compact Riemann surface with negative curvature. This surface can be constructed as the complex upper plane divided by a discrete subgroup of the modular group $PSL(2, \mathbf{R})$, and it is equipped with the Poincaré metric. A classical problem is to determine the lengths ℓ_p of the primitive periodic orbits (p.p.o.), that is, the geodesics on this surface. A quantum problem is to find the spectrum of the Laplace-Beltrami operator $\Delta = y^2(\partial_x^2 + \partial_y^2)$,

$$-\Delta \psi_n(x,y) = E_n \,\psi_n(x,y), \qquad E_n = \frac{1}{4} + k_n^2. \tag{3}$$

Selberg's trace formula establishes a relation between the momenta k_n and the length of the geodesics ℓ_p [9]

$$\sum_{n} h(k_n) = \frac{\mu(D)}{4\pi} \int_{-\infty}^{\infty} dk \ k \ h(k) \tanh(\pi k) + \sum_{\text{p.p.o.}} \ell_p \sum_{n=1}^{\infty} \frac{g(n\ell_p)}{2\sinh(n\ell_p/2)}$$
(4)

where h(k) is a test function, g(k) its the Fourier transform and $\mu(D)$ is the area of the fundamental domain D describing the Riemann surface. Selberg also defined a zeta function in terms of the lenghts ℓ_p as A physics pathway to the Riemann hypothesis

$$Z(s) = \prod_{\text{p.p.o.}} \prod_{m=0}^{\infty} (1 - e^{-\ell_p(s+m)})$$
(5)

in close analogy to the Euler's product formula of the zeta function [4]:

$$\zeta(s) = \prod_{p} \frac{1}{1 - p^{-s}}, \quad \text{Re } s > 1$$
 (6)

where the product is over all the prime numbers p. Selberg zeta function satisfies a RH which can be proved. The trivial zeros of Z(s) are $s_n = -n$ (n = 0, 1, ...), and the non trivial ones are $s_n = \frac{1}{2} + ik_n$. Since k_n are real numbers, any complex zero of Z(s) lies on the line Re s = 1/2. The functions $\zeta(s)$ and Z(s) both satisfy functional equations that relate their values to $\zeta(1 - s)$ and Z(1 - s), respectively. Finally, Selberg's trace formula is reminiscent to the Riemann-Weil explicit formula relating the prime numbers p and the imaginary part of the Riemann zeros γ_n [10]:

$$\sum_{n} h(\gamma_{n}) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} h(k) \frac{\Gamma'}{\Gamma} \left(\frac{1}{4} \frac{ik}{2}\right) + h(\frac{i}{2}) + h(-\frac{i}{2}) - \log \pi \ g(0) - 2 \sum_{p} \log p \sum_{n=1}^{\infty} p^{-n/2} g(n \log p)$$
(7)

where the notations are as in eq.(4). A comparison between eqs.(4) and (7) suggests that prime numbers and primitive geodesics are in one-to-one correspondence, such that: $\ell_p \leftrightarrow \log p$. This correspondence also underlines the Quantum Chaos approach to the RH reviewed later on. However, the analogy between these two formulas fails in two respects: i) the term $1/(2\sinh(n\ell_p/2))$, with the identification $\ell_p = \log p$, only converges to $p^{-n/2}$ for large values of p, and ii) the factor of -2 in the last term of eq.(7) as compared to the factor one, in the last term of eq.(4). The difference in signs of the two terms finds an explanation in Connes spectral realization of Riemann zeros (see below). Finally, we observe that the imaginary part of the Riemann zeros, γ_n , seem to correspond to the momenta k_n rather than to the eigenenergies E_n . This suggest that the Riemann Hamiltonian is probably related to a first order linear operator, as we shall see in the discussion of the H = xp Hamiltonian.

3. Random matrix theory and Quantum Chaos (70's-80' s)

In 1973 Montgomery, asuming the RH, proved that the imaginary part of the Riemann zeros, γ_n , where distributed at random, according the gaussian unitary ensemble distribution (GUE) of Random Matrix Theory (RMT) [11]. This result found strong numerical confirmation by Odlyzko in the 80' s, who computed trillions of Riemann zeros (near the 10^{12} -th zero and near the 10^{20} - th zero) [12]. These phenomenological findings means that the statistical properties of the Riemann zeros are similar to those of the eigenvalues of large hermitean matrices, in particular the property of level repulsion. There are three universality classes of random matrices corresponding to orthogonal (GOE), hermitean (GUE) and symplectic matrices (GSE). The GUE statistics corresponds to random systems where time reversal is broken, which gives a strong indication that the Riemann Hamiltonian H must break this symmetry.

A further step along this direction was taken by Berry, who noticed a formal analogy between the fluctuations of the Riemann zeros and the fluctuations of the energy levels of quantum chaotic system around their average values [8]. The latter fluctuations are given by the semiclassical Gutwiller formula,

$$N_{QC,fl}(E) = \frac{1}{\pi} \sum_{\gamma} \sum_{m=1}^{\infty} \frac{\sin(mET_{\gamma})}{2m\sinh(m\lambda_{\gamma}/2)}$$
(8)

where E is an eigenenergy, γ is a primitive periodic orbit, T_{γ} its period and λ_{γ} its Lyapunov exponent. The sum over m corresponds to the repetitions of the primitive orbits. The fluctuation part of the Riemann zeros is given by

$$N_{R,\rm fl}(E) = -\frac{1}{\pi} \sum_{p} \sum_{m=1}^{\infty} \frac{\sin(mE \log p)}{mp^{m/2}}$$
(9)

where the sum is over the prime numbers p. Comparing (8) and (9), Berry conjectured the existence of a classical chaotic Hamiltonian whose primitive periodic orbits, γ , would be labelled by the prime numbers p, with periods $T_p = \log p$, and instability exponents $\lambda_p = \pm \log p$. Moreover, since each orbit is counted once, the Hamiltonian must break time reversal (otherwise there would be a factor $2/\pi$, in front of eq. (9) instead of $1/\pi$). This analogy is reminiscent to the one existing between Selberg trace formula and the Riemann-Weil formula, and also suffers from a "sign problem" and "asymptotic problem" as observed before. The connection with Quantum Chaos also explained some numerical discrepancies found by Odlyzko between RMT and the statistics of zeros for long range spectral correlations, which are due to the shortest periodic orbits, where universality no longer holds. They were explained by Berry , Keating and Bogomolny [13, 14]. All these results put on a more firm basis the Polya-Hilbert conjecture giving further clues on the structure of the dynamical system behind the Riemann zeros.

4. The Hamiltonian H = xp (1999)

In 1999 Berry and Keating, and Connes suggested that the Riemann zeros are related to the classical Hamiltonian $H_{cl} = xp$, where x and p are the position and momenta of a particle movin in 1D [15, 16]. The classical trayectories of this Hamiltonian are hyperbolas in the phase space

$$x(t) = x_0 e^t, \qquad p(t) = p_0 e^{-t}$$
 (10)

and therefore unbounded, which would then imply a continuous spectrum rather than a discrete spectrum associated to the Riemann zeros. The connection with the latter arises in two possible ways depending on two different regularizations of the phase space. Berry and Keating introduced a minimal length ℓ_x and a minimal momenta ℓ_p , whose product is the Planck quantum $\ell_x \ell_p = 2\pi\hbar$ [15]. In terms of these quantities, they imposed $|x| \ge \ell_x$ and $|p| \ge \ell_p$, so that the trayectories are now bounded. The semiclassical number of states is given by the area below the hyperbola and above the boundaries $|x| = \ell_x$ and $|p| = \ell_p$, and it is given, in units $\hbar = 1$, by

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$$n_{\rm BK}(E) = \frac{A_{\rm BK}}{2\pi} = \frac{E}{2\pi} \left(\log \frac{E}{2\pi} - 1 \right) + 1 \tag{11}$$

Rather surprisingly, this result coincides, asymptotically, with the average number of Riemann zeros up to a height *E* in the critical strip (i.e. $0 < \Re s < 1, 0 < \Im s < E$). The constant in Riemann's formula is actually 7/8, which can be obtained by taking into account a Maslow phase contributing -1/8 to eq. (11), due to the fact that the particle only travels one quadrant of the phase space. Unfortunately, eq.(11) has remained so far heuristic since it is not supported by a quantum mechanical model (see later).

Connes regularization is based on the restrictions $|x| \le \Lambda$ and $|p| \le \Lambda$, where Λ is a cutoff, which is taken to infinity at the end of the calculation [16]. The semiclassical number of states is computed as before yielding,

$$n_{\rm Co}(E) = \frac{A_{\rm Co}}{2\pi} = \frac{E}{2\pi} \log \frac{\Lambda^2}{2\pi} - \frac{E}{2\pi} \left(\log \frac{E}{2\pi} - 1 \right)$$
(12)

The first term on the RHS of this formula diverges in the limit $\Lambda \rightarrow \infty$, which corresponds to a continuum of states. The second term is minus the average number of Riemann zeros, which according to Connes, become missing spectral lines in the continuum. This is the, so called, "absortion" spectral interpretation of the Riemann zeros, as opposed to the standard "emission" spectral interpretation where they form a discrete spectrum. The minus sign in eq.(12) could also be related to the minus sign in the trace formulas discussed earlier. Connes interpretation has however two drawbacks. First of all, the average number of Riemann zeros is not fully obtained in eq(12). The term $E/2\pi \log 1/2\pi$, actually cancells between the first and second summands in this formula. Other objection is that the second term in (12) is simply a finite size correction of discrete energy levels, where no lines are missing, and the same remains true in the continuum limit.

The H = xp model was modified in references [17], adding a non local interaction suggested by a relation of this model to a BCS model of superconductivity with a cyclic renormalization group. The spectrum of this interacting xp-model is a continuum where the Riemann zeros are embedded as bound states. This result reconciles the Berry-Keating and Connes spectral interpretations. However, the non locality of the interaction implies that the Hamiltonian has no classical limit, and consequently its relation to classical chaotic dynamical systems remains unclear. Moreover the prime numbers do not appear in this construction, which as we saw in previous sections, is an important ingredient of the trace formulas.

A different route to H = xp was suggested in reference [18] which will bring us to more familiar territores in Physics.

Landau levels and Riemann zeros (2008)

Let us consider a charged particle moving in a plane under the action of a perpendicular magnetic field and an electrostatic potential with a saddle shape [18]. The Langrangian describing the dynamics is given, in the Landau gauge, by

$$\mathcal{L} = \frac{\mu}{2}(\dot{x}^2 + \dot{y}^2) - \frac{eB}{c}\dot{y}x - e\lambda xy$$
(13)

where μ is the mass, *e* the charge, *B* the magnetic field, *c* the speed of light and λ a coupling constant parameterizing the electrostatic potential. There are two normal modes with real, ω_c , and imaginary, ω_h , angular frequencies, describing cyclotronic and a hyperbolic motions respectively. In the limit where $\omega_c >> \omega_h$, only the Lowest Landau Level (LLL) is relevant and the effective Lagrangian becomes

$$\mathcal{L}_{\text{eff}} = p\dot{x} - |\omega_h| xp, \qquad p = \frac{\hbar y}{\ell^2}, \qquad \ell = \left(\frac{\hbar c}{eB}\right)^{1/2}, \qquad |\omega_h| \sim \frac{\lambda c}{B} \tag{14}$$

. . .

where ℓ is the magnetic length, which is proportional to the radius of the cyclotronic orbits in the LLL. The coordinates x and y, which commute in the 2D model, after the proyection to the LLL become canonical conjugate variables, and the effective Hamiltonian coincides with the xp Hamiltonian introduced by Berry, Keating and Connes, where the energy is measured in units of $\hbar |\omega_h|$. This realization of the xp Hamiltonian allow us to interpret the semiclassical quantization of these authors in the language of the Landau model. In particular, the semiclassical counting of states in the xp model follows from the counting of quantum fluxes in a certain area of the x - y plane. If the plane is infinite, then the number of states in the LLL will is also infinite. To have a finite number of states we put the particle into a box: |x| < L, |y| < L, which reproduces Connes regularizations conditions. The number of semiclassical states with an energy between 0 and E is given by

$$n_{\rm sm}(E) = \frac{E}{2\pi} \log \frac{L^2}{2\pi\ell^2} - \frac{E}{2\pi} \left(\log \frac{E}{2\pi} - 1 \right)$$
(15)

which agrees with Connes eq. (12). The classical energy is given by $E = xy/\ell^2$ (in units of $\hbar |\omega_h|$), and it attains its maximum value at $E_{\text{max}} = L^2/\ell^2$. Plugging this value into (15) yields $n_{\text{sm}}(E_{\text{max}}) = L^2/2\pi\ell^2$, which is the number of quantum fluxes in the first quadrant. This semiclassical results can be derived from the quantization of the model. Indeed, the energy levels follows from the identification of the wave function at the boundaries x = L and y = L (up to a phase)

$$\psi_E(x,L) = e^{ixL/\ell^2} \psi_E(L,x) \Longrightarrow \frac{\Gamma\left(\frac{1}{4} + \frac{iE}{2}\right)}{\Gamma\left(\frac{1}{4} - \frac{iE}{2}\right)} \left(\frac{L^2}{2\ell^2}\right)^{-iE} = 1$$
(16)

Taking the logarithm on the RHS of (16), one gets the smooth part of the Riemann formula, whose asymptotic expansion coincides with eq. (15). The 2D formulation of the xp model allows one to convert the hyperbolic orbits into periodic orbits by means of the boundary condition (16). It would be extremely interesting to derive Berry-Keating regularization in the Landau version of the model. To achieve this goal, one must inject the particle approaching the boundary at $y = \ell_y$, back to the boudary at $x = \ell_x$, so that the orbits become periodic. The Berry-Keating semiclassical arguments suggest that the spectrum will be discrete and associated to the smooth Riemann zeros. Preliminary results suggest that this possibility can indeed be realized. Of course the main problem that remains is the construction of the Hamiltonian giving rise to the exact Riemann zeros. It is not clear at the moment wether this can be done, but preliminary results suggest that the higher Landau levels may play a role [18]. In any case, the Landau model formulation of the H = xp Hamiltonian provides a promissing new avenue where to explore the fascinating problem of a physical interpretation of the Riemann zeros, and perhaps a physicist proof of the RH.

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References

- [1] B. Riemann, "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse," Monat. der Königl. Preuss. Akad. der Wissen. zu Berlin aus der Jahre 1859, 671 (1860).
- [2] E. Bombieri, "Problems of the millennium: The Riemann hypothesis," Clay Mathematics Institute (2000), http://www.claymath.org/millennium/Riemann Hypothesis/Official Problem Description.pdf.
- [3] B. Conrey, "The Riemann hypothesis," AMS Notices, 341 (2003).
- [4] H. M. Edwards, Riemann's zeta function, New York: Academic Press (1974).
- [5] E. C. Titchmarsh, The theory of the Riemann zeta function, Oxford: Clarendon Press (1986).
- [6] M. R. Watkins, http://www.maths.ex.ac.uk/?mwatkins/zeta/physics.htm.
- [7] "Frontiers in Number Theory, Physics, and Geometry I On Random Matrices, Zeta Functions, and Dynamical Systems", Eds: P. Cartier, B. Julia, P. Moussa, P. Vanhove. Springer Verlag, Berlin, 2006.
- [8] Berry M V 1986 Riemann's zeta function: a model for quantum chaos? Quantum Chaos and Statistical Nuclear Physics ed T H Seligman and H Nishioka vol 263, 1-17
- [9] A. Selberg, "Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series," J. Indian Math. Soc. 20, 47 (1956). See also D. Hejhal, "The Selberg Trace Formula for PSL(2, R)", Vol. I, Springer Lecture Notes 548 (1976).
- [10] A. Weil, "Sur les formules explicites de la théorie des nombres premiers," Meddelanden Fran Lunds Univ. Mat. Sem., 252 (1952);
- [11] H. L. Montgomery, "The pair correlation of zeros of the zeta function," in Analytic Number Theory, vol. 24 of AMS Proceedings of Symposia in Pure Mathematics, 181 (1973).
- [12] A. M. Odlyzko, "On the distribution of spacings between zeros of the zeta function," Math. Computation 48, 273 (1987).
- [13] M.V. Berry, "Semiclassical formula for the number variance of the Riemann zeros", Nonlinearity 1 399-407 (1988).

- [14] E. B. Bogomolny and J. P. Keating "Gutzwiller' s trace formula and spectral statistics: beyond the diagonal approximation", Phys. Rev. Lett. 77 1472-5 (1996).
- [15] M. V. Berry and J. Keating, "H = xp and the Riemann zeros" in Supersymmetry and trace formulae: chaos and disorder, I. V. Lerner and J. P. Keating, eds., New York: Plenum (1999). "The Riemann zeros and eigenvalue asymptotics," SIAM Review 41, 236 (1999) and references therein.
- [16] A. Connes, "Trace formula in noncommutative geometry and the zeros of the Riemann zeta function," Sel. Math., New Ser. 5, 29 (1999) [arXiv:math/9811068]. For more details consult the recent book A. Connes and M. Marcolli, Noncommutative geometry, quantum Pelds, and motives, AMS (2008).
- [17] G. Sierra, "H = x p with interaction and the Riemann zeros," Nucl. Phys. B 776, 327 (2007) [arXiv:math-ph/0702034]; "The Riemann zeros and the cyclic Renormalization Group," J. Stat. Mech. 0512, P006 (2005) [arXiv:math.nt/0510572]; "A quantum mechanical model of the Riemann zeros," New J. Phys. 10, 033016 (2008) [arXiv:0712.0705 [math-ph]];
- [18] G. Sierra and P. K. Townsend, "Landau levels and Riemann zeros," Phys. Rev. Lett. 101, 110201 (2008) [arXiv:0805.4079 [math-ph]].