Quantum cohomology of the odd symplectic Grassmannian of lines IG $(2, 2n+1)$

Clélia Pech
clelia.pech@ujf-grenoble.fr

Abstract

Odd symplectic Grassmannians are a generalization of symplectic Grassmannians to odd-dimensional spaces. Here we compute the classical and quantum cohomology of the odd symplectic Grassmannian of lines. Although these varieties are non homogeneous, we obtain Pieri and Giambelli formulas that are very similar to the symplectic case. We notice that their quantum cohomology is semi-simple, which enables us to check Dubrovin's conjecture for this case.

Introduction

The quantum cohomology of homogeneous varieties has been extensively studied (see [13] for references). Other well-known examples are toric varieties, yet apart from these settings, there are only few examples where the quantum cohomology has been explicitly determined. Quasi-homogeneous varieties provide interesting non toric and non homogeneous examples. Among these two Hilbert schemes have been studied, $\mathrm{Hilb}(2, \mathbb{P}^2)$ [6] and $\mathrm{Hilb}(2, \mathbb{P}^1 \times \mathbb{P}^1)$ [11].

In [10] Mihai studied a family of varieties, the odd symplectic flag manifolds, which have many features in common with the symplectic flag manifolds. These varieties are interesting at least for two reasons ; first, they are quasi-homogeneous, and secondly, since they have an action of the algebraic group Sp_{2n+1} (the odd symplectic group), whose properties are closely related to those of Sp_{2n}, they are expected to behave almost like homogeneous spaces and thus be relatively easy to deal with. The classical and quantum cohomology of symplectic Grassmannians has been described in [2] and [1], so one can ask whether it is possible to obtain similar results in the case of odd symplectic Grassmannians.

Here we deal with the case of the odd symplectic Grassmannian of lines IG $(2, 2n+1)$, although some of the results about the classical cohomology hold in a more general setting. In 1.2 and 1.6 we use the natural embeddings of IG $(2, 2n+1)$ in the usual Grassmannian and in the symplectic Grassmannian to compute classical Pieri (see 1.4) and Giambelli (see 1.7) formulas, as well as a presentation of the cohomology ring (see 1.8).

For the quantum cohomology the situation is more complicated. Since these varieties are not convex it is necessary to study carefully the moduli spaces corresponding to 2- and 3-pointed invariants of degree 1 to show that they are unobstructed. This is done in 2.1. Another difficulty is that since the group action is not transitive, an important transversality result, Kleiman's lemma [7] no longer holds. So it will not be possible to force two Schubert varieties to meet transversely by an adequate choice of the defining flags as was done for instance in [4]. Hence the Gromov-Witten invariants associated to Schubert varieties are not always enumerative. To solve this problem we replace Schubert varieties by another family of subvarieties and we use a transversality result of Graber [6] suited for quasi-homogeneous spaces. Finally in 2.5 we obtain a quantum Pieri formula and a presentation of the quantum cohomology ring.

Our results show that there is a lot of similitude with the symplectic case, since the classical and quantum Pieri formulas are almost the same in both cases. The Hasse diagrams are closely related as well (see 1.5). However, Poincaré duality is very different, since the Poincaré dual of a Schubert class is no longer always a single Schubert class (see 1.3). Finally, as an application of the quantum presentation, we show in 2.6 that contrary to the symplectic case (see [3]), the small quantum cohomology ring of the odd symplectic Grassmannian of lines is semi-simple, hence the same result holds for the big quantum cohomology ring. Since it is possible to find exceptional collections for these varieties, we deduce that Dubrovin's conjecture [5] holds.

I wish to thank Laurent Manivel for his help on this subject.

1 Classical cohomology

Let V be a $\mathbb{C}$-vector space of dimension $2n+1$ ($n \leq 2$) and ω be an antisymmetric form of maximal rank on V. We denote its kernel by K. The odd symplectic Grassmannian is

$$\mathrm{IG}_\omega(m,V) := \{\Sigma \in \mathrm{G}(m,V) \mid \Sigma \text{ is isotropic for } \omega\}.$$

It has an action of the odd symplectic group :

$$\mathrm{Sp}(V) := \{g \in \mathrm{GL}(V) \mid \forall u,v \in V \ \omega(gu,gv) = \omega(u,v)\}.$$

Up to isomorphism, $\mathrm{IG}_\omega(m,V)$ does not depend on the $2n+1$-dimensional vector space V nor on the form ω, so we may denote it by $\mathrm{IG}(m,2n+1)$. Similarly, from now on we denote $\mathrm{Sp}(V)$ by Sp_{2n+1}. We recall some basic facts from [10] :

Proposition 1. *1. The odd symplectic Grassmannian* $\mathrm{IG}(m,2n+1)$ *is a smooth subvariety of dimension* $m(2n+1-m) - \frac{m(m-1)}{2}$ *of the usual Grassmannian* $\mathrm{G}(m,2n+1)$.

2. If $1 \leq m \leq n$, $\mathrm{IG}(m,2n+1)$ *has two orbits under the action of the odd symplectic group* Sp_{2n+1} *:*

- *the closed orbit* $\mathbb{O} := \{\Sigma \in \mathrm{IG}(m,2n+1) \mid \Sigma \supset K\}$*, which is isomorphic to the symplectic Grassmannian* $\mathrm{IG}(m-1,2n)$ *;*
- *the open orbit* $\{\Sigma \in \mathrm{IG}(m,2n+1) \mid \Sigma \not\supset K\}$*, which is isomorphic to the dual of the tautological bundle over the symplectic Grassmannian* $\mathrm{IG}(m,2n)$.

For us a quasi-homogeneous space will be an algebraic variety endowed with an action of an algebraic group with only finitely many orbits. Odd symplectic Grassmannians with $m \leq n$ are examples of such spaces. In the sequel we will always assume $m \leq n$.

1.1 Schubert varieties

A $\mathbb{C}$-vector space V of dimension $2n+1$ endowed with an antisymmetric form of maximal rank ω can be embedded in a symplectic space $(\overline{V},\overline{\omega})$ of dimension $2n+2$ such that $\overline{\omega}\mid_V = \omega$. This construction gives rise to a natural embedding $\mathbf{i} : \mathrm{IG}(m,2n+1) \hookrightarrow \mathrm{IG}(m,2n+2)$. Mihai proved in [10] that $\mathbf{i}$ identifies $\mathrm{IG}(m,2n+1)$ with a Schubert subvariety of $\mathrm{IG}(m,2n+2)$. Moreover he showed how to use this embedding to obtain a description of the Schubert subvarieties of $\mathrm{IG}(m,2n+1)$. In 1.1.1 we recall some facts about Schubert varieties in $\mathrm{IG}(m,2n)$, then in 1.1.2 we explain Mihai's description for Schubert varieties in $\mathrm{IG}(m,2n+1)$ and introduce another one using partitions.

1.1.1 Schubert varieties in the symplectic Grassmannian

Here we recall the indexing conventions introduced in [2]. Two kinds of combinatorial objects can be used to index Schubert varieties of the symplectic Grassmannian $\mathrm{IG}(m,2n)$, k-strict partitions (with $k := n-m$) and index sets :

Definition 1. *1. A k-strict partition is a weakly decreasing sequence of integers* $\lambda = (\lambda_1 \geq \cdots \geq \lambda_m \geq 0)$ *such that* $\lambda_j > k \Rightarrow \lambda_j > \lambda_{j+1}$.

2. An index set of length m for the symplectic Grassmannian is a subset $P \subset [1,2n]$ *with m elements such that for all* $i,j \in P$ *we have* $i+j \neq 2n+1$.

Now if $F_\bullet$ is an isotropic flag (i.e a complete flag such that $F_{n-i}^\perp = F_{n+i}$ for all $0 \le i \le n$), to each admissible index set $P = (p_1, \dots, p_m)$ of length m we can associate the Schubert cell

$$X_P^\circ(F_\bullet) := \left\{\Sigma \in \mathrm{IG}(m, 2n) \mid \dim(\Sigma \cap F_{p_j}) = j,\ \forall\, 1 \le j \le m\right\}.$$

Moreover there is a bijection between $(n-m)$-strict partitions λ such that $\lambda_1 \le 2n-m$ and index sets $P \subset [1, 2n]$ of length m, given by

$$\lambda \mapsto P = (p_1, \dots, p_m) \text{ where } p_j = n+k+1-\lambda_j + \#\{i<j \mid \lambda_i + \lambda_j \le 2k+j-i\}$$
$$P \mapsto \lambda = (\lambda_1, \dots, \lambda_m) \text{ where } \lambda_j = n+k+1-p_j + \#\{i<j \mid p_i + p_j > 2n+1\}.$$

The advantage of the representation by k-strict partitions is twofold : it mimics the indexation of Schubert classes of type A Grassmannians by partitions, and the codimension of the Schubert variety associated to a k-strict partition λ is easily computed as $|\lambda| = \sum_{j=1}^m \lambda_j$. In the next paragraph we will describe a similar indexation for the odd symplectic Grassmannian.

1.1.2 Schubert varieties in the odd symplectic Grassmannian

In the odd symplectic Grassmannian $\mathrm{IG}(m, 2n+1)$ we define index sets of length m as m-uples $P = (p_1 \le \dots \le p_m)$ with $1 \le p_j \le 2n+1$ for all j and $p_i + p_j \ne 2n+3$ for all i, j.

Proposition 2 ([10]). *The embedding* $\mathbf{i} : \mathrm{IG}(m, 2n+1) \to \mathrm{IG}(m, 2n+2)$ *identifies* $\mathrm{IG}(m, 2n+1)$ *with the Schubert subvariety of* $\mathrm{IG}(m, 2n+2)$ *associated to the* $(n+1-m)$*-strict partition* λ^0 *such that* $\lambda_1^0 = \dots = \lambda_m^0 = 1$ *(or, equivalently, to the index set* $P^0 = (2n+2-m, \dots, 2n+1)$*).*

Schubert subvarieties of $\mathrm{IG}(m, 2n+1)$ are defined with respect to an isotropic flag $F_\bullet$ in $\mathbb{C}^{2n+1}$, i.e a complete flag which is the restriction of an isotropic flag in $\mathbb{C}^{2n+2}$. Proposition 2 implies that the Schubert varieties in $\mathrm{IG}(m, 2n+1)$ can be indexed by index sets P such that $P \le P^0$ (for the lexicographical order). Now if P is such an index set, we associate to it a $(n-m)$-strict m-uple of weakly decreasing integers $\lambda = (\lambda_1 \ge \dots \ge \lambda_m \ge -1)$ defined by :

$$\lambda_j = 2n+2-m-p_j + \#\{i<j \mid p_i + p_j > 2n+3\} \text{ for all } 1 \le j \le m.$$

Conversely if $\lambda = (\lambda_1 \ge \dots \ge \lambda_m \ge -1)$ is any $(n-m)$-strict m-uple of weakly decreasing integers such that $\lambda_1 \le 2n+1-m$, then the assignement

$$p_j = 2n+2-m-\lambda_j + \#\{i<j \mid \lambda_i + \lambda_j \le 2(n-m)+j-i\} \text{ for all } 1 \le j \le m$$

defines an index set of $[1, 2n+1]$. It is easy to check that with respect to this indexation convention, the Schubert variety $X_\lambda(F_\bullet)$ has codimension $|\lambda|$ in $\mathrm{IG}(m, 2n+1)$.

Remark 1. For the case of the odd symplectic Grassmannian of lines $\mathrm{IG}(2, 2n+1)$, it follows that the indexing partitions can be either

- "usual" $(n-2)$-strict partitions $\lambda = (2n-1 \ge \lambda_1 \ge \lambda_2 \ge 0)$;
- the "partition" $\lambda = (2n-1, -1)$ corresponding to the class of the closed orbit $\mathbb{O}$.

1.2 Embedding in the symplectic Grassmannian

Now we draw some consequences of the embedding of $\mathrm{IG}(2, 2n+1)$ as a Schubert subvariety of a symplectic Grassmannian. Since we know the cohomology of $\mathrm{IG}(2, 2n+2)$, the knowledge of the restriction map $\mathbf{i}^*$ will give us information on the cohomology of $\mathrm{IG}(2, 2n+1)$. Let $F_\bullet$ be an isotropic flag, $Y_{a,b}(F_\bullet)$ a Schubert subvariety of $\mathrm{IG}(2, 2n+2)$ and $\upsilon_{a,b}$ the associated Schubert class, where (a, b) is an $(n-2)$-strict partition. From Proposition 2 we know that $\mathrm{IG}(2, 2n+1)$ is isomorphic to the Schubert subvariety $Y_{1,1}(E_\bullet)$ of $\mathrm{IG}(2, 2n+2)$, where $E_\bullet$ is an isotropic flag which we may assume to be in general position with respect to $F_\bullet$. Then it follows that $Y_{a,b}(F_\bullet)$ and $Y_{1,1}(E_\bullet)$ meet transversally, hence we can compute the restriction $\mathbf{i}^*\upsilon_{a,b}$ by computing the class of the intersection $Y_{a,b} \cap Y_{1,1}$ in $\mathrm{IG}(2, 2n+2)$ using the classical Pieri rules for $\mathrm{IG}(2, 2n+2)$ (see [2]):

$$\upsilon_{a,b} \cdot \upsilon_{1,1} = \begin{cases} \upsilon_{a+1,b+1} & \text{if } a+b \ne 2n-2, 2n-1, \\ \upsilon_{a+1,b+1} + \upsilon_{a+2,b} & \text{if } a+b = 2n-2 \text{ or } 2n-1. \end{cases}$$

Remark 2. In the above formula, we should remove classes indexed by partitions which do not make sense, i.e that are not indexing partitions for the corresponding Grassmannian. For instance, we should remove classes corresponding to partitions that are not k-strict for the suitable value of k, have first part too big... We will adopt this convention throughout the rest of the text.

Now, remembering the identification of $\mathrm{IG}(2, 2n+1)$ with $\upsilon_{1,1}$, express the pushforward $\mathbf{i}_*\tau_{c,d}$ of a class $\tau_{c,d}$ in $\mathrm{IG}(2, 2n+1)$:

$$\mathbf{i}_*\tau_{c,d} = \upsilon_{c+1,d+1}.$$

Using the projection formula $\mathbf{i}_*(\alpha \cdot \mathbf{i}^*\beta) = \mathbf{i}_*\alpha \cdot \beta$, we obtain :

Lemma 1.

$$i^*\upsilon_{a,b} = \begin{cases} \tau_{a,b} & \text{if } a+b \neq 2n-2, 2n-1, \\ \tau_{a,b} + \tau_{a+1,b-1} & \text{if } a+b = 2n-2 \text{ or } 2n-1. \end{cases}$$

In particular we notice that $\mathbf{i}^*$ is surjective and has kernel generated by the class υ_{2n}. So the classical cohomology of $\mathrm{IG}(2, 2n+1)$ is entirely determined by the classical cohomology of $\mathrm{IG}(2, 2n+2)$.

Remark 3. The surjectivity of the restriction map $\mathbf{i}^*$ remains true for any odd symplectic Grassmannian $\mathrm{IG}(m, 2n+1)$. However, combinatorics is much more intricate than in the case $m = 2$.

1.3 Poincaré duality

Denote by $\widetilde{\alpha}$ the Poincaré dual of the cohomology class α (be it in $\mathrm{IG}(2, 2n+1)$ or in $\mathrm{IG}(2, 2n+2)$). Poincaré duality in $\mathrm{IG}(2, 2n+1)$ takes the form :

Proposition 3 (Poincaré duality)**.**

$$\widetilde{\tau_{a,b}} = \begin{cases} \tau_{2n-1-b,2n-2-a} & \text{if } a+b < 2n-2, \\ \tau_{2n-2-b,2n-1-a} + \tau_{2n-1-b,2n-2-a} & \text{if } a+b = 2n-2 \text{ or } 2n-1, \\ \tau_{2n-2-b,2n-1-a} & \text{if } a+b > 2n-1. \end{cases}$$

Proof. We will derive this result from Poincaré duality on $\mathrm{IG}(2, 2n+2)$ using lemma 1. But first we state

Lemma 2. *Let α be a cohomology class in $\mathrm{IG}(2, 2n+2)$. Then $\mathbf{i}^*\widetilde{\alpha} = \widetilde{\alpha_-}$, where we denote by α_- the class in $\mathrm{IG}(2, 2n+1)$ such that $\mathbf{i}_*(\alpha_-) = \alpha$. Notice that if α is a Schubert class, α_- only exists when $\alpha = \upsilon_{a,b}$ with $b \geq 1$ or $(a, b) = (2n, 0)$.*

Proof of the lemma. By definition of Poincaré duality, if α and β are two cohomology classes in $\mathrm{IG}(2, 2n+2)$, then

$$\int_{\mathrm{IG}(2,2n+2)} \alpha \cdot \widetilde{\beta} = \delta_{\alpha,\beta}$$

where δ is the Kronecker symbol. So

$$\int_{\mathrm{IG}(2,2n+2)} (\mathbf{i}_*\alpha_-) \cdot \widetilde{\beta} = \delta_{\alpha,\beta} = \int_{\mathrm{IG}(2,2n+2)} \mathbf{i}_*(\alpha_- \cdot \mathbf{i}^*\widetilde{\beta}) = \delta_{\alpha,\beta}. \tag{1}$$

Now express $\mathbf{i}^*\widetilde{\beta}$ on the dual base in $\mathrm{IG}(2, 2n+1)$: $\mathbf{i}^*\widetilde{\beta} = \sum_\gamma x_{\beta,\gamma}\widetilde{\gamma}$. We get

$$\delta_{\alpha,\beta} = \sum_\gamma x_{\beta,\gamma} \int_{\mathrm{IG}(2,2n+2)} \mathbf{i}_*(\alpha_- \cdot \widetilde{\gamma}) = \sum_\gamma x_{\beta,\gamma}\delta_{\alpha_-,\gamma}.$$

So $x_{\beta,\alpha_-} = \delta_{\alpha,\beta}$, and the result follows. □

To conclude we prove with the projection formula that if α is a class in $\mathrm{IG}(2, 2n+2)$, then $\widetilde{\alpha_-} = (\widetilde{\alpha} \cdot \upsilon_{1,1})_-$. Then using the Poincaré duality formula in $\mathrm{IG}(2, 2n+2)$ proved in [2], an easy calculation gives the result. □

Remark 4. This result is very different from what we get for the usual Grassmannians or even the symplectic or orthogonal ones. Indeed, the dual of a Schubert class is not necessarily a Schubert class ! This fact will have many consequences ; in particular, the Hasse diagram of $\mathrm{IG}\,(2,2n+1)$ (see figure 2) will be much less symmetric that the Hasse diagram of, say, $\mathrm{IG}\,(2,2n+2)$ (see figure 1).

1.4 Pieri formula

To compute the cup product of two cohomology classes in $\mathrm{IG}\,(2,2n+1)$, we need two ingredients : a *Pieri formula* describing the cup product of any Schubert class with a special class (that is, one of the classes τ_1 or $\tau_{1,1}$), and a *Giambelli formula* decomposing any Schubert class as a polynomial in τ_1 and $\tau_{1,1}$. In this paragraph we describe the Pieri formula as well as an alternative rule for multiplying Schubert classes by classes of the form τ_p with $0 \leq p \leq 2n-1$ or $\tau_{2n-1,-1}$.

We start by expressing cohomology classes in $\mathrm{IG}\,(2,2n+1)$ in terms of cohomology classes in $\mathrm{IG}(2,2n+2)$ using lemma 1 :

$$\tau_{c,d} = \begin{cases} \mathbf{i}^*\upsilon_{c,d} & \text{if } c+d \neq 2n-2, 2n-1, \\ \sum_{j=0}^{c-n} (-1)^{c-n-j}\mathbf{i}^*\upsilon_{n-1+j,n-1-j} & \text{if } c+d = 2n-2, \\ \sum_{j=c-n}^{n-1} (-1)^{j-c+n}\mathbf{i}^*\upsilon_{n+j,n-1-j} & \text{if } c+d = 2n-1. \end{cases}$$

Now combining this with the Pieri rule in $\mathrm{IG}(2,2n+2)$, we can prove a Pieri rule for $\mathrm{IG}\,(2,2n+1)$:

Proposition 4 (Pieri formula)**.**

$$\tau_{a,b}\cdot\tau_1 = \begin{cases} \tau_{a+1,b} + \tau_{a,b+1} & \textit{if } a+b \neq 2n-3, \\ \tau_{a,b+1} + 2\tau_{a+1,b} + \tau_{a+2,b-1} & \textit{if } a+b = 2n-3. \end{cases}$$

$$\tau_{a,b}\cdot\tau_{1,1} = \begin{cases} \tau_{a+1,b+1} & \textit{if } a+b \neq 2n-4, 2n-3, \\ \tau_{a+1,b+1} + \tau_{a+2,b} & \textit{if } a+b = 2n-4 \textit{ or } 2n-3. \end{cases}$$

We may also state a rule for multiplying by the Chern classes of the quotient bundle

$$c_p(\mathcal{Q}) = \begin{cases} \tau_p & \text{if } 0 \leq p \leq 2n-1 \text{ and } p \neq 2n-2 \\ \tau_{2n-2} + \tau_{2n-1,-1} & \text{if } p = 2n-2. \end{cases}$$

We prove it the same way as Proposition 4 :

Proposition 5 (another Pieri formula)**.**

$$\tau_{a,b}\cdot\tau_p = \begin{cases} (\upsilon_{a+1,b+1}\cdot\upsilon_p)_- & \textit{if } p \neq 2n-2 \textit{ or } (a+b \neq 2n-1 \textit{ and } (a,b) \neq (2n-1,-1)), \\ (-1)^a\tau_{2n-1,2n-2} & \textit{if } p = 2n-2,\ a+b = 2n-1 \textit{ and } b \neq 0, \\ 0 & \textit{if } p = 2n-2 \textit{ and } ((a,b) = (2n-1,-1) \textit{ or } (2n-1,0)). \end{cases}$$

$$\tau_{a,b}\cdot\tau_{2n-1,-1} = \begin{cases} (-1)^{a-1}\tau_{2n-1,2n-2} & \textit{if } a+b = 2n-1, \\ \tau_{2n-1,a-1} & \textit{if } b = 0 \textit{ and } a \neq 2n-2, \\ \tau_{2n-1,2n-3} & \textit{if } (a,b) = (2n-1,-1), \\ 0 & \textit{else.} \end{cases}$$

Notice that contrary to the symplectic case (and to the case of other homogeneous spaces) we sometimes get negative coefficients for the second Pieri rule. It is a consequence of the fact that we only have a quasi-homogeneous space, so it is not always possible to find representatives of the two Schubert varieties that intersect transversally. So even in degree 0 Gromov-Witten invariants associated to Schubert classes are not always enumerative, contrary to the case of homogeneous spaces. That is why we will have to outline conditions in 2.2 to recover enumerativity for some invariants.

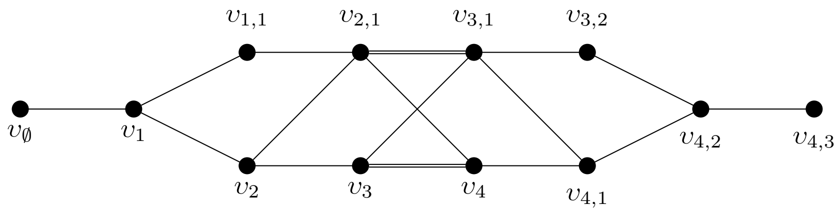

Figure 1: Hasse diagram of $\mathrm{IG}(2,6)$

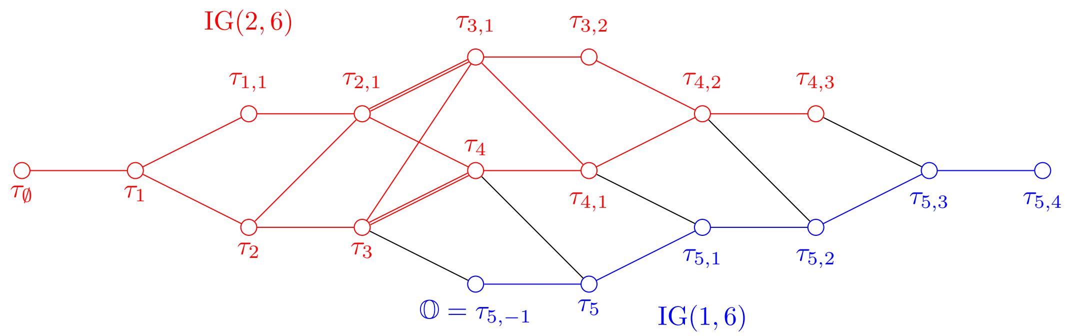

Figure 2: Hasse diagram of $\mathrm{IG}(2,7)$

1.5 The Hasse diagram of $\mathrm{IG}\,(2, 2n+1)$

The Pieri rule from Proposition 4 enables us in particular to compute the multiplication by the hyperplane class τ_1. The corresponding graph is called the Hasse diagram of $\mathrm{IG}\,(2, 2n+1)$. For instance see figure 2 for the Hasse diagram of $\mathrm{IG}(2,7)$. As a comparison, see also the Hasse diagram of the symplectic Grassmannian $\mathrm{IG}(2,6)$ in figure 1, and of $\mathrm{IG}(2,8)$ in figure 3.

Looking at these examples we notice that the Hasse diagram of $\mathrm{IG}(2,7)$ contains the Hasse diagram of $\mathrm{IG}(2,6)$ as a subgraph, the subgraph induced by the remaining vertices being isomorphic to the Hasse diagram of $\mathrm{IG}(1,6)$. Moreover, the Hasse diagram of $\mathrm{IG}(2,8)$ contains the Hasse diagram of $\mathrm{IG}(2,7)$ as a subgraph, the subgraph induced by the remaining vertices being isomorphic to the Hasse diagram of $\mathrm{IG}(1,6)$. This is a general fact. More precisely, we have the following decomposition of the Hasse diagrams of the even and odd symplectic Grassmannian :

Proposition 6. • *The Hasse diagram of* $\mathrm{IG}(2,2n+1)$ *is isomorphic to the disjoint union of :*

1. *the Hasse diagram of* $\mathrm{IG}(2,2n)$*, whose vertices are the classes in* $\mathrm{IG}(2,2n+1)$ *associated*

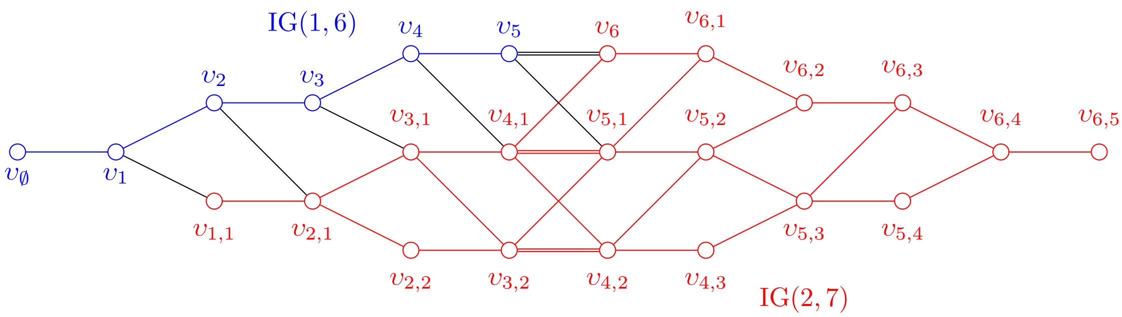

Figure 3: Hasse diagram of $\mathrm{IG}(2,8)$

to the Schubert varieties not contained in the closed orbit ;

2. *the Hasse diagram of the closed orbit* $\mathbb{O} \cong \mathrm{IG}(1, 2n)$ *;*

with parts 1 *and* 2 *linked by the simple edges joining* τ_{2n-3} *to* $\tau_{2n-1,-1}$ *and* $\tau_{2n-,a}$ *to* $\tau_{2n-1,a}$ *for* $0 \leq a \leq 2n-3$.

- *The Hasse diagram of* $\mathrm{IG}(2, 2n)$ *is isomorphic to the disjoint union of :*

1. *the Hasse diagram of* $\mathrm{IG}(2, 2n-1)$*, whose vertices are the classes in* $\mathrm{IG}(2, 2n)$ *associated to the Schubert varieties contained in* $X_{1,1}$ *;*
2. *the Hasse diagram of* $\mathrm{IG}(1, 2n-2)$*, corresponding to the classes* $\tau_\emptyset$ *to* τ_{2n-3} *;*

with parts 1 *and* 2 *linked by the double edge joining* τ_{2n-3} *to* τ_{2n-2} *and the (simple) edges joining* τ_p *to* $\tau_{p,1}$ *for* $1 \leq p \leq 2n-3$.

Proof. We will denote by $\mathcal{H}_{\mathrm{IG}(m,N)}$ the Hasse diagram of $\mathrm{IG}(m, N)$.

- Let G_1 be the subgraph of $\mathcal{H}_{\mathrm{IG}(2,2n+1)}$ induced by the vertices τ_λ for λ such that $\lambda_1 < 2n-1$. We need to prove that $G_1 = \mathcal{H}_{\mathrm{IG}(2,2n)}$. First notice these graphs have the same set of vertices. Then we define a rational map :

$$\begin{array}{rccc} \phi : & \mathrm{IG}(2, 2n+1) & \dashrightarrow & \mathrm{IG}(2, 2n) \\ & \Sigma & \longmapsto & \Sigma/K \end{array}$$

This map is well-defined on the open orbit, which is a dense open subset of $\mathrm{IG}\,(2, 2n+1)$. Looking at incidence conditions we notice that $\phi^* \upsilon_\lambda = \tau_\lambda$ for each Schubert class υ_λ of $\mathrm{IG}(2, 2n)$, and we get :

$$\phi^* \left(\upsilon_1 \upsilon_\lambda\right) = \phi^* \upsilon_1 \phi^* \upsilon_\lambda = \tau_1 \tau_\lambda,$$

hence G_1 and $\mathcal{H}_{\mathrm{IG}(2,2n)}$ have the same edges. Now the vertices of $\mathcal{H}_{\mathrm{IG}(2,2n+1)}$ not contained in G_1 correspond to the classes τ_λ with $\lambda_1 = 2n-1$, that is to the Schubert varieties contained in the closed orbit $\mathbb{O} \cong \mathbb{P}^{2n-1}$. So the graph G_2 they induced is isomorphic to $\mathrm{IG}(1, 2n)$. Finally, the edges joining G_1 and G_2 are determined using the Pieri rule 4.

- For $\mathrm{IG}(2, 2n)$ the result is simply a consequence of the isomorphism between $\mathrm{IG}\,(2, 2n+1)$ and the Schubert subvariety $X_{1,1}$ of $\mathrm{IG}(2, 2n)$ stated in paragraph 1.2, and of the Pieri rule for $\mathrm{IG}(2, 2n)$ proved in [2].

□

Remark 5. This result can be easily generalized to all symplectic Grassmannians $\mathrm{IG}(m, N)$.

1.6 Embedding in the usual Grassmannian

The easiest way to find a Giambelli formula for $\mathrm{IG}\,(2, 2n+1)$ is to use the Giambelli formula on $\mathrm{G}(2, 2n+1)$ and to "pull it back" to $\mathrm{IG}\,(2, 2n+1)$. More precisely, we use the natural embedding :

$$\mathbf{j} : \mathrm{IG}\,(2, 2n+1) \hookrightarrow \mathrm{G}\,(2, 2n+1)\,.$$

This embedding identifies $\mathrm{IG}\,(2, 2n+1)$ with a hyperplane section of $\mathrm{G}(2, 2n+1)$. So using the same arguments as for lemma 1, we can prove :

Lemma 3.
- *If* $a + b < 2n-2$ *then* $\mathbf{j}^* \sigma_{a,b} = \tau_{a,b}$.
- *If* $a + b \geq 2n-2$ *then*

$$\mathbf{j}^* \sigma_{a,b} = \tau_{a,b} + \tau_{a+1,b-1}.$$

This proves that the map $\mathbf{j}^*$ is surjective and that its kernel is generated by the class

$$\sum_{i=0}^{n-1} (-1)^{n-i} \sigma_{n+i,n-i}.$$

1.7 Giambelli formula

With lemma 3 and the Giambelli formula for $\mathrm{G}(2,2n+1)$, we can prove a Giambelli formula with respect to τ_1 and $\tau_{1,1}$. First define $d_r := (\tau_{1^{1+j-i}})_{1\leq i,j\leq r}$, with the convention that $\tau_{1^p}=0$ if $p<0$ or $p>2$. We have :

Proposition 7 (Giambelli formula)**.**

$$\tau_{a,b}=\begin{cases}\tau_{1,1}^{b}d_{a-b} & \text{if } a+b\leq 2n-3,\\ \sum_{q=0}^{p}(-1)^{p-q}\tau_{1,1}^{c-q}d_{2q} & \text{if } (a,b)=(c+1+p,c-1-p),\\ \sum_{q=p}^{2n-2-c}(-1)^{q-p}\tau_{1,1}^{c-q}d_{2q+1} & \text{if } (a,b)=(c+1+p,c-p),\end{cases}$$

where $n-1\leq c\leq 2n-2$ *and* $0\leq p\leq 2n-2-c$.

We can also state a Giambelli formula expressing classes in terms of the $e_p := c_p(\mathcal{Q})$:

Proposition 8 (Another Giambelli formula)**.**

$$\tau_{a,b}=\begin{cases}e_ae_b-e_{a+1}e_{b-1} & \text{if } a+b\leq 2n-3\\ (-1)^{a-n}e_{n-1}^2-e_ae_b+2\sum_{j=1}^{a-n}(-1)^{a-n-j}e_{n-1+j}e_{n-1-j} & \text{if } a+b=2n-2\\ e_ae_b+2\sum_{j=1}^{2n-1-a}(-1)^je_{a+j}e_{b-j} & \text{if } a+b\geq 2n-1.\end{cases}$$

1.8 A presentation for the classical cohomology ring

1.8.1 Presentation in terms of the classes e_p

Proposition 9 (Presentation of $\mathrm{H}^*(\mathrm{IG}(2,2n+1),\mathbb{Z})$)**.** *The ring* $\mathrm{H}^*(\mathrm{IG}(2,2n+1),\mathbb{Z})$ *is generated by the classes* $(e_p)_{1\leq p\leq 2n-1}$ *and the relations are*

$$\det\left(e_{1+j-i}\right)_{1\leq i,j\leq r}=0 \text{ for } 3\leq r\leq 2n, \tag{R1}$$

$$e_n^2+2\sum_{i\geq 1}e_{n+i}e_{n-i}=0. \tag{R2}$$

Proof. First of all, the quotient bundle $\mathcal{Q}$ of $\mathrm{IG}(2,2n+1)$ is the pullback by the restriction map $\mathbf{i}$ of the quotient bundle $\mathcal{Q}^+$ on $\mathrm{IG}(2,2n+2)$. So the $\mathbf{i}^*c_p(\mathcal{Q}^+)=c_p(\mathcal{Q})=e_p$ for $1\leq p\leq 2n$ generate $\mathrm{H}^*(\mathrm{IG}(2,2n+1),\mathbb{Z})$. But $\mathcal{Q}$ having rank $2n-1$, $\mathbf{i}^*c_{2n}(\mathcal{Q}^+)=0$, hence the cohomology ring of $\mathrm{IG}(2,2n+1)$ is generated by the $(e_p)_{1\leq p\leq 2n-1}$. Then we follow the method from [2] to obtain presentations for the isotropic Grassmannians. Consider the graded ring $A:=\mathbb{Z}[a_1,\ldots,a_{2n-1}]$, where $\deg a_i=i$. Set $a_0=1$, and $a_i=0$ if $i<0$ or $i>2n-1$. We also define $d_0:=1$ and $d_r:=\det\left(a_{1+j-i}\right)_{1\leq i,j\leq r}$ for $r>0$. For all $r\geq 0$, set $b_r:=a_r^2+2\sum_{i\geq 1}(-1)^ia_{r+i}a_{r-i}$. Now let $\phi: A\longrightarrow \mathrm{H}^*(\mathrm{IG}(2,2n+1),\mathbb{Z})$ be the degree-preserving morphism of graded rings sending a_i to e_i for all $1\leq i\leq 2n-1$. Since the e_p generate $\mathrm{H}^*(\mathrm{IG}(2,2n+1),\mathbb{Z})$, this morphism is surjective. To prove that relations (R1) and (R2) are satisfied, we must check that $\phi(d_r)=0$ for all $r>2$ and $\phi(b_n)=0$.

(R1) Expanding the determinant d_r with respect to the first column, we get the identity

$$d_r=\sum_{i=1}^{r}(-1)^{i-1}a_id_{r-i}.$$

Hence the identity on formal series :

$$\left(\sum_{i=0}^{2n-1}a_it^i\right)\left(\sum_{i\geq 0}(-1)^id_it^i\right)=1. \tag{2}$$

On $\mathrm{IG}(2,2n+1)$ we have the following short exact sequence of vector bundles

$$0\to\mathcal{S}\to\mathcal{O}_{\mathrm{IG}(2,2n+1)}\to\mathcal{Q}\to 0$$

so $c(\mathcal{S})c(\mathcal{Q})=1$, where c denotes the total Chern class. But

$$c(\mathcal{Q})=\sum_{i=0}^{2n-2}\tau_i t^i,$$

so (2) implies

$$c(\mathcal{S})=\sum_{i\geq 0}(-1)^i\phi(d_i)t^i.$$

Since $\mathcal{S}$ has rank 2, it follows that $\phi(d_r)=0$ for all $r>2$, hence the relations (R1).

(R2) From the presentation of $\mathrm{IG}(2,2n+2)$ in [2] we know that

$$\upsilon_n^2+2\sum_{i\geq 1}(-1)^i\upsilon_{n+i}\upsilon_{n-i}=0$$

in $\mathrm{IG}(2,2n+2)$. Pulling back by $\mathbf{i}$ we get (R2).

Now consider the Poincaré polynomial of $\mathrm{IG}\left(2,2n+1\right)$ from [10] :

$$P\left(\mathrm{IG}(m,2n+1),q\right)=\frac{\prod_{i=1}^{l}\left(q^{2n+2-2i}-1\right)\prod_{i=l+1}^{m}\left(q^{2n+4-2i}-1\right)}{(q^m-1)(q^{m-1}-1)\dots(q-1)}$$

for $m=2l$. Evaluating this polynomial at $q=1$, we get that the rank of $\mathrm{H}^*\left(\mathrm{IG}(2,2n+1)\right)$ is $2n^2$.

As in [2] we will need the following lemma :

Lemma 4. *The quotient of the graded ring* $Z\left[a_1,\dots,a_d\right]$ *with* $\deg a_i=i$ *modulo the relations*

$$\det\left(a_{1+j-i}\right)_{1\leq i,j\leq r}=0, m+1\leq r\leq m+d$$

is a free $\mathbb{Z}$*-module of rank* $\binom{m+d}{d}$.

To prove the previous lemma notice that the above presentation is nothing but the presentation of the cohomology ring of the usual Grassmannian $\mathrm{G}(m,m+d)$. Now to conclude the proof of the proposition we use :

Lemma 5. *Let* $A=\mathbb{Z}\left[a_1,\dots,a_d\right]$ *be a free polynomial ring generated by homogeneous elements* a_i *such that* $\deg a_i=i$*. Let* I *be an ideal in* A *generated by homogeneous elements* $c_1,\dots,c_d$ *in* A *and* $\phi:A/I\longrightarrow H$ *be a surjective ring homomorphism. Assume :*

C1. H *is a free* $\mathbb{Z}$*-module of rank* $\prod_i\left(\frac{\deg c_i}{\deg a_i}\right)$.

C2. for every field K*, the* K*-vector space* $(A/I)\otimes_{\mathbb{Z}}K$ *has finite dimension.*

Then ϕ *is an isomorphism.*

This result was proven in [2]. Apply it for

$$H=\mathrm{H}^*(\mathrm{IG}\left(2,2n+1\right),\mathbb{Z}),\ I=\left(d_3,\dots,d_{2n},b_n\right),\text{ and }A,\phi\text{ as above.}$$

Condition 1 is an immediate consequence of the rank calculation. For condition 2 it is enough to prove that A/I is a quotient of $A/\left(d_3,\dots,d_{2n+1}\right)$. Indeed, by lemma 4, the last module is a free $\mathbb{Z}$-module of finite rank. So we are left with proving that d_{2n+1} belongs to the ideal I. But the following identities of formal series hold :

$$\left(\sum_{i=0}^{2n-1}a_it^i\right)\left(\sum_{i=0}^{2n-1}(-1)^ia_it^i\right)=\sum_{i=0}^{2n-1}(-1)^ib_it^{2i}$$

$$\left(\sum_{i=0}^{2n-1}(-1)^ia_it^i\right)\left(\sum_{i\geq 0}d_it^i\right)=1.$$

Hence we get

$$\sum_{i=0}^{2n-1} a_i t^i = \left(\sum_{i=0}^{2n-1} (-1)^i b_i t^{2i}\right)\left(\sum_{i\geq 0} d_i t^i\right).$$

Modding out by the ideal I, it yields :

$$\sum_{i=0}^{2n-1} a_i t^i \equiv \left(\sum_{i=0}^{n-1}(-1)^i b_i t^{2i} + \sum_{i=n+1}^{2n-1} (-1)^i b_i t^{2i}\right)\left(\sum_{i=0}^{2} d_i t^i + \sum_{i\geq 2n+1} d_i t^i\right).$$

In degree $2n+1$, we get

$$0 \equiv d_{2n+1},$$

which ends the proof of the proposition. □

1.8.2 Presentation in terms of τ_1 and $\tau_{1,1}$

First we will need a presentation for the symplectic Grassmannian $\mathrm{IG}(2,2n)$ in terms of υ_1 and $\upsilon_{1,1}$:

Proposition 10. *The ring* $\mathrm{H}^*(\mathrm{IG}(2,2n),\mathbb{Z})$ *is generated by the classes* υ_1, $\upsilon_{1,1}$ *and the relations are*

$$\frac{1}{\upsilon_1}\det\left(\upsilon_{1^{1+j-i}}\right)_{1\leq i,j\leq 2n-1} = 0$$

$$\det\left(\upsilon_{1^{1+j-i}}\right)_{1\leq i,j\leq 2n} = 0$$

Proof. We will use lemma 5. Set $R := \mathbb{Z}[a_1,a_2]$, where $\deg a_i = i$. We denote by $\phi : R \to \mathrm{H}^*(\mathrm{IG}(2,2n),\mathbb{Z})$ the surjective ring homomorphism given by $a_i \mapsto \tau_{1^i}$. We also use the convention that $a_0 = 1$ and $a_i = 0$ for $i \notin \{0,1,2\}$. For $r \geq 1$, set $\delta_r := \det\left(a_{1+j-i}\right)_{1\leq i,j\leq r}$. We have the recurrence relation

$$\delta_r = a_1\delta_{r-1} - a_2\delta_{r-2}, \tag{3}$$

which is equivalent to the identity of formal series

$$\left(\sum a_i t^i\right)\left(\sum (-1)^i \delta_i t^i\right) = 1.$$

But $\phi(a_i) = \tau_{1^i} = c_i(\mathrm{S}^*)$. Moreover, as

$$0 \to \mathrm{S}^\perp \to \mathcal{O}_{\mathrm{IG}} \to \mathrm{S}^* \to 0,$$

where we denote by S the tautological bundle on IG, we have $c(\mathrm{S}^\perp)c(\mathrm{S}^*) = 1$, hence $\delta_r = c_r\left((\mathrm{S}^\perp)^*\right) = c_r(\mathrm{Q})$ (Q being the quotient bundle on IG). Since Q has rank $2n-2$, we have $\phi(\delta_r) = 0$ for all $r > 2n-2$, and in particular we get $\phi(\delta_{2n-1}) = \phi(\delta_{2n}) = 0$. We can write δ_{2q+1} as

$$\delta_{2q+1} = a_1 P_q(a_1,a_2),$$

where $P_q(a_1,a_2)$ is a homogeneous polynomial of degree $2q$. Now set $\delta'_{2q+1} := P_q(a_1,a_2)$. We want to prove that $\phi(\delta'_{2n-1}) = 0$. For this, since $\mathrm{IG}(2,2n+1)$ is a hyperplane section of the usual Grassmannian $\mathrm{G}(2,2n+1)$, we use Lefschetz's theorem. In particular, we obtain that the multiplication by the hyperplane class υ_1 is surjective from $\mathrm{H}^{2n-2}(\mathrm{IG},\mathbb{Z})$ to $\mathrm{H}^{2n-1}(\mathrm{IG},\mathbb{Z})$. But these vector spaces have the same dimension $n-1$, so it is bijective. As we already know that $\phi(\delta_{2n-1}) = 0$ it implies that $\phi(\delta'_{2n-1}) = 0$. Now let $I := (\delta'_{2n-1}, \delta_{2n})$. We proved that $\phi(I) = 0$ so we may define $\overline{\phi} : R/I \to \mathrm{H}^*(\mathrm{IG}(2,2n),\mathbb{Z})$. Now check that conditions 1 and 2 are satisfied :

(C1) $\mathrm{H}^*(\mathrm{IG}(2,2n),\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank $2n(n-1) = \frac{\deg(d_{2n-1})'\deg(d_{2n})}{\deg a_1 \deg a_2}$.

(C2) For every field K, $(R/I)\otimes_{\mathbb{Z}} K$ is finite-dimensional. Indeed R/I is a quotient of $R/(d_{2n-1}, d_{2n})$, which is isomorphic with $\mathrm{H}^*(\mathrm{G}(2,2n),\mathbb{Z})$, hence a free $\mathbb{Z}$-module of finite rank.

Finally lemma 5 yields that $\overline{\phi}$ is an isomorphism, hence the result. □

Now we deduce a presentation of $\mathrm{H}^*(\mathrm{IG}(2,2n+1),\mathbb{Z})$ using classes τ_1 and $\tau_{1,1}$:

Proposition 11 (another presentation of $\mathrm{H}^*(\mathrm{IG}(2,2n+1),\mathbb{Z})$)**.** *The ring* $\mathrm{H}^*(\mathrm{IG}(2,2n+1),\mathbb{Z})$ *is generated by the classes* τ_1*,* $\tau_{1,1}$ *and the relations are*

$$\det\left(\tau_{1^{1+j-i}}\right)_{1\leq i,j\leq 2n} = 0$$
$$\frac{1}{\tau_1}\det\left(\tau_{1^{1+j-i}}\right)_{1\leq i,j\leq 2n+1} = 0$$

Proof. First notice that τ_1 and $\tau_{1,1}$ generate the cohomology ring of $\mathrm{IG}(2,2n+1)$ since they are the pullbacks of the Chern classes of the dual tautological bundle over $\mathrm{G}(2,2n+1)$ by the surjective restriction map $\mathbf{j}$. Then define $R := \mathbb{Z}[a_1,a_2]$, where $\deg a_i = i$. We denote by $\phi : R \to \mathrm{H}^*(\mathrm{IG}(2,2n+1),\mathbb{Z})$ the surjective ring homomorphism given by $a_i \mapsto \tau_{1^i}$. We also use the convention that $a_0 = 1$ and $a_i = 0$ for $i \notin \{0,1,2\}$. For $r \geq 1$, set $\delta_r := \det\left(a_{1+j-i}\right)_{1\leq i,j\leq r}$. On $\mathrm{G}(2,2n+1)$ we know by the usual presentation (see [12]) that

$$\det\left(\sigma_{1^{1+j-i}}\right)_{1\leq i,j\leq 2n} = 0$$

Now define δ'_{2q+1} as in the proof of Proposition 10. Using the embedding in the symplectic Grassmannian $\mathrm{IG}(2,2n+2)$, we get that $\phi(\delta'_{2n+1}) = 0$. Indeed, we only have to pull back the relation $\frac{1}{\upsilon_1}\det\left(\upsilon_{1^{1+j-i}}\right)_{1\leq i,j\leq 2n+1} = 0$ proven in 10. Finally, set $I = (d_{2n}, d'_{2n=1})$ and apply lemma 5. □

2 Quantum cohomology

Our main goal in this section is to prove a quantum Pieri formula for $\mathrm{IG}(2,2n+1)$. We denote the quantum product of two classes τ_λ and τ_μ as $\tau_\lambda \star \tau_\mu$. The degree of the quantum parameter q is equal to the index of $\mathrm{IG}(2,2n+1)$, so $\deg q = 2n$.

Theorem 1 (Quantum Pieri rule for $\mathrm{IG}(2,2n+1)$)**.**

$$\tau_1 \star \tau_{a,b} = \begin{cases} \tau_{a+1,b} + \tau_{a,b+1} & \textit{if } a+b \neq 2n-3 \textit{ and } a \neq 2n-1, \\ \tau_{a,b+1} + 2\tau_{a+1,b} + \tau_{a+2,b-1} & \textit{if } a+b = 2n-3, \\ \tau_{2n-1,b+1} + q\tau_b & \textit{if } a = 2n-1 \textit{ and } 0 \leq b \leq 2n-3, \\ q(\tau_{2n-1,-1} + \tau_{2n-2}) & \textit{if } a = 2n-1 \textit{ and } b = 2n-2. \end{cases}$$

$$\tau_{1,1} \star \tau_{a,b} = \begin{cases} \tau_{a+1,b+1} & \textit{if } a+b \neq 2n-4,\ 2n-3 \textit{ and } a \neq 2n-1, \\ \tau_{a+1,b+1} + \tau_{a+2,b} & \textit{if } a+b = 2n-4 \textit{ or } 2n-3, \\ q\tau_{b+1} & \textit{if } a = 2n-1 \textit{ and } b \neq 2n-3, \\ q(\tau_{2n-1,-1} + \tau_{2n-2}) & \textit{if } a = 2n-1 \textit{ and } b = 2n-3. \end{cases}$$

The previous theorem is proved in 2.5, and from this a quantum presentation is deduced in 2.6. To prove the quantum Pieri formula, we first study in 2.1 the moduli spaces of stable maps of degree 1 to $\mathrm{IG}(2,2n+1)$ with 2 or 3 marked points. Then in 2.2 we decribe conditions for the Gromov-Witten invariants to have enumerative meaning. Finally, in 2.3 and 2.4 we compute the invariants we need. From now on, we denote $\mathrm{IG}(2,2n+1)$ by IG.

2.1 The moduli spaces $\overline{\mathcal{M}}_{0,2}(\mathrm{IG},1)$ and $\overline{\mathcal{M}}_{0,3}(\mathrm{IG},1)$

If X is a smooth projective variety we denote by $\overline{\mathcal{M}}_{g,n}(X,\beta)$ the moduli space of stable n-pointed maps f in genus g to X with degree β. This moduli space is endowed with n evaluation maps $(ev_i)_{1\leq i\leq n}$ that send a stable map f to its value at the i^{th} marked point. In this section we prove

Proposition 12. *1. The moduli spaces* $\overline{\mathcal{M}}_{0,2}(\mathrm{IG},1)$ *and* $\overline{\mathcal{M}}_{0,3}(\mathrm{IG},1)$ *are smooth (as stacks) and of the expected dimension (respectively* $6n-4$ *and* $6n-3$*).*

2. *The locus* $\mathcal{M}^*_{0,r}(\mathrm{IG},1)$ *in* $\overline{\mathcal{M}}_{0,r}(\mathrm{IG},1)$ *of stable maps with irreducible source is smooth of dimension* $6n-6+r$.

From the obstruction theory of moduli spaces, it follows that to prove the smoothness of $\overline{\mathcal{M}}_{0,2}(\mathrm{IG},1)$, $\overline{\mathcal{M}}_{0,3}(\mathrm{IG},1)$ and $\mathcal{M}^*_{0,r}(\mathrm{IG},1)$, we only need to prove that for each stable map f in these spaces we have $\mathrm{H}^1(f^*\mathrm{T}\,\mathrm{IG})=0$. We refer to [6] for more details.

2.1.1 $\overline{\mathcal{M}}_{0,2}(\mathrm{IG},1)$

Let $f:(C,p_1,p_2)\to\mathrm{IG}$ be a rational map of degree 1 with two marked points. There are two possibilities for C :

- either C is irreducible ;
- or $C=C_0\cup C_1$, where C_0 et C_1 are two (smooth) rational curves meeting in one point q, p_1 and p_2 are in C_0, f contracts C_0 and has degree 1 on C_1.

This gives us five cases for the map f :

1. C is irreducible and $f(C)\not\subset\mathbb{O}$;
2. C is irreducible and $f(C)\subset\mathbb{O}$;
3. C is reducible, $f(C)\not\subset\mathbb{O}$ and $f(p_1)=f(p_2)\notin\mathbb{O}$;
4. C is reducible, $f(C)\not\subset\mathbb{O}$ and $f(p_1)=f(p_2)\in\mathbb{O}$;
5. C is reducible, $f(C)\subset\mathbb{O}$,

where $\mathbb{O}$ is the closed orbit in IG.

Obstruction in cases 1 and 3. As in [6], we use the Sp_{2n+1}-action on IG. This action is transitive on $\mathrm{IG}\setminus\mathbb{O}$, so the tangent bundle T IG is globally generated outside of $\mathbb{O}$. If the map f is such that no irreducible component of C is entirely mapped into $\mathbb{O}$, which is true in cases 1 and 3, then $f^*\mathrm{T}\,\mathrm{IG}$ is generically generated by global sections, so $\mathrm{H}^1(C,f^*\mathrm{T}\,\mathrm{IG})=0$ and thus there is no obstruction.

Obstruction in case 2. We use the tangent exact sequence of the closed orbit

$$0\to\mathrm{T}\,\mathbb{O}\to\mathrm{T}\,\mathrm{IG}_{|\mathbb{O}}\to\mathrm{N}_{\mathbb{O}}\to 0, \tag{4}$$

where we denote by $\mathrm{N}_{\mathbb{O}}$ the normal bundle of the closed orbit. Pulling back by f, one deduces the long exact cohomology sequence

$$0\to\mathrm{H}^0(C,f^*\mathrm{T}\,\mathbb{O})\to\mathrm{H}^0(C,f^*\mathrm{T}\,\mathrm{IG})\to\mathrm{H}^0(C,f^*\mathrm{N}_{\mathbb{O}})\to\mathrm{H}^1(C,f^*\mathrm{T}\,\mathbb{O})\\ \to\mathrm{H}^1(C,f^*\mathrm{T}\,\mathrm{IG})\to\mathrm{H}^1(C,f^*\mathrm{N}_{\mathbb{O}})\to 0. \tag{5}$$

As $\mathbb{O}$ is homogeneous under the Sp_{2n+1}-action, $\mathrm{T}\mathbb{O}$ is generated by global sections, so $\mathrm{H}^1(C,f^*\mathrm{T}\,\mathbb{O})=0$. To compute the obstruction, it is then enough to know $f^*\mathrm{N}_{\mathbb{O}}$.

Lemma 6. $\mathrm{N}_{\mathbb{O}}=\mathrm{S}^{\perp}/\mathrm{S}$, *where we denote by* S *the (restriction to the closed orbit of the) tautological bundle of* IG.

Proof. First notice that $\mathrm{S}=\mathcal{O}\oplus\mathcal{O}(-1)$. Indeed all elements in $\mathbb{O}$ contain the kernel K of ω, the quotient S/K is nothing but $\mathcal{O}(-1)$ and the extension is split. Now consider the tangent exact sequence of IG restricted to $\mathbb{O}$:

$$0\longrightarrow\mathrm{T}\,\mathrm{IG}_{|\mathbb{O}}\longrightarrow\mathrm{T}\,\mathrm{G}_{|\mathbb{O}}\xrightarrow{\phi_\omega}\mathcal{O}(1)\longrightarrow 0$$

We have $\mathrm{T}\,G \cong \mathrm{S} \otimes \mathrm{Q} \cong \mathrm{Hom}(\mathrm{S}, \mathrm{Q})$, where we denote by Q the quotient bundle. It is easy to see that $\mathrm{Hom}(\mathrm{S}, \mathrm{S}^\perp/\mathrm{S})$ is in the kernel of ϕ_ω. It has codimension 1 in $\mathrm{T}\,\mathrm{IG}\,|_{\mathbb{O}}$. Moreover we have $\mathbb{O} \cong \mathbb{P}(\mathbb{C}^{2n+1}/K)$, hence $\mathrm{T}\,\mathbb{O} \cong \mathcal{O}(1) \otimes \mathrm{Q} = \mathrm{Q}(1) \supset \mathrm{S}^\perp/\mathrm{S}(1)$. We get the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & L' & & L & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \mathrm{T}\mathbb{O} & \longrightarrow & \mathrm{TIG}_{|\mathbb{O}} & \longrightarrow & \mathrm{N}_{\mathbb{O}} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathrm{S}^\perp/\mathrm{S}(1) & \xrightarrow{i} & \mathrm{S}^\perp/\mathrm{S} \oplus \mathrm{S}^\perp/\mathrm{S}(1) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where L and L' are line bundles. It follows immediately that i is an injection. We also notice that $L = \det \mathrm{Q} \otimes (\mathrm{S}^\perp/\mathrm{S})^{-1} = L'$, so the diagram becomes

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & L' & = & L & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \mathrm{T}\mathbb{O} & \longrightarrow & \mathrm{TIG}_{|\mathbb{O}} & \longrightarrow & \mathrm{N}_{\mathbb{O}} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \mathrm{S}^\perp/\mathrm{S}(1) & \xrightarrow{i} & \mathrm{S}^\perp/\mathrm{S} \oplus \mathrm{S}^\perp/\mathrm{S}(1) & \longrightarrow & \mathrm{S}^\perp/\mathrm{S} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

hence the isomorphism $\mathrm{N}_{\mathbb{O}} \cong \mathrm{S}^\perp/\mathrm{S}$. □

Now we compute $f^*\mathrm{N}_{\mathbb{O}}$. A line $\mathcal{D}$ in the closed orbit is of the form

$$\mathcal{D} := \mathcal{D}\,(K, W) = \{\Sigma \in \mathrm{IG} \mid K \subset \Sigma \subset W\},$$

where K is the kernel of the antisymmetric form ω. Moreover we have two possibilities for W, whether it is isotropic or not. In the first case, W has in $\mathbb{C}^{2n+1}$ an orthogonal complement U, so for $\Sigma \in \mathcal{D}$, we have $\Sigma^\perp = \Sigma \oplus U$, and $\mathrm{N}_{\mathbb{O}}\,|_{\mathcal{D}}$ is trivial. In the second case $W \subset \Sigma^\perp$, hence the exact sequence of vector spaces

$$0 \to W/\Sigma \to \Sigma^\perp/\Sigma \to \Sigma^\perp/W \to 0. \tag{6}$$

But the bundle with fiber W/Σ over $\mathcal{D}$ is isomorphic to $\mathcal{O}_{\mathcal{D}}\,(1)$. Moreover, an easy remark is that there exists W' of dimension 2 and U' of dimension $2n-4$ such that

- $\mathbb{C}^{2n+1} = W \oplus W' \oplus U'$;
- $U' \perp W \oplus W'$

- $\omega\mid_{W\oplus W'}$ has rank 4.

As $\Sigma^\perp \supset U'$, we have $\mathrm{S}^\perp/W \cong \mathcal{O}_{\mathcal{D}}(-1)\oplus\mathcal{O}^{\oplus(2n-4)}$, so (6) becomes :

$$0\to\mathcal{O}(1)\to \mathrm{S}^\perp/\mathrm{S}\to\mathcal{O}(-1)\oplus\mathcal{O}^{\oplus(2n-4)}\to 0.$$

Now we only have to notice that this exact sequence is split. Then $\mathrm{H}^1(C, f^*\mathrm{N}_{\mathbb{O}}) = \mathrm{H}^1(C, f^*\mathrm{T\,IG}) = 0$.

Obstruction in case 5. The map $f^*\mathrm{T\,IG}\to f^*\mathrm{T\,IG}\mid_{C_1}$ is surjective. Its kernel corresponds to local sections at q that vanish along C_1, which means we have the following exact sequence

$$0\to\mathcal{O}_{C_0}^{\oplus(2n-2)}\otimes\mathcal{I}_q\to f^*\mathrm{T\,IG}\to f^*\mathrm{T\,IG}\mid_{C_1}\to 0, \qquad (7)$$

hence the long exact cohomology sequence

$$\begin{aligned}0\to \mathrm{H}^0\left(C_0,\mathcal{O}_{C_0}^{\oplus(2n-2)}\otimes\mathcal{I}_q\right)&\to \mathrm{H}^0(C,f^*\mathrm{T\,IG})\to \mathrm{H}^0(C_1,f^*\mathrm{T\,IG}\mid_{C_1})\to\\ \to \mathrm{H}^1\left(C_0,\mathcal{O}_{C_0}^{\oplus(2n-2)}\otimes\mathcal{I}_q\right)&\to \mathrm{H}^1(C,f^*\mathrm{T\,IG})\to \mathrm{H}^1(C_1,f^*\mathrm{T\,IG}\mid_{C_1})\to 0. \qquad (8)\end{aligned}$$

As $\mathcal{I}_q\cong\mathcal{O}(-1)$, we have $\mathrm{H}^1\left(C_0,\mathcal{O}_{C_0}^{\oplus(2n-2)}\otimes\mathcal{I}_q\right)=0$. Moreover the proof for case 2 showed that $\mathrm{H}^1(C_1,f^*\mathrm{T\,IG}\mid_{C_1})=0$, so $\mathrm{H}^1(C,f^*\mathrm{T\,IG})=0$.

Obstruction in case 4. The proof is very similar to that of case 5.

Conclusion. This ends the proof of the first item of Proposition 12 for $\overline{\mathcal{M}}_{0,2}(\mathrm{IG},1)$. □

2.1.2 $\overline{\mathcal{M}}_{0,3}(\mathrm{IG},1)$

Let $f:(C,p_1,p_2,p_3)\to\mathrm{IG}$ be a map in $\overline{\mathcal{M}}_{0,3}(\mathrm{IG},1)$. We have four possibilities for C :

- C is irreducible ;
- $C=C_0\cup C_1$, where C_0 and C_1 are two rational curves meeting in one point q, p_1 and p_2 lie on C_0, p_3 lies on C_1, f contracts C_0 and has degree 1 on C_1 ;
- $C=C_0\cup C_1$, where C_0 and C_1 are two rational curves meeting in one point q, p_1, p_2 and p_3 lie on C_0, f contracts C_0 and has degree 1 on C_1 ;
- $C=C_0'\cup C_0\cup C_1$, where C_0' and C_0 meet at q', C_0 and C_1 meet at q, p_1 and p_2 lie on C_0, p_3 lies on C_0, f contracts C_0 and C_0' has degree 1 on C_1.

In the first three situations, arguments similar to those used for $\overline{\mathcal{M}}_{0,2}(\mathrm{IG},1)$ show that there is no obstruction. So we only need to compute the obstruction in the last situation, which gives us three cases :

1. $f(C)\not\subset\mathbb{O}$, $f(q)\notin\mathbb{O}$;
2. $f(C)\not\subset\mathbb{O}$, $f(q)\in\mathbb{O}$;
3. $f(C)\subset\mathbb{O}$.

Obstruction in case 1. As $f^*\mathrm{T\,IG}$ is generated by global sections, we get $\mathrm{H}^1(C,f^*\mathrm{T\,IG})=0$, so there is no obstruction.

Obstruction in case 2. The map $f^*\mathrm{T\,IG} \to f^*\mathrm{T\,IG}\,|_{C_1}$ is surjective with kernel $(f^*\mathrm{T\;IG})\,|_{C_0\cup C_0'}$ $\otimes \mathcal{I}_q$. In addition, the map $(f^*\mathrm{T\;IG})\,|_{C_0\cup C_0'} \otimes\mathcal{I}_q \to (f^*\mathrm{T\;IG})\,|_{C_0} \otimes\mathcal{I}_q$ is surjective with kernel $(f^*\mathrm{T\;IG})\,|_{C_0'} \otimes\mathcal{I}_{q'}$. So we get two exact sequences

$$0 \to (f^*\mathrm{T\;IG})\,|_{C_0\cup C_0'} \otimes\mathcal{I}_q \to f^*\mathrm{T\;IG} \to f^*\mathrm{T\;IG}\,|_{C_1}\to 0, \tag{9}$$

$$0 \to (f^*\mathrm{T\;IG})\,|_{C_0'} \otimes\mathcal{I}_{q'} \to (f^*\mathrm{T\;IG})\,|_{C_0\cup C_0'} \otimes\mathcal{I}_q \to (f^*\mathrm{T\;IG})\,|_{C_0} \otimes\mathcal{I}_q \to 0. \tag{10}$$

In the second exact sequence, we have

$$(f^*\mathrm{T\;IG})\,|_{C_0}= \mathcal{O}_{C_0}^{\oplus(2n-2)},$$

so

$$(f^*\mathrm{T\;IG})\,|_{C_0'} \otimes\mathcal{I}_{q'} = \mathcal{O}_{C_0'}(-1)^{\oplus(2n-2)},$$

hence

$$\mathrm{H}^1\left(C_0 \cup C_0', (f^*\mathrm{T\;IG})\,|_{C_0\cup C_0'} \otimes\mathcal{I}_q\right) = 0.$$

But $\mathrm{H}^1\left(C_1, (f^*\mathrm{T\;IG})\,|_{C_1}\right) = 0$ according to the proof for the case where C is irreducible, so the first exact sequence yields

$$\mathrm{H}^1\left(C, f^*\mathrm{T\;IG}\right) = 0.$$

Obstruction in case 3. The proof is very similar to that of case 5.

Conclusion. This concludes the proof of the first item of Proposition 12. □

2.1.3 $\mathcal{M}^*_{0,r}\,(\mathrm{IG},1)$

Let $f : (C, p_1, \ldots, p_r) \to \mathrm{IG}$ be a map in $\overline{\mathcal{M}}_{0,r}\,(\mathrm{IG},1)$. Since C is assumed to be irreducible no component of the source curve is contracted and we only have two possibilities :

1. $f(C) \not\subset \mathbb{O}$;
2. $f(C) \subset \mathbb{O}$.

In the first case we again use the global generation of T IG outside of the closed orbit, while in the second, the previous calculation of the normal bundle enables us to conclude. □

2.2 Enumerativity of the invariants in $\overline{\mathcal{M}}_{0,2}\,(\mathrm{IG},1)$ and $\overline{\mathcal{M}}_{0,3}\,(\mathrm{IG},1)$

In this section we prove a Kleiman-type lemma for quasi-homogeneous spaces, due to Graber in [6]. First we state Kleiman's lemma :

Lemma 7 ([7])**.** *Let G be a connected algebraic group, X an irreducible algebraic variety over $\mathbb{C}$ with a transitive G-action. Let $i : Y \hookrightarrow X$ be the embedding of an irreducible subvariety and $f : Z \to X$ be a map from an irreducible algebraic scheme. For g in G, let gY denote the translate of Y by g.*

(i) There exists a dense open subset U_1 of G such that, for each g in U_1, either the scheme $f^{-1}(gY)$ is empty or it is equidimensional and its dimension is given by the formula,

$$\dim(f^{-1}(gY)) = \dim(Y) + \dim(Z) - \dim(X).$$

(ii) Assume Y and Z are smooth. Then, there exists a dense open subset U_2 of G such that, for each g in U_2, the scheme $f^{-1}(gY)$ is smooth and reduced.

Now we are going to use the previous lemma to prove a version for quasi-homogeneous spaces. Note that the hypothesis that the space has only finitely many orbits is crucial.

Lemma 8. *Let X be a variety endowed with an action of a connected algebraic group G with only finitely many orbits and Z an irreducible scheme with a morphism $f : Z \to X$. Let Y be a subvariety of X that intersects the orbit stratification properly. Then there exists a dense open subset U of G such that $\forall g \in U$, $f^{-1}(gY)$ is either empty or has pure dimension $\dim Y + \dim Z - \dim X$. Moreover, if X, Y and Z are smooth and we denote by Y_{reg} the subset of Y along which the intersection with the stratification is transverse, then the (possibly empty) open subset $f^{-1}(gY_{\mathrm{reg}})$ is smooth.*

Proof. Let O be a G-orbit. We apply Kleiman's lemma 7 to the following diagram

$$\begin{array}{ccc} & & f^{-1}(O) \\ & & \downarrow f \\ Y \cap O & \xhookrightarrow{\;i\;} & O \end{array}$$

We deduce that there exists a non empty open subset $U_O \subset G$ such that $\forall g \in U_O$, $f^{-1}(gY \cap O)$ is either empty or has pure dimension $\dim f^{-1}(O) + \dim(Y \cap O) - \dim O$. But $\operatorname{codim}_O(Y \cap O) = \operatorname{codim}_X Y$ by the transversality assumption, so if $f^{-1}(gY \cap O)$ is non-empty, then

$$\dim f^{-1}(gY \cap O) = \dim f^{-1}(O) - \operatorname{codim}_X Y \le \dim Y + \dim Z - \dim X.$$

Then the finite intersection $U := \bigcap U_O$ has the required properties.

Now assume X, Y and Z are smooth. Kleiman's lemma applied to the previous diagram shows that for $g \in U'_O$ a non-empty open subset of G, $f^{-1}(gY \cap O)$ is smooth, that is

$$D_f(\mathrm{T}_z\, Z) + \mathrm{T}_{f(z)}\,(Y \cap O) = \mathrm{T}_{f(z)}\, O$$

for any z such that $f(z) \in O$. Moreover on Y_{reg} the intersection with O is transverse, so

$$\mathrm{T}_{f(z)}\, Y + \mathrm{T}_{f(z)}\, O = \mathrm{T}_{f(z)}\, X,$$

hence

$$D_f(\mathrm{T}_z\, Z) + \mathrm{T}_{f(z)}\, Y = \mathrm{T}_{f(z)}\, X,$$

which is the required transversality relation. □

Theorem 2 (Enumerativity of the Gromov-Witten invariants)**.** *Let $r = 2$ or 3 and $Y_1, \ldots, Y_r$ be subvarieties of IG representing cohomology classes $\gamma_1, \ldots, \gamma_r$ of codimension at least 2 that intersect the closed orbit generically transversely and such that $\sum_{i=1}^r \operatorname{codim} \gamma_i = \dim \overline{\mathcal{M}}_{0,r}\,(\mathrm{IG}, 1)$. Then there exists a dense open subset $U \subset \mathrm{Sp}_{2n+1}^r$ such that for all $g_1, \ldots, g_r \in U$, the Gromov-Witten invariant $I_1(\gamma_1, \ldots, \gamma_r)$ is equal to the number of lines of IG incident to the translates $g_1Y_1, \ldots, g_rY_r$.*

Proof. The result is proven by successively applying the transversality lemma 8. First we prove that stable maps with reducible source do not contribute to the Gromov-Witten invariant by applying the lemma to the following diagram :

$$\begin{array}{ccc} & & \overline{\mathcal{M}} \setminus \mathcal{M}^* \\ & & \downarrow \underline{ev} \\ \underline{Y} & \xhookrightarrow{\;i\;} & \mathrm{IG}^r \end{array}$$

where $\underline{Y} = (Y_1, \ldots, Y_r)$, $\underline{ev} = ev_1 \times \cdots \times ev_r$, $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{0,r}\,(\mathrm{IG}, 1)$ and $\mathcal{M}^*$ is the locus of map with irreducible source, which is a dense open subset by Proposition 12.

We should also prove that it is not possible for a line to be incident to one of the subvarieties Y_i in more than one point, since such a line would contribute several times to the invariant. Suppose for example that there exists a line L that intersects Y_1 in a least two points. Then any stable map f whose image curve is L corresponds to a map $\tilde{f}$ in $\overline{\mathcal{M}}_{0,r+1}\,(\mathrm{IG}, 1)$ that contributes to the invariant $I_1(\gamma_1, \gamma_1, \ldots, \gamma_r)$. Since we have already excluded the case of maps with reducible source, it follows that $\tilde{f}$ lies in fact in $\mathcal{M}_{0,r+1}(\mathrm{IG}, 1)$, which has dimension $6n-5+r$ by the second part of Proposition 12. Hence applying Lemma 8 to the following diagram

$$\begin{array}{ccc} & & \mathcal{M}^*_{0,r+1}(\mathrm{IG},1) \\ & & \Big\downarrow \underline{ev} \\ Y_1 \times Y_1 \times \cdots \times Y_r & \overset{i}{\hookrightarrow} & \mathrm{IG}^{r+1} \end{array}$$

and using the fact that codim $\gamma_1 \geq 2$ we conclude that such a line cannot exist.

Now using :

$$\begin{array}{ccc} & & \mathcal{M}^* \\ & & \Big\downarrow \underline{ev} \\ \mathrm{Sing}\ \underline{Y} & \overset{i}{\hookrightarrow} & \mathrm{IG}^r \end{array}$$

where Sing $\underline{Y}$ denotes the singular locus of $\underline{Y}$, we may assume that $\underline{Y}$ is smooth. Moreover, since $Y_1, \ldots, Y_r$ intersect the closed orbit generically transversely, another application of Lemma 8 allows us to assume that this intersection is transverse everywhere. Finally, applying the lemma to

$$\begin{array}{ccc} & & \mathcal{M}^* \\ & & \Big\downarrow \underline{ev} \\ \underline{Y} & \overset{i}{\hookrightarrow} & \mathrm{IG}^r \end{array}$$

we conclude that there exists a dense open subset $U \subset \mathrm{Sp}_{2n+1}^r$ such that for all $g_1, \ldots, g_r \in U$, $\bigcap_{i=1}^r ev_i^{-1}(g_i Y_i)$ is a finite number of reduced points, which equals the number of lines incident to all the $g_i Y_i$. □

Remark 6. Theorem 2 enables us to compute the Gromov-Witten invariants by geometric means. However, Schubert varieties will not be appropriate to perform this calculation. Indeed, the intersection of a Schubert variety and the closed orbit is not even proper. So we will instead use the restrictions of the Schubert varieties of the usual Grassmannian.

2.3 Computation of the invariants in $\overline{\mathcal{M}}_{0,2}(\mathrm{IG}, 1)$

First we state some conditions that will be required for flags defining our varieties.

Notation 1. Denote by

- $\mathbb{F}_n$ the variety of complete flags in $\mathbb{C}^{2n+1}$;
- Λ_n the variety of antisymmetric 2-forms with maximal rank on $\mathbb{C}^{2n+1}$.

Lemma 9. *Assume $n \geq 2$. The set of triples $(F_\bullet, G_\bullet, \omega) \in \mathbb{F}_n \times \mathbb{F}_n \times \Lambda_n$ such that the following holds*

(C1) $\forall 0 \leq p \leq 2n+1$, $\omega_{|F_p}$ *has maximal rank ;*

(C2) $\forall 0 \leq p \leq 2n+1$, $\omega_{|G_p}$ *has maximal rank ;*

(C3) $\forall 0 \leq p, q \leq 2n+1$, $F_p \cap G_q$ *has the expected dimension ;*

(C4)$_i$ $\dim\left(F_{2n+1-i} \cap G_{i+3} \cap F_1^\perp \cap G_1^\perp\right) = 1$ *;* $(\mathbf{0 \leq i \leq 2n-2})$

(C5)$_i$ $\dim F_{2n-i} \cap G_{i+3} \cap G_1^\perp = 1$ *and* $\dim(F_{2n-i} \cap G_{i+3} \cap G_1^\perp)^\perp \cap F_2 = 1$*;* $(\mathbf{0 \leq i \leq 2n-2})$

(C6)$_i$ $\dim F_{2n+1-i} \cap G_{i+2} \cap F_1^\perp = 1$ *and* $\dim(F_{2n+1-i} \cap G_{i+2} \cap F_1^\perp)^\perp \cap G_2 = 1$; $(\mathbf{2 \leq i \leq 2n-4})$

(C7)$_i$ $\dim (F_{2n-i} \cap G_{i+2})^\perp \cap F_2 = 1$; $(\mathbf{2 \leq i \leq 2n-4})$

(C8)$_i$ $\dim (F_{2n-i} \cap G_{i+2})^\perp \cap G_2 = 1$; $(\mathbf{2 \leq i \leq 2n-4})$

(C9) $F_1 \not\subset G_1^\perp$

(C10) $G_1 \not\subset F_1^\perp$

(C11)$_i$ $F_{2n-1-i} \cap G_{i+3} \cap G_1^\perp = 0$; $(\mathbf{0 \leq i \leq 2n-6})$

(C12)$_i$ $F_{2n+1-i} \cap G_{i+1} \cap F_1^\perp = 0$; $(\mathbf{4 \leq i \leq 2n-2})$

is a dense open subset in $\mathbb{F}_n \times \mathbb{F}_n \times \Lambda_n$.

Proof. $\mathbb{F}_n \times \mathbb{F}_n \times \Lambda_n$ is a (quasi-projective) irreducible variety. Moreover all conditions are clearly open. So it is enough to show that each of them is non-empty.

(C1),(C2) et (C3) Obvious.

(C4)$_i$ Since $n \geq 2$, we may choose the flags $F_\bullet$ and $G_\bullet$ such that the subspace $A := F_{2n+1-i} \cap G_{i+3}$ has dimension 3 and A together with the lines $L := F_1$ and $L' := G_1$ are in direct sum. Then there exists a form $\omega \in \Lambda_n$ such that $A \cap L^\perp \cap L'^\perp$ has dimension 1.

(C5)$_i$ As before we may choose $F_\bullet$ and $G_\bullet$ such that $A := F_{2n-i} \cap G_{i+3}$ has dimension 2 and A, $L := G_1$ and $B := F_2$ are complementary. So we may construct $\omega \in \Lambda_n$ such that $(A \cap L^\perp)^\perp \cap B$ has dimension 1. First construct ω_0 on $A \oplus B \oplus L$. Let $a \in A \setminus 0$ and $b \in B \setminus 0$. There exists ω_0 a symplectic form on $A \oplus B$ such that $\omega_0(a, b) \neq 0$. Then we extend ω_0 to ω defined on $A \oplus B \oplus L$ by setting $\omega(a, l) = 0$, $\omega(a', l) \neq 0$ and for instance $\omega(\beta, l) = 0$ for all $\beta \in B$, where l generates L, a, a' generate A.

(C6)$_i$ As in **(C5)$_i$**.

(C7)$_i$ We may choose $F_\bullet$ and $G_\bullet$ such that $L := F_{2n-i} \cap G_{i+2}$ has dimension 1 and is in direct sum with $A := F_2$. But then there exists $\omega \in \Lambda_n$ such that $A \not\subset L^\perp$.

(C8)$_i$ As in **(C7)$_i$**.

(C9) $G_1^\perp$ is a general hyperplane, so it does not contain F_1.

(C10) As in **(C9)**.

(C11)$_i$ $F_{2n-1-i} \cap G_{i+3}$ is a line $G_1^\perp$ is a general hyperplane, so their intersection is zero.

(C12)$_i$ As in **(C11)$_i$**.

□

We can now define the varieties we will use to compute the invariants, which will be restrictions of the Schubert varieties of the usual Grassmannian :

Lemma 10. *Let* $0 \leq j \leq n-1$ *and* $0 \leq i \leq 2n-1-2j$ *be integers. Let*

$$X_{i,j} := \{\Sigma \in G \mid \Sigma \cap F_{j+1} \neq 0, \Sigma \subset F_{2n+1-i-j}\},$$

be a subvariety of $G := G(2, 2n+1)$*, where* $F_\bullet$ *is a complete flag satisfying condition* **(C1)**.

1. $X_{i,j}$ *and* IG *intersect generically transversely.*

2. *Let $Y_{i,j} := X_{i,j} \cap \mathrm{IG}$. We have*

$$[Y_{i,j}]^{\mathrm{IG}} = \begin{cases} \tau_{2n-1-j,i+j} + \tau_{2n-j,i+j-1} & \text{if } j \neq 0 \text{ and } i \neq 2n-1-2j \\ \tau_{2n-j,2n-2-j} & \text{if } j \neq 0 \text{ and } i = 2n-1-2j \\ \tau_{2n-1,i} & \text{if } j = 0 \text{ and } i \neq 2n-1 \\ 0 & \text{if } j = 0 \text{ and } i = 2n-1, \end{cases}$$

where we denote by $[V]^{\mathrm{IG}}$ (respectively by $[V]^G$) the class of the subvariety V in IG (respectively in G).

Proof. 1. In the Schubert cell $C_{i,j} \subset X_{i,j}$, a direct computation shows that $\mathrm{T}_p\, X_{i,j} \not\subset \mathrm{T}_p\, \mathrm{IG}$ as soon as $F_{j+1} \not\subset F^{\perp}_{2n+1-i-j}$, which is true by condition **(C1)**. So $C_{i,j} \cap \mathrm{IG}$ is transverse. Applying again **(C1)**, we notice that $C_{i,j} \cap \mathrm{IG}$ is an open subset of $X_{i,j} \cap \mathrm{IG}$.

2. We have $[X_{i,j}]^G = \sigma_{2n-1-j,i+j}$. Moreover, the previous item implies that $[Y_{i,j}]^G = \sigma_1 [X_{i,j}]^G$. So

$$[Y_{i,j}]^G = \begin{cases} \sigma_{2n-1-j,i+j+1} + \sigma_{2n-j,i+j} & \text{if } j \neq 0 \text{ and } i \neq 2n-1-2j \\ \sigma_{2n-j,2n-1-j} & \text{if } j \neq 0 \text{ and } i = 2n-1-2j \\ \sigma_{2n-1,i+1} & \text{if } j = 0 \text{ and } i \neq 2n-1 \\ 0 & \text{if } j = 0 \text{ and } i = 2n-1. \end{cases}$$

Moreover, $[Y_{i,j}]^G = \mathbf{j}_\star [Y_{i,j}]^{\mathrm{IG}}$, $[Y_{i,j}]^{\mathrm{IG}} = \sum_{p=0}^{\lfloor n-1-\frac{i}{2} \rfloor} \alpha_p \tau_{2n-1-p,i+p}$ and $\mathbf{j}_\star \tau_{a,b} = \sigma_{a,b+1}$ for $a + b \geq 2n - 1$, so we can determine the α_p by identifying both expressions. □

We now assume all genericity conditions **(C1-12)** are satisfied and prove

Proposition 13. *Let $0 \leq i \leq 2n-2$, $0 \leq 2j \leq 2n-2-i$ and $0 \leq 2l \leq i$ be integers. Set $Y_1 := Y_{i,j}(F_\bullet)$ and $Y_2 := Y_{2n-2-i,l}(G_\bullet)$, where the complete flags $F_\bullet$ and $G_\bullet$ as well as the form ω verify the transversality conditions of lemma 9. Then*

1. *The intersections $Y_1 \cap \mathbb{O}$ et $Y_2 \cap \mathbb{O}$ are transverse. Moreover*

$$Y_1 \cap \mathbb{O} = \begin{cases} \emptyset & \text{if } i \text{ or } j \neq 0 \\ \{F_1 \oplus K\} & \text{if } i = j = 0 \end{cases}$$

$$Y_2 \cap \mathbb{O} = \begin{cases} \emptyset & \text{if } i \neq 2n-2 \text{ or } l \neq 0 \\ \{G_1 \oplus K\} & \text{if } i = 2n-2 \text{ and } l = 0 \end{cases}$$

2. *If j or $l \geq 2$, there exists no line passing through Y_1 and Y_2.*

3. *If j, $l \leq 1$, there exists a unique line passing through Y_1 and Y_2.*

Proof. 1. $Y_1 \cap \mathbb{O} = \{\Sigma \in \mathrm{IG} \mid \Sigma \cap F_{j+1} \neq 0, K \subset \Sigma \subset F_{2n+1-i-j}\}$, so if $i + j \neq 0$, then $K \subset F_{2n+1-i-j}$, which, according to **(C1)**, implies that $Y_1 \cap \mathbb{O} = \emptyset$, so the intersection is transverse. Moreover if $i + j = 0$ we get $Y_1 \cap \mathbb{O} = \{F_1 \oplus K\}$. Denote by Σ_0 the point $K \oplus F_1$. To prove transversality at Σ_0 we use the embedding in the usual Grassmannian $G := G(2, 2n+1)$. It is well-known that $\mathrm{T}_{\Sigma_0} G = \mathrm{Hom}\left(\Sigma_0, \mathbb{C}^{2n+1}/\Sigma_0\right)$. Now express $\mathrm{T}_{\Sigma_0} Y_1$ and $\mathrm{T}_{\Sigma_0} \mathbb{O}$ as subspaces of $\mathrm{T}_{\Sigma_0} G$:

$$\mathrm{T}_{\Sigma_0} Y_1 = \{\phi \in \mathrm{T}_{\Sigma_0} G \mid \phi(f_1) = 0\}$$
$$\mathrm{T}_{\Sigma_0} \mathbb{O} = \{\phi \in \mathrm{T}_{\Sigma_0} G \mid \phi(k) = 0\},$$

where f_1 and k generate F_1 and K. We see that these subspaces are complementary in $\mathrm{T}_{\Sigma_0} G$. Computing $\dim Y_1 = 2n-2$ and $\dim \mathbb{O} = 2n-1$ we conclude that they generate $\mathrm{T}_{\Sigma_0} \mathrm{IG}$. We can proceed in a similar fashion for $Y_2 \cap \mathbb{O}$.

2. Let $\mathcal{D} := \mathcal{D}(V,W)$ be a line meeting Y_1 and Y_2. Then we must have $V \subset F_{2n+1-i-j} \cap G_{i+3-l}$. But according to **(C3)**, this subspace is either zero or it has codimension $2n+4-j-l$. So for $j+l \geq 3$, it is zero and there is no line. If $j=2$ and $l=0$ (and symmetrically if $j=0$ and $l=2$), we must have $V \subset F_{2n-1-i} \cap G_{i+3} \cap G_1^{\perp} = 0$, which is impossible by $\mathbf{(C11)}_i$ (respectively by $\mathbf{(C12)}_i$). So for a line to exist we must have j and $l \leq 1$.

3. There are four cases to study :

   a) $j = l = 0$;
   b) $j = 1, l = 0$;
   c) $j = 0, l = 1$;
   d) $j = l = 1$.

   a) Let $A = F_{2n+1-i} \cap G_{i+3}$. We have $\dim A = 3$ by **(C3)**. But $V \subset A$ and $V \subset F_1^{\perp} \cap G_1^{\perp}$ since $F_1, G_1 \subset W$ and $W \subset V^{\perp}$. By $\mathbf{(C4)}_i$, we have $\dim A \cap F_1^{\perp} \cap G_1^{\perp} = 1$, hence $V = A \cap F_1^{\perp} \cap G_1^{\perp}$. So $W \supset V + (F_1 \oplus G_1)$ (F_1 and G_1 are in direct sum **(C3)**). To show equality, it is enough to prove that the sum is direct. If not then there exists a non-zero vector of the form $af_1 + bg_1$ in V, where f_1 and g_1 generate F_1 et G_1. So $af_1 + bg_1 \in A \subset F_{2n+1-i}$, which implies $bg_1 \in F_{2n+1-i}$, hence $b = 0$ or $i = 0$. If $b = 0$, then $V = F_1$, and consequently $F_1 \subset G_1^{\perp}$, which is impossible by **(C9)**. So $i = 0$. But then $af_1 + bg_1 \in G_3$, so $af_1 \in G_3$ and also $a = 0$. Hence $V = G_1 \subset F_1^{\perp}$, which is excluded by **(C9)**.

   b) Let $A = F_{2n-i} \cap G_{i+3}$. By **(C3)**, $\dim A = 2$. By $\mathbf{(C5)}_i$, $\dim A \cap G_1^{\perp} = 1$, so $V = A \cap G_1^{\perp}$. Moreover $\dim V^{\perp} \cap F_2 = 1$. We have $W \supset V + G_1 + V^{\perp} \cap F_2$. To determine W, it is enough to show that the sum is direct. First, $V + G_1$ is direct, because if it was not we would have $V = G_1$, so $G_1 \subset F_{2n-i}$, which is impossible by **(C3)**. Finally the sum $V \oplus G_1 + V^{\perp} \cap F_2$ is direct, or we would have $V^{\perp} \cap F_2 \subset G_{i+3}$. But $\dim F_2 \cap G_{i+3} = 0$ by **(C3)** since $i \leq 2n-4$.

   c) This case is similar to 3b ; the proof uses **(C3)** and $\mathbf{(C6)}_i$.

   d) By **(C3)**, we get $\dim F_{2n-1} \cap G_{i+2} = 1$, so $V = F_{2n-1} \cap G_{i+2}$. We must have $\dim W \cap F_2 \neq 0$. But $V \not\subset F_2$, or else we would get $G_{i+2} \cap F_2 \neq 0$, which is impossible by **(C3)** since $i \leq 2n-4$. Now $W \subset V^{\perp}$ implies $W \cap F_2 \subset V^{\perp} \cap F_2$, which has dimension 1 by $\mathbf{(C7)}_i$. So $W \subset V^{\perp} \cap F_2 \oplus V$. Similarly, using $\mathbf{(C8)}_i$, we get $W \cap G_2 = V^{\perp} \cap G_2$, so $W \supset V \oplus V^{\perp} \cap F_2 + V^{\perp} \cap G_2$. Now we only have to show that this sum is direct. If not, then there exists a non-zero vector of the form $av + bf_2$ in $V^{\perp} \cap G_2$, where v and f_2 generate V and $V^{\perp} \cap F_2$. As $v \in G_{i+2}$, we obtain $bf_2 \in G_{i+2}$, so $b = 0$ because $i \leq 2n-4$. Hence $V^{\perp} \cap G_2 = V$ and consequently $V \subset G_2$ and $\dim F_{2n-i} \cap G_2 \leq 1$, which is impossible since $i \geq 2$.

□

2.4 Computation of some invariants in $\overline{\mathcal{M}}_{0,3}(\mathrm{IG}, 1)$

In the previous section we computed the two-pointed invariants in IG, which is equivalent to compute the quantum terms of the product by the hyperplane class τ_1. Indeed, the divisor axiom ([8]) yields :

$$I_1(\gamma_1, \gamma_2, \tau_1) = I_1(\gamma_1, \gamma_2),$$

where γ_1 and γ_2 are any cohomology classes. Hence to obtain a quantum Pieri rule for $\mathrm{IG}(2, 2n+1)$, we are left to compute the quantum product by $\tau_{1,1}$. So we have to determine all invariants of the form $I_1(\tau_{1,1}, \tau_\lambda, \tau_\mu)$ with $|\lambda| + |\mu| = 6n-5$, that is to compute the number of lines through the following subvarieties :

$$\begin{aligned}
Y_1 &= \{\Sigma \in \mathrm{IG} \mid \Sigma \cap F_{j+1} \neq 0, \Sigma \subset F_{2n+2-i-j}\} \\
Y_2 &= \{\Sigma \in \mathrm{IG} \mid \Sigma \cap G_{l+1} \neq 0, \Sigma \subset G_{i+3-l}\} \\
Y_3 &= \{\Sigma \in \mathrm{IG} \mid \Sigma \subset H\}
\end{aligned}$$

where $0 \leq i \leq 2n-1$, $0 \leq 2j \leq 2n-1-i$ and $0 \leq 2l \leq i$ are integers, $F_\bullet$ and $G_\bullet$ are isotropic flags and H is a hyperplane.

As before we use a genericity result which is proved in a similar way as lemma 9 :

Lemma 11. *Assume $n \geq 2$. The set of 4-uples $(F_\bullet, G_\bullet, H, \omega) \in \mathbb{F}_n \times \mathbb{F}_n \times \mathbb{P}^{2n} \times \Lambda_n$ satisfying the following conditions*

(C1) $\forall 0 \leq p \leq 2n+1$, $\omega_{|F_p}$ *has maximal rank ;*

(C2) $\forall 0 \leq p \leq 2n+1$, $\omega_{|G_p}$ *has maximal rank ;*

(C3) $\omega_{|H}$ *is symplectic ;*

(C4) $\forall 0 \leq p, q \leq 2n+1$, $F_p \cap G_q$ *has the expected dimension ;*

(C5) $\forall 0 \leq p, q \leq 2n+1$, $F_p \cap G_q \cap H$ *has the expected dimension ;*

(C6)$_i$ $\dim\left(F_{2n+2-i} \cap G_{i+3} \cap H \cap F_1^\perp \cap G_1^\perp\right) = 1$ *;* $(\mathbf{1 \leq i \leq 2n-2})$ *;*

(C7)$_i$ $\dim F_{2n+1-i} \cap G_{i+3} \cap H \cap G_1^\perp = 1$ *and* $\dim(F_{2n+1-i} \cap G_{i+3} \cap H \cap G_1^\perp)^\perp \cap F_2 = 1$*;* $(\mathbf{0 \leq i \leq 2n-3})$ *;*

(C8)$_i$ $\dim F_{2n+2-i} \cap G_{i+2} \cap H \cap F_1^\perp = 1$ *and* $\dim(F_{2n+2-i} \cap G_{i+2} \cap H \cap F_1^\perp)^\perp \cap G_2 = 1$*;* $(\mathbf{2 \leq i \leq 2n-1})$ *;*

(C9)$_i$ $\dim\left(F_{2n+1-i} \cap G_{i+2} \cap H\right)^\perp \cap F_2 = 1$ *;* $(\mathbf{2 \leq i \leq 2n-3})$ *;*

(C10)$_i$ $\dim\left(F_{2n+1-i} \cap G_{i+2} \cap H\right)^\perp \cap G_2 = 1$ *;* $(\mathbf{2 \leq i \leq 2n-3})$ *;*

(C11) $F_1 \not\subset G_1^\perp$ *;*

(C12) $G_1 \not\subset F_1^\perp$ *;*

(C13)$_i$ $F_{2n-i} \cap G_{i+3} \cap H \cap G_1^\perp = 0$ *;* $(\mathbf{0 \leq i \leq 2n-5})$ *;*

(C14)$_i$ $F_{2n+2-i} \cap G_{i+1} \cap H \cap F_1^\perp = 0$ *;* $(\mathbf{4 \leq i \leq 2n-1})$ *;*

(C15)$_i$ $F_2 \cap G_{i+3} \cap G_1^\perp = 0$ *;* $0 \leq i \leq 2n-3$ *;*

(C16)$_i$ $G_2 \cap F_{2n+2-i} \cap F_1^\perp = 0$ *;* $2 \leq i \leq 2n-1$.

is a dense open subset of $\mathbb{F}_n \times \mathbb{F}_n \times \mathbb{P}^{2n} \times \Lambda_n$.

Under these assumptions we can prove

Proposition 14. *1. The intersections $Y_i \cap \mathbb{O}$ are transverse. Moreover*

$$Y_1 \cap \mathbb{O} = \begin{cases} \emptyset & \text{if } i+j \geq 2 \\ \{F_1 \oplus K\} & \text{if } i=1 \text{ and } j=0 \\ \{K \oplus L \mid L \subset F_2\} & \text{if } i=0 \text{ and } j=1 \end{cases}$$

$$Y_2 \cap \mathbb{O} = \begin{cases} \emptyset & \text{and } i \neq 2n-2 \text{ or } l \neq 0 \\ \{G_1 \oplus K\} & \text{if } i=2n-2 \text{ and } l=0 \end{cases}$$

$$Y_3 \cap \mathbb{O} = \emptyset$$

2. If j ou $l \geq 2$, there is no line meeting Y_1, Y_2 and Y_3.

3. If j and $l \leq 1$, there is a unique line meeting Y_1, Y_2 and Y_3.

Proof. 1. The case of $Y_2 \cap \mathbb{O}$ has already been treated in the proof of Proposition 13. If $\Sigma \in Y_1 \cap \mathbb{O}$, we must have $K \subset F_{2n+2-i-j}$, so $i+j=1$. If $i=1$ and $j=0$, then $Y_1 \cap \mathbb{O} = \{K \oplus F_1\}$, and transversality is proven as in Proposition 13. If $i=0$ and $j=1$, then $Y_1 \cap \mathbb{O} = \{K \oplus L \mid L \subset F_2\}$. Take $\Sigma_0 = K \oplus < f_2 >$ where f_2 is a non-zero element in F_2. Again we express $\mathrm{T}_{\Sigma_0} Y_1$ and $\mathrm{T}_{\Sigma_0} \mathbb{O}$ as subspaces of $\mathrm{T}_{\Sigma_0} G$, where G is the usual Grassmannian :

$$\mathrm{T}_{\Sigma_0} Y_1 = \{\phi \in \mathrm{T}_{\Sigma_0} G \mid \phi(f_2) \in F_2 / < f_2 >, \phi(k) \perp f_2\}$$
$$\mathrm{T}_{\Sigma_0} \mathbb{O} = \{\phi \in \mathrm{T}_{\Sigma_0} G \mid \phi(k) = 0\},$$

with k a generator of K. We see that the intersection of $\mathrm{T}_{\Sigma_0} Y_1$ and $\mathrm{T}_{\Sigma_0} \mathbb{O}$ has dimension 1. Computing $\dim Y_1 = 2n-1$ and $\dim \mathbb{O} = 2n-1$ we conclude that they generate T_{Σ_0} IG. Finally, $Y_3 \cap \mathbb{O} = \emptyset$ since $K \not\subset H$ by **(C3)**.

2. By **(C5)**, $F_{2n+2-i-j} \cap G_{i+3-l} \cap H = 0$ as soon as $j+l \geq 3$. Moreover if $j=2$ and $l=0$ then we get $W \supset G_1$, hence $V \subset F_{2n-i} \cap G_{i+3} \cap H \cap G_1^{\perp}$. But this space is zero by $\mathbf{(C13)}_i$, so there is no line. By $\mathbf{(C13)}_i$, we get the same result when $j=0$ and $l=2$.

3. There are four cases :

   a) $j=l=0$;
   b) $j=1, l=0$;
   c) $j=0, l=1$;
   d) $j=l=1$.

   a) We have $V = F_{2n+2-i} \cap G_{i+3} \cap H \cap F_1^{\perp} \cap G_1^{\perp}$ by $\mathbf{(C6)}_i$. Moreover $W \supset V + F_1 + G_1$. To obtain equality we only have to show that the sum is direct. First $V \neq F_1$ since $F_1 \not\subset G_1^{\perp}$ by **(C11)**. Finally if $G_1 \subset V \oplus F_1$, as $V \subset F_1^{\perp}$, we would have $G_1 \subset F_1^{\perp}$, which is impossible by **(C12)**.
   b) We have $V = F_{2n+1-i} \cap G_{i+3} \cap H \cap G_1^{\perp}$ by $\mathbf{(C7)}_i$. Moreover $W \subset V + G_1 + F_2 \cap V^{\perp}$. We prove now that this sum is direct. First $V \neq G_1$, or we would have $G_1 \subset H$, which is excluded by **(C5)**. Now $F_2 \cap V^{\perp} \not\subset V \oplus G_1$ since $F_2 \cap G_{i+3} \cap G_1^{\perp} = 0$ for $i \leq 2n-3$ by $\mathbf{(C15)}_i$.
   c) $V = F_{2n+2-i} \cap G_{i+2} \cap H \cap F_1^{\perp}$ by $\mathbf{(C8)}_i$. Moreover $W \supset V + F_1 + G_2 \cap V^{\perp}$ (by $\mathbf{(C9)}_i$ and $\mathbf{(C10)}_i$), and this sum is direct (same argument than in the previous case, using condition $\mathbf{(C16)}_i$).
   d) $V = F_{2n+1-i} \cap G_{i+2} \cap H$, $W \supset V + W \cap F_2 + W \cap G_2 = V + F_2 \cap V^{\perp} + G_2 \cap V^{\perp}$. This sum is direct ; indeed, $F_2 \cap V^{\perp} \neq G_2 \cap V^{\perp}$ car $F_2 \cap G_2 = 0$ by **(C4)** ; in addition $V \not\subset F_2 \cap V^{\perp} \oplus G_2 \cap V^{\perp}$, or we would get $G_2 \cap F_{2n+1-i} \neq 0$, which is impossible by $i \geq 2$.

□

2.5 Quantum Pieri rule

We can now prove Theorem 1 :

Proof. We start with the invariants $I_1(\tau_1, \tau_{a,b}, \tau_{c,d})$, which are equal to the two-pointed invariants $I_1(\tau_{a,b}, \tau_{c,d})$ because of the divisor axiom. The first item of Proposition 13 enables us to apply the enumerativity theorem 2. Then we use the second item of Proposition 13. For $j=l=0$ we get that for all $0 \leq i \leq 2n-2$ we have $I_1(\tau_{2n-1,i}, \tau_{2n-1,2n-2-i}) = 1$. Then setting $j=0$ and $l>0$ we recursively get $I_1(\tau_{2n-1,i}, \tau_{2n-1-l,2n-2-i+l}) = 0$ (for all i and $l>0$). Finally, setting j and $l>0$ we get $I_1(\tau_{2n-1-j,i+j}, \tau_{2n-1-l,2n-2-i+l}) = 0$ (for all i and $j, l > 0$). Hence :

$$I_1(\tau_1, \tau_{a,b}, \tau_{c,d}) = \begin{cases} 1 \text{ if } a = c = 2n-1 \\ 0 \text{ if } a \text{ or } c < 2n-1. \end{cases}$$

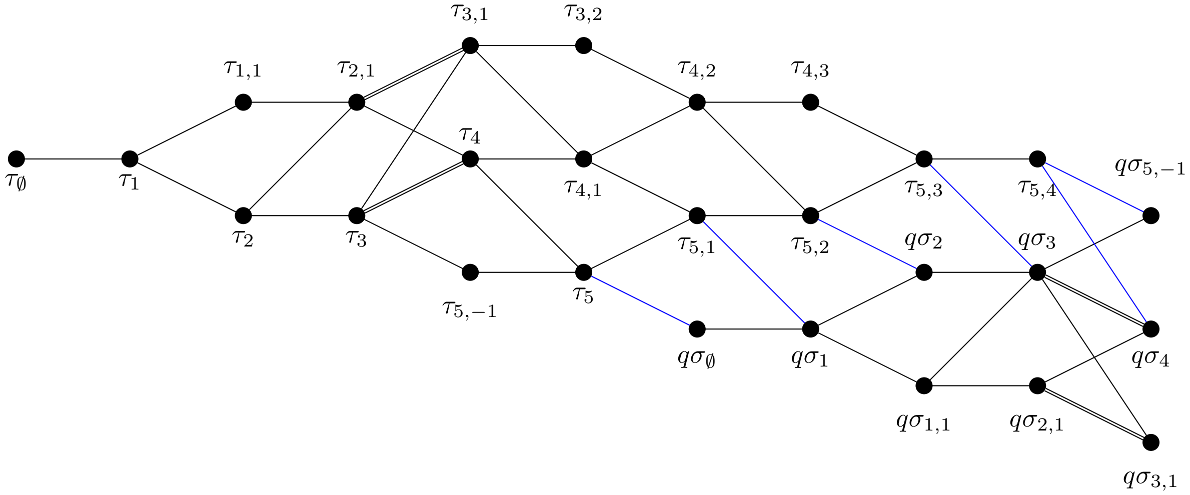

Figure 4: Quantum Hasse diagram of IG(2, 7)

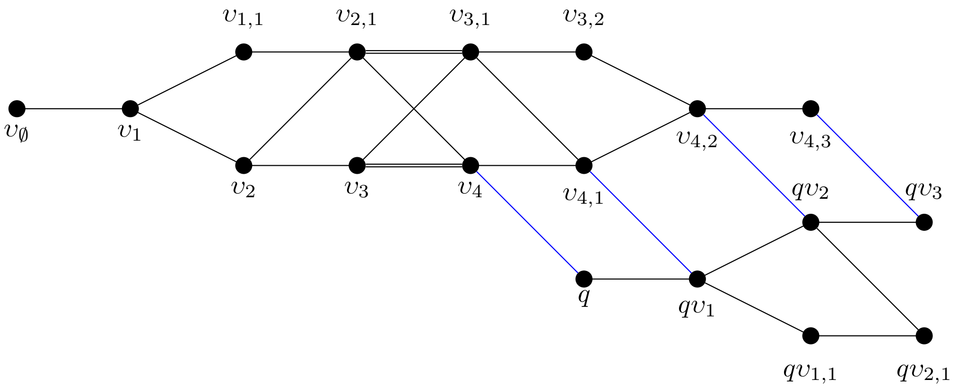

Figure 5: Quantum Hasse diagram of IG(2, 6)

Similarly, Proposition 14 and Theorem 2 imply

$$I_1(\tau_{1,1}, \tau_{a,b}, \tau_{c,d}) = \begin{cases} 1 \text{ if } a = c = 2n-1 \\ 0 \text{ if } a \text{ or } c < 2n-1. \end{cases}$$

Using the classical Pieri rule and Poincaré duality, we get our result. $\square$

Using the quantum Pieri formula we can fill out the Hasse diagram from figure 2 to obtain the quantum Hasse diagram of IG(2, 7) in figure 4. As a comparison see the quantum Hasse diagram of IG(2, 6) in figure 5.

2.6 Quantum presentation, semisimplicity

Proposition 15 (Presentation of $\mathrm{QH}^*(\mathrm{IG}(2,2n+1),\mathbb{Z})$)**.** *The ring* $\mathrm{QH}^*(\mathrm{IG}(2,2n+1),\mathbb{Z})$ *is generated by the classes* τ_1*,* $\tau_{1,1}$ *and the quantum parameter* q*. The relations are*

$$\det\left(\tau_{1^{1+j-i}}\right)_{1\le i,j\le 2n} = 0$$

$$\frac{1}{\tau_1}\det\left(\tau_{1^{1+j-i}}\right)_{1\le i,j\le 2n+1} + q = 0$$

Proof. Siebert and Tian proved in [12] that the quantum relations are obtained by evaluating the classical relations using the quantum product. Define δ_{2n} and δ'_{2n+1} as in the proof of Proposition 10 and denote by $\overline{\delta_{2n}}$ and $\overline{\delta'_{2n+1}}$ the same expressions with the cup product replaced by the quantum product.

Now we consider the quantum products $\Pi_a := (\tau_1)^{2(n-a)} \star (\tau_{1,1})^a$ for $0 \leq a \leq n$. For reasons of degree it has no q-term of degree greater than 1. First we prove that Π_a has no q-term if $a \neq 0, 1$. To prove this, we decompose Π_a for $a > 0$ as

$$\Pi_a = \tau_{1,1} \star \left((\tau_1)^{2(n-a)} (\tau_{1,1})^{a-1} \right).$$

Notice that for degree reasons, $(\tau_1)^{2(n-a)}(\tau_{1,1})^{a-1}$ has no q-term. Moreover, if $a \geq 2$, the classical Pieri formula 4 implies that this product contains only classes $\tau_{c,d}$ with $c < 2n-1$. Then we use the quantum Pieri formula 1 to conclude that there is no q-term in Π_a. We are now left with computing the q-term of Π_0 and Π_1. Set $\alpha_p := (\tau_1)^p$ for $p \leq 2n-1$. α_p has no q-term. We have $\Pi_0 = \tau_1 \star \alpha_{2n-1}$ and $\Pi_1 = \tau_{1,1} \star \alpha_{2n-2}$. We compute recursively the coefficients of τ_p and $\tau_{p-1,1}$ for $p \leq 2n-3$ in α_p using the classical Pieri rule. We find

$$\alpha_p = \tau_p + (p-1)\tau_{p-1,1} + \text{terms with lower first part.}$$

Then

$$\alpha_{2n-2} = \tau_{2n-1,-1} + (2n-2)\tau_{2n-2} + \text{terms with lower first part}$$

and

$$\alpha_{2n-1} = (2n-1)\tau_{2n-1} + \text{terms with lower first part.}$$

Finally we use the quantum Pieri rule to deduce :

$$\begin{aligned} \Pi_0 &= \text{classical terms} + (2n-1)q \\ \Pi_1 &= \text{classical terms} + q. \end{aligned}$$

But

$$\begin{aligned} \overline{\delta_{2n}} &= \Pi_0 - (2n-1)\Pi_1 + \text{linear combination of } \Pi_a\text{'s with } a \geq 2 \\ \overline{\delta'_{2n+1}} &= \Pi_0 - 2n\Pi_1 + \text{linear combination of } \Pi_a\text{'s with } a \geq 2, \end{aligned}$$

hence $\overline{\delta_{2n}} = \delta_{2n}$ and $\overline{\delta'_{2n+1}} = \delta'_{2n+1} - q$. ☐

Now we show that the quantum cohomology ring of IG $(2, 2n+1)$, localized at $q \neq 0$, is semisimple. To do this we use a presentation in terms of the Chern roots of the tautological bundle S, which makes the symmetries more apparent :

Theorem 3. *1. The ring* $\mathrm{QH}^*(\mathrm{IG}(2, 2n+1), \mathbb{Z})$ *is isomorphic to* $R^{\mathfrak{S}_2}$*, where*

$$R = \mathbb{Z}[x_1, x_2, q] / \left(h_{2n}(x_1, x_2), h_n(x_1^2, x_2^2) + q \right)$$

where x_1 and x_2 are the Chern roots of the tautological bundle S *and $h_r(y_1, \ldots, y_p)$ is the r-th complete symmetric function of the variables $y_1, \ldots, y_p$.*

2. $\mathrm{QH}^*(\mathrm{IG}(2, 2n+1), \mathbb{Z})_{q \neq 0}$ *is semisimple.*

Proof. 1. We use the recurrence relation 3 from Proposition 10 to prove that $\delta_r = h_r(x_1, x_2)$ for all r. Then :

$$\delta'_{2n+1} = \frac{h_{2n+1}(x_1, x_2)}{x_1 + x_2} = h_n(x_1^2, x_2^2).$$

2. It is enough to prove the semisimplicity of R localized at $q \neq 0$. We may assume $q = -1$. Using $(x_1 - x_2)h_{2n}(x_1, x_2) = x_1^{2n+1} - x_2^{2n+1}$ and noticing that we must have $x_2 \neq 0$, the first relation implies that $x_1 = \zeta x_2$, where $\zeta \neq 1$ is a $(2n+1)$-th root of unity. Replacing in the second relation $h_n(x_1^2, x_2^2) - 1 = 0$, we get $x_1^{2n} = 1 + \zeta$. Since $\zeta \neq -1$, this equation has $2n$ distinct solutions. So we have $2n$ distinct solutions for x_1, and for each x_1 we have $2n$

distinct solutions for x_2, which gives us (at least) $4n^2$ distinct solutions for the pair (x_1, x_2). But the number of solutions, counted with their multiplicity, should be equal to twice the rank of $\mathrm{H}^*(\mathrm{IG}(2, 2n+1), \mathbb{Z})$, which is equal to $2n^2$. So there are no other solutions, and all solutions are simple. Hence the semisimplicity.

□

Now recall the first part of Dubrovin's conjecture about the quantum cohomology of Fano varieties :

Conjecture (Dubrovin [5]). *Let X be a Fano variety. The big quantum cohomology of X is semisimple if and only its derived category of coherent sheaves $\mathcal{D}^b(Coh(X))$ admits a full exceptional collection.*

Remember that semisimplicity of the small quantum cohomology implies semisimplicity of the big one. So to confirm Dubrovin's conjecture for the case of the odd symplectic Grassmannian of lines it is enough to find a full exceptional collection. But in [9], Kuznetsov computed full exceptional collections for the symplectic Grassmannian of lines. His result can easily be adapted to the odd symplectic case, hence the result.

It should be mentioned that this doesn't work so well for the symplectic Grassmannian of lines. Indeed, although Kuznetsov has found a full exceptional collection for these varieties, Chaput and Perrin proved in [3] that their small quantum cohomology is not semisimple. What happens for the big quantum cohomology is still unknown.

References

[1] A.S. Buch, A. Kresch, and H. Tamvakis. A Giambelli formula for isotropic Grassmannians. *Arxiv preprint math/0811.2781 - arxiv.org*, 2008.

[2] A.S. Buch, A. Kresch, and H. Tamvakis. Quantum Pieri rules for isotropic Grassmannians. *Inventiones Mathematicae*, 178(2):345–405, 2009.

[3] P.E. Chaput and N. Perrin. On the quantum cohomology of adjoint varieties. *Arxiv preprint arXiv:0904.4824*, 2009.

[4] Izzet Coskun. A Littlewood-Richardson rule for two-step flag varieties. *Invent. Math.*, 176(2):325–395, 2009.

[5] Boris Dubrovin. Geometry and analytic theory of Frobenius manifolds. In *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998)*, number Extra Vol. II, pages 315–326 (electronic), 1998.

[6] Tom Graber. Enumerative geometry of hyperelliptic plane curves. *J. Algebraic Geom.*, 10(4):725–755, 2001.

[7] S. Kleiman. The transversality of a general translate. *Compositio Math.*, 28:287–297, 1974.

[8] M. Kontsevich and Y. Manin. Gromov-Witten classes, quantum cohomology, and enumerative geometry. *Communications in Mathematical Physics*, 164(3):525–562, 1994.

[9] Alexander Kuznetsov. Exceptional collections for Grassmannians of isotropic lines. *Proc. Lond. Math. Soc. (3)*, 97(1):155–182, 2008.

[10] I.A. Mihai. Odd symplectic flag manifolds. *Transformation groups*, 12(3):573–599, 2007.

[11] Dalide Pontoni. Quantum cohomology of $\mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$ and enumerative applications. *Trans. Amer. Math. Soc.*, 359(11):5419–5448, 2007.

[12] Bernd Siebert and Gang Tian. On quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator. *Asian J. Math.*, 1(4):679–695, 1997.

[13] H. Tamvakis. Quantum cohomology of homogeneous varieties: a survey. `http://www.math.umd.edu/~harryt/papers/report.07.pdf`.