Some approximation properties of Lupaş q-analogue of Bernstein operators

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Abstract

In this paper, we discuss rates of convergence for the Lupaş q-analogue of Bernstein polynomials $R_{n,q}$. We prove a quantitative variant of Voronovskaja's theorem for $R_{n,q}$.

Key words: q-Bernstein polynomials; Lupaş q-analogue; Voronovskaja-type formulas

1 Introduction

Let $q > 0$. For any $n \in N \cup \{0\}$, the q-integer $[n] = [n]_q$ is defined by

$$[n] := 1 + q + ... + q^{n-1}, \quad [0] := 0;$$

and the q-factorial $[n]! = [n]_q!$ by

$$[n]! := [1]\,[2]\,...\,[n]\,, \qquad [0]! := 1.$$

For integers $0 \leq k \leq n$, the q-binomial is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]!\,[n-k]!}.$$

In the last two decades interesting generalizations of Bernstein polynomials were proposed by Lupaş [6]

$$R_{n,q}(f,x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{\frac{k(k-1)}{2}} x^k (1-x)^{n-k}}{(1-x+qx)...(1-x+q^{n-1}x)}$$

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and by Phillips [10]

$$B_{n,q}\left(f,x\right)=\sum_{k=0}^{n}f\left(\frac{[k]}{[n]}\right)\begin{bmatrix} n \\ k \end{bmatrix}x^{k}\prod_{s=0}^{n-k-1}\left(1-q^{s}x\right).$$

The Phillips q-analogue of the Bernstein polynomials ($B_{n,q}$) attracted a lot of interest and was studied widely by a number of authors. A survey of the obtained results and references on the subject can be found in [8]. The Lupaş operators ($R_{n,q}$) are less known. However, they have an advantage of generating positive linear operators for all $q>0$, whereas Phillips polynomials generate positive linear operators only if $q\in(0,1)$. Lupaş [6] investigated approximating properties of the operators $R_{n,q}(f,x)$ with respect to the uniform norm of $C[0,1]$. In particular, he obtained some sufficient conditions for a sequence $\{R_{n,q}(f,x)\}$ to be approximating for any function $f\in C[0,1]$ and estimated the rate of convergence in terms of the modulus of continuity. He also investigated behavior of the operators $R_{n,q}(f,x)$ for convex functions. In [9] several results on convergence properties of the sequence $\{R_{n,q}(f,x)\}$ is presented. In particular, it is proved that the sequence $\{R_{n,q}(f,x)\}$ converges uniformly to $f(x)$ on $[0,1]$ if and only if $q_n\to 1$. On the other hand, for any $q>0$ fixed, $q\neq 1$, the sequence $\{R_{n,q}(f,x)\}$ converges uniformly to $f(x)$ if and only if $f(x)=ax+b$ for some $a,b\in R$.

In the paper, we investigate the rate of convergence for the sequence $\{R_{n,q}(f,x)\}$ by the modulii of continuity. We discuss Voronovskaja-type theorems for Lupaş operators for arbitrary fixed $q>0$. Moreover, for the Voronovskaja's asymptotic formula we obtain the estimate of the remainder term.

2 Auxiliary results

It will be convenient to use for $x\in[0,1)$ the following transformations

$$v=v\left(q,x\right):=\frac{qx}{1-x+qx},\quad v\left(q^{j},v\right)=\frac{q^{j}v}{1-v+q^{j}v}.$$

Let $0<q<1$. We set

$$b_{nk}(q;x):=\begin{bmatrix} n \\ k \end{bmatrix}\frac{q^{\frac{k(k-1)}{2}}x^{k}(1-x)^{n-k}}{(1-x+qx)...(1-x+q^{n-1}x)},\quad x\in[0,1],$$

$$b_{\infty k}(q;x):=\frac{q^{\frac{k(k-1)}{2}}\left(x/1-x\right)^{k}}{\left(1-q\right)^{k}[k]!\prod\limits_{j=0}^{\infty}\left(1+q^{j}\left(x/1-x\right)\right)},\quad x\in[0,1).$$

It was proved in [6] and [9] that for $q \in (0,1)$ and $x \in [0,1)$,

$$\sum_{k=0}^{n} b_{nk}(q;x) = \sum_{k=0}^{\infty} b_{\infty k}(q;x) = 1.$$

Definition 1 *Lupaş [6].The linear operator $R_{n,q} : C[0,1] \to C[0,1]$ defined by*

$$R_{n,q}(f,x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) b_{nk}(q;x)$$

is called the q-analogue of the Bernstein operator.

Definition 2 *The linear operator defined on $C[0,1]$ given by*

$$R_{\infty,q}(f,x) = \begin{cases} \sum_{k=0}^{\infty} f\left(1-q^k\right) b_{\infty k}(q;x) & \text{if} \quad x \in [0,1), \\ f(1) & \text{if} \quad x = 1. \end{cases}$$

is called the limit q-Lupaş operator.

It follows directly from the definition that operators $R_{n,q}(f,x)$ possess the end-point interpolation property, that is,

$$R_{n,q}(f,0) = f(0), \qquad R_{n,q}(f,1) = f(1)$$

for all $q > 0$ and all $n = 1, 2, ...$

Lemma 3 *We have*

$$b_{nk}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix} \prod_{j=0}^{k-1} v\left(q^j, x\right) \prod_{j=0}^{n-k-1} \left(1 - v\left(q^{k+j}, x\right)\right), \qquad x \in [0,1],$$

$$b_{\infty k}(q;x) = \frac{1}{(1-q)^k [k]!} \prod_{j=0}^{k-1} v\left(q^j, x\right) \prod_{j=0}^{\infty} \left(1 - v\left(q^{k+j}, x\right)\right), \qquad x \in [0,1].$$

It was proved in [6] and [9] that $R_{n,q}(f,x)$, $R_{\infty,q}(f,x)$ reproduce linear functions and $R_{n,q}(t^2,x)$ and $R_{\infty,q}(t^2,x)$ were explicitly evaluated. Using Lemma 3 we may write formulas for $R_{n,q}(t^2,x)$ and $R_{\infty,q}(t^2,x)$ in the compact form.

Lemma 4 *We have*

$$R_{n,q}(1,x) = 1, R_{n,q}(t,x) = x, R_{\infty,q}(1,x) = 1, R_{\infty,q}(t,x) = x,$$

$$R_{n,q}\left(t^2,x\right) = xv(q,x) + \frac{x(1-v(q,x))}{[n]},$$

$$R_{\infty,q}\left(t^2,x\right) = xv(q,x) + (1-q)x(1-v(q,x)) = x - qx(1-v(q,x)).$$

Now define

$$L_{n,q}(f,x) := R_{n,q}(f,x) - R_{\infty,q}(f,x)\,.$$

Lemma 5 *The following recurrence formulae hold*

$$R_{n,q}\left(t^{m+1},x\right) = R_{n,q}\left(t^{m},x\right) - (1-x)\frac{[n-1]^m}{[n]^m}R_{n-1,q}\left(t^{m},v\right), \tag{2.1}$$

$$R_{\infty,q}\left(t^{m+1},x\right) = R_{\infty,q}\left(t^{m},x\right) - (1-x)\,R_{\infty,q}\left(t^{m},v\right), \tag{2.2}$$

$$\begin{aligned} L_{n,q}(t^{m+1},x) &= L_{n,q}(t^{m},x) + (1-x) \\ &\times \left(\left(1-\frac{[n-1]^m}{[n]^m}\right)R_{\infty,q}(t^m,v) - \frac{[n-1]^m}{[n]^m}L_{n-1,q}(t^m,v)\right). \end{aligned} \tag{2.3}$$

Proof. First we prove (2.1). We write explicitly

$$R_{n,q}(t^{m+1},x) = \sum_{k=0}^{n}\frac{[k]^{m+1}}{[n]^{m+1}}\begin{bmatrix} n \\ k \end{bmatrix}\prod_{j=0}^{k-1}v\left(q^j,x\right)\prod_{j=0}^{n-k-1}\left(1-v\left(q^{k+j},x\right)\right) \tag{2.4}$$

and rewrite the first two factor in the following form:

$$\begin{aligned} \frac{[k]^{m+1}}{[n]^{m+1}}\begin{bmatrix} n \\ k \end{bmatrix} &= \frac{[k]^{m}}{[n]^{m}}\left(1-q^k\frac{[n-k]}{[n]}\right)\begin{bmatrix} n \\ k \end{bmatrix} \\ &= \frac{[k]^{m}}{[n]^{m}}\begin{bmatrix} n \\ k \end{bmatrix} - \frac{[n-1]^m}{[n]^m}\frac{[k]^m}{[n-1]^m}\begin{bmatrix} n-1 \\ k \end{bmatrix}q^k. \end{aligned} \tag{2.5}$$

Finally, if we substitute (2.5) in (2.4) we get (2.1):

$$\begin{aligned} R_{n,q}(t^{m+1},x) &= \sum_{k=0}^{n}\frac{[k]^{m}}{[n]^{m}}\begin{bmatrix} n \\ k \end{bmatrix}\prod_{j=0}^{k-1}v\left(q^j,x\right)\prod_{j=0}^{n-k-1}\left(1-v\left(q^{k+j},x\right)\right) \\ &-\frac{[n-1]^m}{[n]^m}(1-x)\sum_{k=0}^{n-1}\frac{[k]^m}{[n-1]^m}\begin{bmatrix} n-1 \\ k \end{bmatrix}\prod_{j=0}^{k-1}v\left(q^j,v(q,x)\right)\prod_{j=0}^{n-k-2}\left(1-v\left(q^{k+j},v(q,x)\right)\right) \\ &= R_{n,q}\left(t^m,x\right) - \frac{[n-1]^m}{[n]^m}(1-x)\,R_{n-1,q}\left(t^m,v(q,x)\right). \end{aligned}$$

Next we prove (2.3)

$$\begin{aligned}L_{n,q}(t^{m+1},x) &= R_{n,q}\left(t^{m+1},x\right) - R_{\infty,q}\left(t^{m+1},x\right)\\ &= R_{n,q}\left(t^{m},x\right) - (1-x)\frac{[n-1]^m}{[n]^m}R_{n-1,q}\left(t^{m},v\left(q,x\right)\right)\\ &\quad - R_{\infty,q}\left(t^{m},x\right) + (1-x)R_{\infty,q}\left(t^{m},v\left(q,x\right)\right)\\ &= L_{n,q}(t^m,x) + (1-x)\\ &\times\left(\left(1-\frac{[n-1]^m}{[n]^m}\right)R_{\infty,q}\left(t^{m},v\left(q,x\right)\right) - \frac{[n-1]^m}{[n]^m}L_{n-1,q}\left(t^{m},v\left(q,x\right)\right)\right).\end{aligned}$$

Formula (2.2) can be obtained from (2.1), by taking the limit as $n\to\infty$. ■

Moments $R_{n,q}\left(t^m,x\right)$, $R_{\infty,q}\left(t^m,x\right)$ are of particular importance in the theory of approximation by positive operators. In what follows we need explicit formulas for moments $R_{n,q}\left(t^3,x\right)$, $R_{\infty,q}\left(t^3,x\right)$.

Lemma 6 *We have*

$$R_{n,q}\left(t^3,x\right) = xv\left(q,x\right) + \frac{x\left(1-v\left(q,x\right)\right)}{[n]^2} - \frac{[n-1]\,[n-2]\,q^2}{[n]^2}x\left(1-v\left(q,x\right)\right)v\left(q^2,x\right),$$
$$R_{\infty,q}\left(t^3,x\right) = xv\left(q,x\right) + (1-q)^2x\left(1-v\left(q,x\right)\right) - q^2x\left(1-v\left(q,x\right)\right)v\left(q^2,x\right).$$

Proof. Note that explicit formulas for $R_{n,q}\left(t^m,x\right)$, $R_{\infty,q}\left(t^m,x\right)$, $m=0,1,2$ were proved in [6], [9]. Now we prove an explicit formula for $R_{n,q}\left(t^3,x\right)$, since formula for $R_{\infty,q}\left(t^3,x\right)$ can be obtained by taking limit as $n\to\infty$. The proof is based on the recurrence formula (2.1). Indeed,

$$\begin{aligned}R_{n,q}\left(t^3,x\right) &= R_{n,q}\left(t^2,x\right) - (1-x)\frac{[n-1]^2}{[n]^2}R_{n-1,q}\left(t^2,v\right)\\ &= xv\left(q,x\right) + \frac{x\left(1-v\left(q,x\right)\right)}{[n]} - (1-x)\frac{[n-1]^2}{[n]^2}v\left(q,x\right)v\left(q^2,x\right)\\ &- (1-x)\frac{[n-1]}{[n]^2}v\left(q,x\right) + (1-x)\frac{[n-1]}{[n]^2}v\left(q,x\right)v\left(q^2,x\right)\\ &= xv\left(q,x\right) + \frac{x\left(1-v\left(q,x\right)\right)}{[n]}\left(1-\frac{q\,[n-1]}{[n]}\right)\\ &- \frac{[n-1]}{[n]^2}\left([n-1]-1\right)qx\left(1-v\left(q,x\right)\right)v\left(q^2,x\right)\\ &= xv\left(q,x\right) + \frac{x\left(1-v\left(q,x\right)\right)}{[n]^2} - \frac{[n-1]\,[n-2]\,q^2}{[n]^2}x\left(1-v\left(q,x\right)\right)v\left(q^2,x\right).\end{aligned}$$

■

In order to prove Voronovskaja type theorem for $R_{n,q}\left(f,x\right)$ we also need explicit formulas and inequalities for $L_{n,q}\left(t^m,x\right)$, $m=2,3,4$.

Lemma 7 *Let $0 < q < 1$. Then*

$$L_{n,q}(t^2, x) = \frac{q^n}{[n]} x\left(1 - v\left(q, x\right)\right), \tag{2.6}$$

$$L_{n,q}(t^3, x) = \frac{q^n}{[n]^2} x\left(1 - v\left(q, x\right)\right) \tag{2.7}$$
$$\times \left[2 - q^n + [n-1]\left(1+q\right) v\left(q^2, x\right) + [n]\, qv\left(q^2, x\right)\right],$$

$$L_{n,q}(t^4, x) = \frac{q^n}{[n]^2} x\left(1 - v\left(q, x\right)\right) M\left(q, v\left(q^2, x\right), v\left(q^3, x\right)\right), \tag{2.8}$$

where M is a function of $\left(q, v\left(q^2, x\right), v\left(q^3, x\right)\right)$.

Proof. First we find a formula for $L_{n,q}\left(t^3, x\right)$. To do this we use the recurrence formula (2.3):

$$\begin{aligned}
& L_{n,q}\left(t^3, x\right) \\
& = L_{n,q}\left(t^2, x\right) + (1 - x) \\
& \times \left[\left(1 - \frac{[n-1]^2}{[n]^2}\right) R_{\infty,q}\left(t^2, v\left(q, x\right)\right) - \frac{[n-1]^2}{[n]^2} L_{n-1,q}\left(t^2, v\left(q, x\right)\right)\right] \\
& = \frac{q^n}{[n]} x\left(1 - v\left(q, x\right)\right) + (1 - x)\left(1 - \frac{[n-1]^2}{[n]^2}\right)\left[\left(1-q\right) v\left(q, x\right) + qv\left(q, x\right) v\left(q^2, x\right)\right] \\
& - (1 - x) \frac{[n-1]^2}{[n]^2} \frac{q^{n-1}}{[n-1]} v\left(q, x\right)\left(1 - v\left(q^2, x\right)\right) \\
& = \frac{q^n}{[n]^2} x\left(1 - v\left(q, x\right)\right) \\
& \times \left[[n] + \left(\frac{[n]^2 - [n-1]^2}{q^{n-1}}\right)\left(1 - q + qv\left(q^2, x\right)\right) - [n-1]\left(1 - v\left(q^2, x\right)\right)\right] \\
& = \frac{q^n}{[n]^2} x\left(1 - v\left(q, x\right)\right) \\
& \times \left[[n] + \left([n-1] + [n]\right)\left(1 - q + qv\left(q^2, x\right)\right) - [n-1]\left(1 - v\left(q^2, x\right)\right)\right] \\
& = \frac{q^n}{[n]^2} x\left(1 - v\left(q, x\right)\right) \\
& \times \left[[n] + 1 - q^{n-1} + 1 - q^n + [n-1]\left(1+q\right) v\left(q^2, x\right) + [n]\, qv\left(q^2, x\right) - [n-1]\right] \\
& = \frac{q^n}{[n]^2} x\left(1 - v\left(q, x\right)\right)\left[2 - q^n + [n-1]\left(1+q\right) v\left(q^2, x\right) + [n]\, qv\left(q^2, x\right)\right].
\end{aligned}$$

The proof of the equation (2.8) is also elementary, but tedious and complicated. Just notice that we use recurrence formula for $L_{n,q}\left(t^4, x\right)$ and clearly each term of the formula contains $\frac{q^n}{[n]^2} x\left(1 - v\left(q, x\right)\right)$. ■

Lemma 8 *We have*

$$L_{n,q}\left((t-x)^2,x\right)=\frac{q^n}{[n]}x\left(1-v\left(q,x\right)\right), \tag{2.9}$$

$$L_{n,q}\left((t-x)^3,x\right)=\frac{q^n}{[n]^2}x\left(1-v\left(q,x\right)\right) \tag{2.10}$$
$$\times\left[2-q^n+[n-1]\left(1+q\right)v\left(q^2,x\right)+[n]\,qv\left(q^2,x\right)-3\left[n\right]x\right],$$

$$L_{n,q}\left((t-x)^4,x\right)\leq K_1\frac{q^n}{[n]^2}x\left(1-v\left(q,x\right)\right), \tag{2.11}$$

where K_1 is a positive constant.

Proof. Proofs of (2.10) and (2.11) are based on (2.7), (2.8) and on the following identities.

$$L_{n,q}\left((t-x)^3,x\right)=L_{n,q}\left(t^3,x\right)-3xL_{n,q}\left((t-x)^2,x\right),$$
$$L_{n,q}\left((t-x)^4,x\right)=L_{n,q}\left(t^4,x\right)-4xL_{n,q}\left((t-x)^3,x\right)-6x^2L_{n,q}\left((t-x)^2,x\right).$$

■

3 Convergence properties

For $f\in C[0,1]$, $t>0$, the modulus of continuity $\omega(f,t)$ and the second modulus of smoothness $\omega_2(f,t)$ of f are defined by

$$\omega(f,t)=\sup_{|x-y|\leq t}|f(x)-f(y)|,$$
$$\omega_2(f,t)=\sup_{0\leq h\leq t}\sup_{0\leq x\leq 1-2h}|f(x+2h)-2f(x+h)+f(x)|.$$

In [9], it is proved that $b_{nk}(q;x)\to b_{\infty k}(q;x)$ uniformly in $x\in[0,1)$ as $n\to\infty$. In the next lemma we show that this convergence is uniform on $(0,q_0]\times[0,1)$ and give some estimates for $|b_{nk}(q;x)-b_{\infty k}(q;x)|$.

Lemma 9 *Let $0<q\leq q_0<1$, $k\geq 0$, $n\geq 1$.*

(i) For any $\varepsilon>0$ there exists $M>0$ such that

$$|b_{nk}(q;x)-b_{\infty k}(q;x)|\leq b_{nk}(q;x)M\left(\varepsilon\right)\frac{\left(q_0+\varepsilon\right)^n}{1-\left(q_0+\varepsilon\right)}+b_{\infty k}(q;x)\frac{q_0^{n-k+1}}{1-q_0}$$

for all $(q,x)\in(0,q_0]\times[0,1)$. In particular, $b_{nk}(q;x)$ converges to $b_{\infty k}(q;x)$ uniformly in $(q,x)\in(0,q_0]\times[0,1)$.

(ii) For any $x \in [0,1)$ we have

$$|b_{nk}(q;x) - b_{\infty k}(q;x)| \leq b_{nk}(q;x)\frac{x}{1-x}\frac{q^n}{1-q} + b_{\infty k}(q;x)\frac{q^{n-k+1}}{1-q}.$$

In particular, $b_{nk}(q;x)$ converges to $b_{\infty k}(q;x)$ uniformly in $(q,x) \in (0,q_0] \times [0,a]$, $0 < a < 1$.

Proof. We only prove part (i), since the proof of (ii) is similar to that of (i). Standard computations show that

$$\begin{aligned}
&|b_{nk}(q;x) - b_{\infty k}(q;x)| \\
&= \Bigg| \begin{bmatrix} n \\ k \end{bmatrix} \prod_{j=0}^{k-1} v\left(q^j, x\right) \prod_{j=0}^{n-k-1} \left(1 - v\left(q^{k+j}, x\right)\right) \\
&- \frac{1}{(1-q)^k\,[k]!} \prod_{j=0}^{k-1} v\left(q^j, x\right) \prod_{j=0}^{\infty} \left(1 - v\left(q^{k+j}, x\right)\right) \Bigg| \\
&= \Bigg| \begin{bmatrix} n \\ k \end{bmatrix} \prod_{j=0}^{k-1} v\left(q^j, x\right) \left(\prod_{j=0}^{n-k-1} \left(1 - v\left(q^{k+j}, x\right)\right) - \prod_{j=0}^{\infty} \left(1 - v\left(q^{k+j}, x\right)\right) \right) \\
&+ \prod_{j=0}^{k-1} v\left(q^j, x\right) \prod_{j=0}^{\infty} \left(1 - v\left(q^{k+j}, x\right)\right) \left(\begin{bmatrix} n \\ k \end{bmatrix} - \frac{1}{(1-q)^k\,[k]!} \right) \Bigg| \\
&\leq b_{nk}(q;x) \left| 1 - \prod_{j=n}^{\infty} \left(1 - v\left(q^j, x\right)\right) \right| + b_{\infty k}(q;x) \left| \prod_{j=n-k+1}^{n} (1 - q^j) - 1 \right|. \qquad (3.1)
\end{aligned}$$

Now using the inequality

$$1 - \prod_{j=1}^{k} (1 - a_j) \leq \sum_{j=1}^{k} a_j, \ \ (a_1, a_2, ..., a_k \in (0,1), \ \ k = 1, 2, ..., \infty),$$

we get from (3.1) that

$$|b_{nk}(q;x) - b_{\infty k}(q;x)| \leq b_{nk}(q;x) \sum_{j=n}^{\infty} v\left(q^j, x\right) + b_{\infty k}(q;x) \sum_{j=n-k+1}^{n} q^j. \qquad (3.2)$$

On the other hand, $\lim_{j\to\infty} \dfrac{v\left(q^{j+1}, x\right)}{v\left(q^j, x\right)} = q < 1$ and observe for any $\varepsilon > 0$ such that $q_0 + \varepsilon < 1$ there exists $n^* \in \mathbb{N}$ such that

$$\frac{v\left(q^{j+1}, x\right)}{v\left(q^j, x\right)} < q_0 + \varepsilon = \frac{(q_0+\varepsilon)^{j+1}}{(q_0+\varepsilon)^j}$$

for all $j > n^*$. Hence, the sequence $v\left(q^j, x\right) / \left(q_0+\varepsilon\right)^j$ is decreasing for large j

and thus uniformly bounded in $(q, x) \in (0, q_0] \times [0, 1)$ by

$$M(\varepsilon) = \max\left\{ \frac{v\left(q^{n^*+1}, x\right)}{(q_0+\varepsilon)^{n^*+1}}, \frac{v\left(q^{n^*}, x\right)}{(q_0+\varepsilon)^{n^*}}, ..., \frac{v(q, x)}{q_0+\varepsilon} \right\}.$$

So, for such $M(\varepsilon) > 0$ we have $|v(q^j, x)| \leq M(\varepsilon)(q_0+\varepsilon)^j$ for all $j = 1, 2, ...$ and $(q, x) \in (0, q_0] \times [0, 1)$.

Now from (3.2) we get the desired inequality

$$|b_{nk}(q;x) - b_{\infty k}(q;x)| \leq b_{nk}(q;x) M(\varepsilon) \frac{(q_0+\varepsilon)^n}{1-(q_0+\varepsilon)} + b_{\infty k}(q;x) \frac{q_0^{n-k+1}}{1-q_0}.$$

■

Before proving the main results notice that the following theorem proved in [9] will allow us to reduce the case $q \in (1, \infty)$ to the case $q \in (0, 1)$.

Theorem 10 *Let $f \in C[0,1]$, $g(x) := f(1-x)$. Then for any $q > 1$,*

$$R_{n,q}(f, x) = R_{n,\frac{1}{q}}(g, 1-x) \qquad \text{and} \quad R_{\infty,q}(f, x) = R_{\infty,\frac{1}{q}}(g, 1-x).$$

Using Lemma 9 we prove the following quantitative result for the rate of local convergence of $R_{n,q}(f, x)$ in terms of the first modulus of continuity.

Theorem 11 *Let $0 < q < 1$ and $f \in C[0,1]$. Then for all $0 \leq x < 1$ we have*

$$|R_{n,q}(f, x) - R_{\infty,q}(f, x)| \leq \frac{2}{1-q}\frac{1}{1-x}\omega(f, q^n).$$

Proof. Consider

$$\Delta(x) := R_{n,q}(f,x) - R_{\infty,q}(f,x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) b_{nk}(q;x) - \sum_{k=0}^{\infty} f\left(1-q^k\right) b_{\infty k}(q;x).$$

Since $R_{n,q}(f,x)$ and $R_{\infty,q}(f,x)$ possess the end point interpolation property $\Delta(0) = \Delta(1) = 0$. For all $x \in (0,1)$ we rewrite Δ in the following form

$$\begin{aligned} \Delta(x) = \sum_{k=0}^{n} &\left[f\left(\frac{[k]}{[n]}\right) - f\left(1-q^k\right)\right] b_{nk}(q;x) \\ &+ \sum_{k=0}^{n}\left[f\left(1-q^k\right) - f(1)\right](b_{nk}(q;x) - b_{\infty k}(q;x)) \\ &- \sum_{k=n+1}^{\infty}\left[f\left(1-q^k\right) - f(1)\right] b_{\infty k}(q;x) =: I_1 + I_2 + I_3. \end{aligned}$$

We start with estimation of I_1 and I_3. Since

$$0 \le \frac{[k]}{[n]} - \left(1-q^k\right) = \frac{1-q^k}{1-q^n} - \left(1-q^k\right) = q^n \frac{1-q^k}{1-q^n} \le q^n,$$
$$0 \le 1 - \left(1-q^k\right) = q^k \le q^n, \qquad k > n,$$

we get

$$|I_1| \le \omega\left(f, q^n\right) \sum_{k=0}^{n} b_{nk}(q;x) = \omega\left(f, q^n\right), \tag{3.3}$$
$$|I_3| \le \omega\left(f, q^n\right) \sum_{k=n+1}^{\infty} b_{\infty k}(q;x) \le \omega\left(f, q^n\right). \tag{3.4}$$

Finally we estimate I_2. Using the property of the modulus of continuity

$$\omega\left(f, \lambda t\right) \le (1+\lambda)\, \omega\left(f, t\right), \quad \lambda > 0$$

and Lemma 9 we get

$$\begin{aligned}
|I_2| &\le \sum_{k=0}^{n} \omega\left(f, q^k\right) \left|b_{nk}(q;x) - b_{\infty k}(q;x)\right| \\
&\le \omega\left(f, q^n\right) \sum_{k=0}^{n} \left(1 + q^{k-n}\right) \left|b_{nk}(q;x) - b_{\infty k}(q;x)\right| \\
&\le 2\omega\left(f, q^n\right) \frac{1}{q^n} \sum_{k=0}^{n} q^k \left|b_{nk}(q;x) - b_{\infty k}(q;x)\right| \\
&\le 2\omega\left(f, q^n\right) \frac{1}{q^n} \sum_{k=0}^{n} q^k \left(b_{nk}(q;x) \frac{x}{1-x} \frac{q^n}{1-q} + b_{\infty k}(q;x) \frac{q^{n-k+1}}{1-q}\right) \\
&\le \frac{2}{1-q} \left(\frac{x}{1-x} + 1\right) \omega\left(f, q^n\right) = \frac{2}{1-q} \frac{1}{1-x} \omega\left(f, q^n\right).
\end{aligned} \tag{3.5}$$

From (3.3), (3.4), and (3.5), we conclude the desired estimation. ■

Corollary 12 *Let $q > 1$ and $f \in C[0,1]$. Then for all $0 < x \le 1$ we have*

$$\left|R_{n,q}(f,x) - R_{\infty,q}(f,x)\right| \le \frac{2q}{q-1} \frac{1}{x} \omega\left(g, q^{-n}\right).$$

Proof. Proof follows from Theorems 11 and 10. ■

Next corollary gives quantitative result for the rate of uniform convergence of $R_{n,q}(f,x)$ in $C[0,a]$ and $C[a,1]$, $0 < a < 1$.

Corollary 13 *Let $f \in C[0,1]$, $0 < a < 1$.*

(1) If $0 < q < 1$, then

$$\left\|R_{n,q}(f) - R_{\infty,q}(f)\right\|_{C[0,a]} \le \frac{2}{1-q} \frac{1}{1-a} \omega\left(f, q^n\right).$$

(2) *If $q > 1$, then*

$$\|R_{n,q}(f) - R_{\infty,q}(f)\|_{C[a,1]} \leq \frac{2q}{q-1}\frac{1}{a}\omega\left(g, q^{-n}\right).$$

In order to prove the estimation in terms of the second modulus of continuity we need the following theorem proved in [16].

Theorem 14 *[16] Let $\{T_n\}$ be a sequence of positive linear operators on $C[0,1]$ satisfying the following conditions:*

(A) *the sequence $\{T_n(t^2)(x)\}$ converges uniformly on $[0,1]$;*

(B) *the sequence $\{T_n(f)(x)\}$ is nonincreasing in n for any convex function f and any $x \in [0,1]$.*

Then there exists an operator T_∞ on $C[0,1]$ such that

$$T_n(f)(x) \to T_\infty(f)(x)$$

as $n \to \infty$ uniformly on $[0,1]$. In addition, the following estimation holds:

$$|T_n(f)(x) - T_\infty(f)(x)| \leq C\omega_2\left(f; \sqrt{\lambda_n(x)}\right),$$

where ω_2 is the second modulus of smoothness, $\lambda_n(x) = |T_n(t^2)(x) - T_\infty(t^2)(x)|$, and C is a constant depending only on $T_1(1)$.

Theorem 15 *Let $0 < q < 1$. Then*

$$\|R_{n,q}(f) - R_{\infty,q}(f)\| \leq c\omega_2\left(f, \sqrt{q^n}\right). \tag{3.6}$$

Moreover,

$$\sup_{0<q\leq 1}\|R_{n,q}(f) - R_{\infty,q}(f)\| \leq c\omega_2\left(f, n^{-1/2}\right), \tag{3.7}$$

where c is a constant.

Proof. From [6], we know that the q-Bernstein operators satisfy condition (B) of Theorem 14. On the other hand

$$0 \leq R_{n,q}\left(t^2, x\right) - R_{\infty,q}\left(t^2, x\right) = \frac{q^n}{[n]}x(1 - v(q,x)) \leq q^n\frac{x(1-x)}{1-x+qx} \leq q^n \tag{3.8}$$

and

$$\sup_{0<q<1}\left|R_{n,q}\left(t^2, x\right) - R_{\infty,q}\left(t^2, x\right)\right| = \sup_{0<q<1}\frac{q^n}{[n]}\frac{x(1-x)}{1-x+qx} = \frac{x(1-x)}{n}.$$

Since

$$\left|R_{n,1}\left(t^2, x\right) - x^2\right| = \frac{x(1-x)}{n},$$

we conclude that

$$\sup_{0<q\leq 1}\left|R_{n,q}\left(t^2, x\right) - R_{\infty,q}\left(t^2, x\right)\right| \leq \frac{x(1-x)}{n} \leq \frac{1}{n}. \tag{3.9}$$

Theorem follows from (3.9), (3.8) and Theorem 14. ■

Theorem 16 *Let $q > 1$. Then*

$$\|R_{n,q}(f) - R_{\infty,q}(f)\| \leq c\omega_2\left(g, \sqrt{q^{-n}}\right).$$

Moreover,

$$\sup_{1\leq q<\infty} \|R_{n,q}(f) - R_{\infty,q}(f)\| \leq c\omega_2\left(g, n^{-1/2}\right), \tag{3.10}$$

where c is a constant.

Proof. The proof is similar to that of Theorem 15. ■

Remark 17 *From (3.7) and (3.10), we conclude that the rate of convergence $\|R_{n,q}(f) - R_{\infty,q}(f)\|$ can be dominated by $c\omega_2\left(f, n^{-1/2}\right)$ uniformly with respect to $q \neq 1$.*

Remark 18 *We may observe here that for $f(x) = x^2$, we have*

$$\|R_{n,q}(f) - R_{\infty,q}(f)\| \asymp q^n \asymp \omega_2\left(f, \sqrt{q^n}\right), \quad 0 < q < 1,$$

where $A(n) \asymp B(n)$ means that $A(n) \ll B(n)$ and $A(n) \gg B(n)$, and $A(n) \ll B(n)$ means that there exists a positive constant c independent of n such that $A(n) \leq cB(n)$. Hence the estimate (3.6) is sharp in the following sense: the sequence q^n in (3.6) cannot be replaced by any other sequence decreasing to zero more rapidly as $n \to \infty$.

4 Voronovskaja type results

Theorem 19 *Let $0 < q < 1$, $f \in C^2[0,1]$. Then there exists a positive absolute constant K such that*

$$\begin{aligned}&\left|\frac{[n]}{q^n}\left(R_{n,q}(f,x) - R_{\infty,q}(f,x)\right) - \frac{f''(x)}{2}x\left(1 - v(q,x)\right)\right| \\ &\leq Kx\left(1 - v(q,x)\right)\omega(f'', [n]^{-\frac{1}{2}}).\end{aligned} \tag{4.1}$$

Proof. Let $x \in (0,1)$ be fixed. We set

$$g(t) = f(t) - \left(f(x) + f'(x)(t-x) + \frac{f''(x)}{2}(t-x)^2\right).$$

It is known that (see [6]) if the function h is convex on $[0,1]$, then

$$R_{n,q}(h,x) \geq R_{n+1,q}(h,x) \geq ... \geq R_{\infty,q}(h,x),$$

and therefore,

$$L_{n,q}(h,x) := R_{n,q}(h,x) - R_{\infty,q}(h,x) \geq 0.$$

Thus $L_{n,q}$ is positive on the set of convex functions on $[0,1]$. But in general $L_{n,q}$ is not positive on $C\,[0,1]$.

Simple calculation gives

$$L_{n,q}\,(g,x) = (R_{n,q}(f,x) - R_{\infty,q}(f,x)) - \frac{q^n}{[n]}\frac{f''(x)}{2}x\,(1-v\,(q,x))\,.$$

In order to prove the theorem, we need to estimate $L_{n,q}\,(g,x)$. To do this, it is enough to choose a function $S\,(t)$ such that the functions $S\,(t) \pm g\,(t)$ are convex on $[0,1]$. Then $L_{n,q}\,(S \pm g, x) \geq 0$, and therefore,

$$|L_{n,q}\,(g(t),x)| \leq L_{n,q}\,(S(t),x)\,.$$

So the first thing to do is to find such function $S\,(t)$. Using the well-known inequality $\omega(f,\lambda\delta) \leq (1+\lambda^2)\omega(f,\delta)$ $(\lambda,\delta>0)$, we get

$$\begin{aligned}|g''(t)| &= |f''(t) - f''(x)| \leq \omega(f'',|t-x|)\\ &= \omega\left(f'', \frac{1}{[n]^{\frac{1}{2}}}\,[n]^{\frac{1}{2}}\,|t-x|\right) \leq \omega\left(f'', \frac{1}{[n]^{\frac{1}{2}}}\right)\left(\left(1+[n]\,(t-x)^2\right)\right).\end{aligned}$$

Define $S\,(t) = \omega\left(f'', [n]^{-\frac{1}{2}}\right)\left[\frac{1}{2}(t-x)^2 + \frac{1}{12}\,[n]\,(t-x)^4\right]$. Then

$$|g''(t)| \leq \frac{1}{6}\omega\left(f'', [n]^{-\frac{1}{2}}\right)\left(3(t-x)^2 + \frac{1}{2}\,[n]\,(t-x)^4\right)''_t = S''(t)$$

Hence the functions $S(t) \pm g(t)$ are convex on $[0,1]$, and therefore,

$$|L_{n,q}\,(g(t),x)| \leq L_{n,q}\,(S(t),x)\,,$$

and

$$L_{n,q}\,(S(t),x) = \frac{1}{6}\omega\left(f'', [n]^{-\frac{1}{2}}\right)\left(\frac{3q^n}{[n]}x\,(1-v\,(q,x)) + \frac{1}{2}\,[n]\,L_{n,q}\left((t-x)^4, x\right)\right).$$

Since by the formula (2.11)

$$L_{n,q}\left((t-x)^4, x\right) \leq K_1\frac{q^n}{[n]^2}x\,(1-v\,(q,x)) \tag{4.2}$$

we have

$$L_{n,q}\,(S(t),x) \leq \frac{1}{6}\omega\left(f'', [n]^{-\frac{1}{2}}\right)\left(3\frac{q^n}{[n]}x\,(1-v\,(q,x)) + \frac{1}{2}\,[n]\,K_1\frac{q^n}{[n]^2}x\,(1-v\,(q,x))\right). \tag{4.3}$$

By (4.2) and (4.3), we obtain (4.1). Theorem is proved. ■

Corollary 20 *Let* $q > 1$, $f \in C^2[0,1]$. *Then there exists a positive absolute* *constant K such that*

$$\left| q^n [n]_{\frac{1}{q}} \left(R_{n,q}(f,x) - R_{\infty,q}(f,x) \right) - \frac{f''(1-x)}{2} v(q,x)(1-x) \right|$$
$$\leq K v(q,x)(1-x)\, \omega(g'', [n]_{\frac{1}{q}}^{-\frac{1}{2}}).$$

Corollary 21 *If $f \in C^2[0,1]$ and $q_n \to 1$ as $n \to \infty$, then*

$$\lim_{q_n \uparrow 1} [n]_{q_n} \left(R_{n,q_n}(f,x) - f(x) \right) = \frac{f''(x)}{2} x(1-x), \tag{4.4}$$
$$\lim_{q_n \downarrow 1} [n]_{\frac{1}{q_n}} \left(R_{n,q_n}(f,x) - f(x) \right) = \frac{f''(1-x)}{2} x(1-x)$$

uniformly on $[0,1]$.

Remark 22 *When $q_n \equiv 1$, (4.4) reduces to the classical Voronovskaja's formula. For the function $f(t) = t^2$, the exact equality*

$$\frac{[n]_q}{q^n} \left(R_{n,q}(t^2,x) - R_{\infty,q}(t^2,x) \right) = x(1 - v(q,x)), \qquad 0 < q < 1,$$
$$q^n [n]_{\frac{1}{q}} \left(R_{n,q}(t^2,x) - R_{\infty,q}(t^2,x) \right) = v(q,x)(1-x), \qquad q > 1,$$

takes place without passing to the limit, but in contrast to the Phillips q-analogue of the Bernstein polynomials the right hand side depends on q. In contrast to the classical Bernstein polynomials and Phillips q-analogue of the Bernstein polynomials the exact equality

$$[n] \left(B_{n,q}(t^2,x) - x^2 \right) = \left(x^2 \right)'' x(1-x)/2$$

does not hold for the Lupaş q-analogue of the Bernstein polynomials.

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