HALF-FLAT STRUCTURES ON DECOMPOSABLE LIE GROUPS

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Abstract. Half-flat SU(3)-structures are the natural initial values for Hitchin's evolution equations whose solutions define parallel G_2-structures. Together with the results of [15], the results of this article completely solve the existence problem of left-invariant half-flat SU(3)-structures on decomposable Lie groups. The proof is supported by the calculation of the Lie algebra cohomology for all indecomposable five-dimensional Lie algebras which refines and clarifies the existing classification of five-dimensional Lie algebras.

1. Introduction

An SU(3)-structure on a six-dimensional real manifold M consists of an almost Hermitian structure (g, J, ω) and a unit $(3,0)$-form Ψ. The structure is called half-flat if it satisfies the exterior system

$$d\,\mathrm{Re}\Psi = 0\,, \quad d(\omega \wedge \omega) = 0.$$

Half-flat SU(3)-structures first appeared as initial values for the Hitchin flow [7] which is still the main motivation for studying their properties. In the physics literature, half-flat SU(3)-structures are considered as internal spaces for string compactifications [6]. For an account of the known results on half-flat structures and further references, the reader may consult for instance [15].

In order to obtain explicit examples and classification results, we assume a high degree of symmetry and consider left-invariant half-flat SU(3)-structures on Lie groups. Due to the left-invariance, the structure is completely determined by the solution of an algebraic system on the Lie algebra, which we refer to as a half-flat SU(3)-structure on a Lie algebra. A precise definition is given in section 2.

So far, the problem of classifying six-dimensional Lie algebras admitting a half-flat SU(3)-structure has been solved for nilpotent Lie algebras [2] and direct sums of three-dimensional Lie algebras [15]. The proof of both results is obtained by the following method. For each case in the known list of the Lie algebras in question, either a certain obstruction condition is applied or an explicit example for a half-flat SU(3)-structure is given.

In this article, the remaining decomposable six-dimensional Lie algebras are considered, separately dealing with direct sums of four- and two-dimensional Lie algebras and direct sums of a five-dimensional Lie algebra and $\mathbb{R}$. In particular, we use the classification of four- and five-dimensional Lie algebras which has been obtained by Mubarakzyanov, [10], [11]. More accessible lists can be found in [12] whose notation we adopt. In the four-dimensional case, a new complete proof of the classification was recently given in [1] including a thorough overview of the literature on four-dimensional Lie algebras.

A general problem of the existing classification lists is the appearance of families of Lie algebras depending on continuous parameters. For instance, the unimodularity of these families depends in many cases on the value of the parameters. In consequence, isomorphism classes with completely different properties are summed up in one "class". In the appendix, we give new lists of four- and five-dimensional Lie algebras in which the existing classes are subdivided according to the dimensions $\mathrm{h}^k(\mathfrak{g})$ of the Lie algebra cohomology groups and the dimension of the center. Not surprisingly, the distinction by Lie algebra cohomology turns out to be useful for our classification problem concerning closed three- and four-forms. The complexity and length of the new lists illustrate the diversity of the class of solvable Lie algebras which is well-known to be rapidly increasing with the dimension. A further subdivision of the classes using finer invariants seems to be possible. In fact, in one case even the existence or non-existence of a left-invariant half-flat SU(3)-structure singles out certain parameter values with identical Lie algebra cohomology.

We summarize the results of this article in the following theorems which are also charted in the tables of the appendix. The nilradical of a Lie algebra $\mathfrak{g}$ is denoted by $\mathrm{Nil}(\mathfrak{g})$.

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Theorem 1.1. *Let $\mathfrak{g}$ be a four-dimensional Lie algebra.*

(a) The direct sum $\mathfrak{g} \oplus \mathbb{R}^2$ admits a half-flat $\mathrm{SU}(3)$-structure if and only if $\mathfrak{g} = \mathfrak{u} \oplus \mathbb{R}$ for a unimodular three-dimensional Lie algebra $\mathfrak{u}$.

(b) The direct sum $\mathfrak{g} \oplus \mathfrak{r}_2$ admits a half-flat $\mathrm{SU}(3)$-structure if and only if

(i) $\mathfrak{g}$ is unimodular and not in $\{ A_{4,5}^{-\frac{1}{2},-\frac{1}{2}}, \mathfrak{h}_3 \oplus \mathbb{R}, \mathbb{R}^4 \}$ or

(ii) $\mathfrak{g}$ is in $\{ A_{4,9}^{-\frac{1}{2}}, A_{4,12}, \mathfrak{r}_2 \oplus \mathfrak{r}_2 \}$.

Theorem 1.2. *Let $\mathfrak{g}$ be an indecomposable five-dimensional Lie algebra.*

(a) If $\mathfrak{g}$ is unimodular, then $\mathfrak{g} \oplus \mathbb{R}$ admits a half-flat $\mathrm{SU}(3)$-structure if and only if

(i) $\mathfrak{g}$ is nilpotent and $\mathfrak{g} \neq A_{5,3}$ or

(ii) $\mathrm{Nil}(\mathfrak{g})$ is four-dimensional, $\mathrm{h}^2(\mathfrak{g}) \geq 2$ and $\mathfrak{g} \neq A_{5,9}^{-1,-1}$ or

(iii) $\mathrm{Nil}(\mathfrak{g})$ is $\mathbb{R}^3$ or $\mathbb{R}^2$.

(b) If $\mathfrak{g}$ is non-unimodular, then $\mathfrak{g} \oplus \mathbb{R}$ admits a half-flat $\mathrm{SU}(3)$-structure if and only if

(i) $\mathrm{Nil}(\mathfrak{g})$ is $\mathfrak{h}_3$ or

(ii) $\mathfrak{g}$ is in $\{ A_{5,19}^{-1,3}, A_{5,19}^{2,-3}, A_{5,30}^{0} \}$.

Theorem 1.1 and Theorem 1.2 are both proved in section 3. Obviously, the unimodular Lie algebras show a tendency to admit a half-flat SU(3)-structure, whereas the existence is obstructed on all non-unimodular solvable Lie algebras apart from two one-parameter families and few exceptions. Recall that unimodular Lie algebras are particularly interesting since unimodularity is a necessary condition for the existence of a cocompact lattice.

We remark that our results are in accordance with the following results concerning the existence of left-invariant hypo SU(2)-structures on five-dimensional Lie groups. The existence problem is solved for the nilpotent case in [4] and, very recently, for the solvable case in [3]. A hypo SU(2)-structure on a five-dimensional Lie algebra $\mathfrak{h}$ induces a half-flat SU(3)-structure on the six-dimensional Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$. Conversely, given a half-flat SU(3)-structure on a six-dimensional Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$, one can define a hypo SU(2)-structure on the five-dimensional Lie algebra $\mathfrak{h}$ if the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$ is orthogonal with respect to the induced euclidean metric. For all five-dimensional hypo structures constructed in [3] we independently found six-dimensional half-flat examples. However, for two five-dimensional, indecomposable Lie algebras which do not admit a hypo SU(2)-structure, namely $A_{5,19}^{-1,3}$ and $A_{5,37}$, we were able to find a half-flat SU(3)-structure on the corresponding six-dimensional Lie algebras $A_{5,19}^{-1,3} \oplus \mathbb{R}$ and $A_{5,37} \oplus \mathbb{R}$ such that the summands are not orthogonal, see Table 4.

In Section 4, we prove a number of secondary results concerning the existence of closed stable forms on Lie algebras and the existence of half-flat structures with indefinite metrics. More precisely, we determine the decomposable Lie algebras which do not admit closed stable forms and those which only admit closed stable forms ρ inducing a para-complex structure. In consequence, the first group of Lie algebras does not admit any half-flat structure and the second group does not admit half-flat $\mathrm{SU}(p,q)$-structures, $p+q=3$.

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2. Preliminaries

Due to the formalism of stable forms [8], it is possible to completely describe an SU(3)-structure by a pair (ω, ρ) of a two-form and three-form satisfying certain compatibilities. In fact, the formalism covers also $\mathrm{SU}(p,q)$-structures, $p+q=3$ and $\mathrm{SL}(3,\mathbb{R})$-structures. A thorough introduction including proofs is for instance given in [5] and [15] and we restrict ourself to a short repetition of the formulas we need.

2.1. **Stable forms in dimension six.** A p-form on a vector space V is called *stable* if its orbit under $\mathrm{GL}(V)$ is open. We only consider the case when V is an oriented six-dimensional real vector space. Let $\kappa : \Lambda^5 V^* \to V \otimes \Lambda^6 V^*$ be the canonical isomorphism $\kappa(\xi) := X \otimes \nu$ with $X \lrcorner \nu = \xi$. By defining

$$K_\rho(v) := \kappa\left((v \lrcorner \rho) \wedge \rho\right) \in V \otimes \Lambda^6 V^*,$$

$$\lambda(\rho) := \frac{1}{6} \mathrm{tr} K_\rho^2 \in \left(\Lambda^6 V^*\right)^{\otimes 2},$$

a quartic invariant λ is associated to each three-form $\rho \in \Lambda^3 V^*$. Since ρ is stable if and only if $\lambda(\rho) \neq 0$, a stable three-form ρ defines a volume form by

$$\phi(\rho) := \sqrt{|\lambda(\rho)|} \in \Lambda^6 V^*,$$

where the positively oriented root is chosen. The endomorphism

$$J_\rho := \frac{1}{\phi(\rho)} K_\rho$$

turns out to be a complex structure if $\lambda(\rho) < 0$ and a para-complex structure if $\lambda(\rho) > 0$. In both cases, a $(3,0)$-form Ψ can be defined by $\mathrm{Re}(\Psi) = \rho$ and $\mathrm{Im}(\Psi) = J_\rho^* \rho$. On one-forms, the (para-)complex structure acts by the formula

$$J_\rho^* \alpha(v)\, \phi(\rho) \quad = \quad \alpha \wedge (v \lrcorner \rho) \wedge \rho, \qquad v \in V,\ \alpha \in V^*. \tag{2.1}$$

A two-form $\omega \in \Lambda^2 V^*$ is stable if and only if it is non-degenerate, i.e.

$$\phi(\omega) := \frac{1}{6}\omega^3 \neq 0.$$

A pair $(\omega, \rho) \in \Lambda^2 V^* \times \Lambda^3 V^*$ of stable forms is called *compatible* if

$$\omega \wedge \rho = 0.$$

Such a pair induces a pseudo-Euclidean metric $g = \omega(J_\rho\,.\,,\,.)$ on V. On one-forms, the induced metric satisfies the identity

$$\alpha \wedge J_\rho^* \beta \wedge \omega^2 \quad = \quad \frac{1}{2} g(\alpha, \beta)\omega^3, \quad \alpha, \beta \in V^*, \tag{2.2}$$

if the pair (ω, ρ) is *normalized* by the condition

$$\phi(\rho) = 2\phi(\omega) \quad \Longleftrightarrow \quad J_\rho^* \rho \wedge \rho = \frac{2}{3}\,\omega^3.$$

2.2. **Half-flat structures on Lie algebras.** By definition, an SU(3)-*structure* on a six-manifold M is a reduction of the frame bundle of M to SU(3). It is well-known that there is a one-to-one correspondence between SU(3)-structures and quadrupels (g, J, ω, Ψ) where (g, J, ω) is an almost Hermitian structure and Ψ is a unit $(3,0)$-form. Due to the formalism of stable forms, SU(3)-structures are also one-to-one with pairs $(\omega, \rho) \in \Omega^2 M \times \Omega^3 M$ such that (ω_p, ρ_p) is for every $p \in M$ a compatible and normalized pair of stable forms on $T_p M$ with $\lambda(\rho_p) < 0$ inducing a Riemannian metric.

Completely analogously, one can deal with indefinite metrics leading to SU(p,q)-*structures*, $p + q = 3$, and even almost para-complex instead of almost complex structures leading to SL$(3,\mathbb{R})$-*structures* with $\lambda(\rho) > 0$. Since we are mainly concerned with the Riemannian case in this article, we refer the reader to [5] and [14] for definitions and properties of these structures and further references.

Unifying all cases, a *half-flat structure* is defined as an everywhere compatible and normalized pair $(\omega, \rho) \in \Omega^2 M \times \Omega^3 M$ of stable forms satisfying the exterior system $d\rho = 0$, $d\omega^2 = 0$.

For a Lie algebra $\mathfrak{g}$ of a Lie group G, the well-known formula

$$d\alpha(X, Y) = -\alpha([X, Y])$$

holds for all $X, Y \in \mathfrak{g}$ and $\alpha \in \mathfrak{g}^*$ when identifying one-forms $\alpha \in \mathfrak{g}^*$ with left-invariant one-forms on G. For an abstract Lie algebra, one can define an exterior derivative d on $\Lambda^* \mathfrak{g}^*$ by this formula. In consequence, the structure of a Lie algebra is equivalently encoded in a dual way by the exterior derivative on one-forms. Since the Jacobi identity holds if and only if d^2 vanishes, $(\Lambda^* \mathfrak{g}^*, d)$ defines a cohomology $H^*(\mathfrak{g})$ called Lie algebra cohomology or Chevalley-Eilenberg cohomology with respect to the trivial representation.

We define a *half-flat structure on a Lie algebra* $\mathfrak{g}$ as a compatible and normalized pair $(\omega, \rho) \in \Lambda^2(\mathfrak{g}^*) \times \Lambda^3(\mathfrak{g}^*)$ of stable forms satisfying the algebraic system

$$d\rho = 0, \quad d\omega^2 = 0.$$

By left-multiplication, a half-flat structure on a Lie algebra can obviously be extended to a half-flat structure on every corresponding Lie group. Hence, it suffices to consider half-flat structures on Lie algebras for the classification results we are interested in.

3. Obstruction theory for half-flat SU(3)-structures

An obstruction to the existence of a half-flat SU(3)-structure on a Lie algebra is proved in [2, Theorem 2] which is based on the vanishing of two cohomology groups of a certain double complex. The idea is simplified in [15, Proposition 4.2]. These obstruction conditions are the essential means in proving the classification results obtained in [2] and [15]. However, for a small number of special cases the obstruction condition fails, although there do not exist half-flat SU(3)-structures. In [2, Lemma 8] and [15, Lemma 4.9], refined obstructions are applied to these special cases. The ideas of both refinements are clarified and generalized in the following proposition.

Proposition 3.1. *Let M be a six-manifold. If there is a point $p \in M$ with a neighborhood U and a one-form $\alpha \in \Omega^1(U)$ with $\alpha_p \neq 0$ which satisfies the identity*

$$(\alpha \wedge J_\rho^* \alpha \wedge \sigma)_p = 0 \tag{3.1}$$

for all closed stable three-forms $\rho \in \Omega^3(U)$ and all closed stable four-forms $\sigma \in \Omega^4(U)$, then M does not admit a half-flat SU(3)*-structure.*

Proof. Suppose (ω, ρ) is a half-flat SU(3)-structure on M. In particular, the restrictions $\rho \in \Omega^3(U)$ and $\sigma = \frac{1}{2}\omega^2 \in \Omega^4(U)$ are closed. However, a one-form α_p in T_p^*M satisfying equation (3.1) is a non-trivial null-vector according to identity (2.2). This proves the proposition by contradiction since the metric of an SU(3)-structure is Riemannian. ☐

The application of this condition seems to be difficult in general, since an explicit solution of the PDE system $d\rho = 0$, $d\sigma = 0$ will be hard to obtain. For the algebraic case of half-flat structures on Lie algebras, however, a solution is easy to find as the exterior system reduces to a linear system. For a Lie algebra $\mathfrak{g}$, let Z^p denote the space of closed p-forms in $\Lambda^p\mathfrak{g}^*$.

Corollary 3.2. *Let $\mathfrak{g}$ be a six-dimensional Lie algebra with a volume form $\nu \in \Lambda^6\mathfrak{g}^*$. If there is a one-form $\alpha \in \mathfrak{g}^*$ satisfying*

$$\alpha \wedge \tilde{J}_\rho^* \alpha \wedge \sigma = 0 \tag{3.2}$$

for all $\rho \in Z^3$ and all $\sigma \in Z^4$, where $\tilde{J}_\rho^*\alpha$ is defined for $X \in \mathfrak{g}$ by

$$\tilde{J}_\rho^*\alpha(X)\,\nu \quad = \quad \alpha \wedge (X \lrcorner \rho) \wedge \rho, \tag{3.3}$$

then $\mathfrak{g}$ does not admit a half-flat SU(3)*-structure.*

Proof. If ρ is a stable form, the one-form $\tilde{J}_\rho^*\alpha$ is a non-trivial multiple of $J_\rho^*\alpha$ by equation (2.1). Thus, the corollary follows immediately when applying Proposition 3.1 to a corresponding Lie group. ☐

Proposition 3.3. *Let $\mathfrak{g}$ be an indecomposable four-dimensional Lie algebra.*

(i) The direct sum $\mathfrak{g} \oplus \mathbb{R}^2$ does not admit a half-flat SU(3)*-structure.*

(ii) The direct sum $\mathfrak{g} \oplus \mathfrak{r}_2$ does not admit a half-flat SU(3)*-structure if $\mathfrak{g}$ is not unimodular and not $A_{4,9}^{-\frac{1}{2}}$ or $A_{4,12}$.*

(iii) The direct sum $A_{4,5}^{-\frac{1}{2},-\frac{1}{2}} \oplus \mathfrak{r}_2$ does not admit a half-flat SU(3)*-structure.*

Proof. We apply Corollary 3.2 for all cases separately according to Table 1. Let $(e^1, \ldots, e^6)$ be a basis of $\mathfrak{g}^* \oplus \mathfrak{h}^*$, $\mathfrak{h} = \mathfrak{r}_2$ or $\mathfrak{h} = \mathbb{R}^2$, such that $(e^1, \ldots, e^4)$ is the standard basis of $\mathfrak{g}^*$ given in Table 1 and (e^5, e^6) is a basis of $\mathfrak{h}^*$. If $\mathfrak{h} = \mathfrak{r}_2$ we always choose (e^5, e^6) such that $de^5 = 0$, $de^6 = e^{56}$. We claim that $\alpha = e^4$ is for all cases except $A_{4,5}^{-\frac{1}{2},-\frac{1}{2}} \oplus \mathfrak{r}_2$ a one-form satisfying the obstruction condition (3.2).

In fact, the equation can be efficiently verified by a computer algebra system as follows. Let ρ be a three-form and σ a four-form involving altogether 35 coefficients when expressed with respect to the induced basis on forms. Due to our distinction of the Lie algebra classes in Table 1, the coefficient equations of $d\rho = d\sigma = 0$ can be solved in a closed form, independently of the parameters in the Lie bracket. Thus, the computer can almost instantaneously provide us with explicit expressions for the general closed three-form $\rho \in Z^3$ and also for the general closed four-form $\sigma \in Z^4$ by eliminating a number of parameters. Now, it is straightforward to compute $\tilde{J}_\rho$ via (3.3) with respect to the basis. The result allows us to verify equation (3.2) for $\alpha = e^4$ and all $\rho \in Z^3$ and all $\sigma \in Z^4$ for each Lie algebra which falls into case (i) or (ii).

Unfortunately, the non-existence of a half-flat SU(3)-structure cannot be proved with this method in case (iii). However, a different obstruction can be established as follows. On the Lie algebra $A_{4,5}^{-\frac{1}{2},-\frac{1}{2}} \oplus \mathfrak{r}_2$, a straightforward calculation yields the identity

$$e^5 \wedge \tilde{J}_\rho^* e^4 \wedge \sigma = -e^4 \wedge \tilde{J}_\rho^* e^5 \wedge \sigma = (e^4 + \sqrt{2}e^5) \wedge \tilde{J}_\rho^*(e^4 + \sqrt{2}e^5) \wedge \sigma$$

for all $\rho \in Z^3$ and all $\sigma \in Z^4$. Suppose that $A_{4,5}^{-\frac{1}{2},-\frac{1}{2}} \oplus \mathfrak{r}_2$ admits a half-flat SU(3)-structure (ρ_0, ω_0). In particular, the forms ρ_0 and $\sigma_0 := \frac{1}{2}\omega_0^2$ are closed and fulfill the previous identity. Hence, if g_0 denotes the induced Euclidean metric, formula (2.2) shows

$$g_0(\mathrm{e}^5, \mathrm{e}^4) = -g_0(\mathrm{e}^4, \mathrm{e}^5) = g_0(\mathrm{e}^4 + \sqrt{2}\mathrm{e}^5, \mathrm{e}^4 + \sqrt{2}\mathrm{e}^5).$$

This is not possible since $\mathrm{e}^4 + \sqrt{2}\mathrm{e}^5$ would be a null-vector and we have proved that there cannot exist a half-flat SU(3)-structure on $A_{4,5}^{-\frac{1}{2},-\frac{1}{2}} \oplus \mathfrak{r}_2$. ☐

Proof of Theorem 1.1. In order to determine all Lie algebras admitting a half-flat SU(3)-structure, we need to prove the existence or non-existence of a half-flat SU(3)-structure in every case. For the direct sums admitting a half-flat SU(3)-structure, an explicit example can be found either in Table 3 or in [15] or in [2]. On the remaining direct sums with decomposable four-dimensional summand, the existence is obstructed in [15]. The non-existence for the direct sums with indecomposable four-dimensional summand is proved in Proposition 3.3. Since we have covered all Lie algebras with a four-dimensional summand, the proof is finished. ☐

Proof of Theorem 1.2. The direct sums of indecomposable five-dimensional Lie algebras and $\mathbb{R}$ can be dealt with completely analogous to the direct sums considered before thanks to Table 2. Again, for all direct sums admitting a half-flat SU(3)-structure, an explicit example can be found either in Table 4 or in [2]. For the remaining direct sums $\mathfrak{g} \oplus \mathbb{R}$, let $(\mathrm{e}^1, \ldots, \mathrm{e}^6)$ be a basis of the dual space such that $(\mathrm{e}^1, \ldots, \mathrm{e}^5)$ is the standard basis given in Table 2 and e^6 is closed. Then, for all remaining cases, Corollary 3.2 can be applied with $\alpha = \mathrm{e}^5$ which has been verified with Maple as explained in the proof of Proposition 3.3. ☐

4. Closed stable three-forms on decomposable Lie algebras

In this section, we turn to the problem of determining the decomposable Lie algebras which do not admit a closed stable three-form ρ with $\lambda(\rho) < 0$. In fact, such Lie algebras do obviously not admit a half-flat $\mathrm{SU}(p,q)$-structure, $p + q = 3$, not even with an indefinite metric. Note that this question has already been answered for direct sums of three-dimensional Lie algebras in [15, Proposition 5.1].

Additionally, we can prove for a number of Lie algebras the non-existence of any closed stable three-form. Thus, there cannot exist a half-flat structure on these Lie algebras, not even a half-flat $\mathrm{SL}(3,\mathbb{R})$-structure.

Proposition 4.1. *Let $\rho \in \Lambda^3\mathfrak{g}^*$ be a closed three-form with quartic invariant $\lambda(\rho)$ on a Lie algebra $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_2$ such that the summand $\mathfrak{g}_4$ is indecomposable.*

(i) If $\mathfrak{g}_2 = \mathbb{R}^2$ and $\mathfrak{g}_4$ not in $\{A_{4,1}, A_{4,5}^{-1,1}, A_{4,9}^{-\frac{1}{2}}, A_{4,12}\}$, then $\lambda(\rho) \geq 0$.
(ii) If $\mathfrak{g}_2 = \mathfrak{r}_2$ and $Nil(\mathfrak{g}_4) = \mathbb{R}^3$ and $\mathrm{h}^*(\mathfrak{g}_4) = (1,0,0,0)$, then $\lambda(\rho) \geq 0$.
(iii) If $\mathfrak{g}_2 = \mathbb{R}^2$ and $\mathrm{h}^*(\mathfrak{g}_4) = (1,0,0,0)$, then $\lambda(\rho) = 0$.

Proof. Let ρ be a closed three-form on $\mathfrak{g}$. In the proof of Proposition 3.3, we explained that the general closed three-form is very straightforward to determine with computer support when $\mathfrak{g}_4$ is one of the classes appearing in Table 1. When calculating the quartic invariant $\lambda(\rho)$ for all Lie algebras in the proposition with the help of Maple, those with $\lambda = 0$ are easily determined. The cases with $\lambda \geq 0$ have been determined by applying the useful Maple function *factor* to $\lambda(\rho)$. ☐

Analogously, we can prove the following proposition.

Proposition 4.2. *Let $\rho \in \Lambda^3\mathfrak{g}^*$ be a closed three-form with quartic invariant $\lambda(\rho)$ on a Lie algebra $\mathfrak{g} = \mathfrak{g}_5 \oplus \mathbb{R}$ such that the summand $\mathfrak{g}_5$ is indecomposable.*

(i) If the column $\lambda \geq 0$ in Table 2 is checked for $\mathfrak{g}_5$, then $\lambda(\rho) \geq 0$.
(ii) If $\mathrm{Nil}(\mathfrak{g}_5) = \mathbb{R}^4$ and $\mathrm{h}^3(\mathfrak{g}_5) = 0$, then $\lambda(\rho) = 0$.

Unfortunately, there seems to be no consistent pattern for the Lie algebras with $\lambda(\rho) \geq 0$ except that the nilradical has to be either $\mathbb{R}^4$ or $\mathfrak{h}_3 \oplus \mathbb{R}$.

5. Appendix

Tables 1 and 2 contain all indecomposable four- and five-dimensional Lie algebras ordered by nilradical. All the Lie algebras are solvable except for the last one $A_{5,40}$. In the first column, the names used in [12] are listed. For the four-dimensional case, a second column is added which contains the names used in [1]. We remark that there is a clear summary of all naming conventions for four-dimensional Lie algebras in [1].

The standard Lie bracket is encoded dually as explained in section 2. In both tables, we denote by $\mathrm{e}^1, \dots, \mathrm{e}^{\dim(\mathfrak{g})}$ a basis of $\mathfrak{g}^*$ and the Lie bracket column contains the images of the basis one-forms under d. We use the abbreviation e^{ij} for $\mathrm{e}^i \wedge \mathrm{e}^j$. In the column $\mathfrak{z}$, we have listed the dimension of the center. The column labeled $\mathrm{h}^*(\mathfrak{g})$ contains the vector $(\mathrm{h}^1(\mathfrak{g}), \dots, \mathrm{h}^{\dim(\mathfrak{g})})$ of the dimensions of the Lie algebra cohomology groups, where $\mathrm{h}^0(\mathfrak{g})$ is omitted since it always equals one. The numbers $\mathrm{h}^*(\mathfrak{g})$ have been calculated with the Maple package *LieAlgebras* which is a native Maple package since version 11. However, the distinction of the parameter values with different cohomology had to be carried out by hand since the functions of the *LieAlgebras* package assume generic parameter values (without further notification).

The following additional information can be read off directly from the column $\mathrm{h}^*(\mathfrak{g})$. A Lie algebra is in fact unimodular if and only if the top cohomology group $\mathrm{H}^{\dim(\mathfrak{g})}(\mathfrak{g})$ does not vanish, see for instance [15, Lemma 2.4]. Thus, we have highlighted the unimodular Lie algebras by a bold and underlined $\mathrm{h}^{\dim(\mathfrak{g})}$. Moreover, since the first cohomology group $H^1(\mathfrak{g})$ equals the annihilator of the derived algebra $[\mathfrak{g}, \mathfrak{g}]$, see for instance [13, Lemma 1.1] it holds $\dim([\mathfrak{g}, \mathfrak{g}]) = \dim(\mathfrak{g}) - \mathrm{h}^1(\mathfrak{g})$. The step length $s(\mathfrak{g})$ of the derived series can be determined as follows. Since the derived Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent for a solvable Lie algebra, it holds $[\mathfrak{g}, \mathfrak{g}] \subset \mathrm{Nil}(\mathfrak{g})$. In most cases, equality follows for dimensional reasons and then we have $s(\mathfrak{g}) = s(\mathrm{Nil}(\mathfrak{g})) + 1$. In the remaining cases, the derived algebra $[\mathfrak{g}, \mathfrak{g}]$ is easily determined due to its low dimension.

Last but not least, we have also charted the results of this article in Tables 1 and 2. In the four-dimensional case, the column labeled half-flat is checked if and only if $\mathfrak{g} \oplus \mathfrak{r}_2$ admits a half-flat SU(3)-structure. Recall that $\mathfrak{g} \oplus \mathbb{R}^2$ never admits a half-flat SU(3)-structure. The two columns labeled $\lambda \geq 0$ are checked if $\lambda(\rho) \geq 0$ for all closed three-forms ρ on $\mathfrak{g} \oplus \mathfrak{r}_2$ or $\mathfrak{g} \oplus \mathbb{R}^2$, respectively. Similarly, the column $\lambda = 0$ is checked if $\lambda(\rho) = 0$ for all closed three-forms ρ on $\mathfrak{g} \oplus \mathbb{R}^2$. In fact, none of the Lie algebras $\mathfrak{g} \oplus \mathfrak{r}^2$ satisfies $\lambda(\rho) = 0$ for all closed three-forms ρ. In the five-dimensional case, the column half-flat is checked if and only if $\mathfrak{g} \oplus \mathbb{R}$ admits a half-flat SU(3)-structure. Analogously, the columns $\lambda \geq 0$ and $\lambda = 0$ are checked if $\lambda(\rho) \geq 0$ or $\lambda(\rho) = 0$, respectively, for all closed three-forms ρ on $\mathfrak{g} \oplus \mathbb{R}$.

In Tables 3 and 4, we list explicit examples (ω, ρ) of half-flat SU(3)-structures. The tables contain all decomposable Lie algebras which admit a half-flat SU(3)-structure and which are not contained in [2] or [15]. For the convenience of the reader, we added an explicit expression for the metric g induced by the pair (ω, ρ). The label ONB indicates that the basis we consider is orthonormal with respect to g. Similarly, OB stands for orthogonal basis and is followed by the length of all non-unit basis one-forms.

Table 1: Indecomposable four-dimensional Lie algebras

$\mathfrak{g}$	[1]	Lie bracket	$\mathfrak{z}$	$\mathrm{h}^*(\mathfrak{g})$	half-flat $\oplus\mathfrak{r}_2$	$\lambda \geq 0$ $\oplus\mathfrak{r}_2$	$\lambda \geq 0$ $\oplus\mathbb{R}^2$	$\lambda = 0$ $\oplus\mathbb{R}^2$
		nilpotent						
$A_{4,1}$	$\mathfrak{n}_4$	$(e^{24}, e^{34}, 0, 0)$	0	(2,2,2,$\underline{\mathbf{1}}$)	✓	–	–	–
		Nilradical $\mathbb{R}^3$						
$A_{4,2}^{\alpha}$	$\mathfrak{r}_{4,\frac{1}{\alpha}}$	$(\alpha e^{14}, e^{24}+e^{34}, e^{34}, 0)$						
		$\alpha \neq -2, -1, 0$	0	(1,0,0,0)	–	✓	✓	✓
		$\alpha = -2$	0	(1,0,1,$\underline{\mathbf{1}}$)	✓	–	✓	–
		$\alpha = -1$	0	(1,1,1,0)	–	–	✓	–
$A_{4,3}$	$\mathfrak{r}_{4,0}$	$(e^{14}, e^{34}, 0, 0)$	1	(2,2,1,0)	–	–	✓	–
$A_{4,4}$	$\mathfrak{r}_4$	$(e^{14}+e^{24}, e^{24}+e^{34}, e^{34}, 0)$	0	(1,0,0,0)	–	✓	✓	✓
$A_{4,5}^{\alpha,\beta}$	$\mathfrak{r}_{4,\alpha,\beta}$ [1]	$(e^{14}, \alpha e^{24}, \beta e^{34}, 0)$						
		$-1 < \alpha \leq \beta \leq 1$, $\alpha\beta \neq 0$, $\beta \neq -\alpha, -(\alpha+1)$	0	(1,0,0,0)	–	✓	✓	✓
		$-1 < \alpha < -\frac{1}{2}$, $\beta = -(\alpha+1)$	0	(1,0,1,$\underline{\mathbf{1}}$)	✓	–	✓	–
		$\alpha = -\frac{1}{2}$, $\beta = -\frac{1}{2}$	0	(1,0,1,$\underline{\mathbf{1}}$)	–	–	✓	–
		$\alpha = -1$, $\beta > 0$, $\beta \neq 1$	0	(1,1,1,0)	–	–	✓	–
		$\alpha = -1$, $\beta = 1$	0	(1,2,2,0)	–	–	–	–
$A_{4,6}^{\alpha,\beta}$	$\mathfrak{r}'_{4,\alpha,\beta}$	$(\alpha e^{14}, \beta e^{24}+e^{34}, e^{42}+\beta e^{34}, 0)$						
		$\alpha > 0$, $\beta \neq 0, -\frac{1}{2}\alpha$	0	(1,0,0,0)	–	✓	✓	✓
		$\alpha > 0$, $\beta = -\frac{1}{2}\alpha$	0	(1,0,1,$\underline{\mathbf{1}}$)	✓	–	✓	–
		$\alpha > 0$, $\beta = 0$	0	(1,1,1,0)	–	–	✓	–
		Nilradical $\mathfrak{h}_3$						
$A_{4,7}$	$\mathfrak{h}_4$	$(2e^{14}+e^{23}, e^{24}+e^{34}, e^{34}, 0)$	0	(1,0,0,0)	–	–	✓	✓
$A_{4,8}$	$\mathfrak{d}_4$	$(e^{23}, e^{24}, e^{43}, 0)$	1	(1,0,1,$\underline{\mathbf{1}}$)	✓	–	✓	–
$A_{4,9}^{\alpha}$	$\mathfrak{d}_{4,\frac{1}{1+\alpha}}$	$((\alpha+1)e^{14}+e^{23}, e^{24}, \alpha e^{34}, 0)$						
		$-1 < \alpha \leq 1$, $\alpha \neq -\frac{1}{2}, 0$	0	(1,0,0,0)	–	–	✓	✓
		$\alpha = -\frac{1}{2}$	0	(1,1,1,0)	✓	–	–	–
		$\alpha = 0$	0	(2,1,0,0)	–	–	✓	–
$A_{4,10}$	$\mathfrak{d}'_{4,0}$	$(e^{23}, e^{34}, e^{42}, 0)$	1	(1,0,1,$\underline{\mathbf{1}}$)	✓	–	✓	–
$A_{4,11}^{\alpha}$	$\mathfrak{d}'_{4,\alpha}$	$(2\alpha e^{14}+e^{23}, \alpha e^{24}+e^{34}, e^{42}+\alpha e^{34}, 0)$, $\alpha > 0$	0	(1,0,0,0)	–	–	✓	✓
		Nilradical $\mathbb{R}^2$						
$A_{4,12}$	$\mathfrak{aff}(\mathbb{C})$	$(e^{13}+e^{24}, e^{41}+e^{23}, 0, 0)$	0	(2,1,0,0)	✓	–	–	–

[1] $A_{4,5}^{\alpha,-\alpha} \cong A_{4,5}^{-1,\frac{1}{\alpha}}$ for all $\alpha \neq 0$ and $A_{4,5}^{-1,\beta} \cong A_{4,5}^{-1,-\beta}$ for all $\beta \neq 0$.

Table 2: Indecomposable five-dimensional Lie algebras

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	half-flat	$\lambda \geq 0$	$\lambda = 0$
	nilpotent					
$A_{5,1}$	$(e^{35}, e^{45}, 0, 0, 0)$	2	(3,6,6,3,**1**)	✓	–	–
$A_{5,2}$	$(e^{25}, e^{35}, e^{45}, 0, 0)$	1	(2,3,3,2,**1**)	✓	–	–
$A_{5,3}$	$(e^{35}, e^{34}, e^{45}, 0, 0)$	2	(2,3,3,2,**1**)	–	–	–
$A_{5,4}$	$(e^{24} + e^{35}, 0, 0, 0, 0)$	1	(4,5,5,4,**1**)	✓	–	–
$A_{5,5}$	$(e^{25} + e^{34}, e^{35}, 0, 0, 0)$	1	(3,4,4,3,**1**)	✓	–	–
$A_{5,6}$	$(e^{25} + e^{34}, e^{35}, e^{45}, 0, 0)$	1	(2,3,3,2,**1**)	✓	–	–
	Nilradical $\mathbb{R}^4$					
$A_{5,7}^{\alpha,\beta,\gamma}$	$(e^{15}, \alpha e^{25}, \beta e^{35}, \gamma e^{45}, 0)$					
[2]	$-1 < \alpha \leq \beta \leq \gamma \leq 1$, $\alpha\beta\gamma \neq 0$, $\beta \neq -\alpha, -(\alpha+1)$, $\gamma \neq -\alpha, -(\alpha+1)$, $-\beta, -(\beta+1)$, $-(\alpha+\beta), -(\alpha+\beta+1)$	0	(1,0,0,0,0)	–	✓	✓
	$\alpha = -1$, $\beta\gamma \neq 0$, $-1 < \beta \leq \gamma$, $\gamma \neq -\beta, -\beta \pm 1$	0	(1,1,1,0,0)	–	✓	–
	$\alpha = -1, \beta = -1$, $\gamma \neq -1, 0, 1, 2$	0	(1,2,2,0,0)	–	✓	–
	$\alpha = -1, \beta = -1, \gamma = -1$	0	(1,3,3,0,0)	–	✓	–
	$\alpha = -1, \beta = -1, \gamma = 1$	0	(1,4,4,1,**1**)	✓	–	–
	$\alpha = -1, \beta = -1, \gamma = 2$	0	(1,2,3,1,0)	–	–	–
	$\alpha = -1$, $0 < \beta < 1$, $\gamma = -\beta$	0	(1,2,2,1,**1**)	✓	–	–
	$\alpha = -1$, $\beta \neq 0, 1$, $\gamma = -\beta - 1$	0	(1,1,2,1,0)	–	–	–
	$\alpha = 1$, $\beta = 1$, $\gamma = -2$	0	(1,0,3,3,0)	–	✓	–
	$-1 < \alpha \leq \beta \leq \gamma \leq 1$, $\alpha\beta\gamma \neq 0$, $\beta \neq -\alpha$, $\gamma = -\alpha - \beta - 1$	0	(1,0,0,1,**1**)	–	✓	✓
	$\alpha \neq -1, 0, 1, \pm\beta, \pm\gamma$, $-1 < \beta \leq -\frac{1}{2}$, $\gamma = -\beta - 1$	0	(1,0,1,1,0)	–	✓	–
	$\alpha = 1$, $\beta \leq -\frac{1}{2}$, $\beta \neq -2, -1$, $\gamma = -\beta - 1$	0	(1,0,2,2,0)	–	✓	–
$A_{5,8}^{\alpha}$	$(e^{25}, 0, e^{35}, \alpha e^{45}, 0)$					
	$-1 < \alpha \leq 1$, $\alpha \neq 0$	1	(2,2,1,0,0)	–	✓	–
	$\alpha = -1$	1	(2,3,3,2,**1**)	✓	–	–
$A_{5,9}^{\alpha,\beta}$	$(e^{15} + e^{25}, e^{25}, \alpha e^{35}, \beta e^{45}, 0)$					
[3]	$\alpha \neq -2, -1, 0$, $\beta \geq \alpha$, $\beta \neq -2, -1, 0, -\alpha$, $-\alpha - 1$, $-\alpha - 2$	0	(1,0,0,0,0)	–	✓	✓
	$\alpha = -2$, $\beta \neq -2, -1, 0, 1, 2$	0	(1,0,1,1,0)	–	✓	–
	$\alpha = -2$, $\beta = -2, 1$	0	(1,0,2,2,0)	–	✓	–
	$\alpha = -2$, $\beta = -1, 2$	0	(1,1,2,1,0)	–	–	–
	$\alpha = -1$, $\beta \neq -2, -1, 0, 1$	0	(1,1,1,0,0)	–	✓	–

[2] $A_{5,7}^{\alpha,-\alpha,\gamma} \cong A_{5,7}^{-1,\frac{1}{\alpha},\frac{\gamma}{\alpha}}$, $A_{5,7}^{\alpha,\beta,-(\alpha+\beta)} \cong A_{5,7}^{\frac{1}{\alpha},\frac{\beta}{\alpha},-(\frac{\beta}{\alpha}+1)}$, $A_{5,7}^{\alpha,\alpha,-(\alpha+1)} \cong A_{5,7}^{1,\frac{1}{\alpha},-(\frac{1}{\alpha}+1)}$, $A_{5,7}^{\alpha,\beta,-(\beta+1)} \cong A_{5,7}^{\frac{\alpha}{\beta},\frac{1}{\beta},-(\frac{1}{\beta}+1)}$

[3] $A_{5,9}^{\alpha,\beta} \cong A_{5,9}^{\beta,\alpha}$, $A_{5,9}^{\alpha,0}$ is decomposable.

Table 2 (continued) : Indecomposable five-dimensional Lie algebras

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	half-flat	$\lambda \geq 0$	$\lambda = 0$
	$\alpha=-1,\ \beta=-1$	0	(1,2,2,1,$\underline{\mathbf{1}}$)	–	✓	–
	$\alpha=-1,\ \beta=1$	0	(1,2,2,0,0)	–	✓	–
	$\alpha<0,\ \alpha\neq-2,-1,\ \beta=-\alpha$	0	(1,1,1,0,0)	–	✓	–
	$\alpha\leq-\frac{1}{2},\ \alpha\neq-2,-1,\ \beta=-\alpha-1$	0	(1,0,1,1,0)	–	✓	–
	$\alpha<-1,\ \alpha\neq-2,\ \beta=-\alpha-2$	0	(1,0,0,1,$\underline{\mathbf{1}}$)	–	✓	✓
$A_{5,10}$	$(e^{25},e^{35},0,e^{45},0)$	1	(2,2,2,1,0)	–	✓	–
$A_{5,11}^{\alpha}$	$(e^{15}+e^{25},e^{25}+e^{35},e^{35},\alpha e^{45},0)$					
	$\alpha\neq-3,-2,-1,0$	0	(1,0,0,0,0)	–	✓	✓
	$\alpha=-3$	0	(1,0,0,1,$\underline{\mathbf{1}}$)	–	✓	✓
	$\alpha=-2$	0	(1,0,1,1,0)	–	✓	–
	$\alpha=-1$	0	(1,1,1,0,0)	–	✓	–
$A_{5,12}$	$(e^{15}+e^{25},e^{25}+e^{35},e^{35}+e^{45},e^{45},0)$	0	(1,0,0,0,0)	–	✓	✓
$A_{5,13}^{\alpha,\beta,\gamma}$	$(e^{15},\alpha e^{25},\beta e^{35}+\gamma e^{45},-\gamma e^{35}+\beta e^{45},0)$					
[4]	$-1<\alpha\leq1,\ \alpha\neq0,\ \beta\neq-\frac{1}{2},0,-\frac{1}{2}\alpha,\ -\frac{1}{2}(\alpha+1),\ \gamma>0$	0	(1,0,0,0,0)	–	✓	✓
	$\alpha=-1,\ \beta>0,\ \beta\neq0,\frac{1}{2},\ \gamma>0$	0	(1,1,1,0,0)	–	✓	–
	$\alpha=-1,\ \beta=0,\ \gamma>0$	0	(1,2,2,1,$\underline{\mathbf{1}}$)	✓	–	–
	$\alpha=-1,\ \beta=\frac{1}{2},\ \gamma>0$	0	(1,1,2,1,0)	–	–	–
	$-1<\alpha\leq1,\ \alpha\neq0,\ \beta=0,\ \gamma>0$	0	(1,1,1,0,0)	–	✓	–
	$\alpha\neq-1,0,1,\ \beta=-\frac{1}{2},\ \gamma>0$	0	(1,0,1,1,0)	–	✓	–
	$\alpha=1,\ \beta=-\frac{1}{2},\ \gamma>0$	0	(1,0,2,2,0)	–	✓	–
	$-1<\alpha\leq1,\ \alpha\neq0,\ \beta=-\frac{1}{2}(\alpha+1),\ \gamma>0$	0	(1,0,0,1,$\underline{\mathbf{1}}$)	–	✓	✓
$A_{5,14}^{\alpha}$	$(e^{25},0,\alpha e^{35}+e^{45},-e^{35}+\alpha e^{45},0)$					
	$\alpha\neq0$	1	(2,2,1,0,0)	–	✓	–
	$\alpha=0$	1	(2,3,3,2,$\underline{\mathbf{1}}$)	✓	–	–
$A_{5,15}^{\alpha}$	$(e^{15}+e^{25},e^{25},\alpha e^{35}+e^{45},\alpha e^{45},0)$					
	$0<\lvert\alpha\rvert\leq1,\ \alpha\neq-1,-\frac{1}{2}$	0	(1,0,0,0,0)	–	✓	✓
	$\alpha=-1$	0	(1,2,2,1,$\underline{\mathbf{1}}$)	✓	–	–
	$\alpha=-\frac{1}{2}$	0	(1,0,1,1,0)	–	✓	–
	$\alpha=0$	1	(2,2,1,0,0)	–	✓	–
$A_{5,16}^{\alpha,\beta}$	$(e^{15}+e^{25},e^{25},\alpha e^{35}+\beta e^{45},-\beta e^{35}+\alpha e^{45},0)$					
[5]	$\alpha\neq-1,-\frac{1}{2},0,\ \beta>0$	0	(1,0,0,0,0)	–	✓	✓

[4] $A_{5,13}^{\alpha,\beta,0}=A_{5,7}^{\alpha,\beta,\beta}$, $A_{5,13}^{\alpha,\beta,\gamma}\cong A_{5,13}^{\alpha,\beta,-\gamma}$, $A_{5,13}^{-1,\beta,\gamma}\cong A_{5,13}^{-1,-\beta,-\gamma}$, $A_{5,13}^{0,\alpha,\beta}$ is decomposable.
[5] $A_{5,16}^{\alpha,\beta}\cong A_{5,16}^{\alpha,-\beta}$, $A_{5,16}^{\alpha,0}=A_{5,9}^{\alpha,\alpha}$

Table 2 (continued) : Indecomposable five-dimensional Lie algebras

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	half-flat	$\lambda \geq 0$	$\lambda = 0$
	$\alpha = -1,\ \beta > 0$	0	(1,0,0,1,$\underline{\mathbf{1}}$)	–	✓	✓
	$\alpha = -\frac{1}{2},\ \beta > 0$	0	(1,0,1,1,0)	–	✓	–
	$\alpha = 0,\ \beta > 0$	0	(1,1,1,0,0)	–	✓	–
$A_{5,17}^{\alpha,\beta,\gamma}$	$(\alpha e^{15} + e^{25}, -e^{15} + \alpha e^{25}, \beta e^{35} + \gamma e^{45}, -\gamma e^{35} + \beta e^{45}, 0)$					
[6]	$\alpha > 0,\ \beta \neq 0, -\alpha,\ 0 < \gamma \leq 1$	0	(1,0,0,0,0)	–	✓	✓
	$\alpha > 0,\ \beta = -\alpha,\ 0 < \gamma < 1$	0	(1,0,0,1,$\underline{\mathbf{1}}$)	–	✓	✓
	$\alpha > 0,\ \beta = -\alpha,\ \gamma = 1$	0	(1,2,2,1,$\underline{\mathbf{1}}$)	✓	–	–
	$\alpha = 0,\ \beta > 0,\ \gamma > 0$	0	(1,1,1,0,0)	–	✓	–
	$\alpha = 0,\ \beta = 0,\ 0 < \gamma < 1$	0	(1,2,2,1,$\underline{\mathbf{1}}$)	✓	–	–
	$\alpha = 0,\ \beta = 0,\ \gamma = 1$	0	(1,4,4,1,$\underline{\mathbf{1}}$)	✓	–	–
$A_{5,18}^{\alpha}$	$(\alpha e^{15} + e^{25} + e^{35}, -e^{15} + \alpha e^{25} + e^{45}, \alpha e^{35} + e^{45}, -e^{35} + \alpha e^{45}, 0)$					
	$\alpha > 0$	0	(1,0,0,0,0)	–	✓	✓
	$\alpha = 0$	0	(1,2,2,1,$\underline{\mathbf{1}}$)	✓	–	–
	Nilradical $\mathfrak{h}_3 \oplus \mathbb{R}$					
$A_{5,19}^{\alpha,\beta}$	$(\alpha e^{15} + e^{23}, e^{25}, (\alpha-1)e^{35}, \beta e^{45}, 0)$					
[7]	$0 < \alpha \leq 2,\ \alpha \neq \frac{1}{2}, 1,\ \beta \neq -1, 0, -2\alpha, -2\alpha+1, -\alpha-1, -\alpha+1$	0	(1,0,0,0,0)	–	✓	–
	$\alpha = -1,\ \beta \neq 0, -1, 2, 3$	0	(1,1,1,0,0)	–	–	–
	$\alpha = -1,\ \beta = -1$	0	(1,2,2,0,0)	–	–	–
	$\alpha = -1,\ \beta = 2$	0	(1,2,2,1,$\underline{\mathbf{1}}$)	✓	–	–
	$\alpha = -1,\ \beta = 3$	0	(1,1,2,1,0)	✓	–	–
	$\alpha = 0,\ \beta > 0$	1	(1,0,1,1,0)	–	✓	–
	$\alpha = 0,\ \beta = 1$	1	(1,1,3,2,0)	–	–	–
	$\alpha = 1,\ \beta \neq -2, -1, 0$	0	(2,1,0,0,0)	–	✓	–
	$\alpha = 1,\ \beta = -2$	0	(2,1,1,2,$\underline{\mathbf{1}}$)	–	–	–
	$\alpha = 1,\ \beta = -1$	0	(2,2,2,1,0)	–	–	–
	$\alpha \neq -1, 0, 1, \frac{1}{2}, 2,\ \beta = -1$	0	(1,1,1,0,0)	–	✓	–
	$\alpha = 2,\ \beta = -1$	0	(1,2,2,0,0)	–	✓	–
	$\alpha \neq -1, 0, 1, \frac{1}{2}, 2,\ \beta = -1 - \alpha$	0	(1,0,1,1,0)	–	–	–
	$\alpha = 2,\ \beta = -3$	0	(1,0,2,2,0)	✓	–	–
	$0 < \alpha \leq 2,\ \alpha \neq \frac{1}{2}, 1,\ \beta = -2\alpha$	0	(1,0,0,1,$\underline{\mathbf{1}}$)	–	✓	–
$A_{5,20}^{\alpha}$	$(\alpha e^{15} + e^{23} + e^{45}, e^{25}, (\alpha-1)e^{35}, \alpha e^{45}, 0)$					

[6] $A_{5,17}^{\alpha,\beta,0} \cong A_{5,13}^{1,\frac{\alpha}{\beta},\frac{1}{\beta}}$ for $\beta \neq 0$, $A_{5,17}^{\alpha,\beta,\gamma} \cong A_{5,17}^{\alpha,\beta,-\gamma} \cong A_{5,17}^{-\alpha,-\beta,\gamma} \cong A_{5,17}^{\frac{\beta}{\gamma},\frac{\alpha}{\gamma},\frac{1}{\gamma}}$ for $\gamma \neq 0$, $A_{5,17}^{\alpha,0,0}$ is decomposable.

[7] $A_{5,19}^{\alpha,\beta} \cong A_{5,19}^{\frac{\alpha}{\alpha-1},\frac{\beta}{\alpha-1}}$ for $\alpha \neq 1$, $A_{5,19}^{0,\beta} \cong A_{5,19}^{0,-\beta}$, $A_{5,19}^{\alpha,0}$ is decomposable.

Table 2 (continued) : Indecomposable five-dimensional Lie algebras

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	half-flat	$\lambda \geq 0$	$\lambda = 0$
	$\alpha \neq -1,\ -\frac{1}{2},\ 0,\ \frac{1}{3},\ \frac{1}{2},\ 1$	0	(1,0,0,0,0)	–	✓	–
	$\alpha = -1,\ \frac{1}{2}$	0	(1,1,1,0,0)	–	✓	–
	$\alpha = -\frac{1}{2},\ \frac{1}{3}$	0	(1,0,1,1,0)	–	–	–
	$\alpha = 0$	1	(2,1,1,2,$\underline{\mathbf{1}}$)	–	–	–
	$\alpha = 1$	0	(2,1,0,0,0)	–	✓	–
$A_{5,21}$	$(2e^{15} + e^{23}, e^{25}, e^{25} + e^{35}, e^{35} + e^{45}, 0)$	0	(1,0,0,0,0)	–	✓	–
$A_{5,22}$	$(e^{23}, 0, e^{25}, e^{45}, 0)$	1	(2,2,2,1,0)	–	–	–
$A_{5,23}^{\alpha}$	$(2e^{15} + e^{23}, e^{25}, e^{25} + e^{35}, \alpha e^{45}, 0)$					
	$\alpha \neq -4, -3, -1, 0$	0	(1,0,0,0,0)	–	✓	–
	$\alpha = -4$	0	(1,0,0,1,$\underline{\mathbf{1}}$)	–	✓	–
	$\alpha = -3$	0	(1,0,1,1,0)	–	–	–
	$\alpha = -1$	0	(1,1,1,0,0)	–	✓	–
$A_{5,24}$ [8]	$(2e^{15} + e^{23} + e^{45}, e^{25}, e^{25} + e^{35}, 2e^{45}, 0)$	0	(1,0,0,0,0)	–	✓	–
$A_{5,25}^{\alpha,\beta}$	$(2\beta e^{15} + e^{23}, \beta e^{25} - e^{35}, e^{25} + \beta e^{35}, \alpha e^{45}, 0)$					
	$\alpha \neq 0,\ \beta \neq 0, -\frac{1}{4}\alpha$	0	(1,0,0,0,0)	–	✓	–
	$\alpha \neq 0,\ \beta = 0$	1	(1,0,1,1,0)	–	✓	–
	$\alpha \neq 0,\ \beta = -\frac{1}{4}\alpha$	0	(1,0,0,1,$\underline{\mathbf{1}}$)	–	✓	–
$A_{5,26}^{\alpha,\varepsilon}$	$(2\alpha e^{15} + e^{23} + \varepsilon e^{45}, \alpha e^{25} - e^{35}, e^{25} + \alpha e^{35}, 2\alpha e^{45}, 0)$					
	$\alpha \neq 0,\ \varepsilon = \pm 1$	0	(1,0,0,0,0)	–	✓	–
	$\alpha = 0,\ \varepsilon = \pm 1$	1	(2,1,1,2,$\underline{\mathbf{1}}$)	–	✓	–
$A_{5,27}$	$(e^{15} + e^{23} + e^{45}, 0, e^{35}, e^{35} + e^{45}, 0)$	0	(2,1,0,0,0)	–	✓	–
$A_{5,28}^{\alpha}$	$(\alpha e^{15} + e^{23}, (\alpha - 1)e^{25}, e^{35}, e^{35} + e^{45}, 0)$					
	$\alpha \neq -2,\ -1,\ -\frac{1}{2},\ 0,\ \frac{1}{2},\ 1$	0	(1,0,0,0,0)	–	✓	–
	$\alpha = -2$	0	(1,0,1,1,0)	–	–	–
	$\alpha = -1,\ \frac{1}{2}$	0	(1,1,1,0,0)	–	–	–
	$\alpha = -\frac{1}{2}$	0	(1,0,0,1,$\underline{\mathbf{1}}$)	–	✓	–
	$\alpha = 0$	1	(1,1,2,1,0)	–	–	–
	$\alpha = 1$	0	(2,1,0,0,0)	–	✓	–
$A_{5,29}$	$(e^{15} + e^{24}, e^{25}, e^{45}, 0, 0)$	1	(2,2,1,0,0)	–	✓	–
	Nilradical $A_{4,1}$					
$A_{5,30}^{\alpha}$	$((\alpha + 1)e^{15} + e^{24}, \alpha e^{25} + e^{34}, (\alpha - 1)e^{35}, e^{45}, 0)$					
	$\alpha \neq -2,\ -1,\ -\frac{1}{3},\ 0,\ \frac{1}{2},\ 1$	0	(1,0,0,0,0)	–	–	–

[8] The parameter in [12] is not necessary.

Table 2 (continued) : Indecomposable five-dimensional Lie algebras

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	half-flat	$\lambda \geq 0$	$\lambda = 0$
	$\alpha = -2,\ \frac{1}{2}$	0	(1,1,1,0,0)	–	–	–
	$\alpha = -1$	1	(1,0,1,1,0)	–	–	–
	$\alpha = -\frac{1}{3}$	0	(1,0,0,1,$\underline{\mathbf{1}}$)	–	–	–
	$\alpha = 0$	0	(1,0,1,1,0)	✓	–	–
	$\alpha = 1$	0	(2,1,0,0,0)	–	–	–
$A_{5,31}$	$(3e^{15} + e^{24}, 2e^{25} + e^{34}, e^{35} + e^{45}, e^{45}, 0)$	0	(1,0,0,0,0)	–	–	–
$A^{\varepsilon}_{5,32}$	$(e^{15} + e^{24} + \varepsilon e^{35}, e^{25} + e^{34}, e^{35}, 0, 0)$, $\varepsilon = \pm 1$	0	(2,1,0,0,0)	–	–	–
	Nilradical $\mathbb{R}^3$					
$A^{\alpha,\beta}_{5,33}$	$(e^{14}, e^{25}, \beta e^{34} + \alpha e^{35}, 0, 0)$					
[9]	$\alpha, \beta \in \mathbb{R}^*$, $(\alpha, \beta) \neq (-1, -1)$	0	(2,1,0,0,0)	–	–	–
	$\alpha = -1$, $\beta = -1$	0	(2,1,1,2,$\underline{\mathbf{1}}$)	✓	–	–
$A^{\alpha}_{5,34}$	$(\alpha e^{14} + e^{15}, e^{24} + e^{35}, e^{34}, 0, 0)$, $\alpha \in \mathbb{R}$	0	(2,1,0,0,0)	–	–	–
$A^{\alpha,\beta}_{5,35}$	$(\beta e^{14} + \alpha e^{15}, e^{24} + e^{35}, -e^{25} + e^{34}, 0, 0)$					
	$(\alpha, \beta) \neq (0, -2)$, $(0, 0)$	0	(2,1,0,0,0)	–	–	–
	$\alpha = 0$, $\beta = -2$	0	(2,1,1,2,$\underline{\mathbf{1}}$)	✓	–	–
$A_{5,38}$	$(e^{14}, e^{25}, e^{45}, 0, 0)$	1	(2,2,1,0,0)	–	–	–
$A_{5,39}$	$(e^{14} + e^{25}, -e^{15} + e^{24}, e^{45}, 0, 0)$	1	(2,2,1,0,0)	–	–	–
	Nilradical $\mathfrak{h}_3$					
$A_{5,36}$	$(e^{14} + e^{23}, e^{24} - e^{25}, e^{35}, 0, 0)$	0	(2,1,0,0,0)	✓	–	–
$A_{5,37}$	$(2e^{14} + e^{23}, e^{24} + e^{35}, -e^{25} + e^{34}, 0, 0)$	0	(2,1,0,0,0)	✓	–	–
	non-solvable, Nilradical $\mathbb{R}^2$					
$A_{5,40}$	$(2e^{12}, -e^{13}, 2e^{23}, e^{24} + e^{35}, e^{14} - e^{25})$	0	(0,1,1,0,$\underline{\mathbf{1}}$)	✓	–	–

Table 3: Direct sums of a four-dimensional and a two-dimensional Lie algebra which admit a half-flat SU(3)-structure and which are not contained in [15]

Lie algebra	Normalized half-flat SU(3)-structure [10]
$A_{4,1} \oplus \mathfrak{r}_2$	$\omega = -e^{16} + e^{25} - e^{34}$, $\rho = e^{123} - e^{145} + e^{156} - e^{246} + e^{345} - 2e^{356}$
	$g = (e^1)^2 + (e^2)^2 + 2(e^3)^2 + (e^4)^2 + (e^5)^2 + 2(e^6)^2 - 2e^1 \cdot e^3 + 2e^4 \cdot e^6$
$B^{\beta} \oplus \mathfrak{r}_2$, $\beta > 0$ [11]	$\omega = e^{15} + e^{24} + e^{36}$, $\rho = e^{123} - e^{146} + e^{256} + e^{345}$, ONB

[9] $A^{\alpha,0}_{5,33}$ and $A^{0,\beta}_{5,33}$ are decomposable.

[10] In each case except B^{β}, the basis $(e^1, \ldots, e^4)$ satisfies the Lie bracket given in Table 1, whereas $de^5 = 0$ and $de^6 = e^{56}$.

[11] The basis (e^1, e^2, e^3, e^4) of B^{β} satisfies the Lie bracket $\big(\beta e^{14} - e^{24}, e^{14}, -\beta e^{34}, 0\big)$. The family B^{β} unifies the cases $A^{-2}_{4,5}$, $A^{\alpha,-(\alpha+1)}_{4,5}$ for $-1 < \alpha < -\frac{1}{2}$ and $A^{\alpha,-\frac{1}{2}\alpha}_{4,6}$ for $\alpha > 0$ in Table 1 since $B^{\beta} \cong A^{\frac{2\beta}{\sqrt{4-\beta^2}}, -\frac{\beta}{\sqrt{4-\beta^2}}}_{4,6}$ for $0 < \beta < 2$, $B^2 \cong A^{-2}_{4,2}$ and $B^{\beta} \cong A^{-\frac{1}{2} - \frac{\sqrt{\beta^2-4}}{2\beta}, -\frac{1}{2} + \frac{\sqrt{\beta^2-4}}{2\beta}}_{4,5}$ for $\beta > 2$.

Lie algebra	Normalized half-flat SU(3)-structure
$A_{4,8} \oplus \mathfrak{r}_2$	$\omega = -e^{14} + e^{16} - e^{24} + e^{25} + e^{34} + e^{35}$, $\rho = 2e^{123} + 4e^{124} + 4e^{134} - 2e^{156} - 2e^{234} + 2e^{236} - e^{245} + 3e^{246} - 3e^{256} + e^{345} + 3e^{346} + 3e^{356} + 12e^{456}$ $g = 2(e^1)^2 + 4(e^2)^2 + 4(e^3)^2 + 57(e^4)^2 + 2(e^5)^2 + 3(e^6)^2 + 4e^1 \cdot e^2 - 4e^1 \cdot e^3 - 18e^1 \cdot e^4 + 2e^1 \cdot e^6 - 4e^2 \cdot e^3 - 26e^2 \cdot e^4 - 2e^2 \cdot e^5 + 4e^2 \cdot e^6 + 26e^3 \cdot e^4 - 2e^3 \cdot e^5 - 4e^3 \cdot e^6 - 18e^4 \cdot e^6$
$A_{4,9}^{-\frac{1}{2}} \oplus \mathfrak{r}_2$	$\omega = e^{16} - 3e^{24} + 2e^{25} + e^{35}$ $\rho = \sqrt{3}\left(e^{124} + 2e^{134} - e^{135} + e^{146} - 2e^{156} + 2e^{236} + 4e^{245} - e^{345} + \frac{29}{2}e^{456}\right)$ $g = (e^1)^2 + 4(e^2)^2 + 4(e^3)^2 + 84(e^4)^2 + 17(e^5)^2 + 29(e^6)^2 - 18e^1 \cdot e^4 + 8e^1 \cdot e^5 + 4e^2 \cdot e^3 + 16e^2 \cdot e^6 - 4e^3 \cdot e^6 - 75e^4 \cdot e^5$
$A_{4,10} \oplus \mathfrak{r}_2$	$\omega = -e^{14} - e^{16} - e^{25} - e^{36}$, $\rho = e^{123} - e^{156} + e^{234} + e^{236} + e^{246} - e^{345} + e^{356} - e^{456}$ $g = (e^1)^2 + (e^2)^2 + (e^3)^2 + 2(e^4)^2 + (e^5)^2 + 3(e^6)^2 + 2e^1 \cdot e^4 + 2e^1 \cdot e^6 + 4e^4 \cdot e^6$
$A_{4,12} \oplus \mathfrak{r}_2$	$\omega = e^{16} - 2e^{23} + e^{25} + e^{34} - e^{36}$, $\rho = e^{123} + 2e^{134} - e^{136} + e^{145} + e^{156} - e^{235} - e^{246} + 2e^{356}$ $g = (e^1)^2 + (e^2)^2 + 9(e^3)^2 + (e^4)^2 + 2(e^5)^2 + 3(e^6)^2 + 4e^1 \cdot e^3 - 2e^1 \cdot e^5 - 2e^2 \cdot e^6 - 8e^3 \cdot e^5 - 2e^4 \cdot e^6$
$\mathfrak{r}_2 \oplus \mathfrak{r}_2 \oplus \mathfrak{r}_2$ [12]	$\omega = e^{12} - e^{23} - e^{25} - e^{35} + e^{46}$, $\rho = e^{124} - e^{126} + 2e^{134} + 3e^{156} - e^{234} + e^{256} + e^{345} + 2e^{356}$ $g = 6(e^1)^2 + (e^2)^2 + 4(e^3)^2 + (e^4)^2 + 3(e^5)^2 + 2(e^6)^2 + 8e^1 \cdot e^3 + 6e^1 \cdot e^5 + 2e^2 \cdot e^3 - 2e^2 \cdot e^5 + 2e^3 \cdot e^5 + 2e^4 \cdot e^6$

Table 4: Direct sums of indecomposable non-nilpotent five-dimensional Lie algebras and the one-dimensional Lie algebra admitting a half-flat SU(3)-structure

Lie algebra	Normalized half-flat SU(3)-structure[13]
$A_{5,7}^{-1,\beta,-\beta} \oplus \mathbb{R}$, $A_{5,7}^{-1,-1,1} \oplus \mathbb{R}$, $A_{5,8}^{-1} \oplus \mathbb{R}$, $A_{5,13}^{-1,0,\gamma} \oplus \mathbb{R}$, $A_{5,14}^{0} \oplus \mathbb{R}$, $A_{5,17}^{0,0,\gamma} \oplus \mathbb{R}$, $A_{5,17}^{0,0,1} \oplus \mathbb{R}$	$\omega = -e^{13} + e^{24} + e^{56}$, $\rho = e^{126} + e^{145} + e^{235} + e^{346}$, ONB
$A_{5,15}^{-1} \oplus \mathbb{R}$, $A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$	$\omega = e^{13} + e^{24} - e^{56}$, $\rho = e^{125} + e^{146} - e^{236} - e^{345}$, ONB
$A_{5,18}^{0} \oplus \mathbb{R}$	$\omega = e^{12} - e^{34} - e^{56}$, $\rho = e^{136} + e^{145} - e^{235} + e^{246}$, ONB
$A_{5,19}^{-1,2} \oplus \mathbb{R}$	$\omega = e^{13} + e^{24} - 2e^{25} - e^{56}$, $\rho = -e^{126} + e^{145} - e^{234} + e^{346} - e^{356}$, $g = (e^1)^2 + 2(e^2)^2 + (e^3)^2 + (e^4)^2 + 2(e^5)^2 + (e^6)^2 - 2e^2 \cdot e^6 - 2e^4 \cdot e^5$
$A_{5,19}^{-1,3} \oplus \mathbb{R}$	$\omega = e^{13} - 2e^{25} - e^{46}$, $\rho = e^{126} - 2e^{145} + e^{234} + 2e^{356}$, OB, $\lvert\lvert e_5 \rvert\rvert^2 = 2$
$A_{5,19}^{2,-3} \oplus \mathbb{R}$	$\omega = e^{12} + 2e^{35} - e^{46}$, $\rho = e^{134} + 2e^{156} + e^{236} + 2e^{245}$, OB, $\lvert\lvert e_5 \rvert\rvert^2 = 2$
$A_{5,30}^{0} \oplus \mathbb{R}$	$\omega = e^{16} + e^{25} + e^{34}$, $\rho = e^{123} + 2e^{145} - e^{156} - e^{246} - e^{345} + e^{356}$, $g = 2(e^1)^2 + (e^2)^2 + (e^3)^2 + 2(e^4)^2 + (e^5)^2 + (e^6)^2 - 2e^1 \cdot e^3 + 2e^4 \cdot e^6$
$A_{5,33}^{-1,-1} \oplus \mathbb{R}$	$\omega = e^{12} - e^{36} - e^{45}$, $\rho = -e^{135} + e^{146} + e^{234} + e^{256}$, ONB

[12] The basis satisfies the Lie bracket $(0, e^{12}, 0, e^{34}, 0, e^{56})$.

[13] In each case, $(e^1, \ldots, e^6)$ denotes a basis such that $e^1, \ldots, e^5$ satisfy the Lie algebra structure given in Table 2 and e^6 is closed.

Lie algebra	Normalized half-flat SU(3)-structure
$A_{5,35}^{0,-2} \oplus \mathbb{R}$	$\omega = e^{16} + e^{25} + 3e^{26} + e^{34}$, $\rho = e^{123} + e^{145} + 2e^{146} + e^{245} + e^{246} + e^{356}$,
	$g = (e^1)^2 + 2(e^2)^2 + (e^3)^2 + (e^4)^2 + (e^5)^2 + 5(e^6)^2 + 2e^1 \cdot e^2 + 4e^5 \cdot e^6$
$A_{5,36} \oplus \mathbb{R}$	$\omega = \frac{1}{12}e^{12} + e^{13} + e^{16} - \frac{1}{4}e^{24} + e^{46} + e^{56}$
	$\rho = -\frac{1}{6}e^{124} + \frac{1}{12}e^{125} - e^{134} - e^{135} + 4e^{146} + 4e^{236} + 3e^{345} + 3e^{456}$
	$g = \frac{5}{12}(e^1)^2 + \frac{1}{12}(e^2)^2 + 12(e^3)^2 + \frac{7}{4}(e^4)^2 + \frac{1}{4}(e^5)^2 + 28(e^6)^2$
	$+\frac{3}{2}e^1 \cdot e^4 - \frac{1}{2}e^1 \cdot e^5 + 2e^2 \cdot e^6 + 24e^3 \cdot e^6 - e^4 \cdot e^5$
$A_{5,37} \oplus \mathbb{R}$	$\omega = -\frac{1}{3}e^{16} + 3e^{24} + e^{35}$
	$\rho = -e^{125} + 3e^{134} + 2e^{146} + e^{236} + 6e^{345} - \frac{13}{3}e^{456}$
	$g = (e^1)^2 + 3(e^2)^2 + 3(e^3)^2 + 3(e^4)^2 + \frac{13}{3}(e^5)^2 + \frac{13}{9}(e^6)^2 + 4e^1 \cdot e^5 - 4e^3 \cdot e^6$
$A_{5,40} \oplus \mathbb{R}$	$\omega = e^{14} + e^{25} + e^{34} - e^{36}$, $\rho = e^{124} - e^{126} - e^{135} + e^{234} + e^{456}$
	$g = (e^1)^2 + (e^2)^2 + (e^3)^2 + 2(e^4)^2 + (e^5)^2 + (e^6)^2 - 2e^4 \cdot e^6$

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