Lagrangian spheres, symplectic surfaces and the symplectic mapping class group

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Abstract

Given a Lagrangian sphere in a symplectic 4-manifold (M, ω) with $b^+ = 1$, we find embedded symplectic surfaces intersecting it minimally. When the Kodaira dimension of (M, ω) is $-\infty$, this result turns out very useful in both the uniqueness and existence problems of Lagrangian spheres. On the uniqueness side, for a symplectic rational manifold, we show that the Torelli part of $Symp(M, \omega)$ acts transitively on homologous Lagrangian spheres; if the rational manifold has Euler number less than 8, we show $Ham(M, \omega)$ already acts transitively. On the existence side, we give a characterization of classes represented by Lagrangian spheres, which enables us to describe the non-Torelli part of the mapping class group.

1 Introduction

For a symplectic 4-manifold (M, ω), symplectic surfaces and Lagrangian surfaces are of complementary dimensions. Thus we can ask what can be said about their intersection pattern. Welschinger investigated this problem for a Lagrangian torus L in [47], where he proves that the class $[L]$ pairs trivially with any effective class, and a symplectic sphere with positive Chern number can be isotoped symplectically away from L.

In the case when L is a Lagrangian sphere and (M, ω) is a symplectic Del Pezzo surface with Euler number at most 7, Evans shows in [12] that it can be displaced from certain symplectic spheres with positive Chern number up to symplectic isotopy, and applies it to prove the uniqueness of Lagrangian isotopy therein.

In section 2 we generalize Evan's displacement result in two ways, the first being

Theorem 1.1. *Let L be a Lagrangian sphere in a symplectic 4-manifold (M, ω), and $A \in H_2(M, \mathbb{Z})$ with $A^2 \geq -1$. Suppose A is represented by a symplectic sphere C. Then C can be isotoped symplectically to another representative of A which intersects L minimally.*

In this paper all surfaces are smooth, embedded, connected, and oriented. We say that two surfaces intersect *minimally* if they intersect transversely at $|k|$ points where k is the homological intersection number.

The second generalization is for symplectic surfaces of arbitrary genus in $b^+ = 1$ manifolds. To state it let $\mathcal{E}_\omega$ be the set of $\omega-$exceptional classes:

$$\{E \in H_2(M, \mathbb{Z}) : E \text{ is represented by an } \omega\text{-symplectic } (-1)\text{-sphere}\}.$$

Theorem 1.2. *Suppose (M, ω) is a symplectic 4-manifold with $b^+ = 1$ and L is a Lagrangian sphere. Assume $A \in H_2(M, \mathbb{Z})$ satisfies $\omega(A) > 0, A^2 > 0$ and $A \cdot E \geq 0$ for all $E \in \mathcal{E}_\omega$. Then there exists a symplectic surface in the class nA intersecting L minimally for large $n \in \mathbb{Z}$.*

One implication of Theorem 1.1 is the uniqueness of Lagrangian spheres in a *symplectic rational manifold.* In this paper, a *rational manifold* is $\mathbb{C}P^2 \# k\overline{\mathbb{C}P}^2$ or $S^2 \times S^2$, and a pair (M, ω) where M is a rational manifold endowed with a symplectic form ω is called a *symplectic rational manifold.* A symplectic rational manifold which is monotone, i.e. $[\omega] = K_\omega$, is called a *symplectic Del Pezzo surface.*

Theorem 1.3. *For any symplectic rational manifold (M, ω) with $\chi(M) \leq 7$, Lagrangian spheres in each homology class are unique up to Lagrangian isotopy.*

This was due to Hind ([19]) in the case of $S^2 \times S^2$, and to Evans ([12]) for symplectic Del Pezzo surfaces with Euler number up to 7. Notice this is equivalent to that the Hamiltonian group $Ham(M, \omega)$ acts transitively on the space of homologous Lagrangian spheres. The proof of Theorem 1.3 will be presented in Section 5.2. We also apply Theorem 1.1 to present an alternative proof of Hind's fundamental uniqueness theorems.

One application of Theorem 1.2 is that we will be able to effectively perform Lagrangian-relative inflation procedure (Section 4). This turns out useful dealing with various classical questions in the Lagrangian-relative context. For example, it is known that Theorem 1.3 is not always true, demonstrated by Seidel's twisted Lagrangian spheres in symplectic Del Pezzo surfaces with $\chi(M) \geq 8$ ([43]). However, combining an idea proposed by Hind and techniques in [35], we are able to prove the following version of Lagrangian uniqueness:

Theorem 1.4. *For any two homologous Lagrangian spheres L_1 and L_2 in a symplectic rational surface (M, ω), there is a $\phi \in Symp_h(M, \omega)$ such that $\phi(L_1) = L_2$.*

In other words, $Symp_h(M,\omega)$, the subgroup of $Symp(M,\omega)$ acting trivially on homology, acts transitively on the space of Lagrangian spheres in a fixed homology class. Evans [13] calculated explicitly the homotopy type of $Symp_h(M,\omega)$ when (M,ω) is a symplectic Del Pezzo surface with $\chi(M) \leq 8$. (also known to M.Pinnsonault). In particular, when $\chi(M) \leq 7$, it is connected thus agreeing with $Ham(M,\omega)$. In our upcoming work [32] we will extend the connectedness to non-monotone cases. In view of these results and Seidel's example, it seems natural to ask the following question,

Question 1.5. *Suppose (M,ω) is a symplectic rational manifold with Euler number bigger than 7. Is it true that the Hamiltonian subgroup is always a proper subgroup of $Symp_h(M,\omega)$, i.e. the Torelli part of the symplectic mapping class group is non-trivial?*

Another key ingredient to the proof of Theorem 1.4 is the classification of *K-Lagrangian* spherical classes in Section 3. A class ξ is called *K-Lagrangian spherical* if ξ is represented by a smooth sphere with $\xi^2 = -2, K(\xi) = 0$.

Let $\kappa(M,\omega)$ be the *Kodaira dimension* of (M,ω) (see for example [26]). κ takes values in the set $\{-\infty, 0, 1, 2\}$ and $\kappa(M,\omega) = -\infty$ is equivalent to that (M,ω) is symplectic rational or ruled. In section 4 we also give an explicit description of K-Lagrangian spherical classes for ruled manifolds. The classification of K-Lagrangian spherical classes, together with the Lagrangian-relative inflation, enables us to further show that the obvious necessary condition for the existence of a Lagrangian sphere in (M,ω) is also sufficient.

Theorem 1.6. *Let (M,ω) be a symplectic 4-manifold with $\kappa = -\infty$. $\xi \in H_2(M,\mathbb{Z})$ is represented by a Lagrangian sphere if and only if ξ is K-Lagrangian spherical and $\omega(\xi) = 0$.*

A nice consequence of Theorem 1.6 is the characterization of the non-Torelli part of the symplectic mapping class group. Recall that each framed Lagrangian sphere L gives rise to a symplectomorphism, well defined up to isotopy (see [43]), which is denoted τ_L and called the Lagrangian Dehn twist along L.

Theorem 1.7. *Let (M,ω) be a symplectic 4-manifold with $\kappa = -\infty$. Then the homological action of $f \in Symp(M,\omega)$ can always be generated by Lagrangian Dehn twists. In other words, there are Lagrangian spheres L_i such that $f_* = (\tau_{L_1})_* \circ (\tau_{L_2})_* \circ \cdots \circ (\tau_{L_r})_*$.*

On the homological level, this theorem should also be viewed as a symplectic version of the classical theorem of M. Noether, which asserts a birational automorphism of $\mathbb{C}P^2$, which is known as a *plane Cremona map*, can be decomposed into a series of *ordinary quadratic transformations* (see [1] for a nice account). The latter is a counterpart of Seidel's Dehn twist in symplectic rational manifolds. It would be interesting to know whether the symplectic version of Noether decomposition indeed holds, that is, whether, up to symplectic isotopy, elements in $Symp(M)$ can be decomposed into Lagrangian Dehn twists when M is symplectic rational, at least when the form is monotone (this is easily verified by Evans' and our results above when $\chi(M) \leq 7$).

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2 Minimal intersection

Proof of Theorem 1.2: In a $4-$manifold with $b^+ = 1$, due to Taubes' SW$\Rightarrow$GT theorem and the Seiberg-Witten wall crossing formula, there are plenty of connected embedded symplectic surfaces ([30], see also [5], [35], [28]) :

Proposition 2.1. *Let (M, ω) be a symplectic 4-manifold with $b^+ = 1$ and canonical class K_ω. Let $A \in H_2(M; \mathbb{Z})$ be a class with $A^2 > 0$ and $\omega(A) > 0$. Assume that $A - PD(K_\omega)$ is ω-positive and has non-negative square. Further assume that $A \cdot E \geq 0$ for all $E \in \mathcal{E}_\omega$, then A has non-vanishing Gromov-Taubes invariant and A is represented by a connected embedded symplectic surface.*

Note that the proof in [27] indeed finds a connected embedded J-holomorphic representative for generic J tamed by ω.

Next recall if K_ω denotes the symplectic canonical class of ω, then one defines $\omega-$symplectic genus of a class A in $H_2(M, \mathbb{Z})$ as:

$$\eta_\omega(A) = \frac{A \cdot A + K_\omega(A) + 2}{2}$$

This is exactly the genus of a connected embedded symplectic surface in class A (if there is one) given by the adjunction formula. It is straightforward that when n is large, under the assumption of Theorem 1.2 the multiple nA has non-trivial GT invariant and is represented by an embedded connected symplectic surface. By abuse of notation, we will still denote this multiple by A.

By fixing the standard round metric on S^2, one has an induced metric on T^*S^2 along with a standard symplectic structure. By $T^*_r S^2$ we mean the open set of cotangent vectors with norm $< r$ in T^*S^2. From Weinstein neighborhood theorem, a Lagrangian S^2 has a symplectic neighborhood which is symplectomorphic to $T_r^*S^2$ for some small $r > 0$. As usual, let $\mathcal{J}_\omega$ be the class of ω-tamed almost complex structures. Define:

$$\begin{aligned}\overline{\mathcal{J}} =& \{J \in \mathcal{J}_\omega : J \text{ is standard in a Weinstein neighborhood} \\ & U \text{ of } S^2,\ U \text{ is symplectomorphic to } T_r^*S^2\}\end{aligned} \tag{2.1}$$

Recall from [12] that we say J is standard in $T_r^*S^2$ if it is the restrction of the pulled back from the map $\eta : T^*S^2 \to (\mathbb{C}^3, J_{std})$ defined by:

$$(u, v) \mapsto u_j cosh(|v|) + \sqrt{-1} v_j sinh(|v|)/|v|)$$

Here T^*S^2 is identified as $\{(u, v \in \mathbb{R}^3 \times \mathbb{R}^3 : |u| = 1, u \cdot v = 0)\}$. All such almost-complex structures are adjusted near the boundary of U in the sense of symplectic field theory. Note that transversality of any closed curve can be attained within class $\overline{\mathcal{J}}$ since they have to pass through $M \backslash U$.

From the dimension formula, $-K_\omega(A) \geq 1 - \eta_\omega(A)$. We start with the case that $-K_\omega(A) = 1 - \eta_\omega(A)$. Since A has non-vanishing Gromov-Taubes invariant with a connected embedded representative, we can perform the procedure "stretch the neck" with respect to J along ∂U and get a sequence of J_{t_i}-curves C_{t_i}, for some sequence of real numbers $t_i \to \infty$. One is referred to [9] for a comprehensive description of neck-stretching in SFT. To minimize our subscripts and for future reference, we simply denote this sequence by $\{C_t\}_{t=0}^\infty$.

If C_t does not intersect L for some $t < \infty$, the theorem follows when $\langle [L], [C] \rangle = 0$ in $H_2(M, \mathbb{Z})$. Now we assume that all C_t intersects L. This assumption will eventually deduce a contradiction when $\langle [L], [C] \rangle = 0$ and is automatically satisfied for $\langle [L], [C] \rangle \neq 0$ in $H_2(M, \mathbb{Z})$. By the compactness theorem proven in [9], we have a leveled curve C_∞ as the Gromov-Hofer limit of $\{C_t\}_{t=0}^\infty$ with 3 levels: the curve in $M \backslash U$, which we call C_W or W-part; the curve in the symplectization of $\partial U = \mathbb{R}P^3$, which we call C_S or S-part; the curve in U, which we call C_U or U-part.

Lemma 2.2. *Under the above assumption, there is a generic set $\mathcal{J}_{reg}$ such that if $J \in \mathcal{J}_{reg}$, the J_t-curves C_t converges to a leveled curve C_∞ with non-empty connected irreducible genus-$g_\omega(A)$ W-part, and all asymptotic Reeb orbits are simple.*

Proof. Since we assume that C_t intersects L, the U-part of C_∞ is non-empty. By maximum principle, we are forced to have also non-empty S- and W-part. From the calculation in [19] (see also [12]), we have a global trivialization on T^*S^2 where simple Reeb orbits have Conley-Zehnder index 2, and $c_1^\Phi = 0$ for all punctured curves in T^*S^2. For the definition of Conley-Zehnder index and relative first Chern class, one is referred to [10][41]. See also [8].

If there are more than one components in W-part, there must be some components B_i with

$$c_1^{\Phi}(B_i) \leq -g_i \tag{2.2}$$

where g_i is the genus of component B_i. Such inequality holds because of our assumption $-K_\omega(A) = 1 - \eta_\omega(A)$, $\sum_j g_j \leq g(C_\infty) = g(C_0) = \eta_\omega(A)$ and $\sum_j c_1^{\Phi}(B_j) = -K_\omega(A)$, where j runs over all C_W components.

On the other hand, The dimension formula of such components B_i in W reads:

$$\text{virdim}\mathscr{M}([B_i]) = -(2 - 2g_i - s^-) + 2c_1^{\Phi}([B_i]) - \sum_{k=1}^{s_i^-}(\mu_{i_k} - \frac{1}{2}\text{dim}S_{i_k}) \tag{2.3}$$

Here s_i^- is the total punctures of B_i, $\text{dim}S_{i_k} = 2$ is the dimension of the Morse-Bott family of Reeb orbits at the punctures, μ_{i_k} the Conley-Zehnder index of corresponding Reeb orbits of the puncture, which equates the Morse index 2. For such a component to have non-negative virtual dimension, we must then have $c_1^{\Phi}(B_i) \geq 1 - g_i$. This contradiction to (2.2) shows that there can be only one component when J is generic. By applying (2.3) again, one sees that such a component must have genus g and all asymptotes are simple. □

Now we look at the corresponding S-part. Since each positive puncture is simple, from the λ-energy consideration, all components have to be trivial cylinders to stay in genus g (cf. [12] Lemma 7.5, [19], [9]).

For U-part, recall first from [12] that it is biholomorphic to the affine quadric

$$Q = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^2 + z_2^2 + z_3^2 = 0\}$$

For any point $p \in Q$, there are exactly two complex lines in Q passing through p, which we call α_p and β_p. Such planes form two families which we call α-planes and β-planes, with α-family intersecting $L = Re(Q)$ positively and β-family negatively. Now from Lemma 6.5 in [12] and Lemma 2.2, the U-part consists of α- and β-planes. Moreover we have:

Lemma 2.3. *Either all planes in U are in α-family or all are in β-family.*

Proof. As is explained in [12] Lemma 7.7, an α-plane and a β-plane do not intersect iff they have the same asymptotic Reeb orbit. This must be the case to avoid self-intersection of the holomorphic building C_∞ which contradicts adjunction formula for C_t at some $t < \infty$. Therefore, if U-part has at least one α-plane and one β-plane, all planes must asymptote to the same Reeb orbit. By [12] Lemma 7.8, when $s^+ > 1$, which corresponds to $s^- > 1$ for W-part, the transversality of puncture evaluation shows that for generic J, C_W does not have all punctures asymptotic to the same Reeb orbit. This finishes the proof of the lemma.

$\square$

Now Theorem 1.2 follows from Lemma 2.3. For example, when $\langle [L],[C]\rangle = k > 0$, C_∞ intersects L transversally at k points via α-planes, hence positively. Therefore for some $t < \infty$, the C_t satisfies all requirements of our symplectic curve. The cases for $\langle [L],[C]\rangle = k \leq 0$ similarly follows. This concludes the proof of Theorem 1.2.

Now we consider the case $-K_\omega(A) > 1-\eta_\omega(A)$. We adapt Welschinger's idea in [47] here. Note that $g_\omega(A) > 0$ when n in the beginning of our proof is large. We consider the class of complex structures $\mathcal{F} \subset \bar{\mathcal{J}}$ which is standard also in a neighborhood U_p such that $U_p \cap L = \emptyset$ around a point $p \notin L$. Then from a small size of Kähler blow-up around p (see [37]), we obtain an artificial exceptional class e_p. Now $-K_\omega(A-e_p) = -K_\omega(A) - 1$ and $\eta_\omega(A-e_p) = \eta_\omega(A)$. From the blow-up formula, Corollary 4.4 in [31], A' also has nontrivial Gromov-Taubes invariant. To check the connectedness, we only need to verify the homological condition $(A-e_p)\cdot E > 0$ for any $E \in \mathcal{E}_{\omega'}$. Note that since A has a connected J-holomorphic representative C with genus > 0 for a generic $J \in \mathcal{F}$. For the homological intersection in our case, we could assume the Kähler blow-up is performed at $p \in C$. Then such a complex blow-up at p one still obtains a connected representative C' with the same genus as C. By positivity of intersections, it is clear that $[C']\cdot E = (A-e_p)\cdot E \geq 0$ for any exceptional class E. The condition on homology and hence the connectedness follows.

By iterating the above process we eventually get a class $\tilde{A}$ with $-K_\omega(\tilde{A}) = 1 - \eta_\omega(\tilde{A})$ with non-trivial GT-invariant and connected representative. Note that the genericity of class $\tilde{A}$ is achieved in class $\mathcal{F}$. Now the stretching the neck process goes along the lines as in the case $-K_\omega(A) = 1 - \eta_\omega(A)$, we could proceed as before to find an embedded J-holomorphic curve in class $\tilde{A}$ intersecting L minimally for $J \in \mathcal{F}$. One then blow down the curves at the artificial exceptional curves.

$\square$

Before giving the proof of Theorem 1.1, we recall the following familiar result. We call a set of symplectic curves with prescribed homology classes in each irreducible components a *symplectic configuration.*

Proposition 2.4. *Let (M,ω) be a symplectic 4-manifold, $A \in H_2(M,\mathbb{Z})$ with $A^2 \geq -1$, then the embedded symplectic spheres in class A are all symplectic isotopic to each other. This is also true for a configuration consisting of such spheres with positive transversal intersections between the irreducible components.*

The proof is a standard argument using Gromov-Witten theory on pseudo-holomorphic spheres and we omit it. For more details see [38].

Proof of Theorem 1.1: Notice that, once we establish that the GT invariant of A is non-trivial with a connected representative, the rest of the proof follows word-by-word as in Theorem 1.2. The existence of such a representative is again classical: when $A^2 = -1$, A is simply an exceptional class for which our assertion is obvious; when $A^2 \geq 0$, from the celebrated theorem of McDuff [33], M is rational or ruled. Pick an almost complex structure $J \in \overline{\mathcal{J}}$ such that C is J-holomorphic. From the automatic transversality theorem, $D\pi|_C$ is surjective in the sense of Gromov's theory. Positivity of intersections and the adjunction formula in turn shows that C is the only J-sphere in its class. Therefore, the existence of C implies the nontriviality of GT invariant as well as gives a connected representative. The isotopy assertion is an immediate consequence of Proposition 2.4.

□

Remark 2.5. One easily sees that the above proof works for finitely many Lagrangian spheres that do not intersect each other. It is not clear to the authors whether the theorem holds when they do intersect. There is some hints that this phenomenon might be related to the wall crossing from the stretching family of almost-complex structures adjusted to one Lagrangian to another.

However, in exactly the same way, one may push off a symplectic configuration. Precisely:

Corollary 2.6. *Let L be a Lagrangian sphere in a symplectic 4-manifold (M, ω), and $D = \{A_1, \cdots, A_n\}$ be a set of spherical classes $A_i \in H_2(M, \mathbb{Z})$. Suppose A_i's have non-vanishing GT-invariants and pair trivially with $[L]$. Then there is a symplectic configuration with class D. Moreover, they can be realized as a J-holomorphic configuration for a J tamed by the symplectic form and are isotopic to each other.*

The proof is immediate: notice that in the proof of Theorem 1.2 or 1.1, one could stretch the neck once and for all classes with non-vanishing GT-invariants. The rest of the argument is straightforward. Note also that such argument also works for configurations with prescribed intersection patterns as long as corresponding irreducible components have non-vanishing GT-invariants with point constraints. For example, suppose class A has non-vanishing GT-invariant with one-point constraint, and B, C with non-vanishing GT-invariants, $B \cdot C = 1$. Then in the complement of a given Lagrangian sphere, one could find a corresponding configuration with the three components intersecting at a single point. Moreover, if all three classes are spherical, such configurations are again connected similar to Proposition 2.4.

Remark 2.7. In principle one should be able to deform a family of symplectic spheres to be disjoint from a given Lagrangian sphere in a weak sense: one needs a family of J_T-spheres to start with, where T is a parameter of almost-complex structures. However, it is more subtle than it appears. As is pointed out to us by R. Hind, if one takes a representative of the generator of $\pi_1(Sympl(S^2, \omega_0))$, the graph of this generator as a path of symplectic curves in $S^2 \times S^2$ cannot be isotoped away from the antidiagonal. The point is that there seems to be some topological obstruction from homotopy theory here, which will be explored in subsequent papers.

Remark 2.8. The existence of C can be replaced by some conditions that are easier to verify when $A^2 \geq 0$. By restricting our concerns to the rational and ruled cases, a theorem [27] says we could achieve the non-triviality of GT invariants as long as $A^2 \geq \eta(A) - 1$, where $\eta(A)$ is the symplectic genus (for the definition see section 3 below). An argument in [27] further shows if the class A is reduced (see section 3 for the rational case and [27] the general case), one only needs the minimal homological conditions asking $A^2 \geq g_\omega(A) - 1$ since $g_\omega(A) = \eta(A)$. However, when $g_\omega(A) > 0$ in this case, we no longer have the assertion of symplectic isotopy but only the existence of such a surface.

3 K-Lag spherical classes of rational manifolds

It is in general difficult to find out whether a spherical class has a Lagrangian sphere representative. We are able to completely solve this problem for rational and ruled manifolds in Section 4.2. In this section we first derive some preliminary results.

We fix some notations: in this section M is $\mathbb{C}P^2 \# n\overline{\mathbb{C}P}^2$. Let $D(M)$ be the image of diffeomorphism group of M in $Aut(H_2(M, \mathbb{Z}))$. We say two classes in $H_2(M; \mathbb{Z})$ are *equivalent* if they are related by $D(M)$.

Let $\mathcal{E}$ and $\mathcal{L}$ be the sets of integral homology classes represented by smoothly embedded spheres of square -1 and -2 respectively.

Let $\mathcal{K}$ be the set of symplectic canonical classes of M. It is shown in [30] that $D(M)$ acts transitively on $\mathcal{K}$. For $K \in \mathcal{K}$ let $D_K(M)$ be the isotropy subgroup.

The set of K-exceptional spherical classes and $K-$Lag spherical classes are defined to be:

$$\mathcal{E}_K = \{E \in \mathcal{E} | K(E) = -1\}$$

$$\mathcal{L}_K = \{\xi \in \mathcal{L} | K(\xi) = 0\}$$

Notice that D_K acts on $\mathcal{E}_K$ and $\mathcal{L}_K$. The K-symplectic cone is defined as $\mathcal{C}_K = \{[\omega] | K_\omega = K\}$. It is shown in [30] that

$$\mathcal{C}_K = \{\tau \in H^2(M;\mathbb{R}) | \tau \cdot E > 0 \text{ for any } E \in \mathcal{E}_K \text{ for some symplectic form } \omega\}.$$

3.1 A review of $D(M)$ and symplectic genus

We recall some familiar facts about $D(M)$ and the notions of symplectic genus of a class $e \in H_2(M, \mathbb{Z})$ in this section.

For $\gamma \in H_2(M, \mathbb{Z})$ with $\gamma^2 = \gamma \cdot \gamma = \pm 1$ or ± 2, there is an automorphism $R(\gamma)$ of the lattice called the reflection along γ,

$$R(\gamma)\beta = \beta - \frac{2(\gamma \cdot \beta)}{\gamma \cdot \gamma}\gamma.$$

If γ is represented by a smoothly embedded sphere, Proposition 2.4 in Chapter III in [15] then says that $R(\gamma) \in D(M)$ when $\gamma \cdot \gamma = \pm 1$ or ± 2. It is shown in [27] that $D(M)$ is generated by a set of spherical reflections $R(\gamma)$.

To define the symplectic genus of $e \in H_2(M, \mathbb{Z})$ introduce

$$\mathcal{K}_e = \{K \in \mathcal{K} | \text{there is a class } \tau \in \mathcal{C}_K \text{ such that } \tau \cdot e > 0\}$$

For $K \in \mathcal{K}_e$ define the $K-$symplectic genus $\eta_K(e)$ to be $\frac{1}{2}(e \cdot K + e^2) + 1$, and the *symplectic genus* of class e by

$$\eta(e) = \max_{K \in \mathcal{K}_e} \eta_K(e).$$

It is proved in [27], Lemma 3.2 that $\eta(e)$ has the following basic properties:

(1) $\eta(e)$ is no bigger than the minimal genus of e, and they are both equal to $\eta_\omega(e)$ if e is represented by an $\omega-$symplectic surface for some symplectic form ω;

(2) Equivalent classes have the same η;

Note that in [27] these properties are stated for classes with positive square, but the proof actually covered all cases.

Suppose An orthogonal basis $\{H, E_1, \cdots E_n\}$ of $H_2(M;\mathbb{Z})$ is called standard if $H^2 = 1$ and $E_i \in \mathcal{E}$. From now on we fix a standard basis.

For any sequence $\{\delta_i\}, i = 0, ..., n$ with $\delta_i = 0$ or 1, let

$$K_{\{\delta_i\}} = -(3H - (-1)^{\delta_1} E_1 - (-1)^{\delta_2} E_2 - \cdots - (-1)^{\delta_n} E_n).$$

Then $K_{\{\delta_i\}} \in \mathcal{K}$. When $\delta_i = 0$ for any i, we simply denote it by K_0, i.e.

$$K_0 = -3H + E_1 + \cdots + E_n.$$

It is clear that $E_i \in \mathcal{E}_{K_0}$. Moreover, for any symplectic form ω with $K_\omega = K_0$, the GT invariant of H and any $E \in \mathcal{E}_{K_0}$ is non-trivial. By the positivity of intersection, we have

Lemma 3.1. *Suppose $\xi = aH - \sum b_iE_i$ is in $\mathcal{E}_{K_0}$, then $a \geq 0$ and $b_i \geq 0$. If $a = 0$, then $\xi = E_i$ for some i.*

There is an analogue for $\mathcal{L}_{K_0}$.

Lemma 3.2. *If $\xi = aH - \sum b_iE_i \in H_2(M, \mathbb{Z})$ with $a > 0$ then $K_{\{\delta_i\}} \in \mathcal{K}_\xi$.*

Proof. Notice that for any $K_{\{\delta_i\}}$, one could easily find $\tau \in \mathcal{C}_{K_{\{\delta_i\}}}$ by requiring $\tau \cdot H \gg 0$, but keeping the corresponding signs of E_i in τ opposite to that of $K_{\{\delta_i\}}$. Such a construction follows from the easy observation that classes in $\mathcal{E}_{K_{\{\delta_i\}}}$ are obtained by changing the corresponding signs of those in $\mathcal{E}_K$ and Theorem 4 of [30].

By possibly even enlarging $\tau \cdot H$ further, since $a > 0$, one could also assure that $\tau \cdot \xi > 0$. Therefore, $K_{\{\delta_i\}} \in \mathcal{K}_\xi$. $\square$

Lemma 3.3. *Suppose $\xi = aH - \sum b_iE_i \in H_2(M, \mathbb{Z})$ is in $\mathcal{L}_{K_0}$, If $a > 0$ then $\eta(\xi) = \eta_{K_0}(\xi)$ and $b_i \geq 0$.*

Proof. For any $\xi \in \mathcal{L}_{K_0}$, $\eta_{K_0}(\xi) = 0$ and the minimal genus is 0 as well. By Lemma 3.2, if $\xi = aH - \sum b_iE_i \in H_2(M, \mathbb{Z})$ with $a > 0$, then $\eta_{K_{\{\delta_i\}}}(\xi)$ is defined. Recall from the minimal genus assumption and the fact that symplectic genus is no bigger than the minimal genus, $0 = \eta_{K_0}(\xi) \geq \eta_{K_{\{\delta_i\}}}(\xi)$ for all choices of $\{\delta_i\}$. But this holds only if $b_i \geq 0$, hence the conclusion follows. $\square$

When $n \geq 3$, a class $\xi = aH - \sum_{i=1}^n b_iE_i$ with $a \geq 0$ and $b_1 \geq b_2 \geq \cdots \geq b_n \geq 0$ is called *reduced* ([16], [22]) if

$$a \geq b_1 + b_2 + b_3.$$

We have the following assertion regarding (-1) and (-2)-classes:

Proposition 3.4 ([27], Lemma 3.4, Lemma 3.6(2))**.** *For e with $e \cdot e = -1$ or -2, $\eta(e) = 0$ if and only if e is not equivalent to a reduced class.*

Moreover, for e with $e \cdot e = -1$, $\eta(e) = 0$ if and only if $e \in \mathcal{E}$, Any class in $\mathcal{E}$ is equivalent to either E_i or $H - E_i - E_j$ for some $1 \leq i, j, k \leq n$. If further $n \neq 2$, it is equivalent to E_i.

Similarly, for e with $e \cdot e = -2$, $\eta(e) = 0$ if and only if $e \in \mathcal{L}$. Any class in $\mathcal{L}$ is equivalent to either $E_i - E_j$ or $H - E_i - E_j - E_k$ for some $1 \leq i, j, k \leq n$. If further $n \neq 3$, it is equivalent to $E_i - E_j$.

3.2 $K-$Lag spherical classes and $D_K(M)$

We say two classes are *$K-$equivalent* if they are related by $D_K(M)$. The action of $D(M)$ on $\mathcal{E}$ and $\mathcal{L}$ is completely understood in Proposition 3.4. But for our purpose, we need to further understand the $K-$equivalence. Due to the transitive action of $D(M)$ on $\mathcal{K}$ (c.f. [30]), we will restrict to the case $K = K_0$ without loss of generality. For $\mathcal{E}_{K_0}$, one has the following:

Proposition 3.5 ([39], Proposition 1.2.12). *Any class in $\mathcal{E}_{K_0}$ is K_0-equivalent to either E_i or $H - E_i - E_j$ for some $1 \leq i, j, k \leq n$. If further $n \neq 2$, it is K_0-equivalent to E_i.*

Corollary 3.6. *Let M be a rational manifold with $b^-(M) = n \geq 2$. If $\{E_i'\}_{i=1}^k$, $k \leq n-2$ is an orthogonal subset of $\mathcal{E}_{K_0}$), then there is $\phi \in D_{K_0}(M)$ such that $\phi(E_i') = E_i$.*

Proof. It is certainly true for $n = 2$. We apply induction on n. From Proposition 3.5, there is $\tilde{\phi} \in D_K(M)$ such that $\tilde{\phi}(E_1') = E_1$. Now we are reduced to the case $n - 1$. □

Remark 3.7. Note that this cannot be done when $k = n - 1$. In the case that $\tilde{\phi}(E_i') = E_i$, $i \leq n-2$ and $\tilde{\phi}(E_{n-1}') = H - E_{n-1} - E_n$, one has to apply diffeomorphisms involving E_i's for $i \leq n - 2$ to send $H - E_{n-1} - E_n$ to E_{n-1} or E_n.

Another consequence of Proposition 3.5 is:

Proposition 3.8. *$D_{K_0}(M)$ is generated by ordinary Cremona transforms when $n \neq 3$. In the case $n = 3$, one also includes the reflections $R(E_i - E_j)$.*

Proof. From Proposition 3.5, for $\phi \in D_{K_0}(M)$, there is $f \in Aut(H_2(M, \mathbb{Z}))$ generated by ordinary Cremona transforms such that $f(\phi_*(E_1)) = E_i$ for some $1 \leq i \leq n$. One then again send E_i back to E_1. Notice that this can be done by ordinary Cremona transforms when $n \neq 3$, but $E_1 - E_i$ is needed for $n = 3$. Denote still the composed map by f. Note also that $f(\phi_*(E_i)) \cdot E_1 = -\delta_{1i}$. One can then again conclude the proof by a simple induction.

□

We may also prove the analogue for the $\mathcal{L}_{K_0}$. Following Evans [12], a class is called *binary* if it is of the form $E_i - E_j$, and *ternary* if it is of the form $H - E_i - E_j - E_k$, $1 \leq i, j, k \leq n$. Clearly, binary and ternary classes are in $\mathcal{L}_{K_0}$. Denote $R(H - E_i - E_j - E_k)$ by Γ_{ijk} and call it an **ordinary Cremona transform**. More explicitly,

$$\Gamma_{ijk}(aH - \sum b_i E_i) = (2a - b_i - b_j - b_k)H - \sum c_l E_l, \tag{3.1}$$

where $c_l = b_l$ if $l \neq i, j, k$, and $c_l = b_l + (a - b_i - b_j - b_k)$ if $l = i, j, k$.

Proposition 3.9. *For $\xi \in \mathcal{L}_{K_0}$, either ξ is K_0-equivalent to a binary or ternary class for some $1 \leq i, j, k \leq n$. If further $n \neq 3$, it is K_0-equivalent to a binary class.*

Proof. Let $\xi = aH - \sum b_i E_i$. When $a = 0$ it is easy to conclude that ξ is binary. Let r be the number of nonzero b_i. An easy calculation verifies the case when

$r \leq 3$. Thus we assume $r > 3$ with $a > 0$ by possibly reversing the signs of ξ (simply do a reflection with respect to ξ). By Lemma 3.3, we may assume that $b_1 \geq b_2 \geq \cdots \geq b_n \geq 0$ from

Now we consider the reflection Γ_{123}. From equation (3.1),

$$\Gamma_{123}(\xi) = (2a - b_1 - b_2 - b_3)H - \sum c_i E_i,$$

where $c_i = b_i$ for $i > 3$.

If $2a - b_1 - b_2 - b_3 < 0$, we could consider the class $-\Gamma_{123}(\xi)$. It is easy to see that this class is in $\mathcal{L}_{K_0}$ since Γ_{123} is represented by a smooth version of Dehn twist. In this case, the leading coefficient of $-\Gamma_{123}(\xi)$ is bigger than 0. However, since $r > 3$, one must have $-c_r = -b_r < 0$, a contradiction to Lemma 3.3.

Moreover, from Lemma 3.4, ξ is not reduced but the first equation of reduced class is satisfied, hence one must have $b_1 + b_2 + b_3 > a$. Combining these facts, we have

$$0 \leq 2a - b_1 - b_2 - b_3 < a.$$

Also notice that $\Gamma_{123}(\xi)$ verifies all conditions of Lemma 3.3, thus $c_i > 0$ still holds. One could then repeat the above process and use induction on the coefficient $H \cdot \xi$ until $r \leq 3$ or $a = 0$.

□

Remark 3.10. The algorithm reducing a K-Lag spherical classes is also valid for exceptional classes. In this case, one gets an explicit K_0-equivalence from an exceptional class to E_i when $n \geq 3$ or possibly $H - E_1 - E_2$ when $n = 2$. This is also used in [39].

3.3 $(K, \alpha)-$Lag spherical classes and $D_{K,\alpha}(M)$

For $\alpha \in \mathcal{C}_K$, we define a $(K, \alpha)-$*Lag spherical class* to be a $K-$Lag spherical class which pairs trivially with α. Reflections $R(\xi)$, for ξ a $(K, \alpha)-$Lag spherical class, are called $(K_0, \alpha)-$*twists*. We also define the subgroup $D_{K,\alpha}(M)$ to be the subgroup of $D_K(M)$ preserving α. One has the following easy observation :

Lemma 3.11. *If $\phi \in D_K$ then*

- *ϕ induces a bijection from $\mathcal{L}_{K,\alpha}$ to $\mathcal{L}_{K,\phi^{-1}(\alpha)}$.*
- *$f \to \phi^{-1} \circ f \circ \phi$ defines an isomorphism from $D_{K,\alpha}$ to $D_{K,\phi^{-1}(\alpha)}$ taking $R(\xi)$ to $R(\phi(\xi))$.*
- *α has a positive lower bound on $\mathcal{E}_K$ which is attained by some K-exceptional class.*

The third assertion is a consequence of Gromov compactness and the well-known fact that, for any $E \in \mathcal{E}_K$, $GT(E) \neq 0$ with respect to any symplectic form ω representing α. We are now ready to prove the following:

Proposition 3.12. *$D_{(K_0,\alpha)}$ is generated by $(K_0, \alpha)-$twists.*

Proof. We will use induction on n. For $n \leq 3$ this is easy to verify directly by listing all exceptional classes.

If $n \geq 3$ choose $\{E_i'\}_{i=1}^{n-2} \subset \mathcal{E}_{K_0}$ such that E_1' has minimal α-area, and E_i' has minimal α-area among exceptional classes orthogonal to E_j for all $j < i$. By Lemma 3.6, there is $\psi \in D_{K_0}(M)$ such that $\psi(E_i') = E_i$. By Lemma 3.11 we can assume that $E_i' = E_i$.

Let $f \in D_{(K_0,\alpha)}$. If one could find a series of $(K_0, \alpha)-$twists such that their composition ϕ satisfies $\phi \circ f(E_1) = E_1$, one can then include ϕ^{-1} into our decomposition of f. Since E_1 is orthogonal to $\phi \circ f(E_i)$ for $i \neq 1$, one can then use induction on these classes. Therefore we will look for such a ϕ in the rest of the proof.

Notice first that

$$\alpha(H - E_i - E_j - E_k) \geq 0, \;\; i > j > k. \tag{3.2}$$

This is clear from the construction since $(H - E_i - E_j) \cdot E_l = 0$, for all $l < k$ and $k \leq n - 2$.

Assume $f(E_1) = aH - \sum b_{r_i} E_{r_i}$. Notice that $f(E_1) \in \mathcal{E}_{K_0}$ and $\alpha(f(E_1)) = \alpha(E_1)$. If $a = 0$ then $f(E_1) = E_k$ for some k and $E_1 - E_k \in \mathcal{L}_{K_0,\alpha}$. In particular, $R(E_1 - E_k) \in D_{K_0,\alpha}$ and we can choose $\phi = R(E_1 - E_k)$.

If $a \neq 0$, by Lemma 3.1, $a > 0$ and $b_i \geq 0$. Suppose $b_{r_1} \geq b_{r_2} \geq \cdots \geq b_{r_n} \geq 0$. Now apply $\Gamma_{r_1 r_2 r_3}$,

$$\Gamma_{r_1 r_2 r_3}(f(E_1)) = f(E_1) + (a - b_{r_1} - b_{r_2} - b_{r_3})(H - E_{r_1} - E_{r_2} - E_{r_3})$$

From Lemma 3.4, $a - b_{r_1} - b_{r_2} - b_{r_3} < 0$. By (3.2), $\alpha(H - E_{r_1} - E_{r_2} - E_{r_3}) \geq 0$, thus

$$\alpha(E_1) = \alpha(f(E_1)) \geq \alpha(\Gamma_{r_1 r_2 r_3}(f(E_1))).$$

By the choice of E_1, we must have $\alpha(H - E_{r_1} - E_{r_2} - E_{r_3}) = 0$. This means that $H - E_{r_1} - E_{r_2} - E_{r_3} \in \mathcal{L}_{K_0,\alpha}$ and $\Gamma_{r_1 r_2 r_3} \in D_{K_0,\alpha}(M)$.

Now from Remark 3.10, by repeating the above operations we eventually have an equivalence between E_1 and E_k for some k. Denote their composition to be $\tilde{\phi}$.

If $k = 1$ we let $\phi = \tilde{\phi}$. If $k \neq 1$, then $\alpha(E_k) = \alpha(E_1)$ and we let $\phi = R(E_1 - E_k) \circ \tilde{\phi}$.

∎

4 Lagrangian spherical classes when $b^+ = 1$

Theorem 1.2 allows us effectively apply a Lagrangian-relative version of inflation procedure in this section. Together with the classification of K-Lag spheres in Proposition 3.9, this in turn gives a classification of classes which admit actual Lagrangian spheres, that is, the Lagrangian spherical classes in symplectic rational manifolds. The discussions of ruled manifolds and cases with $\kappa \geq 0$ are in order. We also give the proof of Theorem 1.7 in Section 4.3.

4.1 Lagrangian relative inflations

The inflation procedure was first introduced by Lalonde [23] and proved useful in many fundamental problems in symplectic geometry (see [24] for example). The version in [35], Lemma 1.1, together with Theorem 1.2, gives

Lemma 4.1 (**Inflation Lemma**). *Let L be a Lagrangian sphere in a symplectic 4-manifold with $b^+ = 1$. Let A be a class in $H_2(M, \mathbb{Z})$ with positive self-intersection number and non-zero Gromov-Taubes invariant $GT(A)$. Assume also that $A \cdot L = 0$. Then given any family ω_t, $0 \leq t \leq 1$, of symplectic forms on M with $\omega_0 = \omega$, there is a family ρ_t of closed forms on M in class $PD(A)$ so that the family*

$$\omega_t + \kappa(t)\rho_t, \quad 0 \leq t \leq 1$$

is symplectic when $\kappa(t) \geq 0$, and L remains Lagrangian.

The proof is straightforward: note that from the proof Lemma 1.1 in [35], ρ_t is supported near a symplectic surface in class A. Therefore, if such a symplectic surface is disjoint from a given Lagrangian submanifold, the Lagrangian remains intact along the inflation procedure. Now Theorem 1.2 provides the symplectic surface as needed.

As the first application, we consider the space of symplectic ball packings in the complement of a Lagrangian. P. Biran and O. Cornea also study such packings in their work on Lagrangian Quantum theory [7] (in which they call it *mixed packing*), where the size of maximal ball embedding is found in some cases.

We denote for $\bar{\lambda} = (\lambda_1, \ldots, \lambda_k)$

$$E^L_{\bar{\lambda},k}(M, \omega) = \{\psi | \psi : \coprod_{i=1}^{k} (B_4(\lambda_i), \omega_{std}) \to M, Im(\psi) \bigcap L = \emptyset\}$$

In the absolute case, D. McDuff in [34] first proved that the space of embeddings of one or two balls in $\mathbb{C}P^2$ is connected. P. Biran [4] then improved the result to 3-7 balls in $\mathbb{C}P^2$. Later McDuff showed in a much greater generality, essentially the case of $b^+ = 1$ in [35]. We observe that the following immediate consequence, simply by substituting Lemma 1.1 in [35] by Lemma 4.1 in appropriate places, and noticing that the classes we inflate always pairs trivially with L:

Corollary 4.2. *If $b^+(M) = 1$ and $L \subset M$ is a Lagrangian sphere, $E^L_{\bar{\lambda},k}(M, \omega)$ is path connected.*

In principle, Lemma 4.1 shows, when $b^+ = 1$, packing problems in the complement of a Lagrangian sphere are largely parallel to the absolute packing problems. For another example, Biran showed in [6] that in any closed symplectic 4-manifold the symplectic packing problem is stable by inflation on a Donaldson hypersurface. In manifolds with $b^+(M) = 1$ and $\omega \in H^2(M, \mathbb{Q})$, the class $n\omega$ has non-zero Gromov-Taubes invariant, which means the Donaldson hypersurface is (stably) stable. Hence Biran's result also generalizes to the complement of a Lagrangian sphere.

4.2 Existence of Lagrangian spheres

In this subsection we present a proof of Theorem 1.6 and discuss some generalizations to manifolds of $b^+ = 1$. From the transitive action of $D(M)$ on $\mathcal{K}$ as in Section 3, we may reduce the canonical classes to K_0 without any loss in generality for manifolds with $\kappa = -\infty$. The definition of K_0 for irrational ruled manifolds will appear in subsection 4.2.2.

4.2.1 Rational manifolds

Proof of Theorem 1.6, rational manifold case: The case of $S^2 \times S^2$ is well-known and we focus on blow-ups of $\mathbb{C}P^2$ below. The condition is clearly necessary. The sufficiency is also clear in the case $n = 2$, for example from the toric picture. For future reference, we also demonstrate our method by proving this case using inflation. Let $PD([\omega]) = 3H - aE_1 - aE_2$ where $a > 0$. One uses a canonical symplectic deformation to shrink the exceptional classes into $PD([\omega']) = 3H - \epsilon E_1 - \epsilon E_2$ so that $\epsilon > 0$ is tiny. By Mcduff's connectedness theorem from [35] one could arrange spheres in classes E_1 and E_2 in a Darboux chart and find a Lagrangian sphere therein. One then inflate along a symplectic surface with class $n(3H - a'E_1 - a'E_2)$ with $a' > a$, $a' \in \mathbb{Q}$ and n a large integer. From Lemma 4.1 such inflation can be chosen to preserve L. A standard rescaling process as in [35] then concludes the proof.

To prove the sufficiency in general, one notices that we could assume ξ to be binary when $n > 3$. This is because, from Proposition 3.8, there is a self-diffeomorphism ϕ of M, which induces a K-twist on homology and sends ξ to a binary class, and we could just consider $\phi_*(\xi)$ in $(M, (\phi^{-1})^*\omega)$. Without loss of generality we could further assume $\xi = E_1 - E_2$.

We first deal with the case when $PD([\omega]) = 3H - \sum b_i E_i \in H^2(M, \mathbb{Q})$ with $b_1 = b_2 > 0$. One then follows a similar scheme as that of Biran's work [5]. By extension continuity of packing, one has a symplectic form $\tilde{\omega}$ with $\tilde{\omega}(E_i) = b_i + \delta_i$ for some $\delta_i > 0$. One first blows down E_i's for $i \geq 3$ and obtain a Lagrangian sphere L of class $E_1 - E_2$ in the blown-down manifold. Now one again performs $n-2$ small blow-ups of rational sizes $\epsilon_i > 0$ away from L and gets back to M with a different symplectic form ω' such that $[\omega'] = 3H - b_1E_1 - b_2E_2 - \sum_{i\geq 3} \epsilon_i E_i$. Since $[\tilde{\omega}]^2 > 0$ and ϵ's are small, $[\omega'] \cdot [\tilde{\omega}] > 0$. Also notice that $\tilde{\omega}$ itself is a symplectic form sharing the same canonical class with ω', it pairs with all K-exceptional spheres positively. One can then inflate along the class $PD([n\tilde{\omega}])$ using Lemma 4.1 for large $n \in \mathbb{N}$. A rescaling of the inflated form will lie in a class $PD([\omega'']) = 3H - b_1E_1 - b_2E_2 - \sum b_i' E_i$ with $b_i' = b_i + \delta_i'$, where $0 < \delta_i' < \delta_i$. Moreover, since $\tilde{\omega}(\xi) = 0$, Theorem 1.2 asserts the inflation can be chosen supported away from the Lagrangian L thus preserves it. By shrinking the blow-ups of E_i's for $i \geq 3$ to the size b_i, the proof is then complete from Mcduff's theorem in [35] asserting that homologous symplectic forms are symplectomorphic in rational manifolds.

For the general case when $PD([\omega]) = 3H - \sum b_i E_i \in H_2(M, \mathbb{R})$, one again applies the continuity of packing. By enlarging the size of E_i's slightly to a rational number we deal with ω' with $PD([\omega']) = 3H - \sum b_i' E_i \in H_2(M, \mathbb{Q})$ while keeping $b_1' = b_2' = \omega'(E_1) = \omega'(E_2)$, one use the existence result in the rational case on ω' and obtain a Lagrangian sphere L. By applying relative inflation again, we have the following lemma:

Lemma 4.3. *Let (M, ω) be a symplectic four manifold with $b^+(M) = 1$, and E_1, E_2 are two exceptional classes with symplectic area $\omega(E_1) = \omega(E_2) = a$. Assume also that $PD([\omega]) - aE_1 - aE_2 \in H_2(M, \mathbb{Q})$. If there is a Lagrangian sphere in class $E_1 - E_2$, then for a symplectic form ω' with $[\omega'] = [\omega] + tPD(E_1 + E_2)$ on M, $t > 0$, there is also a Lagrangian sphere in (M, ω') with class $E_1 - E_2$.*

Proof. Note that such a class can be obtained by restricting to a smaller symplectic ball corresponding to E_1 and E_2, and from Corollary 4.2 they are all isotopic provided the class is given. The proof is straightforward with the above understanding, while one inflates along a symplectic surface disjoint from L with class $q(PD[\omega] - aE_1 - aE_2)$ where $q \gg 0$ and rescale the inflated form correspondingly. □

Note that the above lemma indeed asserts when one "shrinks" two equal exceptional spheres E_1 and E_2, the existence of Lagrangian is preserved and it clearly applies to our case. By shrinking E_1 and E_2 from b_1' to the size

b_1 we obtain ω'' such that $\omega''(E_1) = \omega''(E_2) = b_1 = b_2$ and $\omega''(E_i) = b'_i$ for $i > 2$. Again from Theorem 1.2, symplectic representatives of E_j for $j > 2$ can be chosen disjoint from L. Since these exceptional spheres correspond to embedded balls, one simply restrict them to smaller balls of size b_i and concludes the theorem.

Now the only case left is when $n = 3$ and $\xi = H - E_1 - E_2 - E_3$. In this case we blow up artificially a small point to get an extra exceptional class E_4 and a new symplectic manifold M'. A smooth Dehn twist along a surface in class $H - E_1 - E_2 - E_4$ reduces the problem to our proved case when $n = 4$ and the class is binary, thus one gets Lagrangian sphere L' in class $E_4 - E_3$. By performing again the same Dehn twist, one gets back to M' with a Lagrangian sphere L in class ξ. Theorem 1.2 then applies to L and the exceptional class E_4. Blowing down an exceptional sphere in E_4 thus conclude the proof.

□

4.2.2 Irrational ruled manifolds

We start our discussion with the case of ruled manifolds. Parallel to the rational manifold case, we use our notion of K-Lag spheres, but the classification is much easier. It is clear that a minimal symplectic irrational ruled manifold does not admit Lagrangian spheres. For a non-minimal ruled manifold, from [33][27], it can always be viewed as a blow-up of a product ruled manifold. We thus can choose a *standard basis in irrational ruled manifold* M as $\{\Sigma, F, E_1, \cdots, E_n\}$, where Σ is the class of the base, which is represented by a surface with genus> 0, F the fiber, and E_i's the exceptional classes. The standard canonical class is then $K_0 = -2\Sigma + (2g(\Sigma) - 2)F + \sum E_i$

Suppose $\xi = a\Sigma + bF + \sum c_i E_i$ is represented by a Lagrangian sphere in M. Notice first that if $a \neq 0$, ξ does not have a smooth spherical representative. This follows from the fact that a sphere does not have a positive degree cover over a higher genus curve. With this understood, $\xi^2 = -2$ implies $\xi = kF \pm E_i \pm E_j$ or $\xi = E_i - E_j$, $1 \leq i \neq j \leq n$. From $K_0 \cdot \xi = 0$, we further conclude from a simple calculation that $\xi = F - E_i - E_j$ or $E_i - E_j$. Note that these two cases can be transformed into each other by a H_2-basis change. Hence we only need to prove the existence of Lagrangian spheres for classes $F - E_i - E_j$:

Proof of Theorem 1.6, irrational ruled manifold case: Suppose the base Σ is assembled from a $4g$-sided polygon as in [33], where g is the genus of Σ, and the vertices are identified as a point $x_0 \in \Sigma$. One can then cut along the glued sides to obtain a topological $S^2 \times D^2$ from M. Recall from [33] Lemma 4.13, 4.14 that with a symplectic deformation supported near an arbitrary small neighborhood of x_0, the $S^2 \times D^2$ is equipped with the standard product symplectic form. One can then compactifies it into a $S^2 \times S^2$ with product symplectic form. Moreover, by considering the fibration as a family of pseudo-holomorphic curves as in [33]. By deleting a codimension-2 subset of M containing spheres in exceptional classes, one still obtains a smooth fibration over $\Sigma\backslash\Lambda$, where $\Lambda \subset \Sigma$ is a finit set. Thus, one could always arrange such cut-and-paste operation above to be supported away from exceptional spheres in each E_i.

Now after the above operation, we turned an irrational ruled manifold into a rational ruled manifold with Euler number at least 6, with the class $F - E_i - E_j$ turned into a binary class. Our existence result follows then from the rational manifold case. The only subtlety is to avoid a given small neighborhood of a product factor S^2 (which corresponds to where the deformation takes place). From the proof in the rational case with 2 blow-ups, we indeed use a deformation supported near the exceptional spheres involved in the binary class, then construct a local Lagrangian sphere then use the inflation. Therefore, as long as the local Lagrangian sphere does not touch the given neighborhood, which is easily achieved, then our proof goes through. This concludes our proof.

□

4.2.3 Manifolds with $b^+ = 1$ and $\kappa \geq 0$

The above proof is easily adapted to the following theorem in manifolds of $b^+ = 1$:

Theorem 4.4. *Suppose (M, ω) is a minimal symplectic manifold with $b^+ = 1$, $[\omega] \in H^2(M, \mathbb{Q})$ and $\kappa(M) \geq 0$, $(\bar{M}, \bar{\omega})$ a symplectic blow-up of M, and the canonical injective map is denoted as: $\iota : H_2(M, \mathbb{Z}) \to H_2(\bar{M}, \mathbb{Z})$. Then $\xi \in H_2(\bar{M}, \mathbb{Z})$ is a Lagrangian spherical class if and only if the following holds: Either*

(1) $\xi \in Im(\iota)$ and $\iota^{-1}(\xi)$ is Lagrangian spherical, or

(2) ξ is binary and $\omega(\xi) = 0$.

Proof. The proof of existence of Lagrangian spheres when either (1) or (2) holds is almost identical to that of Theorem 1.6 for the rational manifold case. One shrinks the blow-ups of $\bar{M}$ as before, and inflate along a surface of class $n\omega(\bar{M})$ by adjusting the blow-up sizes if necessary. Such a surface can be chosen disjoint from a Lagrangian sphere in the destinated class (inherited in M in case (1) and locally constructed in case (2)) in $\bar{M}$.

To show the reverse direction, suppose $\xi = \xi' - \sum a_i E_i$ is represented by a Lagrangian sphere, where $\xi' \in Im(\iota)$, E_i's the exceptional classes, $a_i \neq 0$. The Dehn twist along ξ thus send E_1 to $a\xi' - \sum_{i>1} a_i E_i - (a_1^2 - 1)E_1$. Such a class is an exceptional sphere in $\bar{M}$. However, from the uniqueness of the minimal model for symplectic manifolds which are neither rational nor ruled,

$a\xi' - \sum_{i>1} a_i E_i - (a_1^2 - 1)E_1 = E_j$ for some j. This shows ξ' must be 0 and ξ is indeed binary.

$\square$

4.3 Proof of Theorem 1.7

The present subsection is devoted to the proof of Theorem 1.7. As mentioned in the proof of Theorem 1.6, fixing the canonical class causes no loss of generality. Therefore, throughout the proof, (M, ω) will denote a symplectic rational manifold with a standard basis chosen in Section 3 with canonical class K_0. Also denote $b^-(M) = n$.

Proof. It is clear that Theorem 1.6 implies all (K_0, ω)-twists for symplectic rational manifolds are realized by actural symplectic Dehn twists. With this in mind, the rational manifold case is simply a combination of Proposition 3.12 and Theorem 1.6.

The irrational ruled case is similar. Again we do induction on the Euler number of M. Let E be the exceptional class with minimal symplectic area, the induction is immediate if $\phi_*(E)\cdot E = 0$, in which case one simply do a Dehn twist along the Lagrangian sphere $E - \phi_*(E)$. Otherwise it is not hard to see $\phi_*(E) = F - E$. In this case $2\omega(E) = \omega(F)$. The minimality of $\omega(E)$ forces all other exceptional spheres to have the same area as E (since classes A and $F - A$ are both exceptional classes or neither). When there are more than one exceptional spheres in the standard basis, it is clear that one could send $F - E$ back to E, for example by a twist along $E' - E$ followed by another one along $F - E' - E$ where E' is another exceptional standard basis element orthogonal to E. When there is only one exceptional class in the standard basis, it is not hard to verify that no such ϕ_* could preserve the intersection form thus $\phi_*(E) = F - E$ would not hold. This concludes our proof.

$\square$

In particular in the rational manifold case when the blow-ups are of equal size, we clearly have the following corollary:

Corollary 4.5. *If (M, ω) is monotone, the representation of symplectic mapping class group on $H_2(M, \mathbb{Z})$ is generated by reflections along $H - E_i - E_j - E_k$ and $E_i - E_j$; if the blow-ups are of equal size but M is not monotone, the action is generated by $E_i - E_j$ and is identical to the symmetric group S_n permuting the exceptional spheres.*

Remark 4.6. Theorem 4.4 also has its counterpart as Theorem 1.7 which asserts the homological action of a symplectomorphism in $\bar{M}$ is a composition of a homological action of M and Dehn twists along binary spheres when $\kappa(M) \geq 0$.

This follows directly from the uniqueness of minimal models. Combining this with Mcduff's uniqueness theorem of blow-ups, it is not hard to see that the subgroup of homological actions of $Symp(\bar{M})$ is indeed the product of actions of $Symp(M)$ and Dehn twists coming from binary spheres in this case. Notice in contrast that, for all symplectic manifolds with $b^+ = 1$, the part of homological action fixing the homology of its minimal model is generated by Lagrangian Dehn twists. It would be very interesting to know whether for all symplectic manifolds with $b^+ = 1$ that homological actions of a symplectomorphism is always induced by Dehn twists.

5 Uniqueness of Lagrangian spheres in rational manifolds

5.1 Uniqueness up to symplectomorphism

The present subsection is devoted to the proof of Theorem 1.4. We fix some notations first. Let (M_i, ω_i), $i = 1, 2$, be symplectic manifolds and $N_i^j \subset M_i$ for $j \leq k$ submanifolds (open or closed) therein. A *symplectomorphism of a k-tuple* $\phi : (M_1, N_1^j) \to (M_2, N_2^j)$ is a symplecotomorphism $\phi : (M_1, \omega_1) \to (M_2, \omega_2)$ with $\phi(N_1^j) = N_2^j$.

We start with the case of $S^2 \times S^2$.

5.1.1 $S^2 \times S^2$ via symplectic cut

For $S^2 \times S^2$ we have the stronger uniqueness up to isotopy due to Hind. We here offer an argument for the uniqueness up to symplectomorphism using an idea from Hind [20] turning the Lagrangian uniqueness problem to a symplectic uniqueness problem via symplectic cut. Some preparations are in order.

Denote A, $B \in H_2(S^2 \times S^2, \mathbb{Z})$ the classes of two product factors on $S^2 \times S^2$. Consider ω_λ to be the symplectic form with class dual to $A + (1 + \lambda)B$, where $l - 1 < \lambda \leq l$, l an integer. Due to Lalonde-McDuff's theorem, ω_λ is unique up to symplectomorphisms. We have the following claim:

Proposition 5.1. *The space of symplectic $(-2k)$-spheres in a symplectic $S^2 \times S^2$ is connected.*

This is an immediate consequence of Gromov compactness theorem and the following theorem due to M. Abreu and D. McDuff:

Theorem 5.2 ([2], Proposition 2.1, Corollary 2.8)**.** *The space of ω_λ-tamed almost complex structure $\mathcal{J}_\lambda$ admits a stratification $\{U_k\}_{0 \leq k \leq l}$, such that the following holds:*

(1) For any $J \in U_k$, the class $A-kB$ is represented by a unique J-holomorphic sphere;

(2) Each U_k is connected.

Proof of Proposition 5.1: For the space in consideration to be non-empty, $\lambda > k-1$. For two such symplectic spheres C_i, there are almost-complex structures $J_i \in \mathcal{J}_\lambda$ such that C_i is J_i-holomorphic for $i = 0, 1$. Such J_i are in the k^{th} stratum, and from Theorem 5.2 (1), there is a path J_t in U_k connecting J_0 and J_1. Let C_t be the family of J_t-holomorphic spheres with self-intersection $-2k$. This path of symplectic spheres is continuous due to Gromov's compactness and the uniqueness of $J-$holomorphic curves in the class $A - kB$ for each $J \in U_k$.

$\square$

Theorem 5.3 (Hind,[19])**.** *Lagrangian S^2's in $S^2 \times S^2$ with monotone symplectic form are unique up to a symplectomorphism.*

Proof. Given two Lagrangian spheres $L_1, L_2 \subset S^2 \times S^2$, by Weinstein's neighborhood theorem one can fix two symplectic embeddings ϕ_1, ϕ_2: $T^*_r S^2 \to S^2 \times S^2$ for some small $r > 0$. Considering the geodesic flow on S^2 with round metric (see for example [3]) enables us to symplectic cut [25] along the boundary of ϕ_i, $i = 1, 2$, resulting in two symplectic manifold pairs $((S^2 \times S^2, \omega_i), \Sigma_i)$. Here Σ_i are symplectic (-2)-spheres.

It follows from [24] and Proposition 5.1 there is a symplectomorphism of the pair:

$$\iota : (S^2 \times S^2, \Sigma_1) \to (S^2 \times S^2, \Sigma_2)$$

On the other hand, we could perform symplectic sum (see [17]) on (M_i, Σ_i) with a copy of $(S^2 \times S^2, \text{diagonal})$. Gompf pointed out to us that symplectic sum can be achieved without perturbation as an inverse of symplectic cut as follows, which seems to be well-known. Let (M, Σ) and (N, Σ') be two symplectic pairs, where Σ and Σ' are symplectomorphic whose normal bundles have opposite Euler classes. Let P be the (real) projectivization of one of the normal bundles, then $P \times \mathbb{R}$ has a canonical symplectic form with Hamiltonian S^1-action rotation each fiber. The symplectic cut on $P \times \mathbb{R}$ at 0 gives two $\mathbb{R}^2$-bundles on Σ and Σ' with the same Euler number as they are embedded in M and N, respectively. Therefore, the complement of $P \times \{0\}$ can be locally identified with the two normal bundles of Σ and Σ' removing the zero section. With this method, ι is easily seen to be glued with the identity isomorphism of $S^2 \times S^2$, which leads to a symplectomorphism of pairs $\Psi : (S^2 \times S^2, L_1) \to (S^2 \times S^2, L_2)$.

$\square$

5.1.2 Proof of Theorem 1.4

As in previous sections, let $M = \mathbb{C}P^2 \# l\overline{\mathbb{C}P^2}$ be a symplectic rational surface, H and $\{E_i\}_{i=1}^l$ be a basis of $H_2(M, \mathbb{Z})$, where $H^2 = 1$, and E_i's are orthogonal exceptional classes.

One easily reduces the problem to the binary case as in the proof of Theorem 1.6. Let L_1, L_2 be two homologous Lagrangian spheres in symplectic rational manifold (M, ω) with class $\xi \in H_2(M, \mathbb{Z})$. From Proposition 3.9, one obtains an equivalence from ξ to a binary or ternary class by ordinary Cremona transforms. Such an equivalence is realized as a sequence of smooth Dehn twists whose composition is denoted ψ. Pulling back the symplectic form by ψ, it suffices to show the following:

Lemma 5.4. *The binary Lagrangian spheres are unique up to symplectomorphism.*

Proof. Without loss of generality, let $[L_i] = E_1 - E_2$. For each pair (M, L_i), one could blow down a set of (-1)-spheres of classes $\{E^l\}_{l\geq 3}$ and $H - E_1 - E_2$ away from L_i by Theorem 1.1. This yields two $(k+1)$-tuples of $(\tilde{M}_i, L_i, B_i^l)$, $i = 1, 2$, $3 \leq l \leq k+1$. Here $\tilde{M}_i$ are symplectic $S^2 \times S^2$, L_i the Lagrangian spheres, and B_i^l are symplectic embedded balls corresponding to blow-downs of spheres disjoint from L_i in class E^l when $l \leq k$, and to $H - E_1 - E_2$ for B_i^{k+1}. By [24] there is a symplectomorphism $\Psi : \tilde{M}_1 \to \tilde{M}_2$.

From Theorem 5.3, there is a symplectomorphism sending $\Psi(L_1)$ to L_2. Composing these two symplectomorphisms one obtains a symplectomorphism between the pairs $(\tilde{M}_i, L_i)$, which we still denote as Ψ. The connectedness of relative symplectic ball embedding from Corollary 4.2 asserts that $\Psi(B_1^l)$ can be displaced further by an L_2-preserving Hamiltonian isotopy to B_2^l. This gives a symplectomorphism between the $(k+1)$-tuples $(\tilde{M}_i, L_i, B_i^l)$, which in turn descends to a symplectomorphism of pairs between (M, L_i).

For the ternary case, for example, $H - E_1 - E_2 - E_3$, one can argue word-by-word as above on the balls B_i^l, $4 \leq l \leq k$, and instead of Hind's uniqueness, we apply Theorem 1.3. ☐

5.2 Uniqueness up to isotopy

The present subsection is devoted to the proof of Theorem 1.3. Our main tool is Hind's theorem 5.6.

5.2.1 Hind's results on $S^2 \times S^2$ and T^*S^2 and the symplectic mapping class group

Further exploring the symplectic cut approach in 5.1.1, we discuss the interactions between the symplectic mapping class group and Lagrangian isotopy problems.

From the description of the homotopy type of $Symp(S^2 \times S^2, \omega_0 \oplus \omega_0)$ by Gromov [18], Theorem 5.3 shows L_1 and L_2 are Hamiltonian isotopic, which is the original form that Hind proved in [19]:

Theorem 5.5 (Hind)**.** *Lagrangian S^2's in $S^2 \times S^2$ with monotone symplectic form are Hamiltonian isotopic to each other.*

Via Seidel's description of the symplectomorphism group of T^*S^2, we also obtain an alternative proof of Hind's Lagrangian sphere uniqueness in T^*S^2 below:

Theorem 5.6 (Hind, [20])**.** *Any two Lagrangian spheres in (T^*S^2, ω_{std}) are Lagrangian isotopic.*

Proof: We take the natural identification $T_1^*S^2 = (S^2 \times S^2, \omega_0) \backslash \Delta$, where Δ is the diagonal of $S^2 \times S^2$. It suffices to show the Lagrangian connectedness therein. Given Lagrangians L_1, $L_2 \in (S^2 \times S^2, \omega_0) \backslash \Delta$, we claim that there is $\phi \in Symp_c(T_1^*S^2, \omega_{std})$ such that $\phi(L_1) = L_2$, where $Symp_c$ denotes compactly supported symplectomorphism group. Without loss of generality we assume $L_2 = \bar{\Delta}$, which is the antidiagonal, corresponding in turn to the zero section of T^*S^2. By Hind's Lagrangian uniqueness Theorem 5.5, there is $\Psi \in Ham(S^2 \times S^2, \omega_0)$, such that $\Psi(L_1) = L_2$. Ψ does not fix Δ, but $\Psi(\Delta) \cap \bar{\Delta}(= L_2) = \emptyset$ still holds. On the complement of $\bar{\Delta}$ which is canonically identified with a disk bundle over the diagonal, we have a symplectic isotopy $\tilde{\Phi}_t : S^2 \to S^2 \times S^2$, $\tilde{\Phi}_0(S^2) = \Psi(\Delta)$, $\tilde{\Phi}_1(S^2) = \Delta$ promised by a theorem of R. Hind and A. Ivrii [21] so that $\tilde{\Phi}_t(S^2)$ are disjoint from $\bar{\Delta}$. One then extend $\tilde{\Phi}_t$ to a symplectic isotopy of a neighborhood U of $\Psi(\Delta)$ disjoint from $\bar{\Delta}$, which we still denote as $\tilde{\Phi}_t$.

Now we may consider $\tilde{\phi}_t$, a symplectic isotopy on a neighborhood U' of $\Psi(\Delta) \cup \bar{\Delta}$, which is $\tilde{\Phi}_t$ on U and identity near $\bar{\Delta}$. By Banyaga's isotopy extension theorem (see for example [37], Theorem 3.19), $\tilde{\phi}_t$ extends to a global symplectic isotopy ϕ_t of $S^2 \times S^2$. Consider $\phi_1 \circ \Phi \in Ham(S^2 \times S^2)$. Note that it does not descend to a Hamiltonian isotopy of $T_1^*S^2$ connecting L_1 to the zero section, but it indeed descend to a compactly supported symplectomorphism of $(T_1^*S^2, \omega_{std})$ mapping L_1 to the zero section. From Seidel's description of $Symp_c(T_1^*S^2, \omega_{std})$ [42], $\phi = \tau^n \circ \eta_1$, where τ is Seidel's Dehn twist along the zero section, and η_t, $t \in [0, 1]$ with $\eta_0 = id$ is a compactly supported symplectic isotopy. Now it is clear that $\tau^n \circ \eta_t(L_2)$ is a path connecting L_1 to the zero section since τ fixes the zero section, and this concludes our proof. $\square$

5.2.2 Proof of Theorem 1.3

We will denote V_k the k-point blow-up of $S^2 \times S^2$, or equivalently, $(k+1)$-point blow-up of $\mathbb{C}P^2$. We will switch between two points of view without explicitly mentioned.

For $k = 0$, it is the theorem of Hind [19] stating the uniqueness of symplectic isotopy class of Lagrangian S^2 in monotone $S^2 \times S^2$. It is also proven in [12] for the case when the V_k is monotone. In the general case, we apply Theorem 1.1 and approach as [12]. For some of the details one is referred to Section 9 of [12] and 4.2 of [14]. Throughout the proof, depending on the actual k we are looking at, J_0 denotes the complex structure obtained from a generic k-point blow-up of the standard $\mathbb{C}P^1 \times \mathbb{C}P^1$, for $k \leq 3$.

Proof of Theorem 1.3: Without loss of generality, we may assume ω is tamed by J_0. This follows from Proposition 4.8 in [29] that J_0-tamed cone is the same as the symplectic cone in $H^2(V_k, \mathbb{Z})$, as well as Mcduff's connectedness theorem [35].

Following the idea in [12], we will prove Theorem 1.3 by isotoping a configuration of symplectic spheres off a give Lagrangian sphere, and use Hind's theorem (see Theorem 5.6) in the complement of the configuration. Each configuration is described by the set D of the homology classes of the components.

For the binary case, for example, $E_1 - E_2$, we ask the intersections between components equals the homological intersection,

- For V_1, $D = \{H - E_1 - E_2, H\}$
- For V_2, $D = \{H - E_1 - E_2, H - E_3, E_3\}$
- For V_3, $D = \{H - E_1 - E_2, H - E_3 - E_4, E_3, E_4\}$

For the ternary case, for example, $H - E_1 - E_2 - E_3$,

- For V_2, $D = \{H - E_1, H - E_2, H - E_3\}$
- For V_3, $D = \{H - E_1 - E_4, H - E_2, H - E_3\}$

In addition, we ask the three component to intersect at a single point in the ternary case. We call such configurations of symplectic surface a D-*configurations*.

From Lemma 2.6 and the discussion following it, in the complement of a given Lagrangian sphere L, we have a D-configuration consisting of J-holomorphic spheres with some almost complex structure J tamed by ω. With a small perturbation we may assume the irreducible symplectic curves in the configuration intersect ω-orthogonally.

Since J_0 and J_1 are both tamed by ω, they can be joined by a path of ω-tamed almost complex structures $\{J_t\}$. Following the argument in [12], proof of Theorem 2.7, this gives a smooth isotopy of J_t-curves of the configuration we chose. For example in the binary case when $k=2$, the isotopy is given by a family of curves $\{(H-E_1-E_2)(J_t),(H-E_3)(J_t),E_3(J_t)\}$. From the fact that our configuration has trivial π_1, we then obtain an ambient Hamiltonian diffeomorphism Ψ such that Ψ^{-1} takes L disjoint from a standard configuration by a standard Moser's argument. For example again in the binary case when $k=2$, the standard configuration refers to a configuration of J_0-holomorphic curves in class D $\{(H-E_1-E_2)(J_0),(H-E_3)(J_0),E_3(J_0)\}$. Therefore, one could assume the two homologous Lagrangians L_1 and L_2 lies in the complement of the above standard configuration. Now our proof continues by two cases:

Case 1, when $PD([\omega_k])$ is a rational combination:

We need the following characterization for the complement of configurations in class D:

Lemma 5.7. *Let ω_k, $k\leq 4$, be a Kähler form obtained from the Kähler blow-ups of k-balls of $\mathbb{C}P^2$, and J_0 the standard complex structure. Suppose $[\omega_k]$ is dual to a rational linear combination of H and E_i, $i=1,\dots k$, and C is a J_0-holomorphic $D-$configuration. Then the complement of C is a Stein domain and the symplectic completion is symplectomorphic to T^*S^2 with standard symplectic structure.*

Proof. We prove the lemma again in the example of 3 point blow-ups and binary Lagrangian. Let $PD([\omega_3])=aH-E_1-E_2-bE_3$, $a,b\in\mathbb{Q}^+$. Note that the coefficients of E_1 and E_2 have to coincide to ensure the existence of a Lagrangian sphere in class E_1-E_2. Now it is straightforward to verify that for a large integer α, $PD([\alpha\omega_3])$ is represented as an integral combination of $\{H-E_1-E_2,H-E_3,E_3\}$, say, with coefficients $u,v,w\in\mathbb{Z}^+$.

Therefore, one may choose irreducible divisors F_ξ with $\xi\in D$. Associated to the divisor $F=uF_{H-E_1-E_2}+vF_{H-E_3}+wF_{E_3}$, we have a holomorphic line bundle $\mathscr{L}$, on which we can further take an hermitian metric and a compatible connection such that the curvature form is just $\alpha\omega_3$. In this case for any non-zero holomorphic section s of $\mathscr{L}$ with zeros coinciding F, the function $\phi=-log|s|^2$ defines a plurisubharmonic function with $\partial\bar{\partial}\phi=\alpha\omega_3$ on the complement of the configuration thus gives a Stein structure. The last assertion about symplectic completion follows from Lemma 2.1.6 in [14] or Lemma 5 in [44].

□

From Lemma 5.7, the complement of such configuration is symplectomorphic to a Weinstein neighborhood of T^*S^2 whose symplectic completion is exactly T^*S^2, and thus the corresponding Liouville flow remains inside such an open set. For any two Lagrangians L_1, L_2 in the complement of a standard configuration, we obtain an isotopy in T^*S^2 by Hind's theorem 5.6. Furthermore, one could contract the whole isotopy into our complement of the standard configuration using a Liouville flow as in 4.2 of [14]. The endpoints of the contracted isotopy is also connected to L_1 and L_2 by the Liouville flow, therefore, one gets the desired Lagrangian isotopy. In the end, notice the scaling α of the symplectic form does not change Lagrangian property of a submanifold.

Case 2, when $PD([\omega_k])$ is an irrational combination:

We first consider the binary case and assume $[L_i] = E_1 - E_2$, $i = 1, 2$. By rescaling the symplectic form, we could assume the ω-area of E_1 and E_2 is rational. In this case $H - E_1 - E_2$, E_3 and E_4 are regarded as a result of ball packing in $\mathbb{C}P^1 \times \mathbb{C}P^1$, and the representatives are chosen disjoint from L_1 and L_2 by Theorem 1.1.

From the continuity of packing, one embed slightly larger balls of these classes with rational radii which stays disjoint from L_1 and L_2. One then obtains a Lagrangian isotopy from *Case 1* (note the isotopy constructed above entirely lies in the complement of the symplectic configuration) and concludes the theorem by shrinking the balls.

For the ternary case when $k = 3$, M is a 4-point blow-up of $\mathbb{C}P^2$. One could reduce the problem to the binary case by a smooth Dehn twist as in the proof of Theorem 1.6.

For the ternary case when $k = 2$, $[L_i] = H - E_1 - E_2 - E_3$, we artificially blow up a small ball disjoint from L_i's to obtain an exceptional class E_4. One then reduce the problem with more caution. The smooth Dehn twist ϕ along a sphere in class $H - E_1 - E_2 - E_4$ reduces the problem to $[L_i'] = E_4 - E_3$ in $(M, (\phi^{-1})^*(\omega))$. From the choice of D, one then obtains a Lagrangian isotopy between L_i' lying in the complement of an exceptional sphere of class $H - E_1 - E_2$. Now apply ϕ^{-1} to M, one obtains a Lagrangian isotopy between L_1 and L_2 disjoint from an exceptional sphere in class E_4. Blowing down this sphere thus concludes our proof.

□

5.3 Some remarks on uniqueness results in symplectic manifolds

We end the paper with some discussions of our uniqueness results.

- Theorem 1.4 implies the disconnectedness of homologically trivial symplectormophism groups in the cases when there are non-isotopic Lagrangian spheres.

- Evans remarked in [14] that, one can show binary homologous Lagrangian spheres, for example in class $E_1 - E_2$, are smoothly isotopic in monotone symplectic rational surfaces. Indeed, one isotopes a given Lagrangian sphere away from exceptional spheres in classes E_i, $i \geq 3$ from Theorem 1.1. Then one could blow down these spheres and apply Evans' Lagrangian uniqueness in M_1 when the form is monotone. Once the symplectic structure is forgotten, avoiding the exceptional spheres becomes avoiding a point (instead of avoiding a ball in the symplectic case) which is easily achieved. Now this consideration can be immediately generalized to all Lagrangian spherical classes and arbitrary symplectic forms by Proposition 3.9. Therefore a straightforward corollary reads:

  Corollary 5.8. *Homologous Lagrangian spheres in symplectic rational manifolds are smoothly isotopic.*

- (The case of Lagrangian $\mathbb{R}P^2$) The same argument as in 5.1.1, with the (-2)-curves replaced by (-4)-curves, proves Lagrangian $\mathbb{R}P^2$ in $\mathbb{C}P^2$ are symplectomorphic, from Gromov's connectedness of $Symp(\mathbb{C}P^2, \omega_{std})$. Hence we also have:

  Theorem 5.9 (Hind)**.** *Lagrangian $\mathbb{R}P^2$'s in $\mathbb{C}P^2$ are Hamiltonian isotopic to each other.*

  This is indeed the case R. Hind proposed an alternative approach by symplectic cut. Notice that the classification of symplectic mapping class group on small blow-ups of $\mathbb{C}P^2$ with monotone symplectic forms are known due to the work of Evans [13] and Martin Pinsonnault independently. Once the Abrue-McDuff type theorem is checked, along this line one can show that Lagrangian $\mathbb{R}P^2$ in $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$, $k \leq 4$ with monotone blown-up symplectic forms are Hamiltonian isotopic.

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