An Interior Gradient Estimate for a class of Second Order Partial Differential Inequalities.

Mariko Arisawa
Department of Mathematics
University of Wisconsin-Madison
Madison, WI 53706, U.S.A.

1 Introduction.

In this paper, we show a uniform gradient estimate for functions u defined on a domain Ω in Euclidian space $\mathbf{R^N}$ which satisfy a system of N second order partial differential inequalities of the following form

$$-\frac{\partial}{\partial x_i}(\sum_{j=1}^{N} A_{ij}\frac{\partial u}{\partial x_j})(x) + \sum_{j=1}^{N} b_{ij}\frac{\partial u}{\partial x_j}(x) \le C \quad x \in \Omega, \quad 1 \le i \le N. \qquad (1)$$

The type of this system we concern is discussed in our main results Theorems 1.1, 1.2, 1.3, and 1.4 by some conditions for the coefficient matrix $(A_{ij})_{1\le i,j\le N}$ of second order terms and for the coefficient matrix $(b_{ij})_{1\le i,j\le N}$ of first order terms.
Our method relies essentially on the structure of the system (1), and we shall prove an interior uniform gradient estimate for $u \in C^2(\Omega)$ under the assumption that u is uniformly bounded. It is worth remarking here that the technical difficulty increases with the dimension N, and we only give the result for the cases of $N = 2, 3$. (Theorems 1.1, 1.2.) On the other hand, if we assume that u has a compact support in Ω, or that u is periodic, the boundedness of u is not necessary and the same result holds with a simpler assumption on $(A_{ij})_{1\le i,j\le N}$ for any dimensions. (Theorems 1.3, 1.4.) We

may now state our main theorems.

Theorem 1.1.

Let Ω be a domain in $\mathbf{R^2}$, let $A =(A_{ij})_{1\le i,j\le 2}$, where $A_{ij}= A_{ij}(x_1,x_2)\in W^{1,\infty}(\Omega)$ are real valued functions defined in $(x_1,x_2)\in\Omega$ which satisfy the following conditions.

$$\sup_{(x_1,x_2)\in\Omega}|A_{ij}(x_1,x_2)|\le C_1,\qquad \sup_{(x_1,x_2)\in\Omega}|\nabla A_{ij}(x_1,x_2)|\le C_1,\quad 1\le i,j\le 2, \tag{2}$$

$$|detA|^{-1}=|A_{11}A_{22}-A_{12}A_{21}|^{-1}\le C_2, \tag{3}$$

$$A_{11},\quad A_{22}\neq 0, \tag{4}$$

where C_1, $C_2>0$ are constants. Let $b_{ij}= b_{ij}(x_1,x_2)\in W^{1,\infty}(\Omega)$, $(1\le i,j\le 2)$ be real valued functions defined in $(x_1,x_2)\in\Omega$ which satisfy the following conditions.

$$\sup_{(x_1,x_2)\in\Omega}|b_{ij}(x_1,x_2)|\le C_3,\qquad \sup_{(x_1,x_2)\in\Omega}|\nabla b_{ij}(x_1,x_2)|\le C_3,\quad 1\le i,j\le 2, \tag{5}$$

where $C_3>0$ is a constant. Suppose that a real valued function $u(x_1,x_2)\in C^2(\Omega)$ satisfies the following inequalities

$$-\sum_{j=1}^{2}\{\frac{\partial}{\partial x_i}(A_{ij}\frac{\partial u}{\partial x_j})+b_{ij}\frac{\partial u}{\partial x_j}\}(x_1,x_2)\le C_4\quad in\quad (x_1,x_2)\in\Omega,\quad i=1,2. \tag{6}$$

$$\sup_{(x_1,x_2)\in\Omega}|u|\le C_5, \tag{7}$$

where C_4, $C_5>0$ are constants. Then, for any $(y_1,y_2)\in\Omega$, any $\delta>0$ such that

$$K'=[y_1-2\delta,y_1+2\delta]\times[y_2-2\delta,y_2+2\delta]\subset\Omega,$$

for $K=[y_1-\delta,y_1+\delta]\times[y_2-\delta,y_2+\delta]$, there exists a constant $\overline{C}>0$ depending on δ, matrices (A_{ij}), (b_{ij}), and constants C_4, C_5 such that

$$\sup_{(x_1,x_2)\in K}|\nabla u|\le\overline{C}, \tag{8}$$

$$\overline{C}=O(\frac{1}{\delta}).$$

Theorem 1.2.

Let Ω be a domain in $\mathbf{R^3}$, let $A=(A_{ij})_{1\le i,j\le 3}$, where A_{ij} are constants which satisfy the following conditions.

$$\sup_{(x_1,x_2,x_3)\in\Omega}|A_{ij}(x_1,x_2,x_3)|\le C_1 \quad 1\le i,j\le 3, \tag{9}$$

$$|detA|^{-1}\le C_2, \tag{10}$$

$$(A_{11}A_{22}-A_{12}A_{21})(A_{11}A_{33}-A_{13}A_{31})(A_{22}A_{33}-A_{23}A_{32})\neq 0, \tag{11}$$

$$A_{11}A_{22}A_{33}\neq 0, \tag{12}$$

$$(A_{11}A_{22}+A_{21}A_{12})(A_{11}A_{22}-3A_{21}A_{12})>0, \tag{13}$$

$$(A_{11}A_{33}+A_{31}A_{13})(A_{11}A_{33}-3A_{31}A_{13})>0, \tag{14}$$

$$(A_{22}A_{33}+A_{23}A_{32})(A_{22}A_{33}-3A_{23}A_{32})>0, \tag{15}$$

where C_1, $C_2>0$ are constants, let b_{ij} $(1\le i,j\le 3)$ be constants which satisfy

$$\sup_{(x_1,x_2,x_3)\in\Omega}|b_{ij}|\le C_3 \quad 1\le i,j\le 3, \tag{16}$$

where $C_3>0$ is a constant. Suppose that a real valued function $u(x_1,x_2,x_3)\in C^2(\Omega)$ satisfies the following inequalities

$$-\sum_{j=1}^{3}\{\frac{\partial}{\partial x_i}(A_{ij}\frac{\partial u}{\partial x_j})+b_{ij}\frac{\partial u}{\partial x_j}\}(x_1,x_2,x_3)\le C_4, \tag{17}$$

$$in \quad (x_1,x_2,x_3)\in\Omega, \quad 1\le\forall i\le 3,$$

$$\sup_{(x_1,x_2,x_3)\in\Omega}|u|\le C_5, \tag{18}$$

where C_4, C_5 are constants. Then, for any $(y_1,y_2,y_3)\in\Omega$ and for any $\delta>0$ such that

$$\delta|\frac{A_{ij}b_{ji}}{A_{ii}A_{jj}-A_{ij}A_{ji}+2|A_{ij}A_{ji}|}|(1+|\frac{A_{ij}A_{ji}}{A_{ii}A_{jj}}|)<\frac{1}{8} \quad 1\le i\neq j\le 3, \tag{19}$$

$$\delta|\frac{A_{ij}b_{ij}}{A_{ii}A_{jj}-A_{ij}A_{ji}-2|A_{ij}A_{ji}|}|(1+|\frac{A_{ij}A_{ji}}{A_{ii}A_{jj}}|)<\frac{1}{8}\quad 1\le i\ne j\le 3,$$

$$K'=[y_1-2\delta,y_1+2\delta]\times[y_2-2\delta,y_2+2\delta]\times[y_3-2\delta,y_3+2\delta]\subset\Omega,$$

for $K=[y_1-\delta,y_1+\delta]\times[y_2-\delta,y_2+\delta]\times[y_3-\delta,y_3+\delta]$*, there exists a constant* $\overline{C}>0$ *depending on* δ*, matrices* (A_{ij})*,* (b_{ij})*, and constants* C_4*,* C_5 *such that*

$$\sup_{(x_1,x_2)\in K}|\nabla u|\le\overline{C}, \tag{20}$$

$$\overline{C}=O(\frac{1}{\delta}).$$

Theorem 1.3.

Let Ω *be a domain in* $\mathbf{R^N}$*, let* $A=(A_{ij})_{1\le i,j\le N}$*, where* $A_{ij}=A_{ij}\in L^\infty(\Omega)$ *(*$1\le i,j\le N$*) real valued functions defined in* $x\in\Omega$ *which satisfy the following conditionzs.*

$$\sup_{x\in\Omega}|A_{ij}(x)|\le C_1\quad 1\le i,j\le N, \tag{21}$$

$$|detA|^{-1}\le C_2, \tag{22}$$

where C_1*,* C_2 *are constants. Suppose that a real valued function* $u(x)\in C^2(\Omega)$ *such that suppu* $\subset\subset\Omega$ *satisfies the following inequalities*

$$-\frac{\partial}{\partial x_i}(\sum_{j=1}^{N}A_{ij}\frac{\partial u}{\partial x_j})(x)\le C_3\quad in\quad x\in\Omega,\quad 1\le i\le N, \tag{23}$$

where $C_3>0$ *is a constant. Then, there exists a constant* $\overline{C}>0$ *depending on the matrix* (A_{ij}) *and the constant* $C_3>0$ *such that*

$$\sup_{x\in\Omega}|\nabla u(x)|\le\overline{C}. \tag{24}$$

Theorem 1.4.

Let Ω be an N dimensional torus $\mathbf{T^N}= \mathbf{R^N}/\mathbf{Z^N}= [0,1]^N$, let $A= (A_{ij})$, where $A_{ij}= A_{ij}(x)\in L^\infty(\Omega)$ $(1 \le i,j \le N)$ real valued periodic functions defined in $x \in \Omega$ which satisfy the following conditions.

$$\sup_{x\in\Omega} |A_{ij}(x)| \le C_1 \quad 1 \le i,j \le N, \tag{25}$$

$$|detA|^{-1} \le C_2, \tag{26}$$

where C_1, $C_2 > 0$ are constants. Suppose that a real valued function $u(x) \in C^2(\Omega)$ is periodic in Ω and satisfies the following inequalities

$$-\frac{\partial}{\partial x_i}(\sum_{j=1}^{N} A_{ij}\frac{\partial u}{\partial x_j})(x) \le C_3 \quad in \quad x \in \Omega, \quad 1 \le i \le N, \tag{27}$$

where $C_3 > 0$ is a constant. Then, there exists a constant $\overline{C} > 0$ depending on the matrix (A_{ij}) and the constant $C_3 > 0$ such that

$$\sup_{x\in\Omega} |\nabla u(x)| \le \overline{C}. \tag{28}$$

If we do not assume either the condition supp$u\subset\subset \Omega$ in Theorem 1.3, or the periodicity in Theorem 1.4, we need more restrictive conditions for the matrix A as in Theorems 1.1, 1.2. The following counter examples show the contrast between the case of Theorems 1.1, 1.2 and the case of Theorems 1.3, 1.4.

Example 1.5.

Let $N = 2$, and let $A =(A_{ij})_{1\le i,j\le 2}$ be the matrix with $A_{11}= A_{22}= 0$, $A_{21}= A_{12} = 1$. (det$A \neq 0$.) Consider any functions $u(x_1, x_2)\in C^2(\Omega)$ which satisfy the following inequality in $(x_1, x_2) \in \Omega$.

$$-\frac{\partial^2 u}{\partial x_1 \partial x_2} \le C_0.$$

Then, if supp$u\subset\subset\Omega$, from Theorem 1.3 $|\nabla u| \le C$, where the constant $C > 0$ depends only on the matrix A and C_0. However, if we take the function $u(x_1, x_2) = \psi(x_1)$ with arbitrary $\psi \in C^2(\mathbf{R})$ such that supp$u \cap \partial\Omega\neq \emptyset$, although u satisfies the above partial differential inequality, $|\nabla u|$ is not

bounded in general.

Example 1.6.

Let $N = 3$, and let $A = (A_{ij})_{1\le i,j\le 3}$ be the matrix with $A_{11} = A_{12} = A_{21} = A_{23} = 0$, $A_{13} = A_{22} = A_{31} = A_{32} = A_{33} = 1$.

($\det A \neq 0$) Consider any functions $u(x_1, x_2, x_3) \in C^2(\Omega)$ which satisfy the following inequality in $(x_1, x_2, x_3) \in \Omega$.

$$-\frac{\partial^2 u}{\partial x_1 \partial x_3} \le C_0.$$

$$-\frac{\partial^2 u}{\partial x_2^2} \le C_0.$$

$$-\frac{\partial}{\partial x_3}\left(\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + \frac{\partial u}{\partial x_3}\right) \le C_0.$$

Then, if $\mathrm{supp} u \subset\subset \Omega$, from Theorem 1.3 $|\nabla u| \le C$, where the constant $C > 0$ depends only on the matrix A and C_0. However, if we take the function $u(x_1, x_2, x_3) = \psi(x_1, x_2)$ with arbitrary $\psi \in C^2(\mathbf{R^2})$ such that $-\frac{\partial^2 \psi}{\partial x_2^2} < 0$, $\mathrm{supp} u \cap \partial\Omega \neq \emptyset$, although u satisfies the above partial differential inequalities, $|\nabla u|$ is not bounded in general.

Finally, we shall give an example of second-order degenerate elliptic partial differential equation whose regularity of the solution can be shown by our results.

Example 1.7.

Let $N = 2, 3$, Ω be a bounded domain in $\mathbf{R^N}$, $l > 0$, and suppose that u_l is a solution of the following problem

$$l u_l(x) + \sup_{1\le i\le N}\{-\frac{\partial^2 u_l}{\partial x_i^2}(x)\} - V(x) = 0 \quad x \in \Omega, \tag{29}$$

with either Neumann B.C., or State constraint B.C., where $V(x)$ is a Lipschitz continuous function defined in Ω. Then, for any interior domain $\Omega_0 \subset\subset \Omega$, $l u_l(x)$ is Lipschitz continuous in $x \in \Omega_0$ uniformly with respect to $l > 0$. If we assume that a solution u_l of (29) satisfies Periodic B.C., then $u_l(x)$ is Lipschitz continuous in $x \in \Omega$ uniformly with respect to l.

The proof of Example 1.7 will be given later in this paper. The partial differential equation (29) corresponds to an optimal control problem for a stochastic system which equips N controls of one dimensional diffusions at each state. (Remark that $\sum_{i=1}^{N}(-\frac{\partial^2}{\partial x_i^2}) = -\Delta$.) In view of the degeneracy of (29), we cannot apply the usual regularity theory for uniformly elliptic operators (Gilbarg-Trudinger [5], Caffarelli-Cabre [2]) to study the regularity of the solution u_l.

The plan of this paper is as follows. Theorem 1.1 is proved in §2; Theorem 1.2 is proved in §3; and the proofs of Theorems 1.3, 1.4 and Example 1.7 are given in §4. Throughout in the present paper, we conserve the letter $C > 0$ to denote the constants which depend on constants C_i and matrices (A_{ij}), (b_{ij}) in the Theorems 1.1-1.4.

2 Proof of Theorem 1.1.

Lemma 2.1.

Let ϕ be an arbitrarily fixed real valued twice differentiable function defined on the interval $[-2\delta, 2\delta]$ such that $0 \leq \phi \leq 1$, $supp\phi \subset\subset (-2\delta, 2\delta)$, ϕ is even and

$$\phi = 1 \quad on \quad [-\delta, \delta], \quad \phi' \geq 0 \quad on \quad [-2\delta, 0]. \tag{30}$$

Then, the function u in Theorem 1.1 satisfies the following inequalities : for any $(\hat{x}_1, x_2) \in K$,

$$\begin{aligned}
&(A_{11}\frac{\partial u}{\partial x_1})(\hat{x}_1, x_2) \qquad (31)\\
\leq (resp.\geq) \quad &-(A_{12}\frac{\partial u}{\partial x_2})(\hat{x_1}, x_2) + \frac{1}{2\delta}\int_{-\delta}^{\delta}(A_{12}\frac{\partial u}{\partial x_2})(x_1', x_2)dx_1'\\
&-\frac{1}{2\delta}\int_{-\delta}^{\delta}\int_{\hat{x}_1}^{x_1'}(b_{12}\frac{\partial u}{\partial x_2})(x_1'', x_2)dx_1''dx_1'\\
&-(resp.+)\int_{-2\delta}^{2\delta}\phi'(x_1')(A_{12}\frac{\partial u}{\partial x_2})(x_1', x_2)dx_1'\\
&-(resp.+)\int_{-2\delta}^{2\delta}\phi(x_1')(b_{12}\frac{\partial u}{\partial x_2})(x_1', x_2)dx_1' + (resp.-) \quad C;
\end{aligned}$$

for any $(x_1, \hat{x}_2) \in K$,

$$
\begin{aligned}
&(A_{22}\frac{\partial u}{\partial x_2})(x_1, \hat{x}_2) \\
\leq (resp.\ \geq) \quad &-(A_{21}\frac{\partial u}{\partial x_1})(x_1, \hat{x}_2) + \frac{1}{2\delta}\int_{-\delta}^{\delta}(A_{21}\frac{\partial u}{\partial x_1})(x_1, x_2')dx_2' \\
&-\frac{1}{2\delta}\int_{-\delta}^{\delta}\int_{\hat{x}_2}^{x_2'}(b_{21}\frac{\partial u}{\partial x_1})(x_1, x_2'')dx_2''dx_2' \\
&-(resp.+)\int_{-2\delta}^{2\delta}\phi'(x_2')(A_{21}\frac{\partial u}{\partial x_1})(x_1, x_2')dx_2' \\
&-(resp.+)\int_{-2\delta}^{2\delta}\phi(x_2')(b_{21}\frac{\partial u}{\partial x_1})(x_1, x_2')dx_2' + (resp.-) \quad C;
\end{aligned}
\tag{32}
$$

Lemma 2.2.

For the terms in (31) the following estimates hold

$$
\begin{aligned}
&|\int_{-\delta}^{\delta}\int_{\hat{x}_1}^{x_1'}(b_{12}\frac{\partial u}{\partial x_2})(x_1'', x_2)dx_1''dx_1'| \leq C, \\
&|\int_{-2\delta}^{2\delta}\phi'(x_1')(A_{12}\frac{\partial u}{\partial x_2})(x_1', x_2)dx_1'| \leq C, \\
&|\int_{-2\delta}^{2\delta}\phi(x_1')(b_{12}\frac{\partial u}{\partial x_2})(x_1', x_2)dx_1'| \leq C, \\
&|\int_{-\delta}^{\delta}(A_{12}\frac{\partial u}{\partial x_2})(x_1', x_2)dx_1'| \leq C.
\end{aligned}
\tag{33}
$$

For the terms in (32), the following estimates hold

$$
\begin{aligned}
&|\int_{-\delta}^{\delta}\int_{\hat{x}_2}^{x_2'}(b_{21}\frac{\partial u}{\partial x_1})(x_1, x_2'')dx_2''dx_2'| \leq C, \\
&|\int_{-2\delta}^{2\delta}\phi'(x_2')(A_{21}\frac{\partial u}{\partial x_1})(x_1, x_2')dx_2'| \leq C, \\
&|\int_{-2\delta}^{2\delta}\phi(x_2')(b_{21}\frac{\partial u}{\partial x_1})(x_1, x_2')dx_2'| \leq C,
\end{aligned}
\tag{34}
$$

$$|\int_{-\delta}^{\delta}(A_{21}\frac{\partial u}{\partial x_1})(x_1,x_2')dx_2'| \le C.$$

We temporarily admit Lemmas 2.1, 2.2, and give the proof of Theorem 1.1. Let us remark that (31)-(33), (32)-(34) lead the following estimates for any $(\hat{x}_1, x_2)$, $(x_1, \hat{x}_2) \in K$.

$$-(A_{12}\frac{\partial u}{\partial x_2})(\hat{x}_1,x_2) - C \le (A_{11}\frac{\partial u}{\partial x_1})(\hat{x}_1,x_2) \le -(A_{12}\frac{\partial u}{\partial x_2})(\hat{x}_1,x_2) + C \quad (35)$$

$$-(A_{21}\frac{\partial u}{\partial x_1})(x_1,\hat{x}_2) - C \le (A_{22}\frac{\partial u}{\partial x_2})(x_1,\hat{x}_2) \le -(A_{21}\frac{\partial u}{\partial x_1})(x_1,\hat{x}_2) + C \quad (36)$$

From (4), (36),

$$-(A_{12}\frac{\partial u}{\partial x_2})(\hat{x}_1,x_2) = -(\frac{A_{12}}{A_{22}}A_{22}\frac{\partial u}{\partial x_2})(\hat{x}_1,x_2)$$

$$\le (resp. \quad \ge) \quad (\frac{A_{21}A_{12}}{A_{22}}\frac{\partial u}{\partial x_1})(\hat{x}_1,x_2) + (resp.-)C,$$

and by inserting this into (35) we get for any $(\hat{x}_1, x_2) \in K$,

$$|(\frac{A_{11}A_{22} - A_{21}A_{12}}{A_{22}}\frac{\partial u}{\partial x_1})(\hat{x}_1,x_2)| \le C,$$

where $C > 0$ is a constant. From the assumption (3), (4), we have the bound for $\frac{\partial u}{\partial x_1}$, and the same discussion leads the bound for $\frac{\partial u}{\partial x_2}$. Therefore, (8) is proved.

Now, we shall give the proof of the Lemmas 2.1, 2.2.

Proof of Lemma 2.1.
We only give the proof of (31); (32) will be obtained by the same way. First, from (6) for any (x_1, x_2), $(\hat{x}_1, x_2) \in K$, since

$$-(A_{11}\frac{\partial u}{\partial x_1} + A_{12}\frac{\partial u}{\partial x_2})(x_1,x_2) + \int_{\hat{x}_1}^{x_1}(b_{12}\frac{\partial u}{\partial x_2})(x_1',x_2)dx_1'$$

$$= -(A_{11}\frac{\partial u}{\partial x_1} + A_{12}\frac{\partial u}{\partial x_2})(\hat{x}_1,x_2)$$

$$\int_{\hat{x}_1}^{x_1}\{-\frac{\partial}{\partial x_1'}(A_{11}\frac{\partial u}{\partial x_1'}+A_{12}\frac{\partial u}{\partial x_2})(x_1',x_2)+(b_{12}\frac{\partial u}{\partial x_2})(x_1',x_2)\}dx_1'$$

by using (7), (2), (5), the following holds with ϕ stated in Lemma 2.1.

$$\begin{aligned}
&|-(A_{11}\frac{\partial u}{\partial x_1}+A_{12}\frac{\partial u}{\partial x_2})(x_1,x_2)+\int_{\hat{x}_1}^{x_1}(b_{12}\frac{\partial u}{\partial x_2})(x_1',x_2)dx_1'\\
&\qquad +(A_{11}\frac{\partial u}{\partial x_1}+A_{12}\frac{\partial u}{\partial x_2})(\hat{x}_1,x_2)|\\
&\leq \mathrm{sgn}(x_1-\hat{x}_1)\int_{\hat{x}_1}^{x_1}|-\frac{\partial}{\partial x_1'}(A_{11}\frac{\partial u}{\partial x_1'}+A_{12}\frac{\partial u}{\partial x_2})(x_1',x_2)\\
&\qquad +(b_{11}\frac{\partial u}{\partial x_1'}+b_{12}\frac{\partial u}{\partial x_2})(x_1',x_2)|dx_1'+C\\
&=\mathrm{sgn}(x_1-\hat{x}_1)\int_{\hat{x}_1}^{x_1}|C_0+\frac{\partial}{\partial x_1'}(A_{11}\frac{\partial u}{\partial x_1'}+A_{12}\frac{\partial u}{\partial x_2})(x_1',x_2)\\
&\qquad -(b_{11}\frac{\partial u}{\partial x_1'}+b_{12}\frac{\partial u}{\partial x_2})(x_1',x_2)-C_0|dx_1'+C\\
&\leq \mathrm{sgn}(x_1-\hat{x}_1)\times\\
&\int_{\hat{x}_1}^{x_1}C_0+\frac{\partial}{\partial x_1'}(A_{11}\frac{\partial u}{\partial x_1'}+A_{12}\frac{\partial u}{\partial x_2})(x_1',x_2)-(b_{11}\frac{\partial u}{\partial x_1'}+b_{12}\frac{\partial u}{\partial x_2})(x_1',x_2)dx_1'+C\\
&=\mathrm{sgn}(x_1-\hat{x}_1)\int_{\hat{x}_1}^{x_1}\phi(x_1')\{C_0+\frac{\partial}{\partial x_1'}(A_{11}\frac{\partial u}{\partial x_1'}+A_{12}\frac{\partial u}{\partial x_2})(x_1',x_2)\\
&\qquad -(b_{11}\frac{\partial u}{\partial x_1'}+b_{12}\frac{\partial u}{\partial x_2})(x_1',x_2)\}dx_1'+C\\
&\leq\int_{-2\delta}^{2\delta}\phi(x_1')\{\frac{\partial}{\partial x_1'}(A_{11}\frac{\partial u}{\partial x_1'}+A_{12}\frac{\partial u}{\partial x_2})(x_1',x_2)-(b_{11}\frac{\partial u}{\partial x_1'}+b_{12}\frac{\partial u}{\partial x_2})(x_1',x_2)\}dx_1'+C\\
&\leq-\int_{-2\delta}^{2\delta}\phi(x_1')(A_{11}\frac{\partial u}{\partial x_1'}+A_{12}\frac{\partial u}{\partial x_2})(x_1',x_2)dx_1'-\int_{-2\delta}^{2\delta}\phi(x_1')(b_{12}\frac{\partial u}{\partial x_2})(x_1',x_2)dx_1'+C\\
&=-\int_{-2\delta}^{2\delta}\phi'(x_1')(A_{12}\frac{\partial u}{\partial x_2})(x_1',x_2)dx_1'-\int_{-2\delta}^{2\delta}\phi(x_1')(b_{12}\frac{\partial u}{\partial x_2})(x_1',x_2)dx_1'+C.
\end{aligned}$$

Hence, for any (x_1,x_2), $(\hat{x}_1,x_2)\in K$ we get the following inequalities.

$$(A_{11}\frac{\partial u}{\partial x_1}+A_{12}\frac{\partial u}{\partial x_2})(x_1,x_2) \tag{37}$$

$$\leq (resp.\geq) \quad (A_{11}\frac{\partial u}{\partial x_1}+A_{12}\frac{\partial u}{\partial x_2})(\hat{x}_1,x_2)+\int_{\hat{x}_1}^{x_1} b_{12}\frac{\partial u}{\partial x_2}(x_1',x_2)dx_1'$$
$$-(resp.+)\int_{-2\delta}^{2\delta}\phi'(x_1')(A_{12}\frac{\partial u}{\partial x_2})(x_1',x_2)dx_1'$$
$$-(resp.+)\int_{-2\delta}^{2\delta}\phi(x_1')(b_{12}\frac{\partial u}{\partial x_2})(x_1',x_2)dx_1'+(resp.-) \quad C.$$

Next, we integrate both hand sides of the above inequalities with respect to x_1 on $[-\delta,\delta]$, then devide the obtained result by 2δ and we have the following.

$$\frac{1}{2\delta}\int_{-\delta}^{\delta}(A_{12}\frac{\partial u}{\partial x_2})(x_1',x_2)dx_1' \tag{38}$$

$$\leq \quad (resp.\geq) \quad (A_{11}\frac{\partial u}{\partial x_1}+A_{12}\frac{\partial u}{\partial x_2})(\hat{x}_1,x_2)$$
$$+\frac{1}{2\delta}\int_{-\delta}^{\delta}\int_{\hat{x}_1}^{x_1'}(b_{12}\frac{\partial u}{\partial x_2})(x_1'',x_2)dx_1''dx_1'$$
$$-(resp.+)\int_{-2\delta}^{2\delta}\phi'(x_1')(A_{12}\frac{\partial u}{\partial x_2})(x_1',x_2)dx_1'$$
$$-(resp.+)\int_{-2\delta}^{2\delta}\phi(x_1')(b_{12}\frac{\partial u}{\partial x_2})(x_1',x_2)dx_1'+(resp.-) \quad C.$$

From (38), (31) holds clearly.

Proof of Lemma 2.2.
In the two dimensional case, the argument is easy. In fact, to show the first inequalities in (33), we multiply both hand sides of the inequalities (32) by $(A_{22}^{-1}b_{12})(x_1,\hat{x}_2)$ and then integrate the result first with respect to x_1 on $[\hat{x}_1,x_1']$, then with respect to x_1' on $[-\delta,\delta]$, which leads the conclusion because of (2), (4), (5), (7). The other estimates can be obtained similarly, which we do not repeat here.

3 Proof of Theorem 1.2.

Lemma 3.1.

Let ϕ be an arbotrarily fixed real valued twice differentiable function defined on the interval $[-2\delta, 2\delta]$ such that $0 \le \phi \le 1$, $supp\phi \subset\subset (-2\delta, 2\delta)$, ϕ is even and

$$\phi = 1 \quad on \quad [-\delta, \delta], \quad \phi' \ge 0 \quad on \quad [-2\delta, 0]. \tag{39}$$

Then, the function u in Theorem 1.2 satisfies the following inequalities: for any $(\hat{x}_1, x_2, x_3) \in K$,

$$\begin{aligned}
&(A_{11}\frac{\partial u}{\partial x_1})(\hat{x}_1, x_2, x_3) \qquad (40)\\
&\le (resp. \ge) \quad -(A_{12}\frac{\partial u}{\partial x_2} + A_{13}\frac{\partial u}{\partial x_3})(\hat{x}_1, x_2, x_3)\\
&+\frac{1}{2\delta}\int_{-\delta}^{\delta}(A_{12}\frac{\partial u}{\partial x_2} + A_{13}\frac{\partial u}{\partial x_3})(x_1', x_2, x_3)dx_1'\\
&-\frac{1}{2\delta}\int_{-\delta}^{\delta}\int_{\hat{x}_1}^{x_1'}(b_{12}\frac{\partial u}{\partial x_2} + b_{13}\frac{\partial u}{\partial x_3})(x_1'', x_2, x_3)dx_1''dx_1'\\
&-(resp.+)\int_{-2\delta}^{2\delta}\phi'(x_1')(A_{12}\frac{\partial u}{\partial x_2} + A_{13}\frac{\partial u}{\partial x_3})(x_1', x_2, x_3)dx_1'\\
&-(resp.+)\int_{-2\delta}^{2\delta}\phi(x_1')(b_{12}\frac{\partial u}{\partial x_2} + b_{13}\frac{\partial u}{\partial x_3})(x_1', x_2, x_3)dx_1' + (resp.-) \quad C,
\end{aligned}$$

for any $(x_1, \hat{x}_2, x_3) \in K$,

$$\begin{aligned}
&(A_{22}\frac{\partial u}{\partial x_2})(x_1, \hat{x}_2, x_3) \qquad (41)\\
&\le (resp. \ge) \quad -(A_{21}\frac{\partial u}{\partial x_1} + A_{23}\frac{\partial u}{\partial x_3})(x_1, \hat{x}_2, x_3)\\
&+\frac{1}{2\delta}\int_{-\delta}^{\delta}(A_{21}\frac{\partial u}{\partial x_1} + A_{23}\frac{\partial u}{\partial x_3})(x_1, x_2', x_3)dx_2'\\
&-\frac{1}{2\delta}\int_{-\delta}^{\delta}\int_{\hat{x}_2}^{x_2'}(b_{21}\frac{\partial u}{\partial x_1} + b_{23}\frac{\partial u}{\partial x_3})(x_1, x_2'', x_3)dx_2''dx_2'\\
&-(resp.+)\int_{-2\delta}^{2\delta}\phi'(x_2')(A_{21}\frac{\partial u}{\partial x_1} + A_{23}\frac{\partial u}{\partial x_3})(x_1, x_2', x_3)dx_2'\\
&-(resp.+)\int_{-2\delta}^{2\delta}\phi(x_2')(b_{21}\frac{\partial u}{\partial x_1} + b_{23}\frac{\partial u}{\partial x_3})(x_1, x_2', x_3)dx_2' + (resp.-) \quad C.
\end{aligned}$$

for any $(x_1, x_2, \hat{x}_3) \in K$,

$$(A_{33}\frac{\partial u}{\partial x_3})(x_1, x_2, \hat{x}_3) \tag{42}$$

$$\leq (resp. \geq) \quad -(A_{31}\frac{\partial u}{\partial x_1} + A_{32}\frac{\partial u}{\partial x_2})(x_1, x_2, \hat{x}_3)$$

$$+\frac{1}{2\delta}\int_{-\delta}^{\delta}(A_{31}\frac{\partial u}{\partial x_1} + A_{32}\frac{\partial u}{\partial x_2})(x_1, x_2, x_3')dx_3'$$

$$-\frac{1}{2\delta}\int_{-\delta}^{\delta}\int_{\hat{x}_3}^{x_3'}(b_{31}\frac{\partial u}{\partial x_1} + b_{32}\frac{\partial u}{\partial x_2})(x_1, x_2, x_3'')dx_3''dx_3'$$

$$-(resp.+)\int_{-2\delta}^{2\delta}\phi'(x_3')(A_{31}\frac{\partial u}{\partial x_1} + A_{32}\frac{\partial u}{\partial x_2})(x_1, x_2, x_3')dx_3'$$

$$-(resp.+)\int_{-2\delta}^{2\delta}\phi(x_3')(b_{31}\frac{\partial u}{\partial x_1} + b_{32}\frac{\partial u}{\partial x_2})(x_1, x_2, x_3')dx_3' + (resp.-) \quad C.$$

Lemma 3.2.

Let us denote $(x_1', x_2, x_3) = (x_1')$, $(x_1, x_2', x_3) = (x_2')$, $(x_1, x_2, x_3') = (x_3')$, *and* $(x_1'', x_2, x_3) = (x_1'')$, $(x_1, x_2'', x_3) = (x_2'')$, $(x_1, x_2, x_3'') = (x_3'')$. *Then, for the terms in (40), (41), (42), the following estimate hold*

$$|\int_{-\delta}^{\delta}\int_{\hat{x}_i}^{x_i'}\frac{\partial u}{\partial x_j}(x_i'')dx_i''dx_i'| \leq C, \quad 1 \leq i, j \leq 3, \quad i \neq j, \tag{43}$$

$$|\int_{-2\delta}^{2\delta}\phi'(x_i')\frac{\partial u}{\partial x_j}(x_i')dx_i'| \leq C, \quad 1 \leq i, j \leq 3, \quad i \neq j, \tag{44}$$

$$|\int_{-2\delta}^{2\delta}\phi(x_i')\frac{\partial u}{\partial x_j}(x_i')dx_i'| \leq C, \quad 1 \leq i, j \leq 3, \quad i \neq j, \tag{45}$$

$$|\int_{-\delta}^{\delta}\frac{\partial u}{\partial x_j}(x_i')dx_i'| \leq C, \quad 1 \leq i, j \leq 3, \quad i \neq j, \tag{46}$$

The proofs of Lemmas 3.1, 3.2 will be given below. Here, we admit them and give the proof of Theorem 1.2.

By inserting the estimates (43)-(46) in Lemma 3.2 into (40)-(42), we have the following.

$$(A_{11}\frac{\partial u}{\partial x_1})(\hat{x}_1, x_2, x_3) \leq (resp. \geq) \tag{47}$$

$$-(A_{12}\frac{\partial u}{\partial x_2} + A_{13}\frac{\partial u}{\partial x_3})(\hat{x}_1, x_2, x_3) + (resp.-) \quad C \quad \forall(\hat{x}_1, x_2, x_3) \in K,$$

$$(A_{22}\frac{\partial u}{\partial x_2})(x_1, \hat{x}_2, x_3) \leq (resp. \geq) \tag{48}$$

$$-(A_{21}\frac{\partial u}{\partial x_1} + A_{23}\frac{\partial u}{\partial x_3})(x_1, \hat{x}_2, x_3) + (resp.-) \quad C \quad \forall(x_1, \hat{x}_2, x_3) \in K,$$

$$(A_{33}\frac{\partial u}{\partial x_3})(x_1, x_2, \hat{x}_3) \leq (resp. \geq) \tag{49}$$

$$-(A_{31}\frac{\partial u}{\partial x_1} + A_{32}\frac{\partial u}{\partial x_2})(x_1, x_2, \hat{x}_3) + (resp.-) \quad C \quad \forall(\hat{x}_1, x_2, \hat{x}_3) \in K.$$

From (47), for any $(x_1, x_2, \hat{x}_3)$, $(x_1, \hat{x}_2, x_3) \in K$,

$$-(A_{31}\frac{\partial u}{\partial x_1})(x_1, x_2, \hat{x}_3) = -(\frac{A_{31}}{A_{11}}A_{11}\frac{\partial u}{\partial x_1})(x_1, x_2, \hat{x}_3)$$

$$\leq (resp. \geq)(\frac{A_{31}A_{12}}{A_{11}}\frac{\partial u}{\partial x_2} + \frac{A_{31}A_{13}}{A_{11}}\frac{\partial u}{\partial x_3})(x_1, x_2, \hat{x}_3) + (resp.-) \quad C,$$

$$-(A_{21}\frac{\partial u}{\partial x_1})(x_1, \hat{x}_2, x_3) = -(\frac{A_{21}}{A_{11}}A_{11}\frac{\partial u}{\partial x_1})(x_1, \hat{x}_2, x_3)$$

$$\leq (resp. \geq)(\frac{A_{21}A_{12}}{A_{11}}\frac{\partial u}{\partial x_2} + \frac{A_{21}}{A_{13}}A_{11}\frac{\partial u}{\partial x_3})(x_1, \hat{x}_2, x_3) + (resp.-) \quad C.$$

Introducing the above inequalities into (48), (49) we have the following.

$$(\frac{A_{11}A_{22} - A_{21}A_{12}}{A_{11}}\frac{\partial u}{\partial x_2})(x_1, \hat{x}_2, x_3) \tag{50}$$

$$\leq (resp. \geq) \quad (\frac{A_{21}A_{13} - A_{11}A_{23}}{A_{11}}\frac{\partial u}{\partial x_3})(x_1, \hat{x}_2, x_3) + (resp.-) \quad C,$$

$$(\frac{A_{11}A_{33} - A_{31}A_{13}}{A_{11}}\frac{\partial u}{\partial x_3})(x_1, x_2, \hat{x}_3) \tag{51}$$

$$\leq (resp.\geq) \quad (\frac{A_{31}A_{12}-A_{11}A_{32}}{A_{11}}\frac{\partial u}{\partial x_2})(x_1,x_2,\hat{x}_3)+(resp.-) \quad C,$$

From (50),

$$(\frac{A_{31}A_{12}-A_{11}A_{32}}{A_{11}}\frac{\partial u}{\partial x_2})(x_1,x_2,\hat{x}_3)$$

$$\frac{(A_{31}A_{12}-A_{11}A_{32})(A_{11}A_{22}-A_{21}A_{12})}{(A_{11}A_{22}-A_{21}A_{12})A_{11}}\frac{\partial u}{\partial x_2})(x_1,x_2,\hat{x}_3)$$

$$\leq (resp.\geq)$$

$$\frac{(A_{31}A_{12}-A_{11}A_{32})(A_{21}A_{13}-A_{11}A_{23})}{(A_{11}A_{22}-A_{21}A_{12})A_{11}}\frac{\partial u}{\partial x_3})(x_1,x_2,\hat{x}_3)+(resp.-) \quad C,$$

and by introducing the above inequalities into (51), we have the following.

$$\{(A_{11}A_{33}-A_{31}A_{13})(A_{11}A_{22}-A_{21}A_{12})-(A_{31}A_{12}-A_{11}A_{32})(A_{21}A_{13}-A_{11}A_{23})\}\times$$

$$\times\frac{\partial u}{\partial x_3}(x_1,x_2,\hat{x}_3)\leq (resp.\geq) \quad +(resp.-) \quad C.$$

Therefore, from the assumptions (10), (12), we get the bound for $\frac{\partial u}{\partial x_3}$. A similar argument leads to the bounds for $\frac{\partial u}{\partial x_1}$, $\frac{\partial u}{\partial x_2}$, and we have proved (20).

Proof of Lemma 3.1.
We only prove (40); (41), (42) will be obtained in a similar way. First of all, from (17) for any (x_1,x_2,x_3), $(\hat{x}_1,x_2,x_3)\in K$, since

$$-(A_{11}\frac{\partial u}{\partial x_1}+A_{12}\frac{\partial u}{\partial x_2}+A_{13}\frac{\partial u}{\partial x_3})(x_1,x_2,x_3)+\int_{\hat{x}_1}^{x_1}(b_{12}\frac{\partial u}{\partial x_2}+b_{13}\frac{\partial u}{\partial x_3})(x_1',x_2,x_3)dx_1'$$

$$=-(A_{11}\frac{\partial u}{\partial x_1}+A_{12}\frac{\partial u}{\partial x_2}+A_{13}\frac{\partial u}{\partial x_3})(\hat{x}_1,x_2,x_3)$$

$$+\int_{\hat{x}_1}^{x_1}\{-\frac{\partial}{\partial x_1'}(A_{11}\frac{\partial u}{\partial x_1'}+A_{12}\frac{\partial u}{\partial x_2}+A_{13}\frac{\partial u}{\partial x_3})(x_1',x_2,x_3)$$

$$+(b_{12}\frac{\partial u}{\partial x_2}+b_{13}\frac{\partial u}{\partial x_3})(x_1',x_2,x_3)dx_1',$$

by using (18), (9), (16), the following holds with ϕ stated in Lemma 3.1.

$$|-(A_{11}\frac{\partial u}{\partial x_1}+A_{12}\frac{\partial u}{\partial x_2}+A_{13}\frac{\partial u}{\partial x_3})(x_1,x_2,x_3)+\int_{\hat{x}_1}^{x_1}(b_{12}\frac{\partial u}{\partial x_2}+b_{13}\frac{\partial u}{\partial x_3})(x_1',x_2,x_3)dx_1'$$

$$
\begin{aligned}
&+(A_{11}\frac{\partial u}{\partial x_1}+A_{12}\frac{\partial u}{\partial x_2}+A_{13}\frac{\partial u}{\partial x_3})(\hat{x}_1,x_2,x_3)| \\
\leq \operatorname{sgn}(x_1-\hat{x}_1)&\int_{\hat{x}_1}^{x_1}|-\frac{\partial}{\partial x_1'}(A_{11}\frac{\partial u}{\partial x_1}+A_{12}\frac{\partial u}{\partial x_2}+A_{13}\frac{\partial u}{\partial x_3})(x_1',x_2,x_3) \\
&+(b_{11}\frac{\partial u}{\partial x_1'}+b_{12}\frac{\partial u}{\partial x_2}+b_{13}\frac{\partial u}{\partial x_3})(x_1',x_2,x_3)|dx_1'+C \\
\leq \operatorname{sgn}(x_1-\hat{x}_1)&\int_{\hat{x}_1}^{x_1}|C_0+\frac{\partial}{\partial x_1'}(A_{11}\frac{\partial u}{\partial x_1'}+A_{12}\frac{\partial u}{\partial x_2}+A_{13}\frac{\partial u}{\partial x_3})(x_1',x_2,x_3) \\
&-(b_{11}\frac{\partial u}{\partial x_1'}+b_{12}\frac{\partial u}{\partial x_2}+b_{13}\frac{\partial u}{\partial x_3})(x_1',x_2,x_3)-C_0|dx_1'+C \\
\leq \operatorname{sgn}(x_1-\hat{x}_1)&\int_{\hat{x}_1}^{x_1}C_0+\frac{\partial}{\partial x_1'}(A_{11}\frac{\partial u}{\partial x_1'}+A_{12}\frac{\partial u}{\partial x_2}+A_{13}\frac{\partial u}{\partial x_3})(x_1',x_2,x_3) \\
&-(b_{11}\frac{\partial u}{\partial x_1'}+b_{12}\frac{\partial u}{\partial x_2}+b_{13}\frac{\partial u}{\partial x_3})(x_1',x_2,x_3)dx_1'+C \\
&=\operatorname{sgn}(x_1-\hat{x}_1)\times \\
\times&\int_{\hat{x}_1}^{x_1}\phi(x_1')\{C_0+\frac{\partial}{\partial x_1'}(A_{11}\frac{\partial u}{\partial x_1'}+A_{12}\frac{\partial u}{\partial x_2}+A_{13}\frac{\partial u}{\partial x_3})(x_1',x_2,x_3) \\
&-(b_{11}\frac{\partial u}{\partial x_1'}+b_{12}\frac{\partial u}{\partial x_2}+b_{13}\frac{\partial u}{\partial x_3})(x_1',x_2,x_3)\}dx_1'+C \\
\leq&\int_{-2\delta}^{2\delta}\phi(x_1')\{\frac{\partial}{\partial x_1'}(A_{11}\frac{\partial u}{\partial x_1'}+A_{12}\frac{\partial u}{\partial x_2}+A_{13}\frac{\partial u}{\partial x_3})(x_1',x_2,x_3) \\
&-(b_{11}\frac{\partial u}{\partial x_1'}+b_{12}\frac{\partial u}{\partial x_2}+b_{13}\frac{\partial u}{\partial x_3})(x_1',x_2,x_3)dx_1'+C \\
\leq&\int_{-2\delta}^{2\delta}\phi'(x_1')(A_{11}\frac{\partial u}{\partial x_1'}+A_{12}\frac{\partial u}{\partial x_2}+A_{13}\frac{\partial u}{\partial x_3})(x_1',x_2,x_3)dx_1' \\
&-\int_{-2\delta}^{2\delta}\phi(x_1')(b_{12}\frac{\partial u}{\partial x_2}+b_{13}\frac{\partial u}{\partial x_3})(x_1',x_2,x_3)\}dx_1'+C \\
&=-\int_{-2\delta}^{2\delta}\phi'(x_1')(A_{12}\frac{\partial u}{\partial x_2}+A_{13}\frac{\partial u}{\partial x_3})(x_1',x_2,x_3)dx_1' \\
&-\int_{-2\delta}^{2\delta}\phi(x_1')(b_{12}\frac{\partial u}{\partial x_2}+b_{13}\frac{\partial u}{\partial x_3})(x_1',x_2,x_3)\}dx_1'+C
\end{aligned}
$$

Hence, for any (x_1, x_2, x_3), $(\hat{x}_1, x_2, x_3) \in K$, we get the following inequalities.

$$(A_{11}\frac{\partial u}{\partial x_1} + A_{12}\frac{\partial u}{\partial x_2} + A_{13}\frac{\partial u}{\partial x_3})(x_1, x_2, x_3) \tag{52}$$

$$\leq (resp.\ \geq) \quad (A_{11}\frac{\partial u}{\partial x_1} + A_{12}\frac{\partial u}{\partial x_2} + A_{13}\frac{\partial u}{\partial x_3})(\hat{x}_1, x_2, x_3)$$

$$+ \int_{\hat{x}_1}^{x_1} (b_{12}\frac{\partial u}{\partial x_2} + b_{13}\frac{\partial u}{\partial x_3})(x_1', x_2, x_3)dx_1'$$

$$-(resp.+) \quad \int_{-2\delta}^{2\delta} \phi'(x_1')(A_{12}\frac{\partial u}{\partial x_2} + A_{13}\frac{\partial u}{\partial x_3})(x_1', x_2, x_3)dx_1'$$

$$-(resp.+) \quad \int_{-2\delta}^{2\delta} \phi(x_1')(b_{12}\frac{\partial u}{\partial x_2} + b_{13}\frac{\partial u}{\partial x_3})(x_1', x_2, x_3)dx_1' + (resp.-) \quad C$$

Next, we integrate the both hands sides of the above inequalities with respect to x_1 on $[-\delta, \delta]$, then devide the result by 2δ and we have the following.

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} (A_{12}\frac{\partial u}{\partial x_2} + A_{13}\frac{\partial u}{\partial x_3})(x_1', x_2, x_3)dx_1' \tag{53}$$

$$\leq (resp.\ \geq) \quad (A_{11}\frac{\partial u}{\partial x_1} + A_{12}\frac{\partial u}{\partial x_2} + A_{13}\frac{\partial u}{\partial x_3})(\hat{x}_1, x_2, x_3)$$

$$+\frac{1}{2\delta}\int_{-\delta}^{\delta}\int_{\hat{x}_1}^{x_1} (b_{12}\frac{\partial u}{\partial x_2} + b_{13}\frac{\partial u}{\partial x_3})(x_1'', x_2, x_3)\}dx_1''dx_1'$$

$$-(resp.+) \quad \int_{-2\delta}^{2\delta} \phi'(x_1')(A_{12}\frac{\partial u}{\partial x_2} + A_{13}\frac{\partial u}{\partial x_3})(x_1', x_2, x_3)\}dx_1'$$

$$-(resp.+) \quad \int_{-2\delta}^{2\delta} \phi(x_1')(b_{12}\frac{\partial u}{\partial x_2} + b_{13}\frac{\partial u}{\partial x_3})(x_1', x_2, x_3)\}dx_1' + (resp.-) \quad C$$

The above inequality leads (40).

Proof of Lemma 3.2.
We show the estimates (43)-(46) in the following steps 1-4.

Step 1. (Estimate (43).) We consider the particular case when $i = 1$, $j = 2$; the other cases are obtained in a similar way in view of the symmetry of the conditions on (A_{ij}) and (b_{ij}).

$$|\int_{-\delta}^{\delta}\int_{\hat{x}_1}^{x_1'} \frac{\partial u}{\partial x_2}(x_1'', \hat{x}_2, x_3)dx_1''dx_1'| \leq C, \quad \forall \hat{x}_2, , \forall x_3 \in [-\delta, \delta]. \tag{54}$$

First, we integrate both hands sides of the inequalities (41) with respect to x_1 on $[\hat{x}_1, x_1']$, and then with respect to x_1' on $[-\delta, \delta]$. Remark that since u is bounded, the integrals of $\frac{\partial u}{\partial x_3}\ (x_1, \hat{x}_2, x_3)$ with respect to x_1 are estimated by constants. Moreover, remark that by using (42) and the boundedness of u, the integrals of $\frac{\partial u}{\partial x_3}(x_1, x_2, x_3)$ with respect to x_2 and x_1 are estimated by constants. Thus, we get the following inequality from (41) and (42).

$$\int_{-\delta}^{\delta}\int_{\hat{x}_1}^{x_1'} A_{22}\frac{\partial u}{\partial x_2}(x_1'', \hat{x}_2, x_3)dx_1''dx_1'$$

$$(resp.\geq) \quad \int_{-\delta}^{\delta}\int_{\hat{x}_1}^{x_1'} -A_{23}\frac{\partial u}{\partial x_3}(x_1'', \hat{x}_2, x_3)dx_1''dx_1' + (resp.-) \quad C.$$

We denote

$$B(x_3) = \int_{-\delta}^{\delta}\int_{\hat{x}_1}^{x_1'} A_{22}\frac{\partial u}{\partial x_2}(x_1'', \hat{x}_2, x_3)dx_1''dx_1',$$

where $\hat{x}_2 \in [-\delta, \delta]$ is arbitrarily fixed. Then, since

$$-A_{23}\frac{\partial u}{\partial x_3}(x_1'', \hat{x}_2, x_3) = -\frac{A_{23}}{A_{33}}A_{33}\frac{\partial u}{\partial x_3}(x_1'', \hat{x}_2, x_3),$$

by inserting (42) into the above inequalities and by using the boundedness of u, we deduce

$$A_{22}B(x_3) \leq (resp.\geq) \quad \frac{A_{23}A_{32}}{A_{33}}B(x_3) - \frac{1}{2\delta}\frac{A_{23}A_{32}}{A_{33}}\int_{-\delta}^{\delta} B(x_3')dx_3' \qquad (55)$$

$$+\frac{1}{2\delta}\frac{A_{23}b_{32}}{A_{33}}\int_{-\delta}^{\delta}\int_{\hat{x}_3}^{x_3'} B(x_3'')dx_3''dx_3'$$

$$-(resp.+)|\frac{A_{23}}{A_{33}}|A_{32}\int_{-2\delta}^{2\delta}\phi'(x_3')B(x_3')dx_3'$$

$$-(resp.+) \quad |\frac{A_{23}}{A_{33}}|b_{32}\int_{-2\delta}^{2\delta}\phi'(x_3')B(x_3')dx_3' + (resp.-) \quad C.$$

We multiply (55) by $\phi'(x_3)$ and integrate the result with respect to x_3 on $[-2\delta, 2\delta]$. Then, from the assumption on ϕ in (39), we have

$$\frac{A_{22}A_{33} - A_{23}A_{32}}{A_{33}}\int_{-2\delta}^{2\delta}\phi'(x_3')B(x_3')dx_3'$$

$$\leq (resp.\geq) \quad -(resp.+)2|\frac{A_{23}}{A_{33}}|A_{32}\int_{-2\delta}^{2\delta}\phi'(x_3')B(x_3')dx_3'$$

$$-(resp.+)2|\frac{A_{23}}{A_{33}}|b_{32}\int_{-2\delta}^{2\delta}\phi(x_3')B(x_3')dx_3' + (resp.-) \quad C.$$

Here, we shall denote

$$E_1 = (\frac{A_{22}A_{33} - A_{23}A_{32}}{A_{33}} - 2|\frac{A_{23}}{A_{33}}|A_{32})^{-1}|\frac{A_{23}}{A_{33}}|b_{32} \tag{56}$$

$$E_2 = (\frac{A_{22}A_{33}A_{23}A_{32}}{A_{33}} + 2|\frac{A_{23}}{A_{33}}|A_{32})^{-1}|\frac{A_{23}}{A_{33}}|b_{32} \tag{57}$$

From the condition (15), we have the following two cases.

Case 1. The following inequalities hold.

$$\begin{aligned}\frac{A_{22}A_{33} - A_{23}A_{32}}{A_{33}} + 2|\frac{A_{23}}{A_{33}}|A_{32} > 0,\\ \frac{A_{22}A_{33} - A_{23}A_{32}}{A_{33}} - 2|\frac{A_{23}}{A_{33}}|A_{32} > 0.\end{aligned} \tag{58}$$

Case 2. The following inequalities hold.

$$\begin{aligned}\frac{A_{22}A_{33} - A_{23}A_{32}}{A_{33}} + 2|\frac{A_{23}}{A_{33}}|A_{32} < 0,\\ \frac{A_{22}A_{33} - A_{23}A_{32}}{A_{33}} - 2|\frac{A_{23}}{A_{33}}|A_{32} < 0.\end{aligned} \tag{59}$$

So, in Case 1 ((58)),

$$\int_{-2\delta}^{2\delta}\phi'(x_3')B(x_3')dx_3' \le -2E_2\int_{-2\delta}^{2\delta}\phi(x_3')B(x_3')dx_3',$$

$$\int_{-2\delta}^{2\delta}\phi'(x_3')B(x_3')dx_3' \ge 2E_1\int_{-2\delta}^{2\delta}\phi(x_3')B(x_3')dx_3',$$

and in Case 2 ((59)),

$$\int_{-2\delta}^{2\delta}\phi'(x_3')B(x_3')dx_3' \ge -2E_2\int_{-2\delta}^{2\delta}\phi(x_3')B(x_3')dx_3',$$

$$\int_{-2\delta}^{2\delta} \phi'(x_3')B(x_3')dx_3' \leq 2E_1 \int_{-2\delta}^{2\delta} \phi(x_3')B(x_3')dx_3'.$$

By inserting these inequalities into (55), we get the following

$$\frac{A_{22}A_{33}A_{23}A_{32}}{A_{33}}B(x_3) \leq (resp.\geq) \quad -\frac{1}{2\delta}\frac{A_{23}A_{32}}{A_{33}}\int_{-\delta}^{\delta} B(x_3')dx_3' \tag{60}$$

$$+\frac{1}{2\delta}\frac{A_{23}b_{32}}{A_{33}}\int_{-\delta}^{\delta}\int_{\hat{x}_3}^{x_3'} B(x_3'')dx_3''dx_3'$$

$$-(resp.+) \quad \frac{A_{22}A_{33}-A_{23}A_{32}}{A_{33}}E_i\int_{-2\delta}^{2\delta}\phi(x_3')B(x_3')dx_3' + (resp.-) \quad C,$$

where $i = 1$ in Case 1 and $A_{32} \geq 0$, or in Case 2 and $A_{32} \leq 0$; $i = 2$ in Case 1 and $A_{32} \leq 0$, or in Case 2 and $A_{32} \geq 0$.
Next, we investigate both hand sides of (60) with respect to x_3 on $[-\delta, \delta]$, and devide both hands sides of the result by A_{22}.

$$\int_{-\delta}^{\delta} B(x_3')dx_3' \leq (resp.\geq) \quad \frac{A_{23}b_{32}}{A_{22}A_{33}}\int_{-\delta}^{\delta}\int_{\hat{x}_3}^{x_3'} B(x_3'')dx_3''dx_3' \tag{61}$$

$$-(resp.+) \quad \frac{A_{22}A_{33}-A_{23}A_{32}}{A_{33}}\frac{2\delta E_i}{|A_{22}|}\int_{-2\delta}^{2\delta}\phi(x_3')B(x_3')dx_3' + (resp.-) \quad C,$$

where the indices $i = 1, 2$ are similar to (60). By inserting the above inequalities into (60), then deviding both hands sides of the result by $\alpha = \frac{A_{22}A_{33}-A_{23}A_{32}}{A_{33}}$, we get the following.

$$B(x_3) \leq (resp.\geq) \quad \frac{1}{2\delta}\frac{A_{23}b_{32}}{A_{22}A_{33}}\int_{-\delta}^{\delta}\int_{\hat{x}_3}^{x_3'} B(x_3'')dx_3''dx_3' \tag{62}$$

$$-(resp.+) \quad (\mathrm{sgn}\alpha)E_i(1+|\frac{A_{23}A_{32}}{A_{22}A_{33}}|)\int_{-2\delta}^{2\delta}\phi(x_3')B(x_3')dx_3' + (resp.-) \quad C,$$

where the indices $i = 1, 2$ are similar to (60).
We integrate the both hand sides of (62) with respect to x_3 on $[\hat{x}_3, x_3']$, then with respect to x_3' on $[-\delta, \delta]$, which leads the following.

$$\int_{-\delta}^{\delta}\int_{\hat{x}_3}^{x_3'} B(x_3'')dx_3''dx_3' \leq (resp.\geq) \quad -\hat{x}_3\frac{A_{23}b_{32}}{A_{22}A_{33}}\int_{-\delta}^{\delta}\int_{\hat{x}_3}^{x_3'} B(x_3'')dx_3''dx_3'$$

$$-(resp.+) \quad (-2\delta\hat{x}_3)(\mathrm{sgn}\alpha)E_i(1+\frac{A_{23}A_{32}}{A_{22}A_{33}}|)\int_{-2\delta}^{2\delta}\phi(x_3')B(x_3')dx_3' + (resp.-) \quad C,$$

where the indices $i = 1, 2$ are similar to (60). From (15),

$$\frac{1}{A_{22}A_{33}} \leq \max\{\frac{1}{A_{22}A_{33} - A_{23}A_{32} + 2|A_{23}A_{32}|}, \frac{1}{A_{22}A_{33} - A_{23}A_{32} - 2|A_{23}A_{32}|}\},$$

and since $|\hat{x}_3| < \delta$, we have from (19)

$$| - \hat{x}_3 \frac{A_{23}b_{32}}{A_{22}A_{33}}| \leq \frac{1}{2}.$$

Thus, for each cases of $i = 1, 2$, there exist constants $O_1(\delta)= O(\delta)$, $O_2(\delta)= O(\delta)$ respectively, such that

$$\frac{1}{2\delta}\frac{A_{23}b_{32}}{A_{22}A_{33}} \int_{-\delta}^{\delta}\int_{\hat{x}_3}^{x_3'} B(x_3'')dx_3''dx_3'$$

$$\leq (resp. \geq) \quad - (resp.+) \quad O_i(\delta)\int_{-2\delta}^{2\delta}\phi(x_3')B(x_3')dx_3' + (resp.-) \quad C,$$

where the indices $i = 1, 2$ are similar to (60). By inserting the above estimate into (62), we get

$$B(x_3) \leq (resp. \geq) \tag{63}$$

$$-(resp.+) \quad \{O_i(\delta)+(\mathrm{sgn}\alpha)E_i(1+|\frac{A_{23}A_{32}}{A_{22}A_{33}}|\}\int_{-2\delta}^{2\delta}\phi(x_3')B(x_3')dx_3'+(resp.-) \quad C,$$

where the indices $i = 1, 2$ are similar to (60). We multiply (63) by $\phi(x_3) > 0$ and integrate both hand sides of the result with respect to x_3 on $[-2\delta, 2\delta]$. Then, by remarking that

$$2\delta \leq \int_{-2\delta}^{2\delta}\phi(x_3')B(x_3')dx_3' \leq 4\delta,$$

also by remarking that from (19),

$$|4\delta E_i(1 + |\frac{A_{23}A_{32}}{A_{22}A_{33}}|)| < \frac{1}{2} \quad i = 1, 2,$$

and by noticing that $O_i(\delta)= O(\delta)$ for $i = 1, 2$, we obtain

$$|\int_{-2\delta}^{2\delta}\phi(x_3')B(x_3')dx_3'| < C.$$

By inserting the last estimate into (63), we obtain the estimate.

Step 2. (Estimate (44).)
We consider the particular case when $i = 1$, $j = 2$, by separating it into the following two inequalities; the other cases are obtained in a similar way in view of the symmetry of the conditions on $(A_{ij})_{1\le i,j\le 3}$ and $(b_{ij})_{1\le i,j\le 3}$.

$$|\int_{-2\delta}^{0}\phi'(x_1',\hat{x}_2,x_3)\frac{\partial u}{\partial x_2}(x_1',\hat{x}_2,x_3)dx_1'|<C,\quad \forall\hat{x}_2,\quad x_3\in[-\delta,\delta], \tag{64}$$

$$|\int_{0}^{2\delta}\phi'(x_1',\hat{x}_2,x_3)\frac{\partial u}{\partial x_2}(x_1',\hat{x}_2,x_3)dx_1'|<C,\quad \forall\hat{x}_2,\quad x_3\in[-\delta,\delta], \tag{65}$$

It is enough to show (64), because (65) can be proved in the same way. Now, we set

$$C(x_3)=\int_{-2\delta}^{0}\phi'(x_1',\hat{x}_2,x_3)\frac{\partial u}{\partial x_2}(x_1',\hat{x}_2,x_3)dx_1',$$

where $\hat{x}_2\in[-\delta,\delta]$ is arbitrarily fixed. By using the estimate (43) in (41), in the same way as in Step 1, we obtain

$$\begin{aligned}A_{22}C(x_3)\le(resp.\ge)\quad &\frac{A_{23}A_{32}}{A_{33}}C(x_3)-\frac{1}{2\delta}\frac{A_{23}A_{32}}{A_{33}}\int_{-\delta}^{\delta}C(x_3')dx_3' \\ -(resp.+)\quad &|\frac{A_{23}}{A_{33}}|A_{32}\int_{-2\delta}^{2\delta}\phi'(x_3')C(x_3')dx_3' \\ -(resp.+)\quad &|\frac{A_{23}}{A_{33}}|b_{32}\int_{-2\delta}^{2\delta}\phi(x_3')C(x_3')dx_3'+(resp.-)\quad C.\end{aligned} \tag{66}$$

We multiply both hand sides of (66) by $\phi'(x_3)$, then integrate the result with respect to x_3 on $[-2\delta,2\delta]$. From the assumption on ϕ in (39), we get

$$\begin{aligned}&\frac{A_{22}A_{33}-A_{23}A_{32}}{A_{33}}\int_{-2\delta}^{2\delta}\phi'(x_3')C(x_3')dx_3' \\ \le(resp.\ge)\quad &-(resp.+)\quad 2|\frac{A_{23}}{A_{33}}|A_{32}\int_{-2\delta}^{2\delta}\phi'(x_3')C(x_3')dx_3' \\ -(resp.+)\quad &2|\frac{A_{23}}{A_{33}}|b_{32}\int_{-2\delta}^{2\delta}\phi(x_3')C(x_3')dx_3'+(resp.-)\quad C.\end{aligned}$$

From the condition (15), we have the following two cases.
Case 1.
The following inequalities hold.

$$\frac{A_{22}A_{33}-A_{23}A_{32}}{A_{33}}+2|\frac{A_{23}}{A_{33}}|A_{32}>0,$$

$$\frac{A_{22}A_{33}-A_{23}A_{32}}{A_{33}}-2|\frac{A_{23}}{A_{33}}|A_{32}>0.$$

Case 2.
The following inequalities hold.

$$\frac{A_{22}A_{33}-A_{23}A_{32}}{A_{33}}+2|\frac{A_{23}}{A_{33}}|A_{32}<0,$$

$$\frac{A_{22}A_{33}-A_{23}A_{32}}{A_{33}}-2|\frac{A_{23}}{A_{33}}|A_{32}<0.$$

Thus, denoting by

$$E_1=(\frac{A_{22}A_{33}-A_{23}A_{32}}{A_{33}}-2|\frac{A_{23}}{A_{33}}|A_{32})^{-1}|\frac{A_{23}}{A_{33}}|b_{32}, \tag{67}$$

$$E_2=(\frac{A_{22}A_{33}-A_{23}A_{32}}{A_{33}}+2|\frac{A_{23}}{A_{33}}|A_{32})^{-1}|\frac{A_{23}}{A_{33}}|b_{32}, \tag{68}$$

the same argument as in Step 1 to deduce (61) leads to the following inequalities.

$$\frac{A_{22}A_{33}-A_{23}A_{32}}{A_{33}}C(x_3)\leq(resp.\geq)\quad-\frac{1}{2\delta}\frac{A_{23}A_{32}}{A_{33}}\int_{-\delta}^{\delta}C(x_3')dx_3' \tag{69}$$

$$-(resp.+)\quad\frac{A_{22}A_{33}-A_{23}A_{32}}{A_{33}}E_i\int_{-2\delta}^{2\delta}\phi(x_3')C(x_3')dx_3'+(resp.-)\quad C,$$

where $i=1$ in Case 1 and $A_{32}\geq 0$, or in Case 2 and $A_{32}\leq 0$; $i=2$ in Case 1 and $A_{32}\leq 0$, or in Case 3 and $A_{32}\geq 0$.
Next, by integrating both hand sides of (69) with respect to x_3 on $[-\delta,\delta]$, then by deviding the result by A_{22}, we get the following.

$$\int_{-\delta}^{\delta}C(x_3')dx_3'\leq(resp.\geq)$$

$$-(resp.+)\quad 2\delta E_i\frac{A_{22}A_{33}-A_{23}A_{32}}{|A_{22}|A_{33}}\int_{-2\delta}^{2\delta}\phi(x_3')C(x_3')dx_3'+(resp.-)\quad C,$$

where the indices $i=1,2$ are similar to (69). By inserting this inequality into (69) and devide both hand sides of the result by $\alpha=\frac{A_{22}A_{33}-A_{23}A_{32}}{|A_{22}|A_{33}}$, we get the following.

$$C(x_3)\leq(resp.\geq) \tag{70}$$

$$-(resp.+) \quad (\mathrm{sgn}\alpha)E_i(1+|\frac{A_{23}A_{32}}{A_{22}A_{33}}|)\int_{-2\delta}^{2\delta}\phi(x_3')C(x_3')dx_3' + (resp.-) \quad C.$$

By remarking that from (19),

$$|4\delta E_i(1+|\frac{A_{23}A_{32}}{A_{22}A_{33}}|)| < \frac{1}{2} \quad i=1,2,$$

and by using the same argument as in Step 1, we have

$$|\int_{-2\delta}^{2\delta}\phi(x_3')C(x_3')dx_3'| < C.$$

By inserting the last estimate into (70), we obtain the estimate (64); (65) can be obtained by the same way.

Step 3. (Estimate (45).)
We consider the particular case when $i=1$, $j=2$; the other cases are obtained in a similar way in view of the symmetry of the coefficients (A_{ij}) and (b_{ij}).

$$|\int_{-2\delta}^{2\delta}\phi(x_1')\frac{\partial u}{\partial x_2}(x_1')dx_1'| < C. \tag{71}$$

We set

$$D(x_3) = \int_{-2\delta}^{2\delta}\phi(x_1',\hat{x}_2,x_3)\frac{\partial u}{\partial x_2}(x_1',\hat{x}_2,x_3)dx_1',$$

where $\hat{x}_2 \in [-\delta,\delta]$ is arbitrarily fixed. By inserting the estimates (43), (44) into (41), and by using the same argument as in Steps 1,2, we get the following

$$A_{22}D(x_3) \leq (resp. \geq) \quad \frac{A_{23}A_{32}}{A_{33}}D(x_3) - \frac{1}{2\delta}\frac{A_{23}A_{32}}{A_{33}}\int_{-\delta}^{\delta}D(x_3')dx_3' \tag{72}$$

$$-(resp.+) \quad |\frac{A_{23}}{A_{33}}|b_{32}\int_{-2\delta}^{2\delta}\phi(x_3')D(x_3')dx_3' + (resp.-) \quad C.$$

By integrating both hand sides of (72) with respect to x_3 on $[-\delta,\delta]$, then by deviding the result by A_{22}, we have the following.

$$\int_{-\delta}^{\delta}D(x_3')dx_3' \leq (resp. \geq) \tag{73}$$

$$-(resp.+) \quad 2\delta|\frac{A_{23}A_{32}}{A_{33}}|b_{32}\int_{-2\delta}^{2\delta}\phi(x_3')D(x_3')dx_3' + (resp.-) \quad C.$$

By inserting (73) into (72), we obtain

$$\frac{A_{22}A_{33}-A_{23}A_{32}}{A_{33}}D(x_3) \leq (resp.\geq)$$

$$-(resp.+) \quad |\frac{A_{23}}{A_{33}}|b_{32}(1+|\frac{A_{23}A_{32}}{A_{22}A_{33}}|)\int_{-2\delta}^{2\delta}\phi(x_3')D(x_3')dx_3' + (resp.-) \quad C,$$

where $C > 0$ is a constant. We multiply both hand sides of the above inequalities by $\phi(x_3)$, then integrate the result with respect to x_3 on $[-2\delta, 2\delta]$. Since from the assumption (15),

$$\frac{1}{|A_{22}A_{33}-A_{23}A_{32}|} \leq$$

$$\max\{\frac{1}{A_{22}A_{33}-A_{23}A_{32}+2|A_{23}A_{32}|}, \frac{1}{A_{22}A_{33}-A_{23}A_{32}-2|A_{23}A_{32}|}\},$$

remarking that from (19),

$$4\delta|\frac{A_{22}A_{33}-A_{23}A_{32}}{A_{33}}|^{-1}|\frac{A_{23}}{A_{33}}||b_{32}|(1+|\frac{A_{23}A_{32}}{A_{22}A_{33}}|) < \frac{1}{2},$$

we get

$$|\int_{-2\delta}^{2\delta}\phi(x_3')D(x_3')dx_3'| < C.$$

By inserting the above estimates into (72), (73), from (11), we obtain (71).

Step 4. (Estimate (46).)
We consider the particular case when $i = 1$, $j = 2$; the other cases are obtained in a similar way, in view of the symmetry of the coefficients (A_{ij}), (b_{ij}).

$$|\int_{-\delta}^{\delta}\frac{\partial u}{\partial x_2}(x_1')dx_1'| < C. \tag{74}$$

We set

$$E(x_3) = \int_{-\delta}^{\delta}\frac{\partial u}{\partial x_2}(x_1', \hat{x}_2, x_3)dx_1',$$

where $\hat{x}_2$ is arbitrarily fixed. Then, by inserting the estimates (43)-(45) into (41), and by taking the same arguments as in Steps 1-3, we get

$$A_{22}E(x_3) \leq (resp.\geq) \quad \frac{A_{23}A_{32}}{A_{33}}E(x_3) - \frac{1}{2\delta}\frac{A_{23}A_{32}}{A_{33}}\int_{-\delta}^{\delta}E(x_3')dx_3' + (resp.-) \quad C,$$

where $C > 0$ is a constant. By integrating the both hand sides of the above inequalities, we have the estimate

$$|\int_{-\delta}^{\delta} E(x_3')dx_3'| \leq C,$$

and by inserting this into the above inequality, we obtain (74).
From Steps 1-4, we have proved Lemma 3.2.

4 Proofs of Theorems 1.3, 1.4 and Example 1.7.

We begin with the following Lemma.

Lemma 4.1.

For the function u in Theorem 1.3, the following estimates hold.

$$\sup_{x\in\Omega} |(\sum_{j=1}^{N} A_{ij}\frac{\partial u}{\partial x_i})(x)| \leq C \quad 1 \leq i \leq N. \tag{75}$$

From Lemma 4.1, (21), (22), we obtain the gradient estimate (24) in Theorem 1.3.

Now, we prove Lemma 4.1.
Proof of Lemma 4.1.
We only show the estimate for $i = 1$ in (75); and the others are obtained by the same way. Let us denote $(x_1, ..., x_N) = (x_1)$, $(x_1', ..., x_N) = (x_1')$, $(\hat{x}_1, ..., x_N) = (\hat{x}_1)$ for the convenience. For any (x_1), $(\hat{x}_1) \in \Omega$, since

$$-\sum_{j=1}^{N}(A_{1j}\frac{\partial u}{\partial x_j})(x_1)$$

$$= -\sum_{j=1}^{N}(A_{1j}\frac{\partial u}{\partial x_j})(\hat{x}_1) + \int_{\hat{x}_1}^{x_1} -\frac{\partial}{\partial x_1'}\sum_{j=1}^{N}(A_{1j}\frac{\partial u}{\partial x_j})(x_1')dx_1',$$

the following holds from (23).

$$| - \sum_{j=1}^{N} A_{1j} \frac{\partial u}{\partial x_j})(x_1) + \sum_{j=1}^{N} A_{1j} \frac{\partial u}{\partial x_j})(\hat{x}_1)|$$

$$\leq \mathrm{sgn}(x_1 - \hat{x}_1) \int_{\hat{x}_1}^{x_1} | - \frac{\partial}{\partial x_1'} \sum_{j=1}^{N} (A_{1j} \frac{\partial u}{\partial x_j})(x_1')| dx_1'$$

$$= \mathrm{sgn}(x_1 - \hat{x}_1) \int_{\hat{x}_1}^{x_1} |C_0 + \frac{\partial}{\partial x_1'} \sum_{j=1}^{N} (A_{1j} \frac{\partial u}{\partial x_j})(x_1') - C_0| dx_1'$$

$$\leq \mathrm{sgn}(x_1 - \hat{x}_1) \int_{\hat{x}_1}^{x_1} C_0 + \frac{\partial}{\partial x_1'} \sum_{j=1}^{N} (A_{1j} \frac{\partial u}{\partial x_j})(x_1') dx_1' + C,$$

$$\leq \int_{y}^{z} C_0 + \frac{\partial}{\partial x_1'} \sum_{j=1}^{N} (A_{1j} \frac{\partial u}{\partial x_j})(x_1') dx_1' + C,$$

where $(y, x_2, ..., x_N) \in \partial\Omega$, $(z, x_2, ..., x_N) \in \partial\Omega$, $(y \leq z)$ are the intersections of $\partial\Omega$ and the straight line connecting (x_1) with $(\hat{x}_1)$. Hence, from the assumption that $\mathrm{supp} u \subset\subset \Omega$, we get

$$| - \sum_{j=1}^{N} A_{1j} \frac{\partial u}{\partial x_j}(x_1) + \sum_{j=1}^{N} A_{1j} \frac{\partial u}{\partial x_j}(\hat{x}_1)| \leq C.$$

And, by letting $\hat{x}_1$ be on the boundary, we have proved our purpose.

For Theorem 1.4, the same lemma as above holds.

Lemma 4.2.

For the function u in Theorem 1.4, the following estimate holds.

$$\sup_{x \in \Omega} |(A_{ij} \frac{\partial u}{\partial x_i}(x)| \leq C \quad 1 \leq i \leq N. \tag{76}$$

It is not difficult to prove Lemma 4.2, by modifying the proof of Lemma 4.1. Moreover, it is clear that Lemma 4.2 leads Theorem 1.4, and we do not repeat the argument.

Proof of Example 1.7.
The existence and the unqueness of the solution u_l of (29) is established by the viscosity solutions theory. (We refer the viscosity solutions theory to Crandall-Lions [3], Crandall-Ishii-Lions [4].) Thus, by the comparison result, we have

$$-\frac{\partial^2 u_l}{\partial x_i^2} \leq \text{Const.} \quad x \in \Omega, \quad 1 \leq i \leq N,$$

$$lu_l(x) \leq \text{Const.}$$

Therefore, we can apply Theorems 1.1, 1.2, and 1.4 to obtain the result.

Remarks 4.3.

The regularity result in Example 1.7 can be generalized to a class of some controlled stochastic systems which were treated by Krylov [6], Lions [7]. For the special case of (29), the result in fact holds for any dimensions, if we follow our proof for Theorems 1.1, 1.2, and 1.4.

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