# Homotheties and topology of tangent sphere bundles

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**Abstract**

We prove a Theorem on homotheties between two given tangent sphere bundles $S_rM$ of a Riemannian manifold $M, g$, assuming different variable radius functions $r$ and weighted Sasaki metrics induced just by the conformal class of $g$. We show the associated almost complex and symplectic structures on the manifold $TM$, generalizing the well known structure of Sasaki. Finally the characteristic classes of Chern and Stiefel-Whitney are computed for the manifolds $TM$ and $S_rM$.

**Key Words:** tangent sphere bundle, isometry, characteristic classes.

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## 1.1 Introduction

This article consists of a study of the main properties which identify the tangent sphere bundles $S_rM = \{u \in TM : \ \|u\| = r\}$ of a Riemannian manifold $(M, g)$ with variable radius $r$ and induced weighted Sasaki metric $g^{f_1,f_2} = f_1\pi^*g \oplus f_2\pi^*g$, where $f_1, f_2$ are $\mathbb{R}^+$-valued functions on $M$ and $\pi : TM \to M$ is the bundle map. Recall the well known Sasaki metric on $TM$ is just $g^S = g^{1,1}$ induced by the Levi-Civita connection.

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Our main results are as follows. We consider a conformal change $\lambda g$ by some function $\lambda$ on $M$, take the Levi-Civita connections of $g$ and $\lambda g$ and then their lifts to $TM$. We obtain much different weighted metrics (assuming $\lambda$ non-constant). One wishes to compare the $S_rM$, with radius functions $r, s$ and within the same conformal class, through the map $u \overset{h}{\mapsto} \frac{s}{\sqrt{r\lambda}}u$. For $M$ connected and dimension $\geq 3$ we prove:

$$(S_rM, g^{f_1,f_2}) \quad \text{is isometric to} \quad (S_sM, (\lambda g)^{f_1',f_2'}) \quad \text{via } h \tag{1}$$

if and only if $\frac{f_1'}{f_1}\lambda = \frac{s^2}{r^2}\frac{f_2'}{f_2} = 1$, the function $\lambda$ is constant and one of the following conditions holds: (i) $s/r$ is constant or (ii) $rs$ is constant.

Case (ii) is quite interesting since in particular says that, for any positive function $r$ on $M$,

$$(S_rM, g^S) \quad \text{is isometric to} \quad (S_{\frac{1}{r}}M, g^{1,r^4}). \tag{2}$$

Proceeding with the weighted metric $G = g^{f_1,f_2}$ on $TM$, we define a compatible almost Hermitian structure $(G, I^G, \omega^G)$, which is a generalization of the canonical or Sasaki almost Hermitian structure on $TM$. In our case we also allow $\nabla$ to have torsion. Then the integrability equations of $I^G$ and $\omega^G$ reserve distinguished roles for the functions $f_1/f_2$ and $f_1f_2$ respectively, both implying the torsion to be of certain vectorial type[1]. Finally, the two functions only have to be both constant, the curvature of $\nabla$ flat and the torsion zero if and only if we require the defined structure on $TM$ to be Kähler.

We determine the characteristic classes of $TM$, which is a manifold by right. The Chern classes of $(TM, I^G)$ are proved to agree with the Pontryagin classes of $M$; therefore they do not depend on the metric connection $\nabla$. The Stiefel-Whitney characteristic classes of $S_rM$ are also found. In particular we conclude that any tangent sphere bundle of an oriented manifold is spin.

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## 1.2 Differential geometry of the tangent bundle

### 1.2.1 The tangent bundle

Let $M$ be an $m$-dimensional smooth manifold without boundary. Let $\pi : TM \to M$ be the tangent bundle so that $\pi(u) = x,\ \forall u \in T_xM,\ x \in M$. Then $V = \ker \mathrm{d}\pi$ is known as the vertical bundle tangent to $TM$. There is a canonical identification $V = \pi^*TM$ and a short exact sequence over the manifold $TM$:

$$0 \longrightarrow V \longrightarrow TTM \xrightarrow{\mathrm{d}\pi} \pi^*TM \longrightarrow 0. \tag{3}$$

[1]Having in principle no relation, notice the similarity of these equations with the two cases (i) and (ii) above.

A vector field on a manifold is a section $X$ of its tangent bundle. The tangent bundle $TM$ is endowed with a natural vertical vector field, denoted $\xi$, which is succinctly defined by $\xi_u = u$.

Let $\nabla$ be a connection on $M$. Then there is a complement for $V$

$$H = \{X \in TTM:\ \pi^*\nabla_X\xi = 0\}. \tag{4}$$

Indeed $H$ is $m$-dimensional and $\pi^*\nabla.\xi$ is the vertical projection onto $V$. For any vector field $X$ over $TM$ we may always find the unique decomposition ($\nabla^*$ denotes the pull-back connection)

$$X = X^h + X^v = X^h + \nabla^*_X\xi. \tag{5}$$

Now, $\mathrm{d}\pi$ induces a vector bundle isomorphism between $H$ and $\pi^*TM$, by (3), and we have $V = \pi^*TM$. Hence we may define an endomorphism

$$\theta : TTM \longrightarrow TTM \tag{6}$$

sending $X^h$ to the respective $\theta X^h \in V$ and sending $V$ to 0. We also define an endomorphism, denoted $\theta^t$, which gives $\theta^t X^v \in H$ and which annihilates $H$. In particular $\theta^t\theta X^h = X^h$ and $\theta^2 = 0$. Sometimes we call $\theta X^h$ the mirror image of $X^h$ in $V$. The map $\theta$ appears also in [4]. We endow $TTM$ with the direct sum connection $\nabla^* \oplus \nabla^*$, which we sometimes denote by $\nabla^*$. We have in particular that $\nabla^*\theta = \nabla^*\theta^t = 0$.

Notice the canonical section $\xi$ can be mirrored by $\theta^t$ to give a horizontal canonical vector field $\theta^t\xi$. In the torsion free case, the latter is known as the spray of the connection, cf. [7, 10], or the geodesic field, cf. [8]. It has the further property that $\mathrm{d}\pi_u(\theta^t\xi) = u,\ \forall u \in TM$. Away from the zero section, we have a line bundle $\mathbb{R}\xi \subset V$ and therefore a line sub-bundle too of $H$.

### 1.2.2 Natural metrics

Suppose the previous manifold $M$ is furnished with a Riemannian metric $g$ and a linear connection. We also use $\langle\ ,\ \rangle$ in place of the symmetric tensor $g$; this same remark on notation is valid for the pull-back metric on $\pi^*TM$. We recall from [7, 11] the now called Sasaki metric in $TTM = H \oplus V$: it is given by $g^S = \pi^*g \oplus \pi^*g$ (originally, with the Levi-Civita connection). With $g^S$, the map $\theta_| : H \to V$ is an isometric morphism and $\theta^t$ corresponds with the adjoint of $\theta$. We stress that $\langle\ ,\ \rangle$ on $TTM$ always refers to the Sasaki metric.

Let $\varphi_1, \varphi_2$ be any given functions on $M$ and let

$$G = g^{f_1,f_2} = f_1\pi^*g\ \oplus\ f_2\pi^*g \tag{7}$$

with

$$f_1 = \mathrm{e}^{2\varphi_1}, \quad f_2 = \mathrm{e}^{2\varphi_2}. \tag{8}$$

Obviously, we convention all these functions to be composed with $\pi$ on the right hand side when used on the manifold $TM$.

**Remark.** With the canonical vector field $\xi$ we may produce other symmetric bilinear forms over $TM$: first the 1-forms $\eta = \xi^\flat$ and $\mu = \xi^\flat \circ \theta = (\theta^t \xi)^\flat$ and then the three symmetric products of these. Actually one may see that $\mu$ does not depend on a chosen connection which is metric; cf. last remark in section 1.4.1. The classification of all $g$-induced natural metrics on $TM$ may be found e.g. in [1, 2].

### 1.2.3 Metric connections

Let us assume from now on the connection on $M$ is metric, which implies $\nabla^* g^S = 0$. It is well known that $\nabla^{f_1} = \nabla + C_1$, with

$$C_1(X,Y) = X(\varphi_1)Y + Y(\varphi_1)X - \langle X,Y\rangle \mathrm{grad}\,\varphi_1, \tag{9}$$

is a metric connection for $f_1 g$ on $M$, with the same torsion as $\nabla$ since $C$ is symmetric.

For any function $\varphi$, recall the usual identities $X(\varphi) = \mathrm{d}\varphi(X) = \langle \mathrm{grad}\,\varphi, X\rangle$, adopted throughout. On $TM$ we shall use the functions $\partial\varphi(u) = \mathrm{d}\varphi_{\pi(u)}(u)$, $\forall u$. In other words,

$$\partial\varphi = \langle \theta\pi^* \mathrm{grad}\,\varphi, \xi\rangle \tag{10}$$

where $\theta$ is the mirror map (6). And we agree on lifting gradient vector fields only to $H$.

We have that $\nabla^{*,f_1} = \nabla^* + \pi^* C_1$ makes $f_1 \pi^* g$ parallel on $H$. On the vertical side, $\nabla^{*,f_2}$, defined by

$$\nabla^{*,f_2}_X Y = \nabla^*_X Y + \theta\pi^* C_2(X, \theta^t Y) \tag{11}$$

$\forall X, Y$ vector fields on $TM$, makes $f_2 \pi^* g$ parallel. Henceforth, the connection $\nabla^{*,f_1} \oplus \nabla^{*,f_2}$ is metric for $G = g^{f_1,f_2}$.

**Proposition 1.1.** *(i) The torsion of* $\nabla^* \oplus \nabla^*$ *is* $\pi^* T^\nabla + \mathcal{R}^\xi$.
*(ii) The connection* $\nabla^{*;f_2,'}_X Y = \nabla^*_X Y + X(\varphi_2)Y$ *is metric on* $(V, f_2 \pi^* g)$.

The proof of this result is immediate. The vertical part in (i) is defined by $\mathcal{R}^\xi(X,Y) = \pi^* R^\nabla(X,Y)\xi$. We remark it is $\nabla^{*,f_1}$ and the connection in (ii) which enter in the Levi-Civita connection $\nabla^G$ of $G$. Formulas for the curvature are well known, cf. [2, 7].

## 1.3 Conformal change of the metric

### 1.3.1 Homotheties of $TM$

Suppose we have a conformal change of the metric $g$ on the base $M$. With $\lambda = \mathrm{e}^{2\varphi}$ and $\varphi \in \mathrm{C}^\infty_M$ we pass to the metric

$$g' = \lambda g = \lambda \langle\ ,\ \rangle. \tag{12}$$

Let us distinguish by $T'M$ the tangent manifold of $M$ with the metric $g'$, when necessary. Until the rest of the section we restrict to the Levi-Civita connection

$$\nabla = \nabla^g. \tag{13}$$

Notice $TTM = H \oplus V = H' \oplus V$ and we conform to our previous remarks on notation.

Let also $t : M \to \mathbb{R}\backslash\{0\}$ be a smooth function. Then we may consider the isomorphism

$$h : TM \longrightarrow T'M, \qquad h(u) = u' = \mathrm{e}^{-\varphi} t u = \hat{h} u\ . \tag{14}$$

We treat all given scalar functions like $\varphi$ or $t$, depending on the context, as functions composed with $\pi$. This implies, for example,

$$X(\varphi) = \mathrm{d}\varphi(X) = X^h(\varphi). \tag{15}$$

Recall the 1-form $\mu$ on $TM$ given by $\mu(X) = \langle \theta X, \xi \rangle$.

**Proposition 1.2.** *Let $X$ be any vector field on $TM$ and consider the differential map $h_* : TTM \to h^*TT'M$. It satisfies the identities $h_*(X^v) = \hat{h} X^v$ and, more generally,*

$$h_* X = X^{h'} + \hat{h}\big(\frac{X(t)}{t}\xi + X^v + \partial\varphi.\theta X - \mu(X)\theta \mathrm{grad}\,\varphi\big) \tag{16}$$

*where $\theta$ refers to the decomposition $H \oplus V$.*

*Proof.* We known that $\nabla' = \nabla + C$ where $C_X Y = \mathrm{d}\varphi(X)Y + \mathrm{d}\varphi(Y)X - \langle X, Y\rangle \mathrm{grad}\,\varphi$ (depending on the manifold, $X, Y$ denote vector fields). Since $\pi \circ h = \pi$, then $(h_* X)^{h'} = (\mathrm{d}\pi)^{-1}(\mathrm{d}\pi(X))$ and this is the same as $X^{h'}$, the $H'$-part of $X$. Writing $\xi'$ for the very same canonical vector field $\xi$ on $T'M$, so that $h^*\xi' = \xi \circ h = \hat{h}\xi$, and computing,

$$\begin{aligned}
\pi^*\nabla'_{h_*(X)}\xi' &= h^*\pi^*(\nabla + C)_X h^*\xi' \\
&= \pi^*\nabla_X(\hat{h}\xi) + \theta\pi^* C(X, \theta^t(\hat{h}\xi)) \\
&= \mathrm{d}\hat{h}(X)\xi + \hat{h}\nabla^*_X\xi + \hat{h}\theta\pi^* C(X, \theta^t\xi) \\
&= -X(\varphi)\hat{h}\xi + \mathrm{e}^{-\varphi} X(t)\xi + \hat{h}X^v + \hat{h}X(\varphi)\xi + \\
&\qquad\qquad + \hat{h}(\theta^t\xi)(\varphi).\theta X - \hat{h}\langle\theta X, \xi\rangle\theta\mathrm{grad}\,\varphi \\
&= \hat{h}\big(\frac{X(t)}{t}\xi + X^v + \partial\varphi.\theta X - \mu(X)\theta\mathrm{grad}\,\varphi\big)
\end{aligned}$$

we find the vertical part. ■

**Remark.** Notice any tangent vector $X = X^h + X^v = X^{h'} + X^{v'}$ has two decompositions. We have, cf. figure 1,

$$\begin{aligned}
X^{v'} = \nabla'^*_X\xi &= \nabla_X\xi + \theta\pi^* C(X, \theta^t\xi) \\
&= X^v + \partial\varphi.\theta X + X(\varphi)\xi - \mu(X)\theta\mathrm{grad}\,\varphi, \\
X^{h'} = X - X^{v'} &= X^h - \partial\varphi.\theta X - X(\varphi)\xi + \mu(X)\theta\mathrm{grad}\,\varphi,
\end{aligned} \tag{17}$$

Now we suppose $TM$ is endowed with the metric $G = g^{f_1, f_2}$ introduced in previous sections and we let $T'M$ have the metric $G' = (\lambda g)^{f'_1, f'_2}$ (the four weight functions are just smooth, positive and defined on $M$).

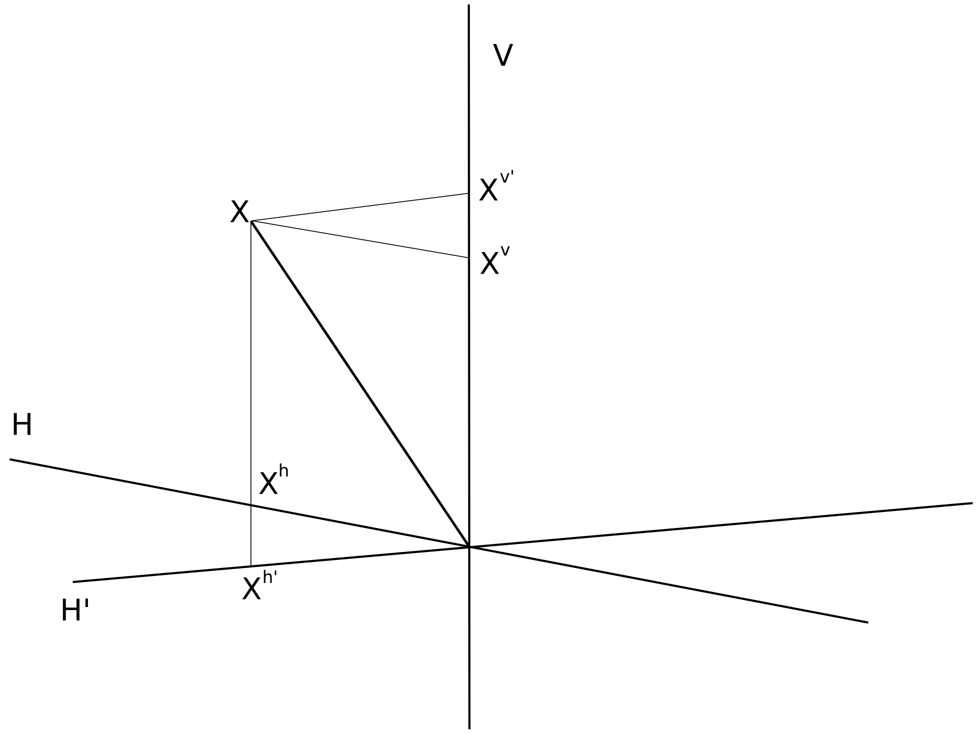


Figure 1: The connection induced projections

**Theorem 1.1.** *The map $h$ is a homothety (ie. $h^*G' = \psi G$ for some function $\psi$) if and only if $t$ and $\lambda$ are constants and satisfy $\frac{f_1'}{f_1}\lambda = t^2\frac{f_2'}{f_2}$. In this case, the latter is the value of $\psi$.*

*Proof.* We write $h_*X = X^{h'} + \hat{h}E(X)$ defining $E$ from (16). Then solving the equation above with vertical vector fields $X_1, X_2$ we immediately find

$$h^*G'(X_1,X_2) = \psi G(X_1,X_2) \;\Leftrightarrow\; \lambda\hat{h}^2 f_2' = \psi f_2 \;\Leftrightarrow\; t^2 f_2' = \psi f_2.$$

In particular, $\psi$ is only defined on $M$. Notice we may write

$$E_{au}(X^h) = aE_u(X^h), \qquad \forall a \in \mathbb{R},$$

because $\xi$ is also hidden linearly in $\partial\varphi$ and $\mu$. Picking two horizontal lifts and having in mind that $t$ and $\psi$ are only defined on $M$, it is then easy to deduce that a necessary condition for $h$ to be a homothety is that $E(X^h) = 0$ for all $H$-horizontal $X$. Now

$$t\langle E(\theta^t\xi),\xi\rangle = t\langle\frac{\partial t}{t}\xi + \partial\varphi.\xi - \|\xi\|^2\theta\mathrm{grad}\,\varphi,\xi\rangle = (\partial t + t\partial\varphi - t\partial\varphi)\|\xi\|^2$$

and thence $\partial t = \mathrm{d}t(\theta^t\xi) = 0$. Choosing any $X$ horizontal and orthogonal to $\theta^t\xi$ (recall $m > 1$), we find $0 = \langle E(X),\theta X\rangle = \partial\varphi\|X\|^2 = 0 \Leftrightarrow \partial\varphi = 0$, as we wished. In particular, $\nabla = \nabla'$. Finally, solving the equation above for horizontal vector fields $X_1, X_2$ we get $f_1'\lambda = \psi f_1$. For generic vectors the result follows. ∎

Completing the Theorem for the case of two conformal changes we have: the map $h$ from $(\lambda_1 g)^{f_1,f_2}$ to $(\lambda_2 g)^{f_1',f_2'}$ is a homothety if and only if

$$t,\frac{\lambda_2}{\lambda_1}\ \text{are constants}\quad\text{and}\quad \frac{f_1'\lambda_2}{f_1\lambda_1}=t^2\frac{f_2'}{f_2}. \tag{18}$$

### 1.3.2 Homotheties of $S_rM$

Let $r,s\in \mathrm{C}^\infty_M(\mathbb{R}^+)$ and recall the tangent sphere bundle of radius $r$

$$S_rM=\{u\in TM:\ \|u\|_g^2=r^2\} \tag{19}$$

submanifold of $TM$, for which we have

$$S_rM\ =\ S_1'M \tag{20}$$

using the metric $\lambda g$ to define $S_s'M$ with $\lambda = r^{-2} = \mathrm{e}^{2\varphi}$. Consider the smooth function $N=r^{-2}\|\xi\|^2$. Then $S_rM=N^{-1}(1)=\{u\in TM:\ G(\xi_u,\xi_u)=1\}$ where $G=g^{f_1,r^{-2}}$ with $f_1$ any positive function. Using Proposition 1.1 to differentiate $N=G(\xi,\xi)$, it is easy to deduce

$$TS_rM=\big\{X\in TTM:\ \langle X,\xi\rangle=rX(r)\big\}. \tag{21}$$

We have to assume $\varphi_2=\varphi=-\ln r$. But of course one just applies $\nabla^*$ to $\|\xi\|^2-r^2=0$ to easily find the same information. Notice $X\in TS_rM\Leftrightarrow\langle X^v,\xi\rangle=rX^h(r)$.

We shall consider a more general setting: with $r$ and $\varphi$ independent.

Let $\lambda g$ be any conformal change of the given metric, $\lambda=\mathrm{e}^{2\varphi}$. Let $s$ be another positive function on $M$ and consider the map $h$ from Proposition 1.2. It restricts to

$$h:S_rM\longrightarrow S_s'M,\qquad\qquad h(u)=\mathrm{e}^{-\varphi}tu=\hat{h}u \tag{22}$$

when we take $t=\frac{s}{r}$.

When is $h$ a homothety for the induced metrics? For a start, only the metrics $G,G'$ constructed as in section 1.3.1 are relevant, i.e. those induced from $\nabla=\nabla^g$ the Levi-Civita connection.

**Remark.** Recall the metric on the right hand side arises from $H'\oplus V$. Since $h_*:T_uS_rM\to T_{\hat{h}u}S_s'M$, it is true that we have

$$rX(r)\ =\ \langle X,\xi\rangle\quad\Leftrightarrow\quad s\,(h_*X)(s)\ =\ \langle h_*X,\hat{h}\xi\rangle'.$$

Indeed, we may write $h_*X=X^{h'}+\hat{h}E(X)$ where $E(X)$ is given in (16):

$$EX=\frac{X(t)}{t}\xi+X^v+\partial\varphi.\theta X-\mu(X)\theta\mathrm{grad}\,\varphi. \tag{23}$$

Also, on vertical vector fields the metrics agree up to the scale, so we find

$$\begin{aligned}
\langle h_*X, \hat{h}\xi\rangle' &= \hat{h}^2\langle EX, \xi\rangle' \\
&= \mathrm{e}^{2\varphi}\mathrm{e}^{-2\varphi}t\langle X(t)\xi + tX^v + t\partial\varphi.\theta X - t\mu(X)\theta\mathrm{grad}\,\varphi, \xi\rangle \\
&= t\big(\frac{X(s)r - sX(r)}{r^2}\|\xi\|^2 + \frac{s}{r}\langle X^v, \xi\rangle + t\partial\varphi.\mu(X) - t\mu(X)\partial\varphi\big) \\
&= t\big(rX(s) - sX(r) + \frac{s}{r}rX(r)\big) \\
&= sX(s) \;=\; s\,(h_*X)(s)
\end{aligned}$$

since on $S_rM$ we have $\|\xi\|^2 = r^2$.

In the next Theorem we prove that each tangent sphere bundle $S_rM$ with metric $G$ induced from that of $TM$ is quite unique, independently of any of the metric transformations above and up to the straightforward coincidences expressed in the corollaries. The reader may notice the impossibility of adapting the arguments used for Theorem 1.1.

We let $\lambda = \mathrm{e}^{2\varphi}$ and $r, s, f_1, f_2, f_1', f_2'$ be any positive functions on $M$.

Until the end of this section we assume $M$ is connected and $\dim M \geq 3$.

**Theorem 1.2.** *Let $S_rM$ have the induced metric $G = g^{f_1,f_2}$ and let $S_s'M$ have the induced metric $G' = (\lambda g)^{f_1',f_2'}$. Then the following are equivalent:*

1. *$h : S_rM \to S_s'M$ is a homothety, ie. $h^*G' = \psi G$ for some function $\psi$.*
2. *$\lambda$ is constant, $\psi$ verifies simultaneously $\psi = \frac{f_1'}{f_1}\lambda = t^2\frac{f_2'}{f_2}$ and one of the following hold:*
   *(i) $t = s/r$ is constant*
   *(ii) $rs$ is constant.*

*For the case of the identity ($\hat{h} = 1$), we have that it is a homothety if and only if $\lambda = t^2$ is a constant and $\frac{f_1'}{f_1} = \frac{f_2'}{f_2}$.*

*Proof.* First we notice

$$\begin{aligned}
G'(h_*X, h_*Y) &= f_1'\langle X^{h'}, Y^{h'}\rangle' + \hat{h}^2 f_2'\langle EX, EY\rangle' \\
&= f_1'\lambda\langle X^h, Y^h\rangle + \hat{h}^2\lambda f_2'\langle EX, EY\rangle.
\end{aligned}$$

Now consider the equation $h^*G'(X,Y) = \psi G(X,Y)$. Choose one vector $X = \xi^\perp$ vertical and orthogonal to $\xi$, and a vector $Y = (\mathrm{grad}\,r)^\perp$ horizontal and orthogonal to $\mathrm{grad}\,r$. Then both $X, Y \in TS_rM$. Indeed, $\langle X, \xi\rangle = 0 = rX(r)$ and $\langle Y, \xi\rangle = 0 = r\langle Y, \mathrm{grad}\,r\rangle = rY(r)$. Then for two vertical vector fields, like $X$, we immediately get the necessary condition $\hat{h}^2\lambda f_2' = \psi f_2 \Leftrightarrow \psi = \frac{s^2}{r^2}\frac{f_2'}{f_2}$. For $X, Y$ we have $EX = X^v$ and $EY = \frac{Y(t)}{t}\xi + \partial\varphi.\theta Y - \mu(Y)\theta\mathrm{grad}\,\varphi$, hence

$$G'(h_*X, h_*Y) = f_2'\lambda\hat{h}^2\big(\partial\varphi\langle X, \theta Y\rangle - \mu(Y)\langle X, \theta\mathrm{grad}\,\varphi\rangle\big) = 0$$

since $G(X,Y) = 0$. Now we choose a point $u \in S_rM$ orthogonal to $\mathrm{grad}\,r$. Then we may take $X = \theta\mathrm{grad}\,r$ and $Y = u \in H$. We have $\langle \theta Y, X\rangle = 0$ and $\mu(Y) = \langle u, u\rangle = r^2$, so our equation yields $\langle X, \theta\mathrm{grad}\,\varphi\rangle = 0$. Equivalently, we must have $\mathrm{grad}\,r \perp \mathrm{grad}\,\varphi$.

Now suppose $\operatorname{grad} r = 0$ on all points of $M$, ie. $r$ is constant. Then $H \subset TS_rM$. Take any non-vanishing $Z_0 \in H$. Then we may further let $Z_0 \in H \cap \{\operatorname{grad} s, \operatorname{grad} \varphi\}^\perp$ (this is not clearly so in dimension 2 since we want $Z_0 \neq 0$, hence the hypothesis on the dimension; although here we may assume that $\operatorname{grad} \varphi$ together with $\operatorname{grad} s$ constitute a basis of $H$ and then try to solve the system of two linear equations and 4 unknowns, in the components of $u$ and $Z_0$ in that basis, given by $Z_0(t)\xi + t\partial\varphi.\theta Z_0 - t\mu(Z_0)\theta \operatorname{grad} \varphi = 0$, for that is all we need). In fact, in dimension $\geq 3$ we may find a point $u$ and a vector $Z_0 \in H_u$ such that $Z_0(s) = Z_0(\varphi) = 0$ and such that $\partial\varphi = 0$, $\mu(Z_0) = 0$. Then on the chosen point $u$ (on the particular $u$ found for the case of dimension 2 if possible), we get $E(Z_0) = 0$ and so $h_*Z_0 = Z_0^{h'}$. Thence our main equation yields the necessary condition $f_1'\lambda = \psi f_1$. Going back a bit, we then consider any point $u$ and any $Z_0 \in H$ perpendicular to $u$, ie. such that $\xi_u \perp \theta Z_0$. Then we deduce

$$G'(h_*Z_0, h_*Z_0) = f_1'\lambda\|Z_0\|^2 + f_2'\lambda\hat{h}^2\Big(\frac{(Z_0(s))^2}{s^2}r^2 + (\partial\varphi)^2\|Z_0\|^2\Big) \;=\; \psi f_1\|Z_0\|^2$$

This immediately implies $Z_0(s) = 0$, $\partial\varphi = 0$. Since $Z_0$ and $u$ may be put in general position, we conclude $s$ and $\varphi$ are constant on $M$, a connected manifold by assumption, and the theorem follows.

So now we admit $\operatorname{grad} r \neq 0$ at some point $x \in M$. Recall $\operatorname{grad} r \perp \operatorname{grad} \varphi$ and let $\epsilon = \|\operatorname{grad} r\|$ and $\delta = \|\operatorname{grad} \varphi\|$.

Thence $u_1 = \frac{r}{\epsilon}\operatorname{grad} r \in S_rM$. Notice $\partial\varphi_{u_1} = \mathrm{d}\varphi(u_1) = 0$. Consider the vector $X_0 = \operatorname{grad} r$ and $X = X_0 + \epsilon\theta X_0$. It is tangent to our sphere bundle in $u_1$ since

$$\langle X, \xi\rangle = \epsilon\frac{r}{\epsilon}\epsilon^2 = r\langle X_0, X_0\rangle = rX(r).$$

And we have that

$$\begin{aligned} h_*X = X^{h'} + \hat{h}EX &= X^{h'} + \hat{h}\Big(\frac{X(t)}{t}\xi_{u_1} + \epsilon\theta X - \mu(X)\theta \operatorname{grad} \varphi\Big) \\ &= X^{h'} + \hat{h}\Big(\frac{X(t)}{t}\frac{r}{\epsilon} + \epsilon\Big)\theta X_0 - \hat{h}r\epsilon\,\theta \operatorname{grad} \varphi. \end{aligned}$$

Consider also the tangent vector at $u_1$, $Z = \theta \operatorname{grad} \varphi$. Then $h_*Z = \hat{h}Z$. And thus $\psi G(X, Z) = \psi f_2\epsilon\langle\theta X_0, Z\rangle = 0$; on the other hand

$$\begin{aligned} h^*G'(X, Z) &= f_2'\lambda\hat{h}^2\langle\Big(\frac{X(t)}{t}\frac{r}{\epsilon} + \epsilon\Big)\theta X_0 - r\epsilon\,\theta \operatorname{grad} \varphi, \theta \operatorname{grad} \varphi\rangle \\ &= -f_2'\lambda\hat{h}^2\epsilon r\delta^2. \end{aligned}$$

This implies $\delta = 0$, ie. $\varphi$ and hence $\lambda = \mathrm{e}^{2\varphi}$ are constants.

Therefore the map $h$ verifies $h_*X = X^h + \hat{h}(\frac{X(t)}{t}\xi + X^v)$, for any vector field $X \in TS_rM$. Now we consider any horizontal vector $X \in \ker \mathrm{d}r \cap \ker \mathrm{d}s$, in particular also tangent to $S_rM$ and orthogonal to $\operatorname{grad} t$ (notice we need $n \geq 2$ again for we are not able to decide the mysterious case of $\dim M = 2$, which may indeed have some further behaviour). Then

$X(t) = 0$ and, just as we had the result $\hat{h}^2\lambda f_2' = \psi f_2$ for vertical vectors, we have the similar for horizontal: $\lambda f_1' = \psi f_1$.

Next, we use two generic tangent vectors $X, Y \in TS_rM$. It is easy to see the conformality equation $h^*G' = \psi G$ is finally equivalent to

$$\langle \frac{X(t)}{t}\xi + X^v, \frac{Y(t)}{t}\xi + Y^v\rangle = \langle X^v, Y^v\rangle,$$

$$\frac{X(t)Y(t)}{t^2}r^2 + \frac{X(t)rY(r)}{t} + \frac{Y(t)rX(r)}{t} = 0$$

or

$$X(t)Y(t)r + X(t)Y(r)t + X(r)Y(t)t \ = \ 0.$$

Notice this last equation only involves the horizontal part of the vectors, so we assume $X, Y$ as such. Now if we take $X$ orthogonal to $\operatorname{grad} t$, ie. satisfying $X(t) = 0$, and take $Y = \operatorname{grad} t$, then we find that $X(r) = 0$ or that $X$ is also orthogonal to $\operatorname{grad} r$. Henceforth, $\operatorname{grad} t$ and $\operatorname{grad} r$ are proportional, ie. lie on the same line. In other terms,

$$\mathrm{d}t = a\mathrm{d}r$$

for some function $a$ on $M$. Clearly the equation above may be written as

$$r\mathrm{d}t \otimes \mathrm{d}t + t\mathrm{d}t \otimes \mathrm{d}r + t\mathrm{d}t \otimes \mathrm{d}r = 0.$$

Hence we have $(ra^2 + 2ta)\mathrm{d}r \otimes \mathrm{d}r = 0$. Recalling $r$ is not constant, we either have $t$ constant or $ra + 2t = 0$. We have both

$$\mathrm{d}t = -\frac{2t}{r}\mathrm{d}r = -\frac{2s}{r^2}\mathrm{d}r \qquad \text{and} \qquad \mathrm{d}t = \frac{r\mathrm{d}s - s\mathrm{d}r}{r^2}.$$

Hence $-2s\mathrm{d}r = r\mathrm{d}s - s\mathrm{d}r \ \Leftrightarrow \ r\mathrm{d}s + s\mathrm{d}r = 0$, from which we find $sr = c^{\mathrm{nt}}$.

Finally all conditions are fulfilled for $h$ to be the expected homothety of ratio $\psi$. The identity map case is trivial. ■

Let $g^S = g^{1,1}$ denote the induced Sasaki metric on the tangent sphere bundle.

**Corollary 1.1.** *$(S_rM, g^S)$ is homothetic to $(S'_sM, (\lambda g)^S)$ via $h$ if and only if $\psi = \lambda = t^2$ and this is a constant. In this case $h$ is the identity, $s = \sqrt{\lambda}r$; in other words $S_rM = S'_sM$. In particular, two tangent sphere bundles both with the induced Sasaki metric are homothetic if and only if they have exactly the same radius function, ie., they coincide.*

**Corollary 1.2.** *Other particular cases are as follows: $(S_rM, g^{f_1,f_2})$ is isometric via $h$ to $(S'_rM, (\lambda g)^{1,f_2})$ if $f_1 = \lambda$ is constant. And $(S_rM, g^{f_1,f_2})$ is isometric to $(S'_1M, (\lambda g)^{1,r^2 f_2})$ if $f_1 = \lambda$ and both $r, f_1$ are constant. Moreover, $(S_rM, g^S)$ is isometric to $(S_1M, g^{1,r^2})$ if $r$ is constant.*

We have used the metric $G = g^{f_1,r^{-2}}$ on $S_rM$. So we study this case separately.

**Corollary 1.3.** *Let $S_rM$ be given the metric $G = g^{1,\frac{1}{r^2}}$ and let $S'_sM$ be with the metric $G' = (\lambda g)^{f'_1,\frac{1}{s^2}}$. Then the following three conditions are equivalent:*

1. *the map $h : S_rM \to S'_sM$ is a homothety.*
2. *the functions verify: $\psi = f'_1\lambda = 1$, $\lambda$ is a constant and $s/r$ or $sr$ is a constant.*
3. *the map $h$ is an isometry.*

*Proof.* Indeed we have $\psi = f'_1\lambda = t^2\frac{r^2}{s^2} = 1$. ∎

**Corollary 1.4.** *Let $r$ be any function on $M$. Then $(S_rM, g^S)$ is isometric to $(S_{\frac{1}{r}}M, g^{1,r^4})$.*

*Proof.* This is due to the second particular case found in the Theorem. We are taking $\lambda = 1$ and $s = \frac{1}{r}$ and indeed $\psi = \frac{f'_1}{f_1}\lambda = 1 = \frac{r^4}{r^4} = t^2\frac{f'_2}{f_2}$ since $t = \frac{1}{r^2}$. Also notice we have $sr$ constant. ∎

One may illustrate this last result by looking for the *isometries* between concentric circles in a plane.

## 1.4 Characteristic classes

### 1.4.1 Almost Hermitian structure on $TM$

The pair $TM, g^S$ admits a compatible almost complex structure, also attributed to Sasaki. It was first studied in [7, 11] and gave origin in [12] to an almost contact structure on the tangent sphere bundle $S_1M$.

We continue the study of $TM$ with the metric $G = g^{f_1,f_2}$ where $f_1 = \mathrm{e}^{2\varphi_1}$ and $f_2 = \mathrm{e}^{2\varphi_2}$. We let $\nabla$ denote a metric connection on $M$ with torsion $T^\nabla$. The almost complex structure of Sasaki may be now written as the bundle endomorphism $I^S = \theta^t - \theta$.

Let

$$\psi = \varphi_2 - \varphi_1, \qquad \overline{\psi} = \varphi_2 + \varphi_1. \tag{24}$$

We then define

$$I^G = \mathrm{e}^{\psi}\theta^t - \mathrm{e}^{-\psi}\theta. \tag{25}$$

It is easy to see the endomorphism $I^G$ is an almost complex structure compatible with the metric $G$. We consider also the associated non-degenerate 2-form $\omega^G$ defined by

$$\omega^G(X, Y) = G(I^G X, Y), \qquad \forall X, Y \in TTM. \tag{26}$$

Since $f_1\mathrm{e}^{\psi} = f_2\mathrm{e}^{-\psi} = \mathrm{e}^{\overline{\psi}}$, it follows that $\omega^G = \mathrm{e}^{\overline{\psi}}\omega^S$ where $\omega^S$ is the 2-form associated to the Sasaki structure $g^S$ and $I^S = \theta^t - \theta$. The next Theorem is shown for completeness of exposition. For the Cartan classification of torsions of metric connections see [3].

**Theorem 1.3** ([5]). *(i) The almost complex structure $I^G$ is integrable if and only if $\nabla$ is flat and has the vectorial type torsion*

$$T^\nabla = \mathrm{d}\psi \wedge 1. \tag{27}$$

*In particular, if $\nabla$ is torsion free, then $I^G$ is integrable if and only if $M$ is Riemannian flat and $f_2/f_1$ =constant.*
*(ii) $(TM, \omega^G)$ is a symplectic manifold if and only if*

$$T^\nabla = \mathrm{d}\overline{\psi} \wedge 1. \tag{28}$$

*In particular, with $\nabla$ the Levi-Civita, $\mathrm{d}\omega^G = 0$ if and only if $f_2 f_1$ =constant.*

We observe that in the strict case of the Sasaki metric we have $T^\nabla = 0$ as *necessary* condition for both integrability of $I^S$ and $\mathrm{d}\omega^S = 0$. In the general case, things are distinguished, as they should, by $\psi$ and $\overline{\psi}$. Clearly we may draw the following conclusion.

**Corollary 1.5** ([5]). *The almost Hermitian structure $(TM, G, I^G, \omega^G)$ is Kähler if and only if $M$ is a Riemannian flat manifold ($T^\nabla = 0$, $R^\nabla = 0$) and $f_1, f_2$ are constants. In this case, $TM$ is flat.*

The last assertion follows indirectly from Proposition 1.1.

**Remark.** Recall $T^*M$ has a natural symplectic structure. It arises as $\mathrm{d}\lambda$ where $\lambda$ is the Liouville form ([8]): the unique 1-form $\lambda$ on $T^*M$ such that on a section $\alpha$

$$\lambda_\alpha = \alpha \circ \pi_* \tag{29}$$

When we introduce the metric, the tangent and co-tangent (sphere) bundles become isometric bundles. And we find that the 1-form $\mu = \xi^\flat \circ \theta = (\theta^t \xi)^\flat$ corresponds with the Liouville form, so it does not depend on the connection. Knowing the torsion of $\nabla^* \oplus \nabla^*$, for any metric connection on $M$ it is easy to deduce, cf. [4] (for any radius function):

$$\mathrm{d}\mu = \omega^S + \mu \circ T^\nabla. \tag{30}$$

The same is to say $\omega^S$ corresponds with the pull-back of the Liouville symplectic form if and only if $T^\nabla = 0$. Hence the Hamiltonian theory of the geodesic flow. We also remark that the geodesic vector field in the sense e.g. of [8] (the vector field $\theta^t \xi$ in our setting) is just the same as the geodesic spray in the sense e.g. of [7, 10].

### 1.4.2 Chern and Stiefel-Whitney classes of $TM$

Let us continue with the structures $G, I^G$ on the tangent bundle, induced from any metric connection $\nabla$, and the same notation from above.

By a deformation retract on the fibres of $\pi : TM \to M$, there is an identification of co-homology spaces $H^*(M) = H^*(TM)$. This is valid for any coefficient ring. In particular

$H^i(TM) = 0,\ \forall i > m$. Let $w_j$ denote the $j$-th Stiefel-Whitney class of $M$ — which is the Stiefel-Whitney class of $TM$ as a vector bundle. Let $w = \sum w_j$ denote the respective total Stiefel-Whitney class.

**Theorem 1.4.** *For any manifold $M$ of dimension $m$, the Euler class of the manifold $TM$ vanishes and the total Stiefel-Whitney class is*

$$w(TTM) = w^2 = \sum_{j=0}^{[m/2]} w_j^2. \tag{31}$$

*Proof.* Being a top degree class, the Euler class must vanish. Since $TTM = \pi^*TM \oplus \pi^*TM$, the Whitney product Theorem and the naturality of the characteristic classes ([9]) immediately give $w(TTM) = w(TM)w(TM) = w^2$. Recall the coefficients are in $\mathbb{Z}_2$. ∎

**Theorem 1.5.** *The Chern classes of the manifold $TM$ with almost complex structure $I^G$ are the Chern classes of the complexified tangent bundle, $TM \otimes_{\mathbb{R}} \mathbb{C} \to M$.*

*Proof.* The complex structure $I^G$ in $TTM$ is equivalent to $I^S$. One complex isomorphism is given by $f : X \mapsto X^h + \mathrm{e}^{\psi} X^v$. Indeed, $\forall X \in TTM$,

$$\begin{aligned} I^S \circ f(X) = (\theta^t - \theta)(X^h + \mathrm{e}^{\psi} X^v) \ &= \ -\theta X^h + \mathrm{e}^{\psi}\theta^t X^v \\ &= \mathrm{e}^{\psi}\theta^t X^v - \mathrm{e}^{\psi}\mathrm{e}^{-\psi}\theta X^h \ = \ f \circ I^G(X). \end{aligned}$$

By the functorial properties, we just have to compute the Chern classes of $I^S$. (Another argument: the homotopy induced by $t\psi,\ t \in [0,1]$, preserves the Chern classes.) Now, the Chern classes of an almost complex manifold $(N, J)$ are the Chern classes of the $\mathbb{C}$-vector bundle $T^+N$, the $+i$-eigenbundle of $J$ where $i = \sqrt{-1}$. In our case,

$$T^+TM = H^c = \pi^*TM^c$$

where $c$ denotes complexification, because of the $\mathbb{C}$-isomorphism induced from $X \in H \mapsto X + i\theta X \in T^+TM$. Indeed $I^S(X + i\theta X) = -\theta X + i\theta^t\theta X = i(X + i\theta X)$. Again by trivial reasons, we thence have $c_j(T^+TM) = c_j(TM^c)$. ∎

We recall the Chern classes $c_{2j}$ define the Pontryagin classes of $M$, cf. [9],

$$p_j(M) = (-1)^j c_{2j}(TM \otimes \mathbb{C}). \tag{32}$$

Moreover, the Chern classes of $(TM, I^G)$ do not depend on the connection $\nabla$.

### 1.4.3 Stiefel-Whitney classes of $S_rM$ with $r$ constant

Now let $m = n + 1$ and let $r > 0$ be a constant. Then the $n$-vector bundle $\kappa = u^\perp \subset V$ is contained in $TSM = H \oplus \kappa$ where we assume e.g. the Sasaki metric. We continue to denote by $w = \sum w_j$ the total Stiefel-Whitney class of $M$.

**Theorem 1.6.** *The total Stiefel-Whitney class of $SM$ is*

$$w(TSM) = \sum_{j=0}^{n} \pi^* w_j^2 \tag{33}$$

*and in particular the Euler class of the manifold $SM$ vanishes.*

*Proof.* We have $w(\pi^*TM) = w(H) = \pi^* w$. And clearly $w(\mathbb{R}\xi) = 1$, so that $w(\pi^*TM) = w(\kappa \oplus \mathbb{R}\xi) = w(\kappa)$. Hence

$$w(TSM) = w(H \oplus \kappa) = w(\pi^*TM)^2 = \pi^* w^2.$$

Notice $w_m(\kappa) = 0$. ∎

**Remark.** (i) We observe that always $w_1(SM) = 0$, as expected because $TM$ is always oriented and $\xi$ induces an orientation on the submanifold.
(ii) If $M$ has a finite good cover, is oriented, and admits a non-vanishing vector field, then we deduce $H^*(SM) = H^*(M) \otimes H^*(S^n)$ by the Theorem of Leray-Hirsh (cf. [6]). In particular $\pi^*$ is an isomorphism $H^i(SM) = H^i(M)$ of co-homology spaces up to degree $i \leq n-1 = m-2$. By contrast, we have proved $\pi^*(w_m) = 0$.

Since $w_2(SM) = w_1^2$, we have the following.

**Corollary 1.6.** *For any oriented Riemannian manifold $M$, the manifold $SM$ is spin.*

Recall $w_2$ is also the obstruction to a 7-manifold admit a $G_2$-structure. We have explicitly constructed a natural $G_2$-structure on $SM$, for any oriented Riemannian 4-manifold $M$, cf. [4] and the references therein.

# References

[1] M. T. K. Abbassi, *Note on the classification theorems of g-natural metrics on the tangent bundle of a Riemannian manifold* $(M, g)$, Comment. Math. Univ. Carolinae 45(4) (2004), 591–596.

[2] M. T. K. Abbassi and G. Calvaruso, *g-Natural Contact Metrics on Unit Tangent Sphere Bundles*, Monatsh. für Mathe. 151 (2006), 89–109.

[3] I. Agricola, *The Srní lectures on non-integrable geometries with torsion*, Archi. Mathe. (Brno), Tomus 42 (2006), Suppl., 5–84.

[4] R. Albuquerque, *On the $G_2$ bundle of a Riemannian 4-manifold*, Jour. Geom. Physics 60 (2010), 924–939.

[5] R. Albuquerque, *Weighted metrics on tangent sphere bundles*, submitted

[6] R. Bott and L. W. Tu, *Differential Forms in Algebraic Topology*, Springer-Verlag Berlin, New York, 1982.

[7] P. Dombrowski, *On the geometry of the tangent bundles*, Jour. Reine Angew. Math. 210 (1962), 73–88.

[8] H. Geiges, *An Introduction to Contact Topology*, Cambridge Studies in Advanced Mathematics 109, Cambridge University Press, 2008.

[9] J. W. Milnor and J. D. Stasheff, *Characteristic Classes*, Annals Math. Stud. 76, Princeton University Press (1974).

[10] T. Sakai, *Riemannian Geometry*, Translations of Mathematical Monographs 149, AMS (1996).

[11] S. Sasaki, *On the differential geometry of tangent bundles of Riemannian manifolds*, Tôhoku Math. Jour. 10 (1958), 338–354.

[12] Y. Tashiro, *On contact structures on tangent sphere bundles*, Tôhoku Math. Jour. 21 (1969), 117–143.