Upper bounds for the bondage number of graphs on topological surfaces

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Abstract

The bondage number $b(G)$ of a graph G is the smallest number of edges of G whose removal from G results in a graph having the domination number larger than that of G. We show that, for a graph G having the maximum vertex degree $\Delta = \Delta(G)$ and embeddable on an orientable surface of genus h and a non-orientable surface of genus k,

$$b(G) \leq \min\{\Delta(G) + h + 2,\ \Delta(G) + k + 1\}.$$

This generalizes known upper bounds for planar and toroidal graphs.

Keywords: Bondage number, Domination number, Topological surface, Embedding on a surface, Euler's formula

1. Introduction

We consider simple finite non-empty graphs. For a graph G, its vertex and edge sets are denoted, respectively, by $V(G)$ and $E(G)$. We also use the following standard notation: $d(v)$ for the degree of a vertex v in G, $\Delta = \Delta(G)$ for the maximum vertex degree of G, $\delta = \delta(G)$ for the minimum vertex degree of G, and $N(v)$ for the neighbourhood of a vertex v in G.

A set $D \subseteq V(G)$ is a *dominating set* if every vertex not in D is adjacent to at least one vertex in D. The minimum cardinality of a dominating set

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of G is the *domination number* $\gamma(G)$. Clearly, for any spanning subgraph H of G, $\gamma(H) \geq \gamma(G)$. The *bondage number* of G, denoted by $b(G)$, is the minimum cardinality of a set of edges $B \subseteq E(G)$ such that $\gamma(G-B) > \gamma(G)$.

The bondage number was introduced by Bauer et al. [1] (see also Fink et al. [4]). Two unsolved classical conjectures for the bondage number of arbitrary and planar graphs are as follows.

Conjecture 1 (Teschner [8])**.** *For any graph G, $b(G) \leq \frac{3}{2}\Delta(G)$.*

Hartnell and Rall [6] and Teschner [9] showed that for the cartesian product $G_n = K_n \times K_n$, $n \geq 2$, the bound of Conjecture 1 is sharp, i.e. $b(G_n) = \frac{3}{2}\Delta(G_n)$. Teschner [8] also proved that Conjecture 1 holds when $\gamma(G) \leq 3$.

Conjecture 2 (Dunbar et al. [3])**.** *If G is a planar graph, then $b(G) \leq \Delta(G) + 1$.*

The planar graphs are precisely the graphs that can be drawn on the sphere with no crossing edges. A topological surface S can be obtained from the sphere S_0 by adding a number of handles or crosscaps. If we add h handles to S_0, we obtain an orientable surface S_h, which is often referred to as the *h-holed torus*. The number h is called the *orientable genus* of S_h. If we add k crosscaps to the sphere S_0, we obtain a non-orientable surface N_k. The number k is called the *non-orientable genus* of N_k. Any topological surface is homeomorphically equivalent either to S_h ($h \geq 0$), or to N_k ($k \geq 1$). For example, S_1, N_1, N_2 are the *torus*, the *projective plane*, and the *Klein bottle*, respectively.

A graph G is *embeddable* on a topological surface S if it admits a drawing on the surface with no crossing edges. Such a drawing of G on the surface S is called an *embedding* of G on S. Notice that there can be many different embeddings of the same graph G on a particular surface S. The embeddings can be distinguished and classified by different properties. The set of faces of a particular embedding of G on S is denoted by $F(G)$.

An embedding of G on the surface S is a *2-cell embedding* if each face of the embedding is homeomorphic to an open disk. In other words, a 2-cell embedding is an embedding on S that "fits" the surface. This is expressed in Euler's formulae (1) and (2) of Theorem 3. For example, a cycle C_n ($n \geq 3$) does not have a 2-cell embedding on the torus, but it has 2-cell embeddings on the sphere and the projective plane. Similarly, a planar graph may have 2-cell and non-2-cell embeddings on the torus.

The following result is usually known as (generalized) *Euler's formula.* We state it here in a form similar to Thomassen [10].

Theorem 3 (Euler's Formula, [10])**.** *Suppose a connected graph G with $|V(G)|$ vertices and $|E(G)|$ edges admits a 2-cell embedding having $|F(G)|$ faces on a topological surface S. Then, either $S = S_h$ and*

$$|V(G)| - |E(G)| + |F(G)| = 2 - 2h, \qquad (1)$$

or $S = N_k$ and

$$|V(G)| - |E(G)| + |F(G)| = 2 - k. \qquad (2)$$

Equation (1) is usually referred to as Euler's formula for an orientable surface S_h of genus h, $h \geq 0$, and Equation (2) is known as Euler's formula for a non-orientable surface N_k of genus k, $k \geq 1$.

The *orientable genus* of a graph G is the smallest integer $h = h(G)$ such that G admits an embedding on an orientable topological surface S of genus h. The *non-orientable genus* of G is the smallest integer $k = k(G)$ such that G can be embedded on a non-orientable topological surface S of genus k. Clearly, in general, $h(G) \neq k(G)$, and the embeddings on $S_{h(G)}$ and $N_{k(G)}$ must be 2-cell embeddings.

Trying to prove Conjecture 2, Kang and Yuan [7] came up with the following upper bound whose simpler topological proof was later discovered by Carlson and Develin [2].

Theorem 4 ([7, 2])**.** *For any connected planar graph G,*

$$b(G) \leq \min\{8,\ \Delta(G) + 2\}.$$

This solves Conjecture 2 in case $\Delta(G) \geq 7$. The upper bound of Theorem 4 is for the sphere S_0 that has orientable genus $h = 0$. The proof of Theorem 4 in [2] is topologically intuitive, uses Euler's formula for the sphere, and allows its authors to establish a partially similar result for the torus.

Theorem 5 ([2])**.** *For any connected toroidal graph G, $b(G) \leq \Delta(G) + 3$.*

Notice that the torus S_1 has orientable genus $h = 1$. As mentioned in [2], it is sufficient to prove the results of Theorems 4 and 5 for connected graphs because the bondage number of a disconnected graph G is the minimum of the bondage numbers of its components.

In this paper, we prove the following result which generalizes the corresponding upper bounds of Theorems 4 and 5 for any orientable or non-orientable topological surface S.

Theorem 6. *For a connected graph G of orientable genus $h = h(G)$ and non-orientable genus $k = k(G)$,*

$$b(G) \leq \min\{\Delta(G) + h + 2,\ \Delta(G) + k + 1\}.$$

The upper bound of Theorem 6 follows from Theorems 8 and 9 proved below in Section 2.

2. The bondage number on orientable and non-orientable surfaces

In this section, we prove Theorem 6 by considering orientable and non-orientable surfaces separately. The proofs are done by using Euler's formulae (1) and (2), counting arguments, and the following result.

Lemma 7 (Hartnell and Rall [6]). *For any edge uv in a graph G, we have $b(G) \leq d(u) + d(v) - 1 - |N(u) \cap N(v)|$. In particular, this implies that $b(G) \leq \delta(G) + \Delta(G) - 1$.*

Having a graph G embedded on a surface S, each edge $e_i = uv \in E(G)$, $i = 1, \ldots, |E(G)|$, can be assigned two weights, $w_i = \frac{1}{d(u)} + \frac{1}{d(v)}$ and $f_i = \frac{1}{m'} + \frac{1}{m''}$, where m' is the number of edges on the boundary of a face on one side of e_i, and m'' is the number of edges on the boundary of the face on the other side of e_i. Notice that, in an embedding on a surface, an edge e_i may be not separating two distinct faces, but instead it can appear twice on the boundary of the same face. For example, every edge of a path P_n ($n \geq 2$) embedded on the sphere is on the boundary of a unique face, and it appears exactly twice on the face boundary walk: once for each side of the edge. Clearly, in this case, $m' = m'' = 2(n-1)$ and $f_i = \frac{2}{m'} = \frac{2}{m''} = \frac{1}{n-1}$.

Notice that weights w_i and f_i, $i = 1, \ldots, |E(G)|$, count the number of vertices of G and faces of its embedding on S as follows:

$$\sum_{i=1}^{|E(G)|} w_i = |V(G)|, \qquad \sum_{i=1}^{|E(G)|} f_i = |F(G)|.$$

Then, by Euler's formula (1), we have

$$\sum_{i=1}^{|E(G)|} (w_i + f_i - 1) = |V(G)| + |F(G)| - |E(G)| = 2 - 2h,$$

or, in other words,

$$\sum_{i=1}^{|E(G)|}\left(w_i+f_i-1-\frac{2-2h}{|E(G)|}\right)=\sum_{i=1}^{|E(G)|}\left(w_i+f_i-1+\frac{2h-2}{|E(G)|}\right)=0.$$

Now, each edge $e_i=uv\in E(G)$, $i=1,\ldots,|E(G)|$, can be associated with the quantity $w_i+f_i-1+\frac{2h-2}{|E(G)|}$ called the *oriented curvature* of the edge. Also, by Euler's formula (2), we have

$$\sum_{i=1}^{|E(G)|}(w_i+f_i-1)=|V(G)|+|F(G)|-|E(G)|=2-k,$$

or, in other words,

$$\sum_{i=1}^{|E(G)|}\left(w_i+f_i-1-\frac{2-k}{|E(G)|}\right)=\sum_{i=1}^{|E(G)|}\left(w_i+f_i-1+\frac{k-2}{|E(G)|}\right)=0.$$

Then, each edge $e_i=uv\in E(G)$, $i=1,\ldots,|E(G)|$, can be associated with the quantity $w_i+f_i-1+\frac{k-2}{|E(G)|}$ called the *non-oriented curvature* of the edge.

Theorem 8. *Let G be a connected graph 2-cell embeddable on an orientable surface of genus $h\geq 0$. Then*

$$b(G)\leq\Delta(G)+h+2. \tag{3}$$

Proof. Suppose G is 2-cell embedded on the h-holed torus S_h. By Lemma 7, if G has any vertices of degree $h+3$ or less, we have $\delta(G)\leq h+3$, and inequality (3) holds. Therefore, we can assume $\Delta(G)\geq\delta(G)\geq h+4$.

Now, suppose the opposite, $b(G)\geq\Delta(G)+h+3$. Then, by Lemma 7, for any edge $e_i=uv$, $i=1,\ldots,|E(G)|$, we have $d(u)+d(v)-1-|N(u)\cap N(v)|\geq b(G)\geq\Delta(G)+h+3$. Then, $d(u)+d(v)\geq\Delta(G)+h+4+|N(u)\cap N(v)|$, and $d(u)\leq\Delta(G)$, $d(v)\leq\Delta(G)$. If either $d(u)$ or $d(v)$ is equal to $h+4$, the other degree must be equal to $\Delta(G)\geq h+4$, and u and v cannot have any common neighbors, so that m' and m'' are both at least 4. Since in this case $|E(G)|\geq\frac{(h+4)(h+5)}{2}$, such an edge $e_i=uv$ has a negative oriented curvature:

$$w_i+f_i-1+\frac{2h-2}{|E(G)|}\leq\frac{2}{h+4}+\frac{2}{4}-1+\frac{2(2h-2)}{(h+4)(h+5)}=\frac{-8+h(3-h)}{2(h+4)(h+5)}<0$$

for any $h \geq 1$, and, in case $h = 0$,

$$w_i + f_i - 1 - \frac{2}{|E(G)|} \leq \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - 1 - \frac{2}{|E(G)|} = \frac{-2}{|E(G)|} < 0.$$

Suppose one of $d(u)$ and $d(v)$ is equal to $h+5$, without loss of generality, $d(u) = h + 5$. Then, $\Delta(G) \geq d(v) \geq \Delta(G) - 1 + |N(u) \cap N(v)|$. If $d(v) = h+4 = \Delta(G)-1$, we are in the previous case. Otherwise, we have $d(v) \geq h+5$, and at most one of m' and m'' can be equal to 3, implying the other is at least 4. Then again, since in this case $|E(G)| \geq \frac{(h+4)(h+4)+2(h+5)}{2} = \frac{h^2+10h+26}{2}$, the edge e_i must have a negative oriented curvature:

$$w_i+f_i-1+\frac{2h-2}{|E(G)|} \leq \frac{2}{h+5}+\frac{1}{3}+\frac{1}{4}-1+\frac{2(2h-2)}{h^2+10h+26} = \frac{-5h^3-3h^2+52h-266}{12(h+5)(h^2+10h+26)} < 0$$

for any $h \geq 1$, and, in case $h = 0$,

$$w_i + f_i - 1 - \frac{2}{|E(G)|} \leq \frac{1}{5} + \frac{1}{5} + \frac{1}{3} + \frac{1}{4} - 1 - \frac{2}{|E(G)|} = -\frac{1}{60} - \frac{2}{|E(G)|} < 0.$$

The only remaining case is when $d(u) \geq h + 6$ and $d(v) \geq h + 6$. Since $m' \geq 3$ and $m'' \geq 3$, and, in this case, $|E(G)| \geq \frac{(h+4)(h+5)+2(h+6)}{2} = \frac{h^2+11h+32}{2}$, the edge e_i must have a negative oriented curvature:

$$w_i+f_i-1+\frac{2h-2}{|E(G)|} \leq \frac{2}{h+6}+\frac{2}{3}-1+\frac{2(2h-2)}{h^2+11h+32} = \frac{-h^3+h^2+28h-72}{3(h+6)(h^2+11h+32)} < 0$$

for any $h \geq 1$, and, in case $h = 0$,

$$w_i + f_i - 1 - \frac{2}{|E(G)|} \leq \frac{1}{6} + \frac{1}{6} + \frac{1}{3} + \frac{1}{3} - 1 - \frac{2}{|E(G)|} = \frac{-2}{|E(G)|} < 0.$$

Summing over all edges $e_i \in E(G)$ yields

$$\sum_{i=1}^{|E(G)|} \left(w_i + f_i - 1 + \frac{2h-2}{|E(G)|} \right) < 0,$$

which is a contradiction to Euler's formula (1) stating

$$\sum_{i=1}^{|E(G)|} \left(w_i + f_i - 1 - \frac{2-2h}{|E(G)|} \right) = |V(G)| + |F(G)| - |E(G)| - (2-2h) = 0.$$

Thus, $b(G) \leq \Delta(G) + h + 2$. □

Theorem 9. *Let G be a connected graph 2-cell embeddable on a non-orientable surface of genus $k \geq 1$. Then*

$$b(G) \leq \Delta(G) + k + 1. \tag{4}$$

PROOF. Suppose G is 2-cell embedded on the sphere with k crosscaps N_k. By Lemma 7, if G has any vertices of degree $k+2$ or less, we have $\delta(G) \leq k+2$, and inequality (4) holds. Therefore, we can assume $\Delta(G) \geq \delta(G) \geq k+3$.

Suppose the opposite, $b(G) \geq \Delta(G) + k + 2$. Then, by Lemma 7, for any edge $e_i = uv$, $i = 1, \ldots, |E(G)|$, we have $d(u) + d(v) - 1 - |N(u) \cap N(v)| \geq b(G) \geq \Delta(G) + k + 2$. Then, $d(u) + d(v) \geq \Delta(G) + k + 3 + |N(u) \cap N(v)|$, and $d(u) \leq \Delta(G)$, $d(v) \leq \Delta(G)$. If either $d(u)$ or $d(v)$ is equal to $k+3$, the other degree must be equal to $\Delta(G) \geq k+3$, and u and v cannot have any common neighbors, so that m' and m'' are both at least 4. Since in this case $|E(G)| \geq \frac{(k+3)(k+4)}{2}$, the non-oriented curvature of the edge $e_i = uv$ is

$$w_i + f_i - 1 + \frac{k-2}{|E(G)|} \leq \frac{2}{k+3} + \frac{2}{4} - 1 + \frac{2(k-2)}{(k+3)(k+4)} = \frac{-4 + k(1-k)}{2(k+3)(k+4)} < 0$$

for any $k \geq 2$, and, in case $k = 1$,

$$w_i + f_i - 1 - \frac{1}{|E(G)|} \leq \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - 1 - \frac{1}{|E(G)|} = \frac{-1}{|E(G)|} < 0.$$

Suppose one of $d(u)$ and $d(v)$, let us say $d(u)$, is equal to $k+4$. Then, $\Delta(G) \geq d(v) \geq \Delta(G) - 1 + |N(u) \cap N(v)|$. If $d(v) = k + 3 = \Delta(G) - 1$, we are in the previous case. Otherwise, we have $d(v) \geq k+4$, and at most one of m' and m'' can be equal to 3, implying the other is at least 4. Then again, since in this case $|E(G)| \geq \frac{(k+3)(k+3)+2(k+4)}{2} = \frac{k^2+8k+17}{2}$, the edge e_i must have a negative non-oriented curvature:

$$w_i + f_i - 1 + \frac{k-2}{|E(G)|} \leq \frac{2}{k+4} + \frac{1}{3} + \frac{1}{4} - 1 + \frac{2(k-2)}{k^2+8k+17} = \frac{-124 - 5k - 12k^2 - 5k^3}{12(k+4)(k^2+8k+17)} < 0$$

for any $k \geq 2$, and, in case $k = 1$,

$$w_i + f_i - 1 - \frac{1}{|E(G)|} \leq \frac{1}{5} + \frac{1}{5} + \frac{1}{3} + \frac{1}{4} - 1 - \frac{1}{|E(G)|} = -\frac{1}{60} - \frac{1}{|E(G)|} < 0.$$

The only remaining case is when $d(u) \geq k+5$ and $d(v) \geq k+5$. Since $m' \geq 3$ and $m'' \geq 3$, and, in this case, $|E(G)| \geq \frac{(k+3)(k+4)+2(k+5)}{2} = \frac{k^2+9k+22}{2}$, the edge e_i must have a negative non-oriented curvature:

$$w_i+f_i-1+\frac{k-2}{|E(G)|} \leq \frac{2}{k+5}+\frac{2}{3}-1+\frac{2(k-2)}{k^2+9k+22} = \frac{-k^3-2k^2+5k-38}{3(k+5)(k^2+9k+22)} < 0$$

for any $k \geq 2$, and, in case $k=1$,

$$w_i + f_i - 1 - \frac{1}{|E(G)|} \leq \frac{1}{6} + \frac{1}{6} + \frac{1}{3} + \frac{1}{3} - 1 - \frac{1}{|E(G)|} = \frac{-1}{|E(G)|} < 0.$$

Summing over all edges $e_i \in E(G)$ yields

$$\sum_{i=1}^{|E(G)|} \left(w_i + f_i - 1 + \frac{k-2}{|E(G)|} \right) < 0,$$

which is a contradiction to Euler's formula (2) stating

$$\sum_{i=1}^{|E(G)|} \left(w_i + f_i - 1 - \frac{2-k}{|E(G)|} \right) = |V(G)| + |F(G)| - |E(G)| - (2-k) = 0.$$

Thus, $b(G) \leq \Delta(G) + k + 1$, and the proof is complete. □

3. Conclusions

The upper bound of Theorem 6 provides a hierarchy of upper bounds that eventually may help solving Conjecture 1. However, it can be seen that the bounds of Theorems 8 and 9 are not tight for larger values of the genera $h = h(G)$ and $k = k(G)$. For example, by adjusting respectively the proofs of Theorems 8 and 9, upper bound (3) can be improved to $b(G) \leq \Delta(G)+h+1$ for $h \geq 8$, and upper bound (4) can be improved to $b(G) \leq \Delta(G) + k$ for $k \geq 3$ and to $b(G) \leq \Delta(G)+k-1$ for $k \geq 6$. It is left to the reader to adjust the proofs and bounds for a particular topological surface of higher genus.

In view of Theorem 4, its proof in [2], and results presented in this paper, it should be reasonable to conjecture that, when $\Delta(G)$ is sufficiently large, the bondage number $b(G)$ is bounded by a certain constant depending only on the properties of topological surfaces where G embeds.

Conjecture 10. *For a connected graph G of orientable genus h and non-orientable genus k, $b(G) \leq \min\{c_h,\, c'_k,\, \Delta(G)+h+2,\, \Delta(G)+k+1\}$, where c_h and c'_k are constants depending, respectively, on the orientable and non-orientable genera of G.*

Since $\delta(G) \leq 5$ for a planar graph G, Fischermann et al. [5] ask whether there exist planar graphs of bondage numbers 6, 7, or 8. A class of planar graphs with the bondage number equal to 6 is shown in [2]. Therefore, in case of planar graphs, we have $6 \leq c_0 \leq 8$. It would be interesting to have an estimation for the constants c_h and c'_k for the torus S_1, projective plane N_1, and Klein bottle N_2.

References

[1] D. Bauer, F. Harary, J. Nieminen, C.L. Suffel, Domination alteration sets in graphs, Discrete Math. 47 (1983) 153–161.

[2] K. Carlson, M. Develin, On the bondage number of planar and directed graphs, Discrete Math. 306 (2006) 820–826.

[3] J.E. Dunbar, T.W. Haynes, U. Teschner, L. Volkmann, Bondage insensitivity and reinforcement, in: T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), Domination in Graphs: Advanced Topics, Marcel Dekker, New York, 1998, pp. 249–259.

[4] J.F. Fink, M.J. Jacobson, L.F. Kinch, J. Roberts, The bondage number of a graph, Discrete Math. 86 (1990) 47–57.

[5] M. Fischermann, D. Rautenbach, L. Volkmann, Remarks on the bondage number of planar graphs, Discrete Math. 260 (2003) 57–67.

[6] B.L. Hartnell, D.F. Rall, Bounds on the bondage number of a graph, Discrete Math. 128 (1994) 173–177.

[7] L. Kang, J. Yuan, Bondage number of planar graphs, Discrete Math. 222 (2000) 191–198.

[8] U. Teschner, A new upper bound for the bondage number of graphs with small domination number, Australas. J. Combin. 12 (1995) 27–35.

[9] U. Teschner, The bondage number of a graph G can be much greater than $\Delta(G)$, Ars Combin. 43 (1996) 81–87.

[10] C. Thomassen, The Jordan-Schönflies theorem and the classification of surfaces, Amer. Math. Monthly 99 (1992) 116-131.