A simple proof of orientability in the colored Boulatov model

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In this short note we use results from the theory of crystallizations to prove that color in group field theories garantees orientability of the piecewise linear pseudo-manifolds associated to each graph generated perturbatively. For the colored Boulatov model the only graphs which represent orientable manifolds are those that have a particular relation between the perturbative order and the number of 2- and 3- bubbles. This relation is the combinatorial requirement of having 3-bubbles which are homeomorphic to 2-spheres.

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I. INTRODUCTION

There has been recently a growth of interest in group field theories [1, 2] and there are many reasons for this to happen. Group field theories (GFT) are at the boundary between Loop Quantum Gravity and Quantum Field Theory over a group manifold and are a generalization of matrix models to higher dimensions [3][4]. It is known that matrix models have a topological expansion in which the genus, the only topological invariant needed to characterize orientable surfaces, plays the role of the parameter of this expansion. Roughly speaking a n-dimensional group field theory has a vertex associated to an n-simplex and a propagator which glues the $(n-1)$-simplices. Feynman diagrams of a n-dimensional group field theory can be interpreted as gluings of simplices and then have the interpretation of piecewise linear (PL) manifolds . However generic GFT in 3 dimensions have the problem that the gluings are too arbitrary, in the sense that the generated simplicial complexes are not even pseudo-manifolds [5], since they present wrapping singularities. As in ordinary ϕ^4 theory, a 3-dimensional group field theory can generate a "8" diagram which however has no obvious geometrical interpretation in the continuum limit. This phenomenon does not happen in 2-dimensions. For this reason a *colored* version of group field theory (cGFT) has been introduced[6–8]. The challenge in these models is to obtain a topological expansion as in the 2-dimensional case [9–12]. It has been shown in [13] that the sphere dominates the partition function. In order to achieve this result, techniques from the theory of crystallizations have been used. In fact, colored n-graphs are well known in mathematics as *gems*: graph-encoded manifolds [14, 15]. In this paper we use the results in this field of mathematics to show that the growth of interest in colored models is not unjustified: colored models generate orientable pseudomanifolds in any number of dimensions. Many of the theorems we will use were known for long time in the context of crystallization and here we report briefly these results. The outcome of this note is that the generation of pseudo-manifolds is due to the color, while the orientability in the colored Boulatov model is due to the presence of two different vertices (clockwise and anti-clockwise). The paper is organized as follows: in section II we recall the colored Boulatov model and its standard interpretation. In section III we review basic results in the field of 3-gems and crystallizations. We will use some of these results in section IV to prove the orientability of simplicial complexes generated perturbatively by the colored Btand that a CGFT graph can be seen either as a stranded graph (using the vertex and the propagators as depicted in Fig. 1) or as a 'oulatov model.

II. THE COLORED BOULATOV MODEL

tand that a CGFT graph can be seen either as a stranded graph (using the vertex and the propagators as depicted in Fig. 1) or as a ' In this section we introduce the main facts behind the colored Boulatov model[16][6]. Consider a compact Lie group H, denote h its elements, e the unit element, and $\int dh$ the integral with respect to the Haar measure of the group.

In 3 dimensions we introduce two fields, $\bar{\psi}^i$ and ψ^i, $i = 0, 1, 2, 3$ be four couples of complex scalar (or Grassmann) fields over three copies of G, $\psi^i : G \times G \times G \to \mathbb{C}$. The index i runs from= 0 to $n + 1$, where n is the number of dimensions, and the ψ and $\bar{\psi}$ are functions of n copies of the group. We denote $\delta^{\Lambda}(h)$ the regularized delta function over G with some cutoff Λ such that $\delta^{\Lambda}(e)$ is finite, but diverges when Λ goes to infinity. In the fermionic version of the theory the indices i can be seen as the dependence of the field from a (global) gauge group $SU(N)$, where $N = n + 1$. A feasible regularization is given, for instance for the group $G = SU(2)$, by

$$\delta^{\Lambda}(h) = \sum_{j=0}^{\Lambda} (2j + 1)\chi^j(h). \tag{1}$$

where $\chi^j(h)$ is the character of h in the representation j. The path integral for the colored Boulatov model over G is:

$$Z(\lambda, \bar{\lambda}) = e^{-F(\lambda,\bar{\lambda})} = \int \prod_{i=0}^{4} d\mu_P(\bar{\psi}^i, \psi^i)\; e^{-S^{int}(\bar{\psi}^i,\psi^i)} \;, \tag{2}$$

where the Gaussian measure P is chosen such that:

$$\int \prod_{i=0}^{4} d\mu_P(\bar{\psi}^i, \psi^i) = 1 \;,$$

and:

$$
\begin{aligned}
&P_{h_0h_1h_2;h_0'h_1'h_2'} = \\
&= \int d\mu_P(\bar\psi^i,\psi^i)\ \bar\psi^i_{h_0h_1h_2}\psi^i_{h_0'h_1'h_2'} = \\
&= \int dh\ \delta^\Lambda\big(h_0h(h_0')^{-1}\big)\delta^\Lambda\big(h_1h(h_1')^{-1}\big)\delta^\Lambda\big(h_2h(h_2')^{-1}\big)\ ,
\end{aligned}
$$

The colored model has two interactions, a "clockwise" and an "anti-clockwise", and one is obtained from the other one by conjugation in the internal group color $SU(N)$, where N is 4 in 3 dimensions, one for each face of the 3-simplex. For convenience we denote $\psi(h,p,q)=\psi_{hpq}$. Invariance under global rotations in the internal color group require at least two interactions:

$$
\begin{aligned}
S^{int} &= \frac{\lambda}{\sqrt{\delta^\Lambda(e)}}\int (dh)^6 \psi^0_{h_{03}h_{02}h_{01}}\psi^1_{h_{10}h_{13}h_{12}}\psi^2_{h_{21}h_{20}h_{23}}\psi^3_{h_{32}h_{31}h_{30}} + \\
&+ \frac{\bar\lambda}{\sqrt{\delta^\Lambda(e)}}\int (dh)^6 \bar\psi^0_{h^{03}h^{02}h^{01}}\bar\psi^1_{h^{10}h^{13}h^{12}}\bar\psi^2_{h^{21}h^{20}h^{23}}\bar\psi^3_{h^{32}h^{31}h^{30}} \qquad (3)
\end{aligned}
$$

where $h_{ij}=h_{ji}$ is symmetric in the two indices and $\lambda\neq\bar\lambda$ in principle. In order to make the notation clearer (already the orientation of the colors is sufficient to distinguish the two vertices), we call "red" the vertex involving the ψ's and "black" the one involving the $\bar\psi$'s. Thus any line coming out of a cGFT vertex has a color i.

The group elements h_{ij} in eq. (3) are associated to the propagators (represented as solid lines), and glue two vertices with opposite orientation. The vertex can be seen as the dual of a tetrahedron and its lines represent the triangles which form the tetrahedron. Each propagators is decomposed into three *parallel* strands which are associated to the

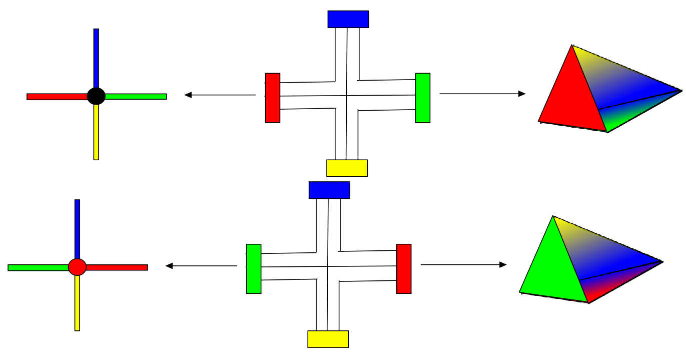

FIG. 1. Colored GFT red and black vertices.

three arguments of the fields, i.e. the 1-dimensional elements of the 1-skeleton of the tetrahedron which bound every face. A colored line represents the gluing of two tetrahedra (of opposite orientations) along triangles of the same color as in Fig. (2).

It is easy to understand that a cGFT graph can be seen either as a stranded graph (using the vertex and the propagators as depicted in Fig. 1) or as a "colored graph" with (colored) solid lines, and two classes of oriented vertices. In this paper we consider only vacuum graphs, i.e. all the vertices of the graphs are 4-valent and we deal only with connected graphs. The lines of a vacuum cGFT graph Γ have two natural orientations given by the fact that only vertices of opposite orientations can be glued. It is easy to see that a vacuum cGFT graph must have the same number of black and red vertices. For any graph Γ, we denote n as the number of vertices, l as the lines of Γ, and we define as *faces* (not to be confused with the faces of the tetrahedron!), $\mathcal{F}_\Gamma$, as any closed strand in the Feynman graph of a GFT. Thus a generic vacuum Feynman amplitude of the theory can be written as:

$$
\mathcal{A} = \frac{(\lambda\bar\lambda)^{\frac{n}{2}}}{[\delta^N(e)]^{\frac{n}{2}}}\int \prod_{l\in\Gamma} dh_l \prod_{f\in\mathcal{F}_\Gamma}\delta^\Lambda_f\Big(\overrightarrow{\prod_{l_0\in f}} h_{l_0}^{\sigma(l_0,f)}\Big), \qquad (4)
$$

where l_0 is a line associated to a face f and $\sigma(l_0,f)$ is alternatively $+1$ or -1 depending on the orientation. In the following we will assume that an orientation is fixed. Because of the properties of $\delta' s$ the orientation does not affect the amplitude. To each colored graph associated to an amplitude of the colored Boulatov model it is possible to associate *bubbles* by removing all the edges of one color. We call $\mathcal{B}_{i_1,\cdots,i_k}$ the set of k-bubbles associated to the deletion of $n-k$ colors. In 3-dimensions, for instance, 3-bubbles have 3-colors (surfaces), 2-bubbles have 2 colors (lines) and so on and so forth. Bubbles play a special role in the theory, since they discriminate manifold from pseudo-manifolds (see next section for the same result in the theory of 3-gems).

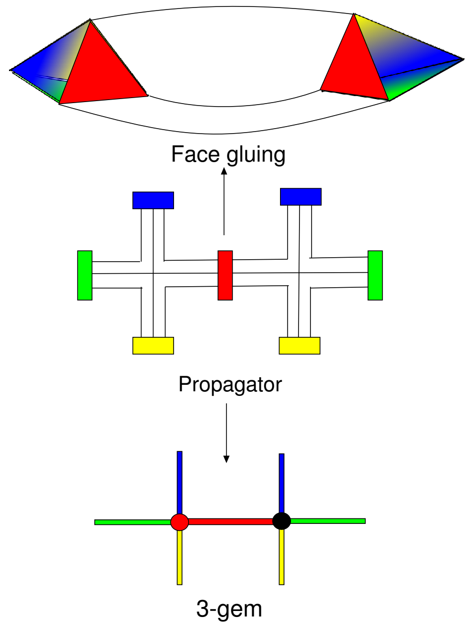

FIG. 2. A gluing using a colored propagator.

III. A SURVEY OF GRAPH-EMBEDDED MANIFOLDS RESULTS

In this section we review some basic results in the field of 3-gems and make a dictionary between the two literatures, as colored group field theory can gain much from the results obtained in all the years of research in such field.

Let Γ be a finite, edge-colored graph, parallel edges allowed. A *k-residue* of Γ, $k \in \mathbf{N}$ is a connected component of subgraph of Γ induced by k color classes (this is what in colored group field theory are called *bubbles*). These graphs represent a piecewice linear manifold in the following sense (a *pseudo*-complex) [18]. A n-regular n-colored graph is a couple $(\Gamma, \gamma)_n$ where n denotes its degree. To a couple $(\Gamma, \gamma)_{n+1}$ there is an associated *pseudo*-complex $K(\Gamma)$ given by the following construction. Take an n-simplex σ^n for each $V(\Gamma)$ and label its vertices Δ_n. If x,y in $V(\Gamma)$ are joined by an edge, then attach the $(n-1)$-faces of their associated simplices. This is the same interpretation given to attaching faces of n-simplices in a n-dimensional group field theory. We denote $|\Gamma|$ the pseudo-complex associated with the colored graph Γ.

Lemma 1 For any PL n-manifold $\mathcal{M}$ there exist a (n+1)-graph Γ such that $|\Gamma| \backsimeq \mathcal{M}$.

We now restrict to the case of 3-dimensions and list some of the basic results[15].

Let Γ be a 4-edge-colored 4-graph and denote by v, e, b, t respectively the number of vertices (0-*residues*), edges (1-*residues*), 2-*residues* and 3-*residues.*

Definition A *3-gem* (a 3 graph-embedded manifold) is a 4-regular properly edge-colored graph such that

$$v + t = b \tag{5}$$

A 4-regular properly edge-colored graph for which (5) does not apply is called *3-gepm* (a 3 graph-embedded pseudo-manifold).

Lemma 2 A necessary and sufficient condition for the graph $(\Gamma, \gamma)_4$ to represent a manifold, is to meet the relation between its 2- and 3- residues (read as it 2- and 3- colored bubbles) and the number of vertices (read as the perturbative order) $v + t = b$.

This Lemma clarifies the reason why 3-gems have to satisfy the relation (5). Let now introduce few definitions which will turn useful later[18]:

Definition A *triball* is a connected, cubic, 3-edge-colored graph $\Gamma_3 \subset \Gamma$ such that its Euler characteristic is the one of the 2-sphere.

Thus we have the relation between its 2-residues b_{Γ_3} and the vertices: $2b_{\Gamma_3} - v = 4$. An important fact is the following:

Lemma 3 A graph $(\Gamma, \gamma)_4$ is a 3-gem iff each of its 3-residue is a triball.

Thus, the condition that graphs have to satisfy in order to be 3-gems is a condition on the topology of its 3-residues. We now discuss *crystallizations* of 3-gems. Let first introduce the *fusion* process. Let be $\mathcal{B}_{ijk}$ and $\mathcal{B}'_{ijk}$

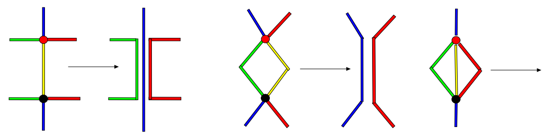

FIG. 3. Fusion moves on a 4-regular 4-edge colored graph of 1-, 2- and 3- dipoles respectively.

two different 3-residues separated by a unique color which, by construction, is different from the color i, j, k. We call $1-$*dipole* this edge connecting the two 3-residues. The generalization to $k-$*dipoles* which connect $(n-k)$-residues is obvious. We call *fusion* the process of contraction of two vertices through the first two combinatorial moves depicted in Fig. 3. Each cancellation of a 1-dipole has the effect of decreasing by one the number of i-residues, where i is the color of the edge which defines the 1-dipole, not changing the number of j-residues, for $j \neq i$. Thus by a succession of 1-dipole cancellation we obtain a 3-gem with 4 triballs. Such a 3-gem is said to be *contracted* and is called a *crystallization* for the associated 3-manifold. It is a fact that *any* closed 3-manifolds has a crystallization, and two closed 3-manifolds are related by a homeomorphic if and only if they are related by creation or contraction of 1- and 2- dipoles with the fusion rules; in this case, the two 3-manifolds are said to be *equivalent* or homeomorphic. Thus it is easy to understand that the fusion rules are the combinatorial equivalent of homeomorphisms. Let now discuss crystallization for generic colored $(n+1)$-graphs. The following results hold:

Theorem 1 For every PL n-manifold $\mathcal{M}$ there exist a crystallization.

Theorem 2 Two n-graphs $|\Gamma_1|$ and $|\Gamma_2|$ are crystallizations of the same manifold $\mathcal{M}$ if one is converted into the other by:
a) Adding or removing a non-degenerate m-dipole with $n-1 > m > 1$;
b) Adding a 1-dipole and deleting another 1-dipole.

A general theorem on the orientability of n-graphs holds:

Theorem 3 (Orientability) Let $(\Gamma, \gamma)_{n+1}$ be any crystallization of an n-manifold $\mathcal{M}$. Then $\mathcal{M}$ is orientable *iff* Γ is bipartite.

These theorems are fundamental in order to have a clear geometrical understanding of graphs generated by a colored group field theory, and will be used in the remainder of this paper. In particular we will use the theorem above to prove the orientability in the colored Boulatov model.

IV. ORIENTABILITY IN CGFT

In this section we prove a Lemma on the orientability of PL manifolds associated to graphs generated by the colored Boulatov model. Orientability of a manifold is a requirement if we want to construct a spin bundle. In 4-dimensions, for instance, the requirement to have a global spin bundle is to have a vanishing first and second Stiefel-Whitney class. While the second can be neglected by constructing local spin bundle and then gluing the charts, the vanishing of the first is a strict requirement and is equivalent to ask the orientability of the manifold[20]. Another important fact is that orientability restricts enormously the class of 3-manifolds which could be generated. As an example, in 2-dimensions the most general decomposition is given by connected sum of spheres, torii and projective planes. Orientability excludes the connected sum of projective planes, which allows the expansion in the ordinary genus we are used to.

Lemma (3-dimensions) Let Γ be a vacuum finite graph generated by the colored Boulatov model. Let $\mathcal{B}_{ijk}$ and $\mathcal{B}_{ij}$ be the set of 3- and 2- bubbles of Γ respectively. Then $|\Gamma|$ represents a closed and orientable 3-manifold iff

$$V + Card\{\mathcal{B}_{ijk}\} = Card\{\mathcal{B}_{ij}\} \tag{6}$$

Proof. This lemma follows directly from the properties of graphs generated by the colored Boulatov model and its interpretation, which is the same of the simplicial construction of *3-gepms*. By Lemma 1 the graph generated is a manifold if and only if the condition (6) is met. Since the graph is finite, the manifold is also closed. Thus what we have to show is that they are orientable. By the theorem on the orientability the 3-gem represents an orientable manifold if and only if the crystallization graph is bipartite. First we note that the graphs generated by colored group field theory are bipartite. Let A and B be the set of clockwise and anti-clockwise vertices of Γ respectively. Since by construction a clockwise vertex has to be contracted with an anticlockwise, then all the edges are between the set A and the set B and none is within the sets, thus the graph is bipartite. Now we have to show that its contraction is still bipartite. However, this fact is trivial because any of the moves in Fig. 3 keeps the bipartiteness of the original graph, thus in particular the fusion of a $1-$ *dipole*. Moreover, since the graph is finite, the crystallization is reached in a finite number of moves.

Part of this Lemma can be generalized to higher dimensions. The construction given in the third section of this note ensures that to each n-dimensional pseudo-complex there is at least a colored $(n+1)$-graph which is homehomorphic to it. It is then easy to see why colored group field theories generates only orientable pseudo-manifolds in any number of dimensions; we state it as a Lemma, even if it clearly follows from the construction given in [14] of n-edge-colored graphs in any number of dimensions, while orientability comes from a generalization to m-dipoles (as in Theorem 2) of the previous proof and the fact that there are two types of vertices:

Lemma (n-dimensions) At any finite order, the vacuum graphs of a colored group field theory are associated with closed and orientable PL pseudo-manifolds.

In the standard interpretation of group field theory as gluing faces of n-simplices this result certifies colored group field theories as the best candidates to obtain reasonable manifolds in the continuum limit.

V. CONCLUSIONS

In this short paper we have used results in the field of 3-gems to prove that all the graphs generated by the colored Boulatov model are related to orientable manifolds. In order to prove it we used new tools which could turn to be very useful in the context of group field theory, more specifically in the *colored* version of it. In fact, color is a fundamental ingredient in all we said. It should be said that what proved here is not an unexpected result[21]. The fact that an orientation for the faces can be chosen with ease was a hint of what proved here. Indeed, as far as the author is concerned, this is the first rigorous proof appeared so far. We should stress that orientability is a fundamental requirement for "reasonable" manifolds. As briefly explained at the beginning of section IV orientability is a necessary ingredient in order to have a spin bundle over a manifold. If the current interpretation of PL complexes associated to group field theory is correct, and since we expect GFT to be models of quantum gravity related to Loop Quantum Gravity, where a $SU(2)$ bundle is required to have spin-networks on the boundary of a covariant path integral, orientability of the low energy manifolds should be merely a consistency check.

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[1] L. Freidel, "Group field theory: An overview," Int. J. Theor. Phys. **44**, 1769 (2005) [arXiv:hep-th/0505016].

[2] D. Oriti, “The group field theory approach to quantum gravity: some recent results”, (2009) [arXiv:0912.2441 [hep-th]].
[3] E. Brezin, C. Itzykson, G. Parisi and J. B. Zuber “Planar Diagrams,” Commun. Math. Phys. **59**, 35 (1978).
[4] F. David, “A Model Of Random Surfaces With Nontrivial Critical Behavior,” Nucl. Phys. B **257**, 543 (1985).
[5] R. Gurau, “Lost in Translation: Topological Singularities in Group Field Theory,” Class. Quant. Grav. **27**, 235023 (2010) [arXiv:1006.0714 [hep-th]].
[6] R. Gurau, “Colored Group Field Theory,”, (2009) [arXiv:0907.2582 [hep-th]].
[7] R. Gurau, “Topological Graph Polynomials in Colored Group Field Theory,” Annales Henri Poincare **11**, 565 (2010) [arXiv:0911.1945 [hep-th]].
[8] J. B. Geloun, J. Magnen and V. Rivasseau, “Bosonic Colored Group Field Theory”, (2010) [arXiv:0911.1719 [hep-th]].
[9] J. B. Geloun, T. Krajewski, J. Magnen and V. Rivasseau, “Linearized Group Field Theory and Power Counting Theorems,” Class. Quant. Grav. **27**, 155012 (2010) [arXiv:1002.3592 [hep-th]].
[10] L. Freidel, R. Gurau and D. Oriti, “Group field theory renormalization - the 3d case: power counting of divergences,” Phys. Rev. D **80**, 044007 (2009) [arXiv:0905.3772 [hep-th]].
[11] J. Magnen, K. Noui, V. Rivasseau and M. Smerlak, “Scaling behaviour of three-dimensional group field theory,” Class. Quant. Grav. **26**, 185012 (2009) [arXiv:0906.5477 [hep-th]].
[12] V. Bonzom and M. Smerlak, Lett. Math. Phys. **93**, 295 (2010) [arXiv:1004.5196 [gr-qc]].
[13] R. Gurau, “The 1/N expansion of colored tensor models” [arXiv:1011.2726v2 [gr-qc]] (2010)
[14] M. Pezzana, “Sulla struttura topologica delle varietà compatte”, Atti Sem. Mat. Fis. Univ. Modena 23, 269-277, (1974)
[15] S. Lins, *Gems, Computers and Attractors for 3-Manifolds, (Series on Knots and Everything, Vol 5)* ISBN: 9810219075/ ISBN-13: 9789810219079
[16] D. V. Boulatov, “A Model of three-dimensional lattice gravity,” Mod. Phys. Lett. A **7**, 1629 (1992) [arXiv:hep-th/9202074].
[17] M. Ferri, C. Gagliardi “ Crystallisation moves,” Pacific Journal of Mathematics Vol. 100, No. 1, (1982)
[18] M. Ferri, C. Gagliardi, L. Grasselli, “A graph-theoretical representation of PL-manifolds - A survey on crystallizations”, Aequationes Mathematicae 31, 121-141 (1986)
[19] S. Lins, A. Mandel, “Graph-encoded 3-manifolds”, Discrete Mathematics 57, 261-284 (1985)
[20] M. Nakahara, *“Geometry, Topology and Physics”, 2ed., IoP, ISBN:0750306068, (2003)*
[21] D. Oriti,R. Gurau, private communications.