Decomposition of spinor groups by the involution σ' in exceptional Lie groups

Toshikazu Miyashita

Introduction

The compact exceptional Lie groups F_4, E_6, E_7 and E_8 have spinor groups as a subgroup as follows.

$$\begin{array}{l} F_4 \supset Spin(9) \supset Spin(8) \supset Spin(7) \supset \cdots \supset Spin(1) \ni 1 \\ \cap \\ E_6 \supset Spin(10) \\ \cap \\ E_7 \supset Spin(12) \supset Spin(11) \\ \cap \\ E_8 \supset Ss(16) \supset Spin(15) \supset Spin(14) \supset Spin(13) \end{array}$$

On the other hand, we know the involution σ' induced an element $\sigma' \in Spin(8) \subset F_4 \subset E_6 \subset E_7 \subset E_8$. Now, in this paper, we determine the group structures of $(Spin(n))^{\sigma'}$ which are the fixed subgroups by the involution σ'. Our results are as follows.

$$\begin{array}{ll} F_4 & (Spin(9))^{\sigma'} \cong Spin(8) \\ E_6 & (Spin(10))^{\sigma'} \cong (Spin(2) \times Spin(8))/\boldsymbol{Z}_2 \\ E_7 & (Spin(11))^{\sigma'} \cong (Spin(3) \times Spin(8))/\boldsymbol{Z}_2 \\ & (Spin(12))^{\sigma'} \cong (Spin(4) \times Spin(8))/\boldsymbol{Z}_2 \\ E_8 & (Spin(13))^{\sigma'} \cong (Spin(5) \times Spin(8))/\boldsymbol{Z}_2 \\ & (Spin(14))^{\sigma'} \cong (Spin(6) \times Spin(8))/\boldsymbol{Z}_2 \end{array}$$

Needless to say, the spinor groups appeared in the first term have relation

$$Spin(2) \subset Spin(3) \subset Spin(4) \subset Spin(5) \subset Spin(6).$$

One of our aims is to find these groups explicitly in the exceptional groups. In the group E_8, we conjecture that

$$(Spin(15))^{\sigma'} \cong (Spin(7) \times Spin(8))/\boldsymbol{Z}_2, \ (Ss(16))^{\sigma'} \cong (Spin(8) \times Spin(8))/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_2),$$

however, we can not realize explicitly.

This paper is closely in connection with the preceding papers [2],[3],[4] and may be a continuation of [2],[3],[4] in some sense.

1. Group F_4

We use the same notation as in [5] (however, some will be rewritten). For example,

the Cayley algebra $\mathfrak{C} = \boldsymbol{H} \oplus \boldsymbol{H}e_4$,

the exceptional Jordan algebra $\mathfrak{J} = \{X \in M(3, \mathfrak{C}) \,|\, X^* = X\}$, the Jordan multiplication $X \circ Y$, the inner product (X, Y) and the elements $E_1, E_2, E_3 \in \mathfrak{J}$,

the group $F_4 = \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}) \,|\, \alpha(X \circ Y) = \alpha X \circ \alpha Y\}$, and the element $\sigma \in F_4 : \sigma X = DXD, D = \mathrm{diag}(1, -1, -1), X \in \mathfrak{J}$ and the element $\sigma' \in F_4 : \sigma' X = D' X D', D' = \mathrm{diag}(-1, -1, 1), X \in \mathfrak{J}$,

the groups $SO(8) = SO(\mathfrak{C})$ and $Spin(8) = \{(\alpha_1, \alpha_2, \alpha_3) \in SO(8) \times SO(8) \times SO(8) \,|\, (\alpha_1 x)(\alpha_2 y) = \overline{\alpha_3 \overline{(xy)}}\}$.

Proposition 1.1. $(F_4)_{E_1} \cong Spin(9)$.

Proof. We define a 9 dimensional $\boldsymbol{R}$-vector space V^9 by

$$V^9 = \{X \in \mathfrak{J} \,|\, E_1 \circ X = 0, \mathrm{tr}(X) = 0\} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \overline{x} & -\xi \end{pmatrix} \,\middle|\, \xi \in \boldsymbol{R}, x \in \mathfrak{C} \right\}$$

with the norm $1/2(X, X) = \xi^2 + \overline{x}x$. Let $SO(9) = SO(V^9)$. Then, we have $(F_4)_{E_1}/\boldsymbol{Z}_2 \cong SO(9)$, $\boldsymbol{Z}_2 = \{1, \sigma\}$. Therefore, $(F_4)_{E_1}$ is isomorphic to $Spin(9)$ as a double covering group of $SO(9)$. (In details, see [5],[8].)

Now, we shall determine the group structure of $(Spin(9))^{\sigma'}$.

Theorem 1.2. $(Spin(9))^{\sigma'} \cong Spin(8)$.

Proof. Let $Spin(9) = (F_4)_{E_1}$. Then, the map $\varphi_1 : Spin(8) \to (Spin(9))^{\sigma'}$,

$$\varphi_1(\alpha_1, \alpha_2, \alpha_3)X = \begin{pmatrix} \xi_1 & \alpha_3 x_3 & \overline{\alpha_2 x_2} \\ \overline{\alpha_3 x_3} & \xi_2 & \alpha_1 x_1 \\ \alpha_2 x_2 & \overline{\alpha_1 x_1} & \xi_3 \end{pmatrix}, \quad X \in \mathfrak{J}$$

gives an isomorphism as groups. (In details, see [3].)

2. Group E_6

We use the same notation as in [5] (however, some will be rewritten). For example,

the complex exceptional Jordan algebra $\mathfrak{J}^C = \{X \in M(3, \mathfrak{C}^C) \,|\, X^* = X\}$, the Freudenthal multiplication $X \times Y$ and the Hermitian inner product $\langle X, Y \rangle$,

the group $E_6 = \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C) \,|\, \alpha X \times \alpha Y = \tau\alpha\tau(X \times Y), \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}$, and the natural inclusion $F_4 \subset E_6$, any element ϕ of the Lie algebra $\mathfrak{e}_6$ of the group E_6 is uniquely expressed as $\phi = \delta + i\widetilde{T}, \delta \in \mathfrak{f}_4, T \in \mathfrak{J}_0$, where $\mathfrak{J}_0 = \{T \in \mathfrak{J} \,|\, \mathrm{tr}(T) = 0\}$.

Proposition 2.1. $(E_6)_{E_1} \cong Spin(10)$.

Proof. We define a 10 dimensional $\boldsymbol{R}$-vector space V^{10} by

$$V^{10} = \{X \in \mathfrak{J}^C \,|\, 2E_1 \times X = -\tau X\} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \overline{x} & -\tau\xi \end{pmatrix} \,\middle|\, \xi \in C, x \in \mathfrak{C} \right\}$$

with the norm $1/2\langle X, X\rangle = (\tau\xi)\xi + \overline{x}x$. Let $SO(10) = SO(V^{10})$. Then, we have $(E_6)_{E_1}/\boldsymbol{Z}_2 \cong SO(10)$, $\boldsymbol{Z}_2 = \{1, \sigma\}$. Therefore, $(E_6)_{E_1}$ is isomorphic to $Spin(10)$ as a double covering group of $SO(10)$. (In details, see [5],[8].)

Lemma 2.2. *For $\nu \in Spin(2) = U(1) = \{\nu \in C \,|\, (\tau\nu)\nu = 1\}$, we define a C-linear transformation $\phi_1(\nu)$ of $\mathfrak{J}^C$ by*

$$\phi_1(\nu)X = \begin{pmatrix} \xi_1 & \nu x_3 & \nu^{-1}\overline{x}_2 \\ \nu\overline{x}_3 & \nu^2\xi_2 & x_1 \\ \nu^{-1}x_2 & \overline{x}_1 & \nu^{-2}\xi_3 \end{pmatrix}, \quad X \in \mathfrak{J}^C.$$

Then, $\phi_1(\nu) \in ((E_6)_{E_1})^{\sigma'}$.

Lemma 2.3. *Any element ϕ of the Lie algebra $((\mathfrak{e}_6)_{E_1})^{\sigma'}$ of the group $((E_6)_{E_1})^{\sigma'}$ is expressed by*

$$\phi = \delta + it(E_2 - E_3)^{\sim}, \quad \delta \in ((\mathfrak{f}_4)_{E_1})^{\sigma'} = \mathfrak{so}(8),\ t \in \boldsymbol{R}.$$

In particular, we have

$$\dim(((\mathfrak{e}_6)_{E_1})^{\sigma'}) = 28 + 1 = 29.$$

Now, we shall determine the group structure of $(Spin(10))^{\sigma'}$.

Theorem 2.4. $(Spin(10))^{\sigma'} \cong (Spin(2) \times Spin(8))/\boldsymbol{Z}_2, \boldsymbol{Z}_2 = \{(1,1), (-1,\sigma)\}$.

Proof. Let $Spin(10) = (E_6)_{E_1}$, $Spin(2) = U(1) \subset ((E_6)_{E_1})^{\sigma'}$(Lemma 2.2) and $Spin(8) = ((F_4)_{E_1})^{\sigma'} \subset ((E_6)_{E_1})^{\sigma'}$ (Theorem 1.2, Proposition 2.1). Now, we define a map $\varphi : Spin(2) \times Spin(8) \to (Spin(10))^{\sigma'}$ by

$$\varphi(\nu, \beta) = \phi_1(\nu)\beta.$$

Then, φ is well-defined : $\varphi(\nu, \beta) \in (Spin(10))^{\sigma'}$. Since $\phi_1(\nu)$ and β are commutative, φ is a homomorphism. $\mathrm{Ker}\,\varphi = \{(1,1), (-1,\sigma)\}$. Since $(Spin(10))^{\sigma'}$ is connected and $\dim(\mathfrak{spin}(2) \oplus \mathfrak{spin}(8)) = 1 + 28 = 29 = \dim((\mathfrak{spin}(10)^{\sigma'}))$ (Lemma 2.3), φ is onto. Thus, we have the isomorphism $(Spin(2) \times Spin(8))/\boldsymbol{Z}_2 \cong (Spin(10))^{\sigma'}$.

3. Group E_7

We use the same notation as in [6](however, some will be rewritten). For example,

the Freudenthal C-vector space $\mathfrak{P}^C = \mathfrak{J}^C \oplus \mathfrak{J}^C \oplus C \oplus C$, the Hermitian inner product $\langle P, Q\rangle$,

for $P, Q \in \mathfrak{P}^C$, the C-linear map $P \times Q : \mathfrak{P}^C \to \mathfrak{P}^C$,

the group $E_7 = \{\alpha \in \mathrm{Iso}_C(\mathfrak{P}^C) \,|\, \alpha(X \times Y)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q\rangle = \langle P, Q\rangle\}$, the natural inclusion $E_6 \subset E_7$ and elements $\sigma, \sigma' \in F_4 \subset E_6 \subset E_7$, $\lambda \in E_7$,

any element $\varPhi$ of the Lie algebra $\mathfrak{e}_7$ of the group E_7 is uniquely expressed as $\varPhi = \varPhi(\phi, A, -\tau A, \nu), \phi \in \mathfrak{e}_6, A \in \mathfrak{J}^C, \nu \in i\boldsymbol{R}$.

In the following, the group $((Spin(10))^{\sigma'})_{F_1(x)}$ is defined by

$$((Spin(10))^{\sigma'})_{F_1(x)} = \{\alpha \in (Spin(10))^{\sigma'} \,|\, \alpha F_1(x) = F_1(x) \;\; \textit{for all}\; x \in \mathfrak{C}\},$$

where $F_1(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \overline{x} & 0 \end{pmatrix} \in \mathfrak{J}$.

Proposition 3.1. $\quad ((Spin(10))^{\sigma'})_{F_1(x)} \cong Spin(2)$.

Proof. Let $Spin(10) = (E_6)_{E_1}$ and $Spin(2) = U(1) = \{\nu \in C \,|\, (\tau\nu)\nu = 1\}$. We consider the map $\phi_1 : Spin(2) \to ((Spin(10))^{\sigma'})_{F_1(x)}$ defined in Section 2. Then, ϕ_1 is well-defined : $\phi_1(\nu) \in ((Spin(10))^{\sigma'})_{F_1(x)}$. We shall show that ϕ_1 is onto. From $((Spin(10))^{\sigma'})_{F_1(x)} \subset (Spin(10))^{\sigma'}$, we see that for $\alpha \in ((Spin(10))^{\sigma'})_{F_1(x)}$, there exist $\nu \in Spin(2)$ and $\beta \in Spin(8)$ such that $\alpha = \varphi(\nu, \beta)$ (Theorem 2.4). Further, from $\alpha F_1(x) = F_1(x)$ and $\phi_1(\nu) F_1(x) = F_1(x)$, we have $\beta F_1(x) = F_1(x)$. Hence, $\beta = (1, 1, 1)$ or $(1, -1, -1) = \sigma$ by the principle of triality. Hence, $\alpha = \phi_1(\nu)$ or $\phi_1(\nu)\sigma$. However, in the latter case, from $\sigma = \phi_1(-1)$, we have $\alpha = \phi_1(\nu)\phi_1(-1) = \phi_1(-\nu)$. Therefore, ϕ_1 is onto. $\mathrm{Ker}\,\phi_1 = \{1\}$. Thus, we have the isomorphism $Spin(2) \cong ((Spin(10))^{\sigma'})_{F_1(x)}$.

We define C-linear maps $\kappa, \mu : \mathfrak{P}^C \to \mathfrak{P}^C$ respectively by

$$\begin{aligned} &\kappa(X, Y, \xi, \eta) = (-\kappa_1 X, \kappa_1 Y, -\xi, \eta), \;\; \kappa_1 X = (E_1, X)E_1 - 4E_1 \times (E_1 \times X), \\ &\mu(X, Y, \xi, \eta) = (2E_1 \times Y + \eta E_1, 2E_1 \times X + \xi E_1, (E_1, Y), (E_1, X)). \end{aligned}$$

Their explicit forms are

$$\kappa(X, Y, \xi, \eta) = \Big(\begin{pmatrix} -\xi_1 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \overline{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & -\eta_2 & -y_1 \\ 0 & -\overline{y}_1 & -\eta_3 \end{pmatrix}, -\xi, \eta \Big),$$

$$\mu(X, Y, \xi, \eta) = \Big(\begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta_3 & -y_1 \\ 0 & -\overline{y}_1 & \eta_2 \end{pmatrix}, \begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi_3 & -x_1 \\ 0 & -\overline{x}_1 & \xi_2 \end{pmatrix}, \eta_1, \xi_1 \Big).$$

We define subgroups $(E_7)^{\kappa,\mu}$, $((E_7)^{\kappa,\mu})_{(0,E_1,0,1)}$ and $((E_7)^{\kappa,\mu})_{(0,E_1,0,1),(0,-E_1,0,1)}$ of E_7 by

$$(E_7)^{\kappa,\mu} = \{\alpha \in E_7 \,|\, \kappa\alpha = \alpha\kappa, \mu\alpha = \alpha\mu\},$$

$$((E_7)^{\kappa,\mu})_{(0,E_1,0,1)} = \{\alpha \in (E_7)^{\kappa,\mu} \,|\, \alpha(0, E_1, 0, 1) = (0, E_1, 0, 1)\},$$

$$((E_7)^{\kappa,\mu})_{(0,E_1,0,1),(0,-E_1,0,1)} = \left\{\alpha \in (E_7)^{\kappa,\mu} \;\middle|\; \begin{array}{l} \alpha(0, E_1, 0, 1) = (0, E_1, 0, 1) \\ \alpha(0, -E_1, 0, 1) = (0, -E_1, 0, 1) \end{array} \right\},$$

and also define subgroups $((E_7)^{\kappa,\mu})_{(E_1,0,1,0)}$ and $((E_7)^{\kappa,\mu})_{(E_1,0,1,0),(E_1,0,-1,0)}$ of E_7 by

$$((E_7)^{\kappa,\mu})_{(E_1,0,1,0)} = \{\alpha \in (E_7)^{\kappa,\mu} \,|\, \alpha(E_1, 0, 1, 0) = (E_1, 0, 1, 0)\},$$

$$((E_7)^{\kappa,\mu})_{(E_1,0,1,0),(E_1,0,-1,0)} = \left\{\alpha \in (E_7)^{\kappa,\mu} \;\middle|\; \begin{array}{l} \alpha(E_1, 0, 1, 0) = (E_1, 0, 1, 0) \\ \alpha(E_1, 0, -1, 0) = (E_1, 0, -1, 0) \end{array} \right\}.$$

Proposition 3.2. (1) $((E_7)^{\kappa,\mu})_{(E_1,0,1,0)} = ((E_7)^{\kappa,\mu})_{(0,E_1,0,1)}$.

(2) $((E_7)^{\kappa,\mu})_{(E_1,0,1,0),(E_1,0,-1,0)} = ((E_7)^{\kappa,\mu})_{(0,E_1,0,1),(0,-E_1,0,1)}$.

Proof. (1) For $\alpha \in ((E_7)^{\kappa,\mu})_{(E_1,0,1,0)}$, we have $\alpha(0, E_1, 0, 1) = \alpha\mu(E_1, 0, 1, 0) = \mu\alpha(E_1, 0, 1, 0) = \mu(E_1, 0, 1, 0) = (0, E_1, 0, 1)$. Hence, $\alpha \in ((E_7)^{\kappa,\mu})_{(0,E_1,0,1)}$. The converse is also proved.

(2) It is proved in a way similar to (1).

Proposition 3.3. $\quad ((E_7)^{\kappa,\mu})_{(0,E_1,0,1),(0,-E_1,0,1)} \cong Spin(10)$.

Proof. If $\alpha \in E_7$ satisfies $\alpha(0, E_1, 0, 1) = (0, E_1, 0, 1)$ and $\alpha(0, -E_1, 0, 1) = (0, -E_1, 0, 1)$, then we have $\alpha(0, 0, 0, 1) = (0, 0, 0, 1)$ and $\alpha(0, E_1, 0, 0) = (0, E_1, 0, 0)$. From the first condition, we see that $\alpha \in E_6$. Moreover, from the second condition, we have $\alpha \in (E_6)_{E_1} = Spin(10)$. The proof of the converse is trivial because κ, μ are defined by using E_1.

Proposition 3.4. $\quad ((E_7)^{\kappa,\mu})_{(0,E_1,0,1)} \cong Spin(11)$.

Proof. We define an 11 dimensional $\boldsymbol{R}$-vector space V^{11} by

$$\begin{aligned} V^{11} &= \{P \in \mathfrak{P}^C \,|\, \kappa P = P, \mu\tau\lambda P = P, P \times (0, E_1, 0, 1) = 0\} \\ &= \left\{\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \overline{x} & -\tau\xi \end{pmatrix}, \begin{pmatrix} \eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\eta \right) \middle|\, x \in \mathfrak{C}, \xi \in C, \eta \in i\boldsymbol{R} \right\} \end{aligned}$$

with the norm

$$(P, P)_\mu = \frac{1}{2}(\mu P, \lambda P) = (\tau\eta)\eta + \overline{x}x + (\tau\xi)\xi.$$

Let $SO(11) = SO(V^{11})$. Then, we have $((E_7)^{\kappa,\mu})_{(0,E_1,0,1)}/\boldsymbol{Z}_2 \cong SO(11), \boldsymbol{Z}_2 = \{1, \sigma\}$. Therefore, $((E_7)^{\kappa,\mu})_{(0,E_1,0,1)}$ is isomorphic to $Spin(11)$ as a double covering group of $SO(11)$. (In details, see [6],[8].)

Now, we shall consider the following group

$$
\begin{aligned}
&((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)} \\
&\quad = \Big\{\alpha \in (Spin(11))^{\sigma'} \;\Big|\; \begin{array}{l} \alpha(0,F_1(y),0,0) \\ = (0,F_1(y),0,0) \end{array} \text{ for all } y \in \mathfrak{C}\Big\}.
\end{aligned}
$$

Lemma 3.5. *The Lie algebra* $((\mathfrak{spin}(11))^{\sigma'})_{(0,F_1(y),0,0)}$ *of the group* $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ *is given by*

$$
\begin{aligned}
&((\mathfrak{spin}(11))^{\sigma'})_{(0,F_1(y),0,0)} \\
&= \Big\{\Phi\Big(i\begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & -\epsilon \end{pmatrix}^{\sim}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \tau\rho \end{pmatrix}, -\tau\begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \tau\rho \end{pmatrix}, 0\Big) \;\Big|\; \epsilon \in \boldsymbol{R}, \rho \in C\Big\}.
\end{aligned}
$$

In particular, we have

$$
\dim(((\mathfrak{spin}(11))^{\sigma'})_{(0,F_1(y),0,0)}) = 3.
$$

Lemma 3.6. *For* $a \in \boldsymbol{R}$*, the maps* $\alpha_k(a) : \mathfrak{P}^C \to \mathfrak{P}^C, k = 1,2,3$ *defined by*

$$
\alpha_k(a)\begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} (1+(\cos a - 1)p_k)X - 2(\sin a)E_k \times Y + \eta(\sin a)E_k \\ 2(\sin a)E_k \times X + (1+(\cos a-1)p_k)Y - \xi(\sin a)E_k \\ ((\sin a)E_k, Y) + (\cos a)\xi \\ (-(\sin a)E_k, X) + (\cos a)\eta \end{pmatrix}
$$

belong to the group E_7*, where* $p_k : \mathfrak{J}^C \to \mathfrak{J}^C$ *is defined by*

$$
p_k(X) = (X, E_k)E_k + 4E_k \times (E_k \times X), \quad X \in \mathfrak{J}^C.
$$

$\alpha_1(a), \alpha_2(b), \alpha_3(c)$ $(a,b,c \in \boldsymbol{R})$ *commute with each other.*

Proof. For $\Phi_k(a) = \Phi(0, aE_k, -aE_k, 0) \in \mathfrak{e}_7$, we have $\alpha_k(a) = \exp\Phi_k(a) \in E_7$. Since $[\Phi_k(a), \Phi_l(b)] = 0, k \neq l, \alpha_k(a)$ and $\alpha_l(b)$ are commutative.

Lemma 3.7. $\qquad ((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}/Spin(2) \simeq S^2.$
In particular, $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ *is connected.*

Proof. We define a 3 dimensional $\boldsymbol{R}$-vector space W^3 by

$$
\begin{aligned}
W^3 &= \{P \in \mathfrak{P}^C \,|\, \kappa P = -P, \mu\tau\lambda P = -P, \sigma' P = P, P \times (E_1, 0, 1, 0) = 0\} \\
&= \Big\{P = \Big(\begin{pmatrix} i\xi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & -\tau\eta \end{pmatrix}, -i\xi, 0\Big) \Big| \xi \in \boldsymbol{R}, \eta \in C\Big\}
\end{aligned}
$$

with the norm

$$
(P,P)_\mu = -\frac{1}{2}(\mu P, \lambda P) = \xi^2 + (\tau\eta)\eta.
$$

Then, $S^2 = \{P \in W^3 \,|\, (P,P)_\mu = 1\}$ is a 2 dimensional sphere. The group $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ acts on S^2. We shall show that this action is transitive. To show this, it is sufficient to show that any element $P \in S^2$ can be transformed to $(-iE_1, 0, i, 0) \in S^2$ under the action of $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$. Now, for a given

$$P = \Big(\begin{pmatrix} i\xi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & -\tau\eta \end{pmatrix}, -i\xi, 0 \Big) \in S^2,$$

choose $a \in \boldsymbol{R}, 0 \le a < \pi/2$ such that $\tan 2a = -\dfrac{2i\xi}{\tau\eta - \eta}$ (if $\tau\eta - \eta = 0$, then let $a = \pi/4$). Operate $\alpha_{23}(a) := \alpha_2(a)\alpha_3(a) = \exp(\varPhi(0, a(E_2 + E_3), -a(E_2 + E_3), 0))$ $\in ((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$(Lemmas 3.5, 3.6) on P. Then, we have the ξ-term of $\alpha_{23}(a)P$ is $-((\cos 2a)(i\xi) + 1/2(\sin 2a)(\tau\eta - \eta)) = 0$. Hence,

$$\alpha_{23}(a)P = \Big(0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & -\tau\zeta \end{pmatrix}, 0, 0\Big) = P_1, \quad \zeta \in C,\ (\tau\zeta)\zeta = 1.$$

From $(\tau\zeta)\zeta = 1, \zeta \in C$, we can put $\zeta = e^{i\theta}, 0 \le \theta < 2\pi$. Let $\nu = e^{-i\theta/2}$, and operate $\phi_1(\nu) \in ((Spin(10))^{\sigma'})_{F_1(x)}$ (Lemma 2.2) ($\subset ((Spin(11)^{\sigma'})_{(0,F_1(x),0,0)})$ on P_1. Then,

$$\phi_1(\nu)P_1 = (0, E_2 - E_3, 0, 0) = P_2.$$

Moreover, operate $\phi_1(e^{i\pi/4})$ on P_2,

$$\phi_1(e^{i\pi/4})P_2 = (0, i(E_2 + E_3), 0, 0) = P_3.$$

Operate again $\alpha_{23}(\pi/4)$ on P_3. Then, we have

$$\alpha_{23}(\pi/4)P_3 = (-iE_1, 0, i, 0).$$

This shows the transitivity. The isotropy subgroup of $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ at $(-iE_1, 0, i, 0)$ is $((Spin(10))^{\sigma'})_{F_1(y)}$ (Propositions 3.2(2), 3.3, 3.4) $= Spin(2)$. Thus, we have the homeomorphism $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}/Spin(2) \simeq S^2$.

Proposition 3.8. $\quad ((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)} \cong Spin(3)$.

Proof. Since $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ is connected (Lemma 3.7), we can define a homomorphism $\pi : ((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)} \to SO(3) = SO(W^3)$ by

$$\pi(\alpha) = \alpha | W^3.$$

$\mathrm{Ker}\,\pi = \{1, \sigma\} = \boldsymbol{Z}_2$. Since $\dim(((\mathfrak{spin}(11))^{\sigma'})_{(0,F_1(y),0,0)}) = 3$ (Lemma 3.5) $= \dim(\mathfrak{so}(3))$, π is onto. Hence, $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}/\boldsymbol{Z}_2 \cong SO(3)$. Therefore, $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ is isomorphic to $Spin(3)$ as a double covering group of $SO(3)$.

Lemma 3.9. *The Lie algebra* $(\mathfrak{spin}(11))^{\sigma'}$ *of the group* $(Spin(11))^{\sigma'}$ *is given by*

$$\begin{aligned}
&(\mathfrak{spin}(11))^{\sigma'} \\
&\quad = \Big\{ \varPhi\Big(D + i \begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & -\epsilon \end{pmatrix}^{\sim}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \tau\rho \end{pmatrix}, -\tau \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \tau\rho \end{pmatrix}, 0 \Big) \\
&\qquad \Big| \, D \in \mathfrak{so}(8), \epsilon \in \boldsymbol{R}, \rho \in C \Big\}.
\end{aligned}$$

In particular, we have

$$\dim((\mathfrak{spin}(11))^{\sigma'}) = 28 + 3 = 31.$$

Now, we shall determine the group structure of $(Spin(11))^{\sigma'}$.

Theorem 3.10. $(Spin(11))^{\sigma'} \cong (Spin(3) \times Spin(8))/\boldsymbol{Z}_2, \boldsymbol{Z}_2 = \{(1,1), (-1,\sigma)\}$.

Proof. Let $Spin(11) = ((E_7)^{\kappa,\mu})_{(0,E_1,0,1)}$, $Spin(3) = ((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ and $Spin(8) = ((F_4)_{E_1})^{\sigma'} \subset ((E_6)_{E_1})^{\sigma'} = (((E_7)^{\kappa,\mu})_{(E_1,0,1,0),(E_1,0,-1,0)})^{\sigma'} \subset (((E_7)^{\kappa,\mu})_{(E_1,0,1,0)})^{\sigma'}$ (Theorem 1.2, Propositions 3.2, 3.3, 3.4). Now, we define a map $\varphi : Spin(3) \times Spin(8) \to (Spin(11))^{\sigma'}$ by

$$\varphi(\alpha, \beta) = \alpha\beta.$$

Then, φ is well-defined : $\varphi(\alpha,\beta) \in (Spin(11))^{\sigma'}$. Since $[\varPhi_D, \varPhi_3] = 0$ for $\varPhi_D = \varPhi(D,0,0,0) \in \mathfrak{spin}(8)$, $\varPhi_3 \in \mathfrak{spin}(3) = ((\mathfrak{spin}(11))^{\sigma'})_{(0,F_1(y),0,0)}$ (Proposition 3.8), we have $\alpha\beta = \beta\alpha$. Hence, φ is a homomorphism. $\operatorname{Ker}\varphi = \{(1,1),(-1,\sigma)\} = \boldsymbol{Z}_2$. Since $(Spin(11))^{\sigma'}$ is connected and $\dim(\mathfrak{spin}(3) \oplus \mathfrak{spin}(8)) = 3$(Lemma 3.5) $+ 28 = 31 = \dim((\mathfrak{spin}(11))^{\sigma'})$ (Lemma 3.9), φ is onto. Thus, we have the isomorphism $(Spin(3) \times Spin(8))/\boldsymbol{Z}_2 \cong (Spin(11))^{\sigma'}$.

Proposition 3.11. $(E_7)^{\kappa,\mu} \cong Spin(12)$.

Proof. We define a 12 dimensional $\boldsymbol{R}$-vector space V^{12} by

$$\begin{aligned}
V^{12} &= \{ P \in \mathfrak{P}^C \,|\, \kappa P = P, \mu\tau\lambda P = P \} \\
&= \Big\{ \Big(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \overline{x} & -\tau\xi \end{pmatrix}, \begin{pmatrix} \eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\eta \Big) \Big| \, x \in \mathfrak{C}, \xi, \eta \in C \Big\}
\end{aligned}$$

with the norm

$$(P,P)_\mu = \frac{1}{2}(\mu P, \lambda P) = (\tau\eta)\eta + \overline{x}x + (\tau\xi)\xi.$$

Let $SO(12) = SO(V^{12})$. Then, we have $(E_7)^{\kappa,\mu}/\boldsymbol{Z}_2 \cong SO(12)$, $\boldsymbol{Z}_2 = \{1, \sigma\}$. Therefore, $(E_7)^{\kappa,\mu}$ is isomorphic to $Spin(12)$ as a double covering group of $SO(12)$. (In details, see [6],[8].)

Now, we shall consider the following group

$$((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)} = \left\{\alpha \in (Spin(12))^{\sigma'} \;\middle|\; \begin{array}{l}\alpha(0,F_1(y),0,0)\\ = (0,F_1(y),0,0)\end{array} \text{ for all } y \in \mathfrak{C}\right\}.$$

Lemma 3.12. *The Lie algebra* $((\mathfrak{spin}(12))^{\sigma'})_{(0,F_1(y),0,0)}$ *of the group* $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$ *is given by*

$$((\mathfrak{spin}(12))^{\sigma'})_{(0,F_1(y),0,0)} = \left\{\Phi\Big(i\begin{pmatrix}\epsilon_1 & 0 & 0\\ 0 & \epsilon_2 & 0\\ 0 & 0 & \epsilon_3\end{pmatrix}^{\sim}, \begin{pmatrix}0 & 0 & 0\\ 0 & \rho_2 & 0\\ 0 & 0 & \rho_3\end{pmatrix}, -\tau\begin{pmatrix}0 & 0 & 0\\ 0 & \rho_2 & 0\\ 0 & 0 & \rho_3\end{pmatrix}, -\frac{3}{2}i\epsilon_1\Big) \;\middle|\; \epsilon_i \in \boldsymbol{R}, \epsilon_1+\epsilon_2+\epsilon_3 = 0, \rho_i \in C\right\}.$$

In particular, we have

$$\dim(((\mathfrak{spin}(12))^{\sigma'})_{(0,F_1(y),0,0)}) = 6.$$

Lemma 3.13. *For* $t \in \boldsymbol{R}$*, the map* $\alpha(t) : \mathfrak{P}^C \to \mathfrak{P}^C$ *defined by*

$$\alpha(t)(X,Y,\xi,\eta) = \left(\begin{pmatrix}e^{2it}\xi_1 & e^{it}x_3 & e^{it}\overline{x}_2\\ e^{it}\overline{x}_3 & \xi_2 & x_1\\ e^{it}x_2 & \overline{x}_1 & \xi_3\end{pmatrix}, \begin{pmatrix}e^{-2it}\eta_1 & e^{-it}y_3 & e^{-it}\overline{y}_2\\ e^{-it}\overline{y}_3 & \eta_2 & y_1\\ e^{-it}y_2 & \overline{y}_1 & \eta_3\end{pmatrix}, e^{-2it}\xi, e^{2it}\eta\right)$$

belongs to the group $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$.

Proof. For $\Phi = \Phi(2itE_1 \vee E_1, 0, 0, -2it) \in ((\mathfrak{spin}(12))^{\sigma'})_{(0,F_1(y),0,0)}$ (Lemma 3.12), we have $\alpha(t) = \exp\Phi \in ((Spin(12)^{\sigma'})_{(0,F_1(y),0,0)}$.

Lemma 3.14. $$((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}/Spin(3) \simeq S^3.$$
In particular, $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$ *is connected.*

Proof. We define a 4 dimensional $\boldsymbol{R}$-vector space W^4 by

$$\begin{aligned} W^4 &= \{P \in \mathfrak{P}^C \,|\, \kappa P = -P, \mu\tau\lambda P = -P, \sigma' P = P\}\\ &= \left\{P = \left(\begin{pmatrix}\xi & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\ 0 & \eta & 0\\ 0 & 0 & -\tau\eta\end{pmatrix}, \tau\xi, 0\right) \;\middle|\; \xi, \eta \in C\right\}\end{aligned}$$

with the norm
$$(P,P)_\mu = -\frac{1}{2}(\mu P, \lambda P) = (\tau\xi)\xi + (\tau\eta)\eta.$$
Then, $S^3 = \{P \in W^4 \,|\, (P,P)_\mu = 1\}$ is a 3 dimensional sphere. The group $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$ acts on S^3. We shall show that this action is transitive. To

show this, it is sufficient to show that any element $P \in S^3$ can be transformed to $(E_1, 0, 1, 0) \in S^3$ under the action of $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$. Now, for a given

$$P = \left(\begin{pmatrix} \xi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & -\tau\eta \end{pmatrix}, \tau\xi, 0 \right) \in S^3,$$

choose $t \in \boldsymbol{R}$ such that $e^{2it}\xi \in i\boldsymbol{R}$. Operate $\alpha(t)$ (Lemma 3.13) on P. Then, we have

$$\alpha(t)P = P_1 \in S^2 \subset S^3.$$

Now, since $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ $(\subset ((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)})$ acts transitively on S^2 (Lemma 3.7), there exists $\beta \in ((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ such that

$$\beta P_1 = (-iE_1, 0, i, 0) = P_2.$$

Operate again $\alpha(\pi/4)$ on P_2. Then, we have

$$\alpha(\pi/4)P_2 = (E_1, 0, 1, 0).$$

This shows the transitivity. The isotropy subgroup of $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$ at $(E_1, 0, 1, 0)$ is $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ (Propositions 3.2(1), 3.4, 3.11) $= Spin(3)$. Thus, we have the homeomorphism $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}/Spin(3) \simeq S^3$.

Proposition 3.15. $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)} \cong Spin(4)$.

Proof. Since $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$ is connected (Lemma 3.14), we can define a homomorphism $\pi : ((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)} \to SO(4) = SO(W^4)$ by

$$\pi(\alpha) = \alpha | W^4.$$

$\mathrm{Ker}\,\pi = \{1, \sigma\} = \boldsymbol{Z}_2$. Since $\dim((\mathfrak{spin}(12))^{\sigma'})_{(0,F_1(y),0,0)}) = 6$ (Lemma 3.12) $= \dim(\mathfrak{so}(4))$, π is onto. Hence, $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}/\boldsymbol{Z}_2 \cong SO(4)$. Therefore, $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$ is isomorphic to $Spin(4)$ as a double covering group of $SO(4)$.

Lemma 3.16. *The Lie algebra $(\mathfrak{spin}(12))^{\sigma'}$ of the group $(Spin(12))^{\sigma'}$ is given by*

$$(\mathfrak{spin}(12))^{\sigma'}$$
$$= \left\{ \varPhi\left(D + i\begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}^{\sim}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, -\tau\begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, -i\frac{3}{2}\epsilon_1 \right) \right.$$
$$\left. \middle| \; D \in \mathfrak{so}(8), \epsilon_i \in \boldsymbol{R}, \epsilon_1 + \epsilon_2 + \epsilon_3 = 0, \rho_i \in C \right\}.$$

In particular, we have

$$\dim((\mathfrak{spin}(12))^{\sigma'}) = 28 + 6 = 34.$$

Now, we shall determine the group structure of $(Spin(12))^{\sigma'}$.

Theorem 3.17. $(Spin(12))^{\sigma'} \cong (Spin(4) \times Spin(8))/\boldsymbol{Z}_2, \boldsymbol{Z}_2 = \{(1,1), (-1,\sigma)\}$.

Proof. Let $Spin(12) = (E_7)^{\kappa,\mu}$, $Spin(4) = ((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$ and $Spin(8) = ((F_4)_{E_1})^{\sigma'} \subset ((E_6)_{E_1})^{\sigma'} = (((E_7)^{\kappa,\mu})_{(E_1,0,1,0),(E_1,0,-1,0)})^{\sigma'} \subset ((E_7)^{\kappa,\mu})^{\sigma'}$(Theorem 1.2, Propositions 3.2, 3.3, 3.11, 3.15). Now, we define a map $\varphi : Spin(4) \times Spin(8) \to (Spin(12))^{\sigma'}$ by

$$\varphi(\alpha, \beta) = \alpha\beta.$$

Then, φ is well-defined : $\varphi(\alpha,\beta) \in (Spin(12))^{\sigma'}$. Since $[\Phi_D, \Phi_4] = 0$ for $\Phi_D = \Phi(D,0,0,0) \in \mathfrak{spin}(8)$, $\Phi_4 \in \mathfrak{spin}(4) = ((\mathfrak{spin}(12))^{\sigma'})_{(0,F_1(y),0,0)}$ (Proposition 3.15), we have $\alpha\beta = \beta\alpha$. Hence, φ is a homomorphism. $\mathrm{Ker}\,\varphi = \{(1,1), (-1,\sigma)\} = \boldsymbol{Z}_2$. Since $(Spin(12))^{\sigma'}$ is connected and $\dim(\mathfrak{spin}(4) \oplus \mathfrak{spin}(8)) = 6(\text{Lemma 3.12}) + 28 = 34 = \dim((\mathfrak{spin}(12))^{\sigma'})$ (Lemma 3.16), φ is onto. Thus, we have the isomorphism $(Spin(4) \times Spin(8))/\boldsymbol{Z}_2 \cong (Spin(12))^{\sigma'}$.

4. Group E_8

We use the same notation as in [2],[4](however, some will rewritten). For example,

C-Lie algebra ${\mathfrak{e}_8}^C = {\mathfrak{e}_7}^C \oplus \mathfrak{P}^C \oplus \mathfrak{P}^C \oplus C \oplus C \oplus C$ and C-linear transformations $\lambda, \widetilde{\lambda}$ of ${\mathfrak{e}_8}^C$,

the groups ${E_8}^C = \{\alpha \in \mathrm{Iso}_C({\mathfrak{e}_8}^C) \,|\, \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2]\}$ and $E_8 = ({E_8}^C)^{\tau\widetilde{\lambda}} = \{\alpha \in {E_8}^C \,|\, \tau\widetilde{\lambda}\alpha = \alpha\tau\widetilde{\lambda}\}$.

For $\alpha \in E_7$, the map $\widetilde{\alpha} : {\mathfrak{e}_8}^C \to {\mathfrak{e}_8}^C$ is defined by

$$\widetilde{\alpha}(\Phi, P, Q, r, u, v) = (\alpha\Phi\alpha^{-1}, \alpha P, \alpha Q, r, u, v).$$

Then, $\widetilde{\alpha} \in E_8$ and we identify α with $\widetilde{\alpha}$. The group E_8 contains E_7 as a subgroup by

$$E_7 = \{\widetilde{\alpha} \in E_8 \,|\, \alpha \in E_7\} = (E_8)_{(0,0,0,0,1,0)}.$$

We define a C-linear map $\widetilde{\kappa} : {\mathfrak{e}_8}^C \to {\mathfrak{e}_8}^C$ by

$$\widetilde{\kappa} = \mathrm{ad}(\kappa, 0, 0, -1, 0, 0) = \mathrm{ad}(\Phi(-2E_1 \vee E_1, 0, 0, -1), 0, 0, -1, 0, 0),$$

and 14 dimensional C-vector spaces $\mathfrak{g}_{-2}$ and $\mathfrak{g}_2$ by

$$\begin{aligned}
\mathfrak{g}_{-2} &= \{R \in {\mathfrak{e}_8}^C \,|\, \widetilde{\kappa}R = -2R\}, \\
&= \{(\Phi(0, \zeta E_1, 0, 0), (\xi_1 E_1, \eta_2 E_2 + \eta_3 E_3 + F_1(y), \xi, 0), 0, 0, u, 0) \\
&\qquad |\, \zeta, \xi_1, \eta_i, \xi, u \in C, y \in \mathfrak{C}^C\}, \\
\mathfrak{g}_2 &= \{R \in {\mathfrak{e}_8}^C \,|\, \widetilde{\kappa}R = 2R\}, \\
&= \{(\Phi(0, 0, \zeta E_1, 0), 0, (\xi_2 E_1 + \xi_3 E_3 + F_1(x), \eta_1 E_1, 0, \eta), 0, 0, v) \\
&\qquad |\, \zeta, \xi_i, \eta_1, \eta, v \in C, x \in \mathfrak{C}^C\}.
\end{aligned}$$

Further, we define two C-linear maps $\widetilde{\mu}_1 : {\mathfrak{e}_8}^C \to {\mathfrak{e}_8}^C$ and $\delta : \mathfrak{g}_2 \to \mathfrak{g}_2$ by

$$\widetilde{\mu}_1(\varPhi, P, Q, r, u, v) = (\mu_1 \varPhi {\mu_1}^{-1}, i\mu_1 Q, i\mu_1 P, -r, v, u),$$

where

$$\mu_1(X, Y, \xi, \eta) = \Bigl(\begin{pmatrix} i\eta & x_3 & \overline{x}_2 \\ \overline{x}_3 & i\eta_3 & -iy_1 \\ x_2 & -i\overline{y}_1 & i\eta_2 \end{pmatrix}, \begin{pmatrix} i\xi & y_3 & \overline{y}_2 \\ \overline{y}_3 & i\xi_3 & -ix_1 \\ y_2 & -i\overline{x}_1 & i\xi_2 \end{pmatrix}, i\eta_1, i\xi_1 \Bigr),$$

and

$$\begin{aligned} &\delta(\varPhi(0, 0, \zeta E_1, 0), 0, (\xi_2 E_2 + \xi_3 E_3 + F_1(x), \eta_1 E_1, 0, \eta), 0, 0, v) \\ &\quad = (\varPhi(0, 0, -vE_1, 0), 0, (\xi_2 E_2 + \xi_3 E_3 + F_1(x), \eta_1 E_1, 0, \eta), 0, 0, -\zeta). \end{aligned}$$

In particular, the explicit form of the map $\widetilde{\mu}_1 : \mathfrak{g}_{-2} \to \mathfrak{g}_2$ is given by

$$\begin{aligned} &\widetilde{\mu}_1(\varPhi(0, \zeta E_1, 0, 0), (\xi_1 E_1, \eta_2 E_2 + \eta_3 E_3 + F_1(y), \xi, 0), 0, 0, u, 0) \\ &\quad = (\varPhi(0, 0, \zeta E_1, 0), 0, (-\eta_3 E_2 - \eta_2 E_3 + F_1(y), -\xi E_1, 0, -\xi_1), 0, 0, u). \end{aligned}$$

The composition map $\delta\widetilde{\mu}_1 : \mathfrak{g}_{-2} \to \mathfrak{g}_2$ of $\widetilde{\mu}_1$ and $\delta\widetilde{\mu}_1$ is denoted by $\widetilde{\mu}_\delta$:

$$\begin{aligned} &\widetilde{\mu}_\delta(\varPhi(0, \zeta E_1, 0, 0), (\xi_1 E_1, \eta_2 E_2 + \eta_3 E_3 + F_1(y), \xi, 0), 0, 0, u, 0) \\ &\quad = (\varPhi(0, 0, -uE_1, 0), 0, (-\eta_3 E_2 - \eta_2 E_3 + F_1(y), -\xi E_1, 0, -\xi_1), 0, 0, -\zeta). \end{aligned}$$

Now, we define the inner product $(R_1, R_2)_\mu$ in $\mathfrak{g}_{-2}$ by

$$(R_1, R_2)_\mu = \frac{1}{30} B_8(\widetilde{\mu}_\delta R_1, R_2),$$

where B_8 is the Killing form of ${\mathfrak{e}_8}^C$. The explicit form of $(R, R)_\mu$ is given by

$$(R, R)_\mu = -4\zeta u - \eta_2\eta_3 + \overline{y}y + \xi_1\xi,$$

for $R = (\varPhi(0, \zeta E_1, 0, 0), (\xi_1 E_1, \eta_2 E_2 + \eta_3 E_3 + F_1(y), \xi, 0), 0, 0, u, 0) \in \mathfrak{g}_{-2}$. Hereafter, we use the notation $(V^C)^{14}$ instead of $\mathfrak{g}_{-2}$.

We define $\boldsymbol{R}$-vector spaces V^{14}, V^{13} and $(V')^{12}$ respectively by

$$\begin{aligned} V^{14} &= \{R \in (V^C)^{14} \,|\, \widetilde{\mu}_\delta \tau \widetilde{\lambda} R = -R\} \\ &= \{R = (\varPhi(0, \zeta E_1, 0, 0), (\xi E_1, \eta E_2 - \tau\eta E_3 + F_1(y), \tau\xi, 0), 0, 0, -\tau\zeta, 0) \\ &\qquad | \, \zeta, \xi, \eta \in C, y \in \mathfrak{C}\} \end{aligned}$$

with the norm

$$(R, R)_\mu = \frac{1}{30} B_8(\widetilde{\mu}_\delta R, R) = 4(\tau\zeta)\zeta + (\tau\eta)\eta + \overline{y}y + (\tau\xi)\xi,$$

$$
\begin{aligned}
V^{13} &= \{R \in V^{14} \,|\, (R, (\varPhi_1, 0, 0, 0, 1, 0))_\mu = 0\} \\
&= \{R = (\varPhi(0, \zeta E_1, 0, 0), (\xi E_1, \eta E_2 - \tau\eta E_3 + F_1(y), \tau\xi, 0), 0, 0, -\zeta, 0) \\
&\qquad |\, \zeta \in \boldsymbol{R}, \xi, \eta \in C, y \in \mathfrak{C}\}
\end{aligned}
$$

with the norm

$$
(R, R)_\mu = \frac{1}{30} B_8(\widetilde{\mu}_\delta R, R) = 4\zeta^2 + (\tau\eta)\eta + \overline{y}y + (\tau\xi)\xi,
$$

$$
\begin{aligned}
(V')^{12} &= \{R \in V^{13} \,|\, (R, (\varPhi_1, 0, 0, 0, -1, 0))_\mu = 0\} \\
&= \{R = (0, (\xi E_1, \eta E_2 - \tau\eta E_3 + F_1(y), \tau\xi, 0), 0, 0, 0, 0) \\
&\qquad |\, \xi, \eta \in C, y \in \mathfrak{C}\}
\end{aligned}
$$

with the norm

$$
(R, R)_\mu = \frac{1}{30} B_8(\widetilde{\mu}_\delta R, R) = (\tau\eta)\eta + \overline{y}y + (\tau\xi)\xi,
$$

where $\varPhi_1 = \varPhi(0, E_1, 0, 0)$. We use the notation $(V')^{12}$ to distinguish from the $\boldsymbol{R}$-vector space V^{12} defined in Section 3. The space $(V')^{12}$ above can be identified with the $\boldsymbol{R}$-vector space

$$
\begin{aligned}
&\{P \in \mathfrak{P}^C \,|\, \kappa P = -P, \mu\tau\lambda P = -P\} \\
&= \{P = (\xi E_1, \eta E_2 - \tau\eta E_3 + F_1(y), \tau\xi, 0) \in \mathfrak{P}^C \,|\, \xi, \eta \in C, y \in \mathfrak{C}\}
\end{aligned}
$$

with the norm $(P, P)_\mu = -\dfrac{1}{2}(\mu P, \lambda P) = (\tau\eta)\eta + \overline{y}y + (\tau\xi)\xi$.

Now, we define a subgroup G_{14} of ${E_8}^C$ by

$$
G_{14} = \{\alpha \in {E_8}^C \,|\, \widetilde{\kappa}\alpha = \alpha\widetilde{\kappa},\ \widetilde{\mu}_\delta \alpha R = \alpha \widetilde{\mu}_\delta R,\ R \in (V^C)^{14}\}.
$$

Lemma 4.1. *The Lie algebra $\mathfrak{g}_{14}$ of the group G_{14} is given by*

$$
\begin{aligned}
\mathfrak{g}_{14} &= \{R \in {\mathfrak{e}_8}^C \,|\, \widetilde{\kappa}(\mathrm{ad}R) = (\mathrm{ad}R)\widetilde{\kappa},\ (\widetilde{\mu}_\delta(\mathrm{ad}R))R' = ((\mathrm{ad}R)\widetilde{\mu}_\delta)R',\ R' \in (V^C)^{14}\} \\
&= \Bigg\{\Bigg(\varPhi\Bigg(D + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & d_1 \\ 0 & -\overline{d}_1 & 0 \end{pmatrix}^{\sim} + \begin{pmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & t_1 \\ 0 & \overline{t}_1 & \tau_3 \end{pmatrix}^{\sim}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_2 & a_1 \\ 0 & \overline{a}_1 & \alpha_3 \end{pmatrix}, \\
&\quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta_2 & b_1 \\ 0 & \overline{b}_1 & \beta_3 \end{pmatrix}, \nu\Bigg), \Bigg(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & p_1 \\ 0 & \overline{p}_1 & \rho_3 \end{pmatrix}, \begin{pmatrix} \rho_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \rho\Bigg), \Bigg(\begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
&\quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & z_1 \\ 0 & \overline{z}_1 & \zeta_3 \end{pmatrix}, \zeta, 0\Bigg), r, 0, 0\Bigg) \,\Bigg|\, D \in \mathfrak{so}(8)^C,\ \tau_i, \alpha_i, \beta_i, \nu, \rho_i, \rho, \zeta_i, \zeta, r \in C, \\
&\quad \tau_1 + \tau_2 + \tau_3 = 0,\ d_1, t_1, a_1, b_1, p_1, z_1 \in \mathfrak{C}^C,\ \tau_1 + \frac{2}{3}\nu + 2r = 0\Bigg\}.
\end{aligned}
$$

In particular, we have

$$\dim_C(\mathfrak{g}_{14}) = 28 + 63 = 91.$$

Proposition 4.2. $\qquad G_{14} \cong Spin(14, C)$.

Proof. Let $SO(14, C) = SO((V^{14})^C)$. Then, we have $G_{14}/\boldsymbol{Z}_2 \cong SO(14, C), \boldsymbol{Z}_2 = \{1, \sigma\}$. Therefore, G_{14} is isomorphic to $Spin(14, C)$ as a double covering group of $SO(14, C)$. (In details, see [2].)

We define subgroups ${G_{14}}^{com}, {G_{13}}^{com}$ and ${G_{12}}^{com}$ of the group E_8 by

$$\begin{aligned} {G_{14}}^{com} &= \{\alpha \in G_{14} \,|\, \tau\widetilde{\lambda}\alpha = \alpha\tau\widetilde{\lambda}\}, \\ {G_{13}}^{com} &= \{\alpha \in {G_{14}}^{com} \,|\, \alpha(\varPhi_1, 0, 0, 0, 1, 0) = (\varPhi_1, 0, 0, 0, 1, 0)\}, \\ {G_{12}}^{com} &= \{\alpha \in {G_{13}}^{com} \,|\, \alpha(\varPhi_1, 0, 0, 0, -1, 0) = (\varPhi_1, 0, 0, 0, -1, 0)\}, \end{aligned}$$

respectively.

Lemma 4.3. $\alpha \in (E_7)^{\kappa,\mu} = Spin(12)$ *satisfies*

$$\alpha\varPhi(0, E_1, 0, 0)\alpha^{-1} = \varPhi(0, E_1, 0, 0), \quad \textit{and} \quad \alpha\varPhi(0, 0, E_1, 0)\alpha^{-1} = \varPhi(0, 0, E_1, 0).$$

Proof. We consider an 11 dimensional sphere $(S')^{11}$ by

$$\begin{aligned} (S')^{11} &= \{P' \in (V')^{12} \,|\, (P, P)_\mu = 1\} \\ &= \{P' = (\xi E_1, \eta E_2 - \tau\eta E_3 + F_1(y), \tau\xi, 0) \\ &\qquad\qquad |\, \xi, \eta \in C,\ y \in \mathfrak{C}, (\tau\eta)\eta + \overline{y}y + (\tau\xi)\xi = 1\}. \end{aligned}$$

Since the group $Spin(12)$ acts on $(S')^{11}$, we can put

$$\alpha(E_1, 0, 1, 0) = (\xi E_1, \eta E_2 - \tau\eta E_3 + F_1(y), \tau\xi, 0) \ \in (S')^{11}.$$

Now, since $1/2\varPhi(0, E_1, 0, 0) = (E_1, 0, 1, 0) \times (E_1, 0, 1, 0)$, we have

$$\begin{aligned} 1/2\alpha\varPhi(0, E_1, 0, 0)\alpha^{-1} &= \alpha((E_1, 0, 1, 0) \times (E_1, 0, 1, 0))\alpha^{-1} \\ &= \alpha(E_1, 0, 1, 0) \times \alpha(E_1, 0, 1, 0) \\ &= (\xi E_1, \eta E_2 - \tau\eta E_3 + F_1(y), \tau\xi, 0) \times (\xi E_1, \eta E_2 - \tau\eta E_3 + F_1(y), \tau\xi, 0) \\ &= 1/2\varPhi(0, ((\tau\eta)\eta + \overline{y}y + (\tau\xi)\xi)E_1, 0, 0). \end{aligned}$$

Since $\alpha(E_1, 0, 1, 0) \in (S')^{11}$, we have $(\tau\eta)\eta + \overline{y}y + (\tau\xi)\xi = 1$. Thus, we obtain $\alpha(E_1, 0, 1, 0) \times \alpha(E_1, 0, 1, 0) = 1/2\varPhi(0, E_1, 0, 0)$, that is, $\alpha\varPhi(0, E_1, 0, 0)\alpha^{-1} = \varPhi(0, E_1, 0, 0)$. Since $\alpha \in Spin(12) \subset E_7$ satisfies $\alpha\tau\lambda = \tau\lambda\alpha$, we have also $\alpha\varPhi(0, 0, E_1, 0)\alpha^{-1} = \varPhi(0, 0, E_1, 0)$.

Proposition 4.4. $\qquad {G_{12}}^{com} = Spin(12)$.

Proof. Now, let $\alpha \in {G_{12}}^{com}$. From $\alpha(\varPhi_1, 0, 0, 0, 1, 0) = (\varPhi_1, 0, 0, 0, 1, 0)$ and $\alpha(\varPhi_1, 0, 0, 0, -1, 0) = (\varPhi_1, 0, 0, 0, -1, 0)$, we have $\alpha(0, 0, 0, 0, 1, 0) = (0, 0, 0, 0, 1, 0)$.

Hence, since $\alpha \in {G_{12}}^{com} \subset E_8$, we see that $\alpha \in E_7$. We first show that $\kappa\alpha = \alpha\kappa$. Since ${G_{12}}^{com} \subset E_7$, it suffices to consider the actions on $\mathfrak{P}^C$. Since $\alpha \in {G_{12}}^{com}$ satisfies $\widetilde{\kappa}\alpha = \alpha\widetilde{\kappa}$, from

$$\widetilde{\kappa}\alpha P = \kappa\alpha P - \alpha P \quad \text{and} \quad \alpha\widetilde{\kappa}P = \alpha\kappa P - \alpha P, \quad P \in \mathfrak{P}^C,$$

we have $\kappa\alpha = \alpha\kappa$. Next, we show that $\mu\alpha = \alpha\mu$. Again, from $\alpha(\varPhi_1, 0, 0, 0, 1, 0) = (\varPhi_1, 0, 0, 0, 1, 0)$ and $\alpha(\varPhi_1, 0, 0, 0, -1, 0) = (\varPhi_1, 0, 0, 0, -1, 0)$, we have $\alpha(\varPhi_1, 0, 0, 0, 0, 0) = (\varPhi_1, 0, 0, 0, 0, 0)$. Hence, since $\alpha \in E_7$, we have $\alpha\varPhi_1\alpha^{-1} = \varPhi_1$, that is, $\alpha\varPhi(0, E_1, 0, 0)\alpha^{-1} = \varPhi(0, E_1, 0, 0)$. Consequently

$$\begin{aligned}
&\alpha(\varPhi(0, 0, E_1, 0), 0, 0, 0, 0, 1) = \alpha(-\widetilde{\mu}_\delta(\varPhi(0, E_1, 0, 0), 0, 0, 0, 1, 0)) \\
&\quad = -\widetilde{\mu}_\delta\alpha(\varPhi(0, E_1, 0, 0), 0, 0, 0, 1, 0) = -\widetilde{\mu}_\delta(\varPhi(0, E_1, 0, 0), 0, 0, 0, 1, 0) \\
&\quad = (\varPhi(0, 0, E_1, 0), 0, 0, 0, 0, 1).
\end{aligned}$$

Similarly, we have $\alpha(\varPhi(0, 0, E_1, 0), 0, 0, 0, 0, -1) = (\varPhi(0, 0, E_1, 0), 0, 0, 0, 0, -1)$. Hence we have $\alpha(\varPhi(0, 0, E_1, 0), 0, 0, 0, 0, 0) = (\varPhi(0, 0, E_1, 0), 0, 0, 0, 0, 0)$. Moreover, from $\alpha \in E_7$, we have $\alpha\varPhi(0, 0, E_1, 0)\alpha^{-1} = \varPhi(0, 0, E_1, 0)$. Hence put together with $\alpha\varPhi(0, E_1, 0, 0)\alpha^{-1} = \varPhi(0, E_1, 0, 0)$, we have $\alpha\varPhi(0, E_1, E_1, 0)\alpha^{-1} = \varPhi(0, E_1, E_1, 0)$, that is, $\alpha\mu\alpha^{-1} = \mu$. Thus, we have $\mu\alpha = \alpha\mu$. Therefore, $\alpha \in (E_7)^{\kappa,\mu} = Spin(12)$.

Conversely, let $\alpha \in Spin(12)$. For $R \in {\mathfrak{e}_8}^C$,

$$\begin{aligned}
\widetilde{\kappa}\alpha R &= [(\kappa, 0, 0, -1, 0, 0), (\alpha\varPhi\alpha^{-1}, \alpha P, \alpha Q, r, u, v)] \\
&= ([\kappa, \alpha\varPhi\alpha^{-1}], \kappa\alpha P - \alpha P, \kappa\alpha Q + \alpha Q, 0, -2u, 2v)
\end{aligned}$$

and

$$\begin{aligned}
\alpha\widetilde{\kappa}R &= \alpha[((\kappa, 0, 0, -1, 0, 0), (\varPhi, P, Q, r, u, v)] \\
&= [\alpha(\kappa, 0, 0, -1, 0, 0), \alpha(\varPhi, P, Q, r, u, v)] \\
&= ([\alpha\kappa\alpha^{-1}, \alpha\varPhi\alpha^{-1}], \alpha\kappa\alpha^{-1}(\alpha P) - \alpha P, \alpha\kappa\alpha^{-1}(\alpha Q) + \alpha Q, 0, -2u, 2v).
\end{aligned}$$

From $\kappa\alpha = \alpha\kappa$, we have $[\alpha\kappa\alpha^{-1}, \alpha\varPhi\alpha^{-1}] = [\kappa, \alpha\varPhi\alpha^{-1}]$. Thus, we have $\widetilde{\kappa}\alpha R = \alpha\widetilde{\kappa}R$, that is, $\widetilde{\kappa}\alpha = \alpha\widetilde{\kappa}$. Next, from $\mu\alpha = \alpha\mu$ and Lemma 4.3, we have $\mu_1(\alpha\varPhi_1\alpha^{-1}){\mu_1}^{-1} = \alpha(\mu_1\varPhi_1{\mu_1}^{-1})\alpha^{-1} = \alpha\varPhi(0, 0, E_1, 0)\alpha^{-1} = \varPhi(0, 0, E_1, 0)$. Hence, for $R = (\zeta\varPhi_1, P, 0, 0, u, 0) \in (V^C)^{14}$,

$$\begin{aligned}
\widetilde{\mu}_\delta\alpha R &= \widetilde{\mu}_\delta(\zeta\alpha\varPhi_1\alpha^{-1}, \alpha P, 0, 0, u, 0) \\
&= (\varPhi(0, 0, -uE_1, 0), 0, i\mu_1\alpha P, 0, 0, -\zeta)
\end{aligned}$$

and

$$\begin{aligned}
\alpha\widetilde{\mu}_\delta R &= \alpha(\varPhi(0, 0, -uE_1, 0), 0, i\mu_1 P, 0, 0, -\zeta) \\
&= (\alpha\varPhi(0, 0, -uE_1, 0)\alpha^{-1}, 0, i\alpha\mu_1 P, 0, 0, -\zeta) \\
&= (\varPhi(0, 0, -uE_1, 0), 0, i\alpha\mu_1 P, 0, 0, -\zeta).
\end{aligned}$$

Hence, from $\mu\alpha = \alpha\mu$, we have $\widetilde{\mu}_\delta\alpha R = \alpha\widetilde{\mu}_\delta R, R \in (V^C)^{14}$. From Lemma 4.3, we have $\alpha(\varPhi_1, 0,0,0,0,0) = (\varPhi_1,0,0,0,0,0)$. Moreover, since $\alpha \in E_7$, we have $\alpha(0,0,0,0,1,0) = (0,0,0,0,1,0)$ and $\alpha(0,0,0,0,-1,0) = (0,0,0,0,-1,0)$. Hence, we have $\alpha(\varPhi_1,0,0,0,1,0) = (\varPhi_1,0,0,0,1,0)$ and $\alpha(\varPhi_1,0,0,0,-1,0) = (\varPhi_1,0,0,0,-1,0)$.

Therefore, $\alpha \in {G_{12}}^{com}$. Thus, the proof of the proposition is completed.

Lemma 4.5. *The Lie algebras* ${\mathfrak{g}_{14}}^{com}$ and ${\mathfrak{g}_{13}}^{com}$ *of the groups* ${G_{14}}^{com}$ and ${G_{13}}^{com}$ *are given respectively by*

$$
\begin{aligned}
{\mathfrak{g}_{14}}^{com} &= \{R \in \mathfrak{g}_{14} \,|\, \tau\widetilde{\lambda}(\mathrm{ad}R) = (\mathrm{ad}R)\tau\widetilde{\lambda}\} \\
&= \Big\{\Big(\varPhi\Big(D + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & d_1 \\ 0 & -\overline{d}_1 & 0 \end{pmatrix}^{\sim} + i\begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & t_1 \\ 0 & \overline{t}_1 & \epsilon_3 \end{pmatrix}^{\sim}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & a_1 \\ 0 & \overline{a}_1 & \rho_3 \end{pmatrix}, \\
&\qquad -\tau\begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & a_1 \\ 0 & \overline{a}_1 & \rho_3 \end{pmatrix}, \nu\Big), \Big(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & z_1 \\ 0 & \overline{z}_1 & \zeta_3 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \zeta\Big), \\
&\qquad -\tau\lambda\Big(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & z_1 \\ 0 & \overline{z}_1 & \zeta_3 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \zeta\Big), r, 0, 0\Big) \\
&\qquad |\, D \in \mathfrak{so}(8), \epsilon_i \in \boldsymbol{R},\, \rho_i, \zeta_i, \zeta \in C,\, \nu, r \in i\boldsymbol{R},\, \epsilon_1 + \epsilon_2 + \epsilon_3 = 0, \\
&\qquad i\epsilon_1 + \frac{2}{3}\nu + 2r = 0,\, d_1, t_1 \in \mathfrak{C},\, a_1, z_1 \in \mathfrak{C}^C\Big\}. \\
{\mathfrak{g}_{13}}^{com} &= \{R \in {\mathfrak{g}_{14}}^{com} \,|\, (\mathrm{ad}R)(\varPhi_1, 0,0,0,1,0) = 0\} \\
&= \Big\{\Big(\varPhi\Big(D + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & d_1 \\ 0 & -\overline{d}_1 & 0 \end{pmatrix}^{\sim} + i\begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & t_1 \\ 0 & \overline{t}_1 & \epsilon_3 \end{pmatrix}^{\sim}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & a_1 \\ 0 & \overline{a}_1 & \rho_3 \end{pmatrix}, \\
&\qquad -\tau\begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & a_1 \\ 0 & \overline{a}_1 & \rho_3 \end{pmatrix}, \nu\Big), \Big(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & z_1 \\ 0 & \overline{z}_1 & -\tau\zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\zeta_1\Big), \\
&\quad -\tau\lambda\Big(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & z_1 \\ 0 & \overline{z}_1 & -\tau\zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\zeta_1\Big), 0, 0, 0\Big) \\
&\qquad |\, D \in \mathfrak{so}(8), \epsilon_i \in \boldsymbol{R},\, \rho_i, \zeta_i \in C,\, \nu \in i\boldsymbol{R}, \epsilon_1 + \epsilon_2 + \epsilon_3 = 0, \\
&\qquad i\epsilon_1 + \frac{2}{3}\nu = 0,\, d_1, t_1, z_1 \in \mathfrak{C},\, a_1 \in \mathfrak{C}^C\Big\}.
\end{aligned}
$$

In particular, we have

$$\dim({\mathfrak{g}_{14}}^{com}) = 28 + 63 = 91, \quad \dim({\mathfrak{g}_{13}}^{com}) = 28 + 50 = 78.$$

Lemma 4.6. (1) *For* $a \in \mathfrak{C}$, *we define a* C*-linear transformation* $\epsilon_{13}(a)$ *of* ${\mathfrak{e}_8}^C$ *by*

$$\epsilon_{13}(a) = \exp(\mathrm{ad}(0, (F_1(a), 0,0,0), (0, F_1(a), 0, 0), 0, 0, 0)).$$

Then, $\epsilon_{13}(a) \in {G_{13}}^{com}$ *(Lemma* 4.5*). The action of* $\epsilon_{13}(a)$ *on* V^{13} *is given by*

$$\epsilon_{13}(a)(\varPhi(0,\zeta E_1,0,0),(\xi E_1,\eta E_2-\tau\eta E_3+F_1(y),\tau\xi,0),0,0,-\zeta,0)$$
$$=(\varPhi(0,\zeta' E_1,0,0),(\xi' E_1,\eta' E_2-\tau\eta' E_3+F_1(y'),\tau\xi',0),0,0,-\zeta',0),$$

$$\begin{cases} \zeta'=\zeta\cos|a|-\dfrac{(a,y)}{2|a|}\sin|a| \\ \xi'=\xi \\ \eta'=\eta \\ y'=y+\dfrac{2\zeta a}{|a|}\sin|a|-\dfrac{2(a,y)a}{|a|^2}\sin^2\dfrac{|a|}{2}. \end{cases}$$

(2) *For* $t \in \boldsymbol{R}$*, we define a* C*-linear transformation* $\theta_{13}(t)$ *of* ${\mathfrak{e}_8}^C$ *by*

$$\theta_{13}(t)=\exp(\mathrm{ad}(0,(0,-tE_1,0,-t),(tE_1,0,t,0),0,0,0)).$$

Then, $\theta_{13}(t) \in {G_{13}}^{com}$ *(Lemma* 4.5*). The action of* $\theta_{13}(t)$ *on* V^{13} *is given by*

$$\theta_{13}(t)(\varPhi(0,\zeta E_1,0,0),(\xi E_1,\eta E_2-\tau\eta E_3+F_1(y),\tau\xi,0),0,0,-\zeta,0)$$
$$=(\varPhi(0,\zeta' E_1,0,0),(\xi' E_1,\eta' E_2-\tau\eta' E_3+F_1(y'),\tau\xi',0),0,0,-\zeta',0),$$

$$\begin{cases} \zeta'=\zeta\cos t-\dfrac{1}{4}(\tau\xi+\xi)\sin t \\ \xi'=\dfrac{1}{2}(\xi-\tau\xi)+\dfrac{1}{2}(\xi+\tau\xi)\cos t+2\zeta\sin t \\ \eta'=\eta \\ y'=y. \end{cases}$$

Lemma 4.7. $\qquad {G_{13}}^{com}/{G_{12}}^{com} \simeq S^{12}.$

In partiular, ${G_{13}}^{com}$ *is connected.*

Proof. Let $S^{12}=\{R \in V^{13} \,|\, (R,R)_\mu=1\}$. The group ${G_{13}}^{com}$ acts on $(S^C)^{12}$. We shall show that this action is transitive. To prove this, it suffices to show that any $R \in S^{12}$ can be transformed to $1/2(\varPhi_1,0,0,0,-1,0) \in S^{12}$. Now for a given

$$R=(\varPhi(0,\zeta E_1,0,0),(\xi E_1,\eta E_2-\tau\eta E_3+F_1(y),\tau\xi,0),0,0,-\zeta,0) \in S^{12},$$

choose $a \in \mathfrak{C}$ such that $|a|=\pi/2,(a,y)=0$. Operate $\epsilon_{13}(a) \in {G_{13}}^{com}$ (Lemma 4.6.(1)) on R. Then, we have

$$\epsilon_{13}(a)R=(0,(\xi E_1,\eta E_2-\tau\eta E_3+F_1(y'),\tau\xi,0),0,0,0,0)=R_1 \in (S')^{11} \subset S^{12},$$

where $(S')^{11}=\{R \in (V')^{12} \,|\, (R,R)_\mu=1\}$. Here, since the group $Spin(12)(\subset {G_{13}}^{com})$ acts transitively on $S^{11}=\{P \in V^{12} \,|\, (P,P)_\mu=1\}$, there exists $\beta \in Spin(12)$ such that $\beta P=(0,E_1,0,1)$ for any $P \in S^{11}$. Hence we have

$$\begin{aligned} \beta R_1 &= \beta(0,P',0,0,0,0)=(0,\beta P',0,0,0,0) \\ &= (0,\beta\mu P,0,0,0,0)=(0,\mu\beta P,0,0,0,0) \\ &= (0,\mu(0,E_1,0,1),0,0,0,0)=(0,(E_1,0,1,0),0,0,0,0)=R_2 \in (S')^{11}, \end{aligned}$$

where $P \in S^{11}$.
Finally, operate $\theta_{13}(-\pi/2) \in {G_{13}}^{com}$ (Lemma 4.6.(2)) on R_2. Then, we have

$$\theta_{13}(-\pi/2)R_2 = \frac{1}{2}(\Phi_1, 0, 0, 0, -1, 0).$$

This shows the transitivity. The isotropy subgroup at $1/2(\Phi_1, 0, 0, 0, -1, 0)$ of ${G_{13}}^{com}$ is obviously ${G_{12}}^{com}$. Thus, we have the homeomorphism ${G_{13}}^{com}/{G_{12}}^{com} \simeq S^{12}$.

Proposition 4.8. $\qquad {G_{13}}^{com} \cong Spin(13)$.

Proof. Since the group ${G_{13}}^{com}$ is connected (Lemma 4.7), we can define a homomorphism $\pi : {G_{13}}^{com} \to SO(13) = SO(V^{13})$ by

$$\pi(\alpha) = \alpha|V^{13}.$$

$\mathrm{Ker}\,\pi = \{1, \sigma\} = \boldsymbol{Z}_2$. Since $\dim({\mathfrak{g}_{13}}^{com}) = 78$ (Lemma 4.5) $= \dim(\mathfrak{so}(13))$, π is onto. Hence, ${G_{13}}^{com}/\boldsymbol{Z}_2 \cong SO(13)$. Therefore, ${G_{13}}^{com}$ is isomorphic to $Spin(13)$ as a double covering group of $SO(13) = SO(V^{13})$.

Proposition 4.9. $\qquad {G_{14}}^{com} \cong Spin(14)$.

Proof. Since the group ${G_{14}}^{com}$ acts on V^{14} and ${G_{14}}^{com}$ is connected(Proposition 4.2), we can define a homomorphism $\pi : {G_{14}}^{com} \to SO(14) = SO(V^{14})$ by

$$\pi(\alpha) = \alpha|V^{14}.$$

$\mathrm{Ker}\,\pi = \{1, \sigma\} = \boldsymbol{Z}_2$. Since $\dim({\mathfrak{g}_{14}}^{com}) = 91$ (Lemma 4.5) $= \dim(\mathfrak{so}(14))$, π is onto. Hence, ${G_{14}}^{com}/\boldsymbol{Z}_2 \cong SO(14)$. Therefore, ${G_{14}}^{com}$ is isomorphic to $Spin(14)$ as a double covering group of $SO(14) = SO(V^{14})$.

Now, we shall consider the following group

$$\begin{aligned}&((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}\\ &\quad = \left\{\alpha \in (Spin(13))^{\sigma'} \;\middle|\; \begin{matrix}\alpha(0,(0,F_1(y),0,0),0,0,0,0)\\ = (0,(0,F_1(y),0,0),0,0,0,0)\end{matrix} \text{ for all } y \in \mathfrak{C}\right\}.\end{aligned}$$

Lemma 4.10. *The Lie algebra* $((\mathfrak{spin}(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ *of the group* $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ *is given by*

$$((\mathfrak{spin}(13))^{\sigma'})_{(0,F_1(y),0,0)^-} = \{R \in (\mathfrak{spin}(13))^{\sigma'} | (\mathrm{ad}R)(0,(0,F_1(y),0,0),0,0,0,0) = 0\}$$

$$= \left\{\left(\Phi\left(i\begin{pmatrix}\epsilon_1 & 0 & 0\\ 0 & \epsilon_2 & 0\\ 0 & 0 & \epsilon_3\end{pmatrix}^{\sim}, \begin{pmatrix}0 & 0 & 0\\ 0 & \rho_2 & 0\\ 0 & 0 & \rho_3\end{pmatrix}, -\tau\begin{pmatrix}0 & 0 & 0\\ 0 & \rho_2 & 0\\ 0 & 0 & \rho_3\end{pmatrix}, \nu\right),\right.\right.$$

$$\Big(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & 0 \\ 0 & 0 & -\tau\zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\zeta_1\Big), -\tau\lambda\Big(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & 0 \\ 0 & 0 & -\tau\zeta_2 \end{pmatrix},$$
$$\begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\zeta_1\Big), 0, 0, 0\Big)$$
$$\mid \epsilon_i \in \boldsymbol{R},\ \rho_i, \zeta_i, \in C,\ \nu \in i\boldsymbol{R}, \epsilon_1 + \epsilon_2 + \epsilon_3 = 0,\ i\epsilon_1 + \frac{2}{3}\nu = 0\Big\},$$

In particular, we have

$$\dim(((\mathfrak{spin}(13))^{\sigma'})_{(0,F_1(y),0,0)^-}) = 10.$$

Lemma 4.11. $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}/Spin(4) \simeq S^4.$
In particular, $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ *is connected.*

Proof. We define a 5 dimensional $\boldsymbol{R}$-vector spaces W^5 by

$$\begin{aligned} W^5 &= \{R \in V^{13} \mid \sigma' R = R\} \\ &= \{R = (\Phi(0, \zeta E_1, 0, 0), (\xi E_1, \eta E_2 - \tau\eta E_3, \tau\xi, 0), 0, 0, -\zeta, 0) \mid \zeta \in \boldsymbol{R}, \xi, \eta \in C\} \end{aligned}$$

with the norm

$$(R, R)_\mu = \frac{1}{30} B_8(\widetilde{\mu}_\delta R, R) = 4\zeta^2 + (\tau\eta)\eta + (\tau\xi)\xi.$$

Then, $S^4 = \{R \in W^5 \mid\ (R, R)_\mu = 1\}$ is a 4 dimensional sphere. The group $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ acts on S^4. We shall show that this action is transitive. To prove this, it suffices to show that any $R \in S^4$ can be transformed to $1/2(\Phi_1, 0, 0, 0, -1, 0) \in S^4$ under the action of $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$. Now, for a given

$$R = (\Phi(0, \zeta E_1, 0, 0), (\xi E_1, \eta E_2 - \tau\eta E_3, \tau\xi, 0), 0, 0, -\zeta, 0) \in S^4,$$

choose $t \in \boldsymbol{R}, 0 \le t < \pi$ such that $\tan t = \dfrac{4\zeta}{\xi + \tau\xi}$ (if $\xi + \tau\xi = 0$, let $t = \pi/2$). Operate $\theta_{13}(t) \in ((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ (Lemmas 4.6.(2), 4.10) on R. Then, we have

$$\theta_{13}(t)R = (0, (\xi' E_1, \eta E_2 - \tau\eta E_3, \tau\xi', 0), 0, 0, 0, 0) = R_1 \in S^3 \subset S^4.$$

Since the group $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)} (\subset ((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-})$ acts transitively on S^3 (Lemma 3.14), there exists $\beta \in ((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$ such that

$$\beta R_1 = (0, (E_1, 0, 1, 0), 0, 0, 0, 0) = R_2 \in S^3.$$

Finally, operate $\theta_{13}(-\pi/2) \in ((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ on R_2. Then, we have

$$\theta_{13}(-\pi/2)R_2 = \frac{1}{2}(\Phi_1, 0, 0, 0, -1, 0).$$

This shows the transitivity. The isotropy subgroup at $1/2(\varPhi_1, 0, 0, 0, -1, 0)$ of $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ is $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$ (Lemma 4.7) $= Spin(4)$. Thus, we have the homeomorphism $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}/Spin(4) \simeq S^4$.

Proposition 4.12. $\qquad ((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-} \cong Spin(5)$.

Proof. Since $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ is connected (Lemma 4.11), we can define a homomorphism $\pi : ((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-} \to SO(5) = SO(W^5)$ by

$$\pi(\alpha) = \alpha|W^5.$$

$\operatorname{Ker}\pi = \{1, \sigma\} = \boldsymbol{Z}_2$. Since $\dim(((\mathfrak{spin}(13))^{\sigma'})_{(0,F_1(y),0,0)^-}) = 10$ (Lemma 4.10) $= \dim(\mathfrak{so}(5))$, π is onto. Hence, $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}/\boldsymbol{Z}_2 \cong SO(5)$. Therefore, $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ is isomorphic to $Spin(5)$ as a double covering group of $SO(5)$.

Lemma 4.13. *The Lie algebra* $(\mathfrak{spin}(13))^{\sigma'}$ *of the group* $(Spin(13))^{\sigma'}$ *is given by*

$$(\mathfrak{spin}(13))^{\sigma'}$$
$$= \Big\{\Big(\varPhi\Big(D + i\begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}^{\sim}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, -\tau\begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, \nu\Big),$$
$$\Big(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & 0 \\ 0 & 0 & -\tau\zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\zeta_1\Big), -\tau\lambda\Big(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & 0 \\ 0 & 0 & -\tau\zeta_2 \end{pmatrix},$$
$$\begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\zeta_1\Big), 0, 0, 0\Big)$$
$$\mid D \in \mathfrak{so}(8), \epsilon_i \in \boldsymbol{R},\ \rho_i, \zeta_i \in C, \nu \in i\boldsymbol{R},\ \epsilon_1 + \epsilon_2 + \epsilon_3 = 0,\ i\epsilon_1 + \frac{2}{3}\nu = 0\Big\}.$$

In particular, we have

$$\dim((\mathfrak{spin}(13))^{\sigma'}) = 28 + 10 = 38.$$

Now, we shall determine the group structure of $(Spin(13))^{\sigma'}$.

Theorem 4.14. $(Spin(13))^{\sigma'} \cong (Spin(5) \times Spin(8))/\boldsymbol{Z}_2, \boldsymbol{Z}_2 = \{(1, 1), (-1, \sigma)\}$.

Proof. Let $Spin(13) = {G_{13}}^{com}, Spin(5) = ((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ and $Spin(8) = ((F_4)_{E_1})^{\sigma'} \subset ((E_6)_{E_1})^{\sigma'} \subset ((E_7)^{\kappa,\mu})^{\sigma'} \subset ({G_{13}}^{com})^{\sigma'}$ (Theorem 1.2, Propositions 4.4, 4.8). Now, we define a map $\varphi : Spin(5) \times Spin(8) \to (Spin(13))^{\sigma'}$ by

$$\varphi(\alpha, \beta) = \alpha\beta.$$

Then, φ is well-defined : $\varphi(\alpha, \beta) \in (Spin(13))^{\sigma'}$. Since $[R_D, R_5] = 0$ for $R_D = (\varPhi(D, 0, 0, 0), 0, 0, 0, 0, 0) \in \mathfrak{spin}(8), R_5 \in \mathfrak{spin}(5) = ((\mathfrak{spin}(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ (Proposition 4.12), we have $\alpha\beta = \beta\alpha$. Hence, φ is a homomorphism. $\operatorname{Ker}\varphi = \{(1, 1), (-1, \sigma)\}$

$= \boldsymbol{Z}_2$. Since $(Spin(13))^{\sigma'}$ is connected and $\dim(\mathfrak{spin}(5) \oplus \mathfrak{spin}(8)) = 10$(Lemma 4.10)$+ 28 = 38 = \dim((\mathfrak{spin}(13))^{\sigma'})$ (Lemma 4.13), φ is onto. Thus, we have the isomorphism $(Spin(5) \times Spin(8))/\boldsymbol{Z}_2 \cong ((Spin(13))^{\sigma'}$.

Now, we shall consider the following group

$$
\begin{aligned}
&((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-} \\
&\quad = \left\{ \alpha \in ((Spin(14))^{\sigma'} \;\middle|\; \begin{array}{l} \alpha(0,(0,F_1(y),0,0),0,0,0,0) \\ = (0,(0,F_1(y),0,0),0,0,0,0) \end{array} \text{ for all } y \in \mathfrak{C} \right\}.
\end{aligned}
$$

Lemma 4.15. *The Lie algebra* $((\mathfrak{spin}(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ *of the group* $((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ *is given by*

$$
\begin{aligned}
&((\mathfrak{spin}(14))^{\sigma'})_{(0,F_1(y),0,0)^-} = \{ R \in (\mathfrak{spin}(14))^{\sigma'} \,|\, (\mathrm{ad}R)(0,(0,F_1(y),0,0),0,0,0,0) = 0 \} \\
&= \Bigg\{ \Bigg(\Phi\Bigg(i \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}^{\sim}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, -\tau \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, \nu \Bigg), \\
&\quad \Bigg(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \zeta \Bigg), -\tau\lambda \Bigg(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix}, \\
&\quad \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \zeta \Bigg), r, 0, 0 \Bigg) \\
&\quad \,|\, \epsilon_i \in \boldsymbol{R},\, \rho_i, \zeta_i, \zeta \in C,\, \nu, r \in i\boldsymbol{R},\, \epsilon_1 + \epsilon_2 + \epsilon_3 = 0,\, i\epsilon_1 + \frac{2}{3}\nu + 2r = 0 \Bigg\}.
\end{aligned}
$$

In particular, we have

$$
\dim(((\mathfrak{spin}(14))^{\sigma'})_{(0,F_1(y),0,0)^-}) = 15.
$$

Lemma 4.16. *For* $t \in \boldsymbol{R}$, *we define a* C*-linear transformation* $\theta_{14}(t)$ *of* ${\mathfrak{e}_8}^C$ *by*

$$
\theta_{14}(t) = \exp(\mathrm{ad}(0,(0,itE_1,0,it),(itE_1,0,it.0),0,0,0)).
$$

Then, $\theta_{14}(t) \in ((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ *(Lemma* 4.15*). The action of* $\theta_{14}(t)$ *on* V^{14} *is given by*

$$
\begin{aligned}
&\theta_{14}(t)(\Phi(0,\zeta E_1,0,0),(\xi E_1,\eta E_2 - \tau\eta E_3 + F_1(y),\tau\xi,0),0,0,-\tau\zeta,0) \\
&\quad = (\Phi(0,\zeta' E_1,0,0),(\xi' E_1,\eta' E_2 - \tau\eta' E_3 + F_1(y'),\tau\xi',0),0,0,-\tau\zeta',0),
\end{aligned}
$$

$$
\left\{
\begin{array}{l}
\zeta' = \dfrac{1}{2}(\zeta + \tau\zeta) + \dfrac{1}{2}(\zeta - \tau\zeta)\cos t - \dfrac{i}{4}(\xi + \tau\xi)\sin t \\
\xi' = \dfrac{1}{2}(\xi - \tau\xi) + \dfrac{1}{2}(\xi + \tau\xi)\cos t - i(\zeta - \tau\zeta)\sin t \\
\eta' = \eta \\
y = y'.
\end{array}
\right.
$$

Lemma 4.17. $((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}/Spin(5) \simeq S^5$.

In particular, $((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ *is connected.*

Proof. We define a 6 dimensional $\boldsymbol{R}$-vector space W^6 by

$$\begin{aligned} W^6 &= \{R \in V^{14} \,|\, \sigma' R = R\} \\ &= \{R = (\Phi(0, \zeta E_1, 0, 0), (\xi E_1, \eta E_2 - \tau\eta E_3, \tau\xi, 0), 0, 0, -\tau\zeta, 0) \,|\, \zeta, \xi, \eta \in C\} \end{aligned}$$

with the norm

$$(R, R)_\mu = \frac{1}{30} B_8(\widetilde{\mu}_\delta R, R) = 4(\tau\zeta)\zeta + (\tau\eta)\eta + (\tau\xi)\xi.$$

Then, $S^5 = \{R \in W^6 \,|\, (R, R)_\mu = 1\}$ is a 5 dimensional sphere. The group $((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ acts on S^5. We shall show that this action is transitive. To prove this, it suffices to show that any $R \in S^5$ can be transformed to $1/2(i\Phi_1, 0, 0, 0, i, 0) \in S^5$ under the action of $((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$. Now, for a given

$$R = (\Phi(0, \zeta E_1, 0, 0), (\xi E_1, \eta E_2 - \tau\eta E_3, \tau\xi, 0), 0, 0, -\tau\zeta, 0) \in S^5,$$

choose $t \in \boldsymbol{R}, 0 \le t < \pi$ such that $\tan t = -\dfrac{2i(\zeta - \tau\zeta)}{\xi + \tau\xi}$ (if $\xi + \tau\xi = 0$, let $t = \pi/2$). Operate $\theta_{14}(t) \in ((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ (Lemmas 4.15, 4.16) on R. Then, we have

$$\theta_{14}(t)R = (\Phi(0, (\zeta' E_1, 0, 0), (\xi' E_1, \eta E_2 - \tau\eta E_3, \tau\xi', 0), 0, 0, -\zeta', 0) = R_1 \in S^4 \subset S^5.$$

Since the group $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-} (\subset ((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-})$ acts transitively on S^4 (Lemma 4.11), there exists $\beta \in ((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ such that

$$\beta R_1 = \frac{1}{2}(\Phi_1, 0, 0, 0, -1, 0) = R_2 \in S^3.$$

Moreover, operate $\theta_{14}(\pi/2)$ and $\alpha(\pi/4)$ (Lemma 3.13) in order,

$$\theta_{14}(\pi/2)R_2 = (0, (-iE_1, 0, i, 0), 0, 0, 0, 0) = R_3,$$

and

$$\alpha(\pi/4)R_3 = (0, (E_1, 0, 1, 0), 0, 0, 0, 0) = R_4.$$

Finally, operate $\theta_{14}(-\pi/2) \in ((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ on R_4. Then, we have

$$\theta_{14}(-\pi/2)R_4 = \frac{1}{2}(i\Phi_1, 0, 0, 0, i, 0).$$

This shows the transitivity. The isotropy subgroup at $1/2(i\Phi_1, 0, 0, 0, i, 0)$ of $((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ is $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ (Proposition 4.8) $= Spin(5)$. Thus, we have the homeomorphism $((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}/Spin(5) \simeq S^5$.

Proposition 4.18. $((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-} \cong Spin(6)$.

Proof. Since $((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ is connected (Lemma 4.17), we can define a homomorphism $\pi : ((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-} \to SO(6) = SO(W^6)$ by

$$\pi(\alpha) = \alpha|W^6.$$

$\mathrm{Ker}\,\pi = \{1,\sigma\} = \boldsymbol{Z}_2$. Since $\dim(((\mathfrak{spin}(14))^{\sigma'})_{(0,F_1(y),0,0)^-}) = 15$ (Lemma 4.15) $= \dim(\mathfrak{so}(6))$, π is onto. Hence, $((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}/\boldsymbol{Z}_2 \cong SO(6)$. Therefore, $((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ is isomorphic to $Spin(5)$ as a double covering group of $SO(6)$.

Lemma 4.19. *The Lie algebra* $(\mathfrak{spin}(14))^{\sigma'}$ *of the group* $((Spin(14))^{\sigma'}$ *is given by*

$$\begin{aligned}
&(\mathfrak{spin}(14))^{\sigma'}\\
&= \Big\{\Big(\Phi\Big(D + i\begin{pmatrix}\epsilon_1 & 0 & 0\\ 0 & \epsilon_2 & 0\\ 0 & 0 & \epsilon_3\end{pmatrix}^{\sim}, \begin{pmatrix}0 & 0 & 0\\ 0 & \rho_2 & 0\\ 0 & 0 & \rho_3\end{pmatrix}, -\tau\begin{pmatrix}0 & 0 & 0\\ 0 & \rho_2 & 0\\ 0 & 0 & \rho_3\end{pmatrix}, \nu\Big),\\
&\Big(\begin{pmatrix}0 & 0 & 0\\ 0 & \zeta_2 & 0\\ 0 & 0 & \zeta_3\end{pmatrix}, \begin{pmatrix}\zeta_1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, 0, \zeta\Big), -\tau\lambda\Big(\begin{pmatrix}0 & 0 & 0\\ 0 & \zeta_2 & 0\\ 0 & 0 & \zeta_3\end{pmatrix},\\
&\begin{pmatrix}\zeta_1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, 0, \zeta\Big), r, 0, 0\Big)\\
&\quad | \, D \in \mathfrak{so}(8), \epsilon_i \in \boldsymbol{R},\ \rho_i, \zeta_i, \zeta \in C,\ \nu \in i\boldsymbol{R}, \epsilon_1 + \epsilon_2 + \epsilon_3 = 0,\ i\epsilon_1 + \frac{2}{3}\nu + 2r = 0\Big\}.
\end{aligned}$$

In particular, we have

$$\dim((\mathfrak{spin}(14))^{\sigma'}) = 28 + 15 = 43.$$

Now, we shall determine the group structure of $(Spin(14))^{\sigma'}$.

Theorem 4.20. $(Spin(14))^{\sigma'} \cong (Spin(6) \times Spin(8))/\boldsymbol{Z}_2, \boldsymbol{Z}_2 = \{(1,1), (-1,\sigma)\}$.

Proof. Let $Spin(14) = {G_{14}}^{com}, Spin(6) = ((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ and $Spin(8) = ((F_4)_{E_1})^{\sigma'} \subset ((E_6)_{E_1})^{\sigma'} \subset ((E_7)^{\kappa,\mu})^{\sigma'} \subset ({G_{13}}^{com})^{\sigma'} \subset ({G_{14}}^{com})^{\sigma'}$ (Theorem 1.2, Propositions 4.8, 4.9). Now, we define a map $\varphi : Spin(6) \times Spin(8) \to (Spin(14))^{\sigma'}$ by

$$\varphi(\alpha, \beta) = \alpha\beta.$$

Then, φ is well-defined : $\varphi(\alpha,\beta) \in (Spin(14))^{\sigma'}$. Since $[R_D, R_6] = 0$ for $R_D = (\Phi(D,0,0,0), 0,0,0,0,0) \in \mathfrak{spin}(8), R_6 \in \mathfrak{spin}(6) = ((\mathfrak{spin}(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ (Proposition 4.18), we have $\alpha\beta = \beta\alpha$. Hence, φ is a homomorphism. $\mathrm{Ker}\,\varphi = \{(1,1), (-1,\sigma)\}$ $= \boldsymbol{Z}_2$. Since $(Spin(14))^{\sigma'}$ is connected and $\dim(\mathfrak{spin}(6) \oplus \mathfrak{spin}(8)) = 15$(Lemma 4.15)+

$28 = 43 = \dim((\mathfrak{spin}(14))^{\sigma'})$ (Lemma 4.19), φ is onto. Thus, we have the isomorphism $(Spin(6) \times Spin(8))/\boldsymbol{Z}_2 \cong ((Spin(14))^{\sigma'}$.

Acknowledgment. The author records here his warmest gratitude to Professor Ichiro Yokota, who recommended to write this paper and has constantly encouraged him.

References

[1] T. Imai and I. Yokota, Simply connected compact simple Lie group $E_{8(-248)}$ of type E_8, J. Math. Kyoto Univ., 21 (1981), 741-762.

[2] T. Miyashita and I. Yokota, 2-graded decompositions of exceptional Lie algebra $\mathfrak{g}$ and group realizations of $\mathfrak{g}_{\text{ev}}, \mathfrak{g}_0$, Part III, $G = E_8$, Japanese J. Math., 26(2000), 31-51.

[3] T. Miyashita and I. Yokota, Fixed points subgroups $G^{\sigma,\sigma'}$ by two involutive automorphisms σ, σ' of compact exceptional Lie group $G = F_4, E_6$ and E_7,(2001), preprint.

[4] T. Miyashita and I. Yokota, An explicit isomorphism between $(\mathfrak{e}_8)^{\sigma,\sigma'}$ and $\mathfrak{so}(8) \oplus \mathfrak{so}(8)$ as Lie algebras, (2001), preprint.

[5] I. Yokota, Realizations of involutive automorphisms σ and G^σ of exceptional linear Lie groups G, Part I, $G = G_2, F_4$ and E_6, Tsukuba J. Math., 4(1990), 185 - 223.

[6] I. Yokota, Realizations of involutive automorphisms σ and G^σ of exceptional linear Lie groups G, Part II, $G = E_7$, Tsukuba, J. Math., 14(1990), 379-404.

[7] I. Yokota, Realizations of involutive automorphisms σ and G^σ of exceptional linear Lie groups G, Part III, $G = E_8$, Tsukuba J. Math., 14 (1991), 301-314.

[8] I. Yokota, Exceptional simple Lie groups (in Japanese), Gendaisuugakusya, Kyoto, 1992.

Toshikazu Miyashita
Tohbu High School
Agata, Tohbu, 389-0517, Japan
E-mail: serano@janis.or.jp