The number of convex pentagons and hexagons in an n-triangular net

Jun-Ming Zhu
Department of Mathematics, East China Normal University,
Dongchuan Road 500, Shanghai 200241, China

E-mail: junming_zhu@163.com

Abstract

In this paper, we obtain the counting formulaes of convex pentagons and convex hexagons, respectively, in an n-triangular net by solving the corresponding recursive formulaes.

Key words: n-triangular net, convex pentagon, convex hexagon, regular hexagon
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1 Introduction

In the paper [2], the author obtained the counting formulaes of triangles and quadrilaterals in an n-triangular net, respectively, by induction. In this paper, we get the counting formulaes of convex pentagons and hexagons in an n-triangular net, respectively, by solving two corresponding recursive formulaes. We first give the following definition.

Definition 1.1 Divide each edge of a (regular) triangular into n ($n \geq 1$) equal parts, and then construct $n-1$ segments between the dividing points on two edges parallel to the third one. Then the graph we get is called an n-(regular) triangular net.

See fig. 1, fig. 2 and fig. 3 in the following.

If $n = 1$, then the n-triangular net reduces to a triangular.

By the property of affine transformation, we know that the numbers of convex pentagons and convex hexagons in an n-triangular net are only dependent on n but independent from the shape and the size of the triangular.

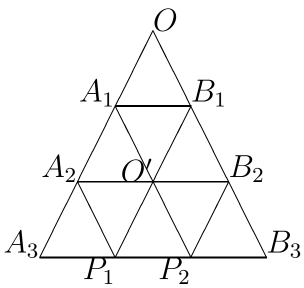

fig. 1: 3-triangular net OA_3B_3

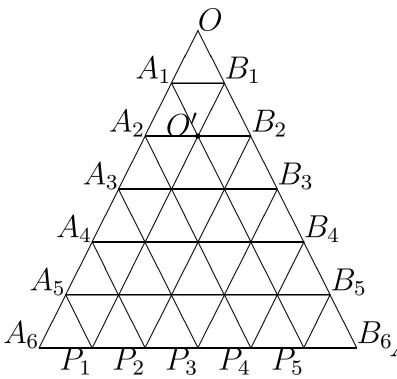

fig. 2: 6-triangular net OA_6B_6

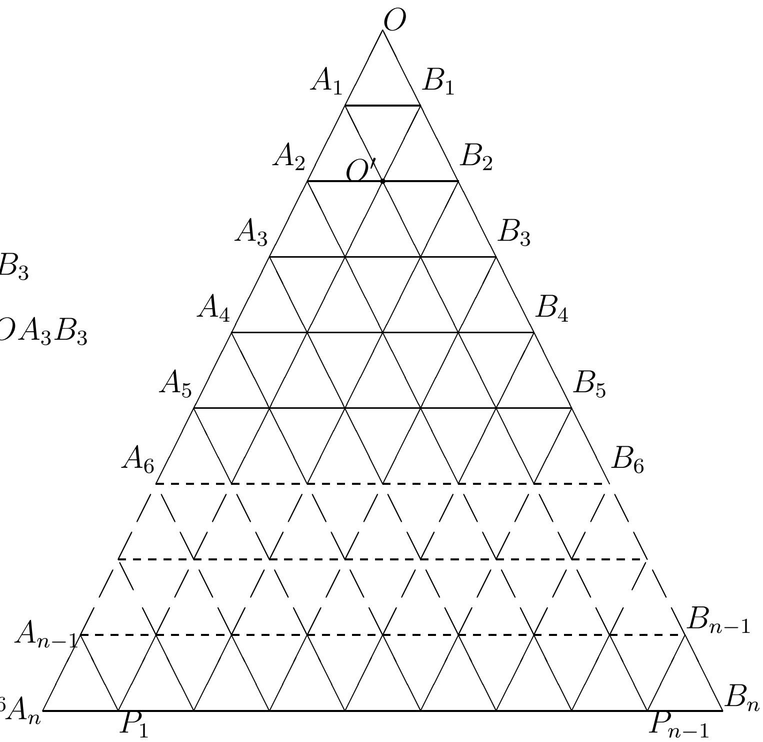

fig. 3

2 The counting formulae of convex pentagons in an n-triangular net

Theorem 2.1. *The number $P(n)$ of convex pentagons in an n-triangular net is*

$$P(n) = \begin{cases} \frac{1}{10}(12k^5 + 25k^4 + 5k^3 - 10k^2 - 2k), & n = 2k+1 \ (k = 0, 1, 2, \cdots), \\ \frac{1}{10}(12k^5 - 5k^4 - 15k^3 + 5k^2 + 3k), & n = 2k \ (k = 1, 2, \cdots). \end{cases} \tag{1}$$

Proof. Without loss of generosity, we suppose that the divided triangular is regular. Then there is only one acute interior angle in each convex pentagon.

Observing the figures above, we know that $P(n)$ can be expressed as $2P(n-1)$ (all the pentagons in the $(n-1)$-triangular net $A_1A_nP_{n-1}$ or $B_1P_1B_n$), subtracting $P(n-2)$ (pentagons in the $(n-2)$-triangular net $O'P_1P_{n-1}$), and then adding the number, denoted by $f(n)$, of the pentagons of which there are vertexes both on OA_n and on OB_n. This is

$$P(n) = 2P(n-1) - P(n-2) + f(n). \tag{2}$$

Obviously, we have $P(1) = 0$, $P(2) = 0$, $P(3) = 3$ and $f(1) = 0$, $f(2) = 0$, $f(3) = 3$.

Now it is crucial to get the expression of $f(n)$. Note that, from the definition of $f(n)$, $f(n) - f(n-1)$ is the number of pentagons of which there are vertexes on OA_n, OB_n and A_nB_n and of which the unique acute vertex must be on OA_n, OB_n or A_nB_n.

The number of all the pentagons of which there are vertexes on OA_n, OB_n and A_nB_n and of which the acute vertex is one of $O,\ A_n$ and B_n is

$$(1+2+3+\cdots+(n-2))\times 3.$$

The number of all the pentagons of which there are vertexes on OA_n, OB_n and A_nB_n and of which the acute vertex is one of $A_1,\ A_2,\ \cdots,\ A_{n-1}, B_1,\ B_2,\ \cdots,\ B_{n-1}$ and $P_1,\ P_2,\cdots,\ P_{n-1}$ is

$$1+2+\cdots+(k-1)+(k-1)+\cdots+2+1 \quad (\text{if } \ n=2k),$$

or

$$1+2+\cdots+(k-2)+(k-1)+(k-2)+\cdots+2+1 \quad (\text{if } \ n=2k+1).$$

So we have

$$\begin{aligned} f(2k)-f(2k-1) &= (1+2+3+\cdots+(2k-2))\times 3 \\ &\quad +1+2+\cdots+(k-1)+(k-1)+\cdots+2+1 \end{aligned}$$

and

$$\begin{aligned} f(2k+1)-f(2k) &= (1+2+3+\cdots+(2k-1))\times 3 \\ &\quad +1+2+\cdots+(k-2)+(k-1)+(k-2)+\cdots+2+1 \\ &\qquad . \end{aligned}$$

This is

$$\begin{aligned} f(2k) = f(2k-1)+3(3k^2-5k+2), \\ f(2k+1) = f(2k)+3(3k^2-2k). \end{aligned}$$

Iterating the above formulaes gives

$$\begin{aligned} f(2) &= f(1)+3(3\cdot 1^2-5\cdot 1+2), \\ f(3) &= f(2)+3(3\cdot 1^2-2\cdot 1), \\ f(4) &= f(3)+3(3\cdot 2^2-5\cdot 2+2), \\ f(5) &= f(4)+3(3\cdot 2^2-2\cdot 2), \\ &\cdots\cdots\cdots\cdots, \\ f(2k) &= f(2k-1)+3(3\cdot k^2-5\cdot k+2), \\ f(2k+1) &= f(2k)+3(3\cdot k^2-2\cdot k). \end{aligned}$$

Overlay the above formulaes and note that $f(1)=0$ to get

$$\begin{aligned} f(2k+1) = \tfrac{3}{2}(4k^3-k^2-k), \\ f(2k) = \tfrac{3}{2}(4k^3-7k^2+3k). \end{aligned}$$

From (2), we have $P(n) - P(n-1) = P(n-1) - P(n-2) + f(n)$. Then

$$P(2k) - P(2k-1) = P(2k-1) - P(2k-2) + f(2k),\ (k = 1, 2, \cdots\cdots),$$
$$P(2k+1) - P(2k) = P(2k) - P(2k-1) + f(2k+1),\ (k = 0, 1, 2, \cdots\cdots).$$

Overlaying again, we have

$$\begin{aligned} P(2k+1) - P(2k) &= 3k^4 + 2k^3 - \frac{3}{2}k^2 - \frac{1}{2}k, \\ P(2k) - P(2k-1) &= 3k^4 - 4k^3 + k. \end{aligned}$$

Overlaying a third time, we get

$$P(2k+1) = \frac{6}{5}k^5 + \frac{5}{2}k^4 + \frac{1}{2}k^3 - k^2 - \frac{1}{5}k,$$
$$P(2k) = \frac{6}{5}k^5 - \frac{1}{2}k^4 - \frac{3}{2}k^3 + \frac{1}{2}k^2 + \frac{3}{10}k,$$

which completes the proof.

□

The difference equation (2) is fundamental in this paper. This is a linear recurrence relation of order 2 and can also be solved using the method usually used in solving recursive formulaes (see, for example, [1, p.218–234, §7.2–7.3]). Our method is different from that of [2]. Obviously, formulaes of this form can also be used to get the counting formulaes of triangles and quadrilaterals in [2] and seem to be more understandable. We will also use the equation of this form to get number of convex hexagons in an n-triangular net in the following.

3 The counting formulae of convex hexagons in an n-triangular net

Theorem 3.1. *The number $H(n)$ of convex hexagons in an n-triangular net is*

$$H(n) = \begin{cases} \frac{1}{60}(8k^6 + 24k^5 + 25k^4 + 10k^3 - 3k^2 - 4k), & n = 2k+1 (k = 0, 1, 2, \cdots), \\ \frac{1}{60}(8k^6 - 5k^4 - 3k^2), & n = 2k (k = 1, 2, \cdots). \end{cases} \quad (3)$$

Proof. Just as the analysis in the proof of theorem 2.1, we have

$$H(n) = 2H(n-1) - H(n-2) + g(n), \quad (4)$$

where $g(n)$ denote the number of the hexagons of which there are vertexes both on OA_n and on OB_n. Obviously, we have $H(1) = 0$, $H(2) = 0$, $H(3) = 1$ and $g(1) = 0$, $g(2) = 0$, $g(3) = 1$.

The equation (4) is similar to the equation (2). So we solve it in the same way as (2). We have

$$\begin{aligned}
g(2k+1) &= g(2k) \\
&\quad +1+2+3+4+\cdots+(2k-4)+(2k-3)+(2k-2)+(2k-1) \\
&\quad +2+3+4+\cdots+(2k-4)+(2k-3)+(2k-2)+(2k-2) \\
&\quad +3+4+\cdots+(2k-4)+(2k-3)+(2k-3)+(2k-3) \\
&\quad +\cdots\cdots\cdots\cdots \\
&\quad \underbrace{+(k-2)+(k-1)+k+(k+1)+(k+2)+(k+2)+\cdots+(k+2)}_{\text{The number is } k+2} \\
&\quad \underbrace{+(k-1)+k+(k+1)+(k+1)+(k+1)+\cdots+(k+1)}_{\text{The number is } k+1} \\
&\quad \underbrace{+k+k+k+k+\cdots+k}_{\text{The number is } k} \\
&\quad +\cdots\cdots\cdots\cdots \\
&\quad +2+2 \\
&\quad +1 \\
&= g(2k)+1^2+2^2+\cdots+(2k-1)^2 \\
&\quad -(+1+2+3+4+\cdots+(2k-4)+(2k-3)+(2k-2) \\
&\quad +1+2+3+4+\cdots+(2k-4) \\
&\quad +\cdots\cdots\cdots\cdots \\
&\quad +1+2+3+4 \\
&\quad +1+2) \\
&= g(2k)+\frac{k}{2}(4k^2-3k+1),
\end{aligned}$$

and

$$\begin{aligned}
g(2k) &= g(2k-1) \\
&\quad +1+2+3+4+\cdots+(2k-5)+(2k-4)+(2k-3)+(2k-2) \\
&\quad +2+3+4+\cdots+(2k-5)+(2k-4)+(2k-3)+(2k-3) \\
&\quad +3+4+\cdots+(2k-5)+(2k-4)+(2k-4)+(2k-4) \\
&\quad +\cdots\cdots\cdots\cdots \\
&\quad \underbrace{+(k-2)+(k-1)+k+(k+1)+(k+1)+\cdots+(k+1)}_{\text{The number is } k+1} \\
&\quad \underbrace{+(k-1)+k+k+k+k+\cdots+k}_{\text{The number is } k} \\
&\quad \underbrace{+(k-1)+(k-1)+(k-1)+\cdots+(k-1)}_{\text{The number is } k-1} \\
&\quad +\cdots\cdots\cdots\cdots
\end{aligned}$$

$$
\begin{aligned}
& +2+2 \\
& +1 \\
= \;& g(2k-1) + 1^2 + 2^2 + \cdots + (2k-2)^2 \\
& -(+1+2+3+4+\cdots+(2k-5)+(2k-4)+(2k-3) \\
& +1+2+3+4+\cdots+(2k-5) \\
& +\cdots\cdots\cdots\cdots \\
& +1+2+3 \\
& +1) \\
= \;& g(2k-1) + \frac{k-1}{2}(4k^2-5k+2).
\end{aligned}
$$

So we have

$$
\begin{aligned}
g(2k) &= \frac{1}{2}k(k-1)(2k^2-2k+1), \\
g(2k+1) &= k^4,
\end{aligned}
$$

and then

$$
\begin{aligned}
H(2k) - H(2k-1) = \tfrac{k}{30}(12k^4 - 15k^3 + 5k^2 - 2) \\
H(2k+1) - H(2k) = \tfrac{k}{30}(12k^4 + 15k^3 + 5k^2 - 2).
\end{aligned}
$$

By overlaying, we get (3). This completes the proof. □

References

[1] R. A. Brualdi, *Introductory Combinatorics*, 4th ed., Person Education, Inc. 2005.

[2] Y. X. Zhu, The number of convex pentagons and hexagons in an n-triangular net, *Bull. Math.*, **46** (2007), no. 8, 51–52 (in Chinese).