AN EXTENSION OF STERN'S CONGRUENCE

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ABSTRACT. Let $\{E_n\}$ be the Euler numbers. In the paper we determine $E_{2^m k+b} - E_b$ modulo 2^{m+7}, where k, m are positive integers with $m \geq 4$ and $b \in \{0, 2, 4, \dots\}$.

1. Introduction.

The Euler numbers $\{E_n\}$ are given by

$$E_0 = 1, \quad E_{2n-1} = 0, \quad \sum_{r=0}^{n} \binom{2n}{2r} E_{2r} = 0 \quad (n = 1, 2, 3, \dots).$$

The first few Euler numbers are shown below:

$$E_0 = 1, \ E_2 = -1, \ E_4 = 5, \ E_6 = -61, \ E_8 = 1385, \ E_{10} = -50521,$$
$$E_{12} = 2702765, \ E_{14} = -199360981, \ E_{16} = 19391512145.$$

Let $\mathbb{N}$ be the set of positive integers. For $k, m \in \mathbb{N}$ and $b \in \{0, 2, 4, \dots\}$, in 1875 Stern ([2]) proved the following Stern's congruence:

$$(1.1) \qquad\qquad E_{2^m k+b} \equiv E_b + 2^m k \pmod{2^{m+1}}.$$

There are many modern proofs of (1.1). See for example [1,3,5,6].

Let $b \in \{0, 2, 4, \dots\}$ and $k, m \in \mathbb{N}$. In [4] the first author showed that

$$(1.2) \qquad\qquad E_{2^m k+b} \equiv E_b + 2^m k \pmod{2^{m+2}} \quad \text{for} \quad m \geq 2,$$

$$(1.3) \qquad\qquad E_{2^m k+b} \equiv E_b + 5 \cdot 2^m k \pmod{2^{m+3}} \quad \text{for} \quad m \geq 3$$

and

$$(1.4) \qquad E_{2^m k+b} \equiv \begin{cases} E_b + 5 \cdot 2^m k \pmod{2^{m+4}} & \text{if } b \equiv 0, 6 \pmod 8, \\ E_b - 3 \cdot 2^m k \pmod{2^{m+4}} & \text{if } b \equiv 2, 4 \pmod 8 \end{cases} \quad \text{for} \quad m \geq 4.$$

From [4, Corollary 3.2] we know that for any nonnegative integer k and positive integer n,

$$(1.5) \qquad \frac{3^{2k+1}+1}{4}E_{2k} \equiv \sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}\frac{3^{2r+1}+1}{4}E_{2r} \pmod{2^{3n}}.$$

In the paper we use (1.5) to obtain a congruence for $E_{2^m k+b} - E_b$ modulo 2^{m+7}, where $k, m \in \mathbb{N}$, $m \geq 4$ and $b \in \{0, 2, 4, \dots\}$, see Theorem 2.1. As a consequence, we have

$$(1.6) \qquad E_{2^m k+b} \equiv E_b - 2^m k((b+1)^2+10) \pmod{2^{m+6}} \quad \text{for} \quad m \geq 6.$$

Throughout the paper we use $\mathbb{Z}_2$ to denote the set of rational numbers whose denominator is odd.

2. The main result.

Theorem 2.1. *Let $b \in \{0, 2, 4, \dots\}$ and $k, m \in \mathbb{N}$ with $m \geq 4$. Then*

$$E_{2^m k+b} \equiv E_b - 2^m k((b+1)^2+42-32(-1)^{\frac{b(b+2)}{8}}+2^m k((-1)^{\frac{b}{2}}+2(-1)^{\frac{b(b+2)}{8}})) \pmod{2^{m+7}}.$$

Proof. Set $f(k) = \frac{3^{2k+b+1}+1}{4}E_{2k+b}$ and $F(k) = 2^{-3k}\sum_{r=0}^{k}\binom{k}{r}(-1)^r f(r)$. From (1.5) (with $n = k$) or [4, Lemma 2.3] we know that $F(k) \in \mathbb{Z}_2$. By the binomial inversion formula we have $f(k) = \sum_{r=0}^{k}\binom{k}{r}(-2)^{3r}F(r)$ and so

$$f(2^{m-1}k) = F(0) + \sum_{r=1}^{2^{m-1}k}\binom{2^{m-1}k}{r}(-2)^{3r}F(r)$$

$$= F(0) + \sum_{r=1}^{2^{m-1}k}\frac{2^{m-1}k}{r}\binom{2^{m-1}k-1}{r-1}(-2)^{3r}F(r).$$

For $r \geq 3$ we have $\frac{2^{r-2}}{r} \in \mathbb{Z}_2$ and $F(r) \in \mathbb{Z}_2$. Thus

$$\frac{2^{m-1}\cdot 2^{3r}F(r)}{r} = 2^{m+2r+1}\cdot\frac{2^{r-2}}{r}F(r) \equiv 0 \pmod{2^{m+7}}.$$

Hence,

(2.1)
$$\frac{3^{2^m k+b+1}+1}{4}E_{2^m k+b}$$
$$= f(2^{m-1}k) \equiv F(0) - 2^{m+2}kF(1) + 2^{m+4}k(2^{m-1}k-1)F(2)$$
$$= \frac{3^{b+1}+1}{4}E_b - 2^{m-1}k(f(0)-f(1)) - 2^{m-2}k(f(0)-2f(1)+f(2)) + 2^{2m+3}k^2 F(2)$$
$$\equiv \frac{3^{b+1}+1}{4}E_b - 2^{m-2}k(3f(0)-4f(1)+f(2)) \pmod{2^{m+7}}.$$

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Putting $n = 3$ in (1.5) we see that

$$\frac{3^{b+1}+1}{4}E_b \equiv \sum_{r=0}^{2}(-1)^{2-r}\binom{b/2-1-r}{2-r}\binom{b/2}{r}\frac{3^{2r+1}+1}{4}E_{2r}$$

$$= \frac{(\frac{b}{2}-1)(\frac{b}{2}-2)}{2} - \frac{b}{2}(\frac{b}{2}-2)\cdot 7\cdot(-1) + \frac{\frac{b}{2}(\frac{b}{2}-1)}{2}\cdot 61\cdot 5$$

$$= 40b^2 - 84b + 1 \pmod{2^9}.$$

Thus,

(2.2)
$$3f(0) - 4f(1) + f(2)$$
$$\equiv 3(40b^2 - 84b + 1) - 4(40(b+2)^2 - 84(b+2) + 1) + 40(b+4)^2 - 84(b+4) + 1$$
$$= 336 - 320b \equiv -176 - 64b \pmod{2^9}.$$

For $r \geq 3$ we see that $4^{r-3}/r \in \mathbb{Z}_2$ and so

$$\binom{2^{m-2}k}{r}80^r = 2^{m+4+2r}\cdot\frac{4^{r-3}}{r}\cdot k\cdot 5^r\binom{2^{m-2}k-1}{r-1} \equiv 0 \pmod{2^{m+9}}.$$

Thus,

$$3^{2^m k} = (1+80)^{2^{m-2}k} = \sum_{r=0}^{2^{m-2}k}\binom{2^{m-2}k}{r}80^r$$

$$\equiv 1 + 5k\cdot 2^{m+2} + 25k\cdot 2^{m+5}(2^{m-2}k-1)$$

$$= 1 - 195\cdot 2^{m+2}k \equiv 1 + 61\cdot 2^{m+2}k \pmod{2^{m+9}}.$$

This together with (2.1) and (2.2) yields

$$\frac{3^{b+1}(1+61\cdot 2^{m+2}k)+1}{4}E_{2^m k+b}$$

$$\equiv \frac{3^{2^m k+b+1}+1}{4}E_{2^m k+b} \equiv \frac{3^{b+1}+1}{4}E_b + 2^{m-2}k(176+64b) \pmod{2^{m+7}}.$$

Thus,

(2.3) $$\frac{3^{b+1}+1}{4}(E_{2^m k+b} - E_b) \equiv -55\cdot 3^b\cdot 2^m k E_{2^m k+b} + 2^m k(44+16b) \pmod{2^{m+7}}.$$

By [3, Corollary 7.9] or [4, Corollary 3.5] we have

(2.4) $$E_b \equiv \begin{cases} 3b^2 - 11b + 1 \pmod{2^7} & \text{if } 4\mid b, \\ b^2 - 23b + 41 \pmod{2^7} & \text{if } 4\mid b-2. \end{cases}$$

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Thus,

$$(2.5) \qquad E_{2^m k+b} \equiv \begin{cases} 3b^2 - 11b + 1 + 2^m k(6b - 11) \pmod{2^7} & \text{if} \quad 4 \mid b, \\ b^2 - 23b + 41 + 2^m k(2b - 23) \pmod{2^7} & \text{if} \quad 4 \mid b - 2. \end{cases}$$

As

$$3^b = (1+8)^{b/2} = 1 + \binom{b/2}{1} 8 + \binom{b/2}{2} 8^2 + \cdots \equiv 8b^2 - 12b + 1$$

$$\equiv \begin{cases} -12b + 1 \pmod{2^7} & \text{if } 4 \mid b, \\ 20b - 31 \pmod{2^7} & \text{if } 4 \mid b - 2, \end{cases}$$

by (2.5) we have

$$3^b E_{2^m k+b} \equiv \begin{cases} (3b^2 - 11b + 1 + 2^m k(6b - 11))(-12b + 1) \pmod{2^7} & \text{if} \quad 4 \mid b, \\ (b^2 - 23b + 41 + 2^m k(2b - 23))(20b - 31) \pmod{2^7} & \text{if} \quad 4 \mid b - 2. \end{cases}$$

If $4 \mid b$, then

$$(3b^2 - 11b + 1)(-12b + 1) = -36b^3 + 135b^2 - 23b + 1 \equiv 7b^2 - 23b + 1 \pmod{128}$$

and

$$(6b - 11)(-12b + 1) \equiv -11 \equiv -3 \pmod 8.$$

If $4 \mid b - 2$, then $32 \mid (b - 2)(b + 2)$ and so

$$(b^2 - 23b + 41)(20b - 31) = ((b - 2)^2 - 19(b - 2) - 1)(20(b + 2) - 71)$$
$$\equiv -20(b + 2) - 71(b^2 - 23b + 41)$$
$$\equiv -7b^2 + 13b - 7 \pmod{128}$$

and

$$(2b - 23)(20b - 31) = (2(b - 2) - 19)(20b - 31) \equiv 19 \cdot 31 \equiv -3 \pmod 8.$$

Thus

$$3^b E_{2^m k+b} \equiv \begin{cases} 7b^2 - 23b + 1 - 3 \cdot 2^m k \pmod{2^7} & \text{if} \quad 4 \mid b, \\ -7b^2 + 13b - 7 - 3 \cdot 2^m k \pmod{2^7} & \text{if} \quad 4 \mid b - 2. \end{cases}$$

Substituting this into (2.3) we obtain
$$(2.6)$$
$$\frac{3^{b+1} + 1}{4} (E_{2^m k+b} - E_b) \equiv \begin{cases} 2^m k(-b^2 + b - 11 - 3 \cdot 2^m k) \pmod{2^{m+7}} & \text{if} \quad 4 \mid b, \\ 2^m k(b^2 + 5b + 45 - 3 \cdot 2^m k) \pmod{2^{m+7}} & \text{if} \quad 4 \mid b - 2. \end{cases}$$

It is easily seen that

$$\frac{3^{b+1} + 1}{4} = \frac{3(1 + 8)^{b/2} + 1}{4} \equiv \frac{3(1 + \frac{b}{2} \cdot 8 + \binom{b/2}{2} \cdot 8^2) + 1}{4} = 6b^2 - 9b + 1 \pmod{2^7}.$$

If $4 \mid b$, one can easily check that

$$(6b^2 - 9b + 1)(b^3 + 3b^2 + 9b + 1) \equiv 1 \pmod{2^7}, \quad b^3 + 3b^2 + 9b + 1 \equiv 1 + b \pmod 8$$

and

$$(-b^2 + b - 11)(b^3 + 3b^2 + 9b + 1) \equiv b^3 - b^2 - 2b - 11 \pmod{2^7}.$$

Thus, by (2.6) we have

$$
\begin{aligned}
E_{2^m k + b} - E_b &\equiv 2^m k(-b^2 + b - 11 - 3 \cdot 2^m k)(b^3 + 3b^2 + 9b + 1) \\
&\equiv 2^m k(b^3 - b^2 - 2b - 11 - 3(b+1) \cdot 2^m k) \\
&\equiv \begin{cases} 2^m k(-(b+1)^2 - 10 - 3 \cdot 2^m k) \pmod{2^7} & \text{if } 8 \mid b, \\ 2^m k(-(b+1)^2 - 10 + 64 + 2^m k) \pmod{2^7} & \text{if } 8 \mid b - 4. \end{cases}
\end{aligned}
$$

If $4 \mid b - 2$, one can easily check that

$$
\begin{aligned}
(6b^2 - 9b + 1)((b+2)^3 - 3(b+2)^2 + 9(b+2) + 3) &\equiv 1 \pmod{2^7}, \\
(b+2)^3 - 3(b+2)^2 + 9(b+2) + 3 &\equiv b + 5 \pmod 8
\end{aligned}
$$

and

$$(b^2 + 5b + 45)((b+2)^3 - 3(b+2)^2 + 9(b+2) + 3) \equiv b^3 + 5b^2 - 22b + 61 \pmod{2^7}.$$

Thus, by (2.6) we have

$$
\begin{aligned}
E_{2^m k + b} - E_b &\equiv 2^m k(b^2 + 5b + 45 - 3 \cdot 2^m k)((b+2)^3 - 3(b+2)^2 + 9(b+2) + 3) \\
&\equiv 2^m k(b^3 + 5b^2 - 22b + 61 - 3(b+5) \cdot 2^m k) \pmod{2^7}.
\end{aligned}
$$

Since

$$
\begin{aligned}
b^3 + 5b^2 &- 22b + 61 + (b+1)^2 + 10 \\
&= b^3 + 6b^2 - 20b + 72 = (b+2)^3 - 32(b-2) \\
&\equiv (b+2)^3 \equiv \begin{cases} 0 \pmod{128} & \text{if } b \equiv 6 \pmod 8, \\ 64 \pmod{128} & \text{if } b \equiv 2 \pmod 8, \end{cases}
\end{aligned}
$$

we see that

$$
\begin{aligned}
E_{2^m k + b} - E_b &\equiv 2^m k(b^3 + 5b^2 - 22b + 61 - 3(b+5) \cdot 2^m k) \\
&\equiv \begin{cases} 2^m k(-(b+1)^2 - 10 - 2^m k) \pmod{2^{m+7}} & \text{if } b \equiv 6 \pmod 8, \\ 2^m k(-(b+1)^2 - 10 + 64 + 3 \cdot 2^m k) \pmod{2^{m+7}} & \text{if } b \equiv 2 \pmod 8. \end{cases}
\end{aligned}
$$

Note that $(-1)^{\frac{b(b+2)}{8}} = 1$ or -1 according as $b \equiv 0, 6 \pmod 8$ or $b \equiv 2, 4 \pmod 8$. Combining all of the above we deduce the result.

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Remark 2.1 Let $k \in \{0, 1, 2, \ldots\}$ and $b \in \{0, 2, 4, \ldots\}$. Taking $m = 3$ in [3, Corollary 7.8] we deduce

$$E_{8k+b} \equiv \binom{k}{2} E_{16+b} - k(k-2) E_{8+b} + \binom{k-1}{2} E_b \pmod{1024}.$$

Hence

(2.7) $$E_{8k+b} - E_b \equiv \binom{k}{2} (E_{16+b} - E_b) - k(k-2)(E_{8+b} - E_b) \pmod{1024}.$$

By [4, Corollary 3.7] we have
(2.8)
$$E_{4k} \equiv \frac{1 + 176k + 128k^2}{1 + 220k - 160k^2} \pmod{512} \quad \text{and} \quad E_{4k+2} \equiv \frac{-7 - 208k + 128k^2}{7 - 68k + 96k^2} \pmod{512}.$$

Corollary 2.1. *Let $k, m \in \mathbb{N}$ with $m \geq 4$. Then*

$$E_{2^m k} \equiv 1 - 2^m k(11 + 3 \cdot 2^m k) \pmod{2^{m+7}}$$

and

$$E_{2^m k + 2} \equiv -1 + 2^m k(45 + 3 \cdot 2^m k) \pmod{2^{m+7}}.$$

Corollary 2.2. *Let $b \in \{0, 2, 4, \ldots\}$ and $k, m \in \mathbb{N}$ with $m \geq 4$. Then*

$$E_{2^m k + b} \equiv E_b - 2^m k((b+1)^2 + 10 - (-1)^{\frac{b}{2}} 2^m k) \pmod{2^{m+6}}.$$

Corollary 2.3. *Let $b \in \{0, 2, 4, \ldots\}$ and $k, m \in \mathbb{N}$ with $m \geq 5$. Then*

$$E_{2^m k + b} \equiv \begin{cases} E_b - 11 \cdot 2^m k \pmod{2^{m+5}} & \text{if} \quad b \equiv 0, 14 \pmod{16}, \\ E_b + 13 \cdot 2^m k \pmod{2^{m+5}} & \text{if} \quad b \equiv 2, 12 \pmod{16}, \\ E_b - 3 \cdot 2^m k \pmod{2^{m+5}} & \text{if} \quad b \equiv 4, 10 \pmod{16}, \\ E_b + 5 \cdot 2^m k \pmod{2^{m+5}} & \text{if} \quad b \equiv 6, 8 \pmod{16}. \end{cases}$$

Proof. By Corollary 2.2 we have

$$E_{2^m k + b} \equiv E_b - 2^m k((b+1)^2 + 10) \pmod{2^{m+5}}.$$

Thus the result follows.

In the end we pose the following conjecture.

Conjecture 2.1. *Let $m, n \in \mathbb{N}$, $m \geq n$ and $b \in \{0, 2, 4, \ldots, 2^{m+n-1} - 2\}$. Then*

$$E_{2^m k + b} - E_b \equiv E_{2^m k + 2^{m+n-1} - 2 - b} - E_{2^{m+n-1} - 2 - b} \pmod{2^{m+n}}.$$

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