MODULI OF VORTICES AND GRASSMANN MANIFOLDS

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Abstract. We use the framework of Quot schemes to give a novel description of the moduli spaces of stable n-pairs, also interpreted as gauged vortices on a closed Riemann surface Σ with target $\mathrm{Mat}_{r\times n}(\mathbb{C})$, where $n \geq r$. We then show that these moduli spaces embed canonically into certain Grassmann manifolds, and thus obtain natural Kähler metrics of Fubini–Study type; these spaces are smooth at least in the local case $r = n$. For abelian local vortices we prove that, if a certain "quantization" condition is satisfied, the embedding can be chosen in such a way that the induced Fubini–Study structure realizes the Kähler class of the usual L^2 metric of gauged vortices. We also give a detailed description of the moduli spaces in the nonabelian local case.

1. Introduction

Gauged vortices are configurations of static, stable fields arising in various classical field theories on a Riemann surface Σ. These objects were first studied as topological solitons of the abelian Higgs model, for which vortex solutions have a distinctive particle-like behavior — they are labelled by divisors on Σ, which specify the precise locations of the cores of individual objects superposing nonlinearly to yield each vortex configuration [JT]. In this setting, there is typically a moduli space of all vortices with a given topology, modelled on the space of effective divisors with a fixed degree. This is a smooth manifold endowed with a complex structure induced from the one specified on Σ. More recently, models for vortices with nontrivial internal structure have been considered, but in the various generalizations it has remained a challenge to understand the corresponding moduli spaces in a satisfactory way.

We shall focus on vortices on a closed Riemann surface Σ with target (or internal) space consisting of the vector space $\mathrm{Mat}_{r\times n}(\mathbb{C})$ of complex $r \times n$ matrices, where $n \geq r$. These have been called nonabelian vortices in the literature, even though the special situation $r = 1$ corresponds to an abelian gauge theory. If $n > r$, one sometimes speaks of semilocal vortices, whereas $n = r$ is known as the local case. The geometric framework is as follows. Let e^2 be any positive real number. Assume that we fix a Kähler form ω_Σ on Σ, as well as a Hermitian metric on a complex vector bundle $E \longrightarrow \Sigma$ of rank r. A vortex is a pair (A, ϕ) consisting of a unitary connection A on the bundle, together with a section ϕ of the direct sum $E^{\oplus n} \longrightarrow \Sigma$,

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satisfying the vortex equations

$$\bar{\partial}_A \phi = 0, \tag{1.1}$$

$$*F_A + e^2 \mu \circ \phi = 0. \tag{1.2}$$

Here, $\bar{\partial}_A$ denotes the holomorphic structure [DK] on $E^{\oplus n}$ defined by the connection A and the complex structure on Σ, $*$ is the Hodge star of the Kähler metric associated to ω_Σ, $F_A := \mathrm{d}A + \frac{1}{2}[A, A]$ is the curvature of A and μ denotes a moment map

$$\mu : \mathrm{Mat}_{r\times n}(\mathbb{C}) \longrightarrow \mathfrak{u}(r)^* \cong \mathfrak{u}(r)$$

of the Hamiltonian action of the reduced structure group $\mathrm{U}(r)$ on the fibers of $E^{\oplus n} \longrightarrow \Sigma$ by multiplication on the left. We use the Killing form on $\mathfrak{u}(r)$ to identify the Lie algebra with its dual. Notice that μ is specified only up to addition of scalar matrices, and following standard conventions we shall write

$$\mu(w) = -\frac{\sqrt{-1}}{2}(ww^\dagger - \tau I_r),$$

where τ is a fixed real number and I_r is the $r \times r$ identity matrix.

The vortex equations (1.1)–(1.2) first appeared in the work [BDW2] of Bertram, Daskalopoulos and Wentworth computing the Gromov–Witten invariants of Grassmannians: the moduli space of holomorphic maps from a compact Riemann surface to a Grassmannian embeds into the moduli space of stable holomorphic n-pairs. The latter can be identified with the space of gauge-equivalence classes of solutions to the vortex equations above, under suitable stability criteria depending on the parameter τ and the topology. This is an example of what is generically known as the Hitchin–Kobayashi correspondence, which goes back to [UY]. Among other things, the authors of [BDW2] described how the moduli space of vortices changes birationally when the parameter τ crosses certain critical values, a phenomenon familiar from earlier work of Thaddeus on moduli of stable pairs [Th]. There is also a useful description of the moduli spaces via infinite-dimensional symplectic reduction (in the spirit of [AB]), which naturally produces a Kähler structure from the L^2 inner product on the space of fields; for abelian vortices, this was described by García-Prada in [Ga]. By now, a whole body of rather well-established technology that reproduces results of this type has been developed for objects that are analogous to vortices on the gauge-theory side of the Hitchin–Kobayashi correspondence, and n-pairs on the other side. The objects on the algebraic-geometric side are often referred to by the name of augmented bundles, of which Higgs bundles and coherent systems are other important examples; we refer the reader to [BDGW] for a clear overview.

Physicists have also been interested in the generalized vortex equations (1.1)–(1.2). Their solutions realize certain configurations of branes in string theory on the one hand, and also feature in models for confinement in QCD [EINOS, To]. Here, one focus of interest is to obtain descriptions of the moduli spaces as explicit as possible, including concrete parametrizations, as well as to understand natural

Hamiltonian systems on the moduli spaces or their cotangent bundles. Much of the work done assumes $\Sigma = \mathbb{C}$, for which there is nothing like a Hitchin–Kobayashi correspondence, but alternative constructions have been proposed which rely on certain mathematical conjectures. More recently, Baptista presented a rigorous description of the moduli space of local vortices when Σ is compact, describing a stratification of the moduli spaces in terms of spaces of internal structures [Ba1]. From Baptista's description, holomorphic matrices representing vortex solutions up to unitary gauge transformations can be readily constructed. From our perspective, his work has the slight disadvantage of depending on auxiliary structure, namely the choice of an inner product on $\mathbb{C}^n$, and it is also difficult to see how the different strata are patched together.

In this paper, we make use of the Hitchin–Kobayashi correspondence of [BDW2] to describe moduli spaces of solutions to the vortex equations (1.1)–(1.2) modulo gauge equivalence,

$$\mathcal{M}_\Sigma = \mathcal{M}_\Sigma(n, r, d), \tag{1.3}$$

where $d = \deg(E)$ is the degree of $E \longrightarrow \Sigma$, in terms of certain Quot schemes parametrizing holomorphic n-pairs. The idea of Quot (or quotient) schemes goes back to Grothendieck [Gr] and has had numerous applications to moduli problems. Given a coherent sheaf and a polynomial, the Quot scheme is a projective scheme of finite type that parametrizes all quotients of the given sheaf for which the Hilbert polynomial [EH] is the given polynomial.

Starting with an ample line bundle $\mathcal{L} \longrightarrow \Sigma$, we shall produce a holomorphic embedding of the moduli space $\mathcal{M}_\Sigma$ into a Grassmann manifold; it follows that $\mathcal{M}_\Sigma$ is projective. A Hermitian structure on $\mathcal{L}$ then induces a Kähler metric of Fubini–Study type on the moduli space. The perspective of Quot schemes has the advantage of being global in nature, and also well-suited to address general questions such as smoothness. We shall also see how it allows a straightforward calculation of the dimension. These properties can also be recovered from more general results scattered in the literature on Gromov–Witten invariants [OT, BDW2].

The simplest example of our class of embeddings into Grassmann manifolds occurs when we set $n = r = 1$; more background on the geometry of the moduli space of vortices in this well-studied case shall be given in Section 5.1 below. Then we have $\mathcal{M}_\Sigma \cong \mathrm{Sym}^d(\Sigma)$, where $d = \deg(E)$ is the vortex number [Br1]. In this setting, one might hope that a suitable choice of hermitian metric on $\mathcal{L}$ will induce a Fubini–Study metric which is related to the usual L^2 metric on the moduli space of vortices. We shall show that, if a certain quantization condition holds, then it turns out that the two corresponding Kähler classes are cohomologous; this is the content of our Theorem 5.1 below. The Kähler class of the moduli space of local abelian vortices was calculated in [MN].

In the last section, we will give a description of the moduli space of local vortices ($n = r$), describing in detail the fibers of a natural map

$$\Phi : \mathcal{M}_\Sigma \longrightarrow \mathrm{Sym}^d(\Sigma)$$

to the space of effective divisors of degree d on Σ. The fibers over reduced divisors are products of projective spaces. For nonreduced divisors, the fibers are more complicated but can be constructed by performing Hecke modifications on vector bundles over Σ, as we shall see. This description is more intrinsic than the one provided by Baptista [Ba1] in terms of internal structures.

2. Stability and the Hitchin–Kobayashi correspondence

Let Σ be a compact connected Riemann surface of genus g. Fix a Kähler form ω_Σ on Σ, so ω_Σ is a positive $(1,1)$-form; it is automatically closed. We will denote by

$$\mathrm{Vol}(\Sigma) := \int_\Sigma \omega_\Sigma \tag{2.1}$$

the total area of the surface determined by ω_Σ.

We briefly sketch the results in [BDW2] establishing the Hitchin–Kobayashi correspondence between solutions of (1.1)–(1.2) up to gauge transformations, and stable n-pairs (E, s) up to isomorphism. We begin by recalling the following

Definition 2.1. *An* n-pair *on the Riemann surface* Σ *is a pair of the form* (E, s)*, where* $E \longrightarrow \Sigma$ *is a holomorphic vector bundle and* $s \in H^0(\Sigma, E^{\oplus n})$*. Two* n*-pairs* (E, s) *and* (E', s') *are said to be* isomorphic *if there is an isomorphism of holomorphic vector bundles* $\psi : E^{\oplus n} \longrightarrow E'^{\oplus n}$ *over* Σ *such that* $\psi^* s' = s$*.*

In this paper, we will denote by $r = \mathrm{rk}(E)$ the rank of a fixed class of vector bundles E over Σ, when no confusion will arise.

The basic mechanism of the correspondence is modeled on Donaldson's famous proof of the Narasimhan–Seshadri theorem [D]. Suppose that we are given an n–pair (E, s). A holomorphic vector bundle $E \longrightarrow \Sigma$ with a Hermitian structure has a unique connection A preserving the Hermitian structure whose $(0,1)$-part coincides with the Dolbeault operator defining the holomorphic structure; this connection A is known as the *Chern connection* [A, pp. 191–192, Proposition 5], [CCL, p. 273]. For a C^∞ section ϕ of $E^{\oplus n} \longrightarrow \Sigma$, the pair (A, ϕ) is a solution of (1.1) if and only if ϕ is holomorphic. So we start by taking $\phi = s \in H^0(\Sigma, E^{\oplus n})$. Complex gauge transformations preserve equation (1.1), and one can ask whether the complex gauge orbit through this initial pair $(A, \phi = s)$ contains a solution of equation (1.2), which itself is invariant only under unitary gauge transformations. The answer is that this occurs if and only if the n–pair (E, s) is τ–stable in a sense that we will explain shortly, for the value of τ appearing in equation (1.2). This solution is unique up to unitary gauge transformations, and therefore we obtain an injective map from the moduli space of τ-stable n–pairs to the moduli space of vortices.

Conversely, a vortex (A, ϕ) in this geometric setting determines an n–pair: E is the bundle where each component of ϕ takes values, with holomorphic structure on E determined by the connection A and the complex structure on Σ. Clearly, one obtains isomorphic n-pairs (E, ϕ) when the original vortex (A, ϕ) undergoes unitary gauge transformations, and one can check that they are still τ–stable.

The stability condition that is appropriate to relate n-pairs and vortices was spelled out in [BDW2, BDGW], using the analysis for stable pairs in [Br2]. Fixing τ, one says that an n-pair (E, ϕ) is τ-stable if the following two conditions hold:

(i) $4\pi \deg(E')/\mathrm{rk}(E') \;<\; \tau e^2 \,\mathrm{Vol}(\Sigma)$ for all holomorphic subbundles $E' \subseteq E$, and
(ii) $4\pi \deg(E/E_s)/\mathrm{rk}(E/E_s) \;>\; \tau e^2 \,\mathrm{Vol}(\Sigma)$ for all holomorphic subbundles $E_s \subsetneq E$ containing all the component sections of s.

($\mathrm{Vol}(\Sigma)$ is defined in (2.1); unlike [BDW2, BDGW], we do not require this area to be normalized.) Notice that, when $E' = E$, condition (i) is necessary for vortex solutions to exist for a given τ: this follows from integrating equation (1.2) over Σ.

Now suppose that $n \geq r = \mathrm{rk}(E)$, and that $\phi \in H^0(\Sigma, E^{\oplus n})$ has maximal rank generically on Σ. Then there is no proper subbundle of E containing all the components of ϕ, and the second condition above is empty. Going through the argument in the proof of Proposition 3.14 in [BDW2], one can show that, under the same assumptions, the inequality

$$\tau e^2 \,\mathrm{Vol}(\Sigma) > 4\pi \deg(E) \tag{2.2}$$

is equivalent to the first condition for τ-stability. Throughout this paper, when the topology of $E \longrightarrow \Sigma$ has been fixed, as well as a Kähler structure on Σ, we shall only deal with the vortex equations (1.1)–(1.2) with values of τ satisfying (2.2). Then we can focus purely on n-pairs and their algebraic geometry to describe the moduli spaces in (1.3).

3. Holomorphic sections of a direct sum

We shall from now on take the algebraic-geometric point of view on the moduli space of vortices provided by the Hitchin-Kobayashi correspondence explained in Section 2. In the present section, the only relevant geometric structures are the complex structure on the closed surface Σ and the holomorphic structures on vector bundles over it.

As before, let $E \longrightarrow \Sigma$ be a holomorphic vector bundle of rank r. Choose an integer $n \geq r$. Let

$$s \,\in\, H^0(\Sigma,\, E^{\oplus n}) \,\cong\, H^0(\Sigma,\, E)^{\oplus n} \tag{3.1}$$

be a holomorphic section. Let $s_i \in H^0(\Sigma, E)$ be the image of s for the projection $E^{\oplus n} \longrightarrow E$ to the i–th factor. So, $s = (s_1, \ldots, s_n)$. Let

$$f_s : \mathcal{O}_\Sigma^{\oplus n} \longrightarrow E \tag{3.2}$$

be the homomorphism defined by $(x\,; c_1, \cdots, c_n) \longmapsto \sum_{i=1}^n c_i \cdot s_i(x)$, where $x \in \Sigma$ and $c_i \in \mathbb{C}$. The image $\mathrm{im}(f_s)$ of f_s is a coherent sheaf which is torsion-free because it is contained in the torsion-free sheaf E. Therefore, $\mathrm{im}(f_s)$ is locally free; equivalently, it is a holomorphic vector bundle. However, $\mathrm{im}(f_s)$ need not be a subbundle of E.

Definition 3.1. Let

$$H^0(\Sigma, E^{\oplus n})_0 \subset H^0(\Sigma, E^{\oplus n})$$

be the subset consisting of sections s as in (3.1) such that the rank of the vector bundle $\mathrm{im}(f_s)$ is r (the rank of E).

It is easy to see that $H^0(\Sigma, E^{\oplus n})_0$ is a Zariski open subset of $H^0(\Sigma, E^{\oplus n})$, and that it is closed under multiplication with $\mathbb{C}^*$ (but it can be empty). Note that it corresponds to the subset of holomorphic sections defining stable n-pairs (E, s), as described in Section 2: if $s \in H^0(\Sigma, E^{\oplus n})_0$, then outside a finite subset of Σ, the image of f_s coincides with E. Therefore, the quotient $E/\mathrm{im}(f_s)$ is either zero, or it is a torsion sheaf supported at finitely many points.

Take any $s \in H^0(\Sigma, E^{\oplus n})_0$. Let

$$\mathcal{K} := \ker(f_s) \subset \mathcal{O}_\Sigma^{\oplus n}$$

be the kernel of the homomorphism f_s in (3.2); it is a subbundle of $\mathcal{O}_\Sigma^{\oplus n}$ because $\mathcal{O}_\Sigma^{\oplus n}/\mathcal{K} = \mathrm{im}(f_s)$ is torsion-free. Consider the dual homomorphism

$$(\mathcal{O}_\Sigma^{\oplus n})^* \cong \mathcal{O}_\Sigma^{\oplus n} \longrightarrow \mathcal{K}^*$$

to the inclusion map $\mathcal{K} \hookrightarrow \mathcal{O}_\Sigma^{\oplus n}$. So $\mathcal{K}^*$ is a quotient bundle of $\mathcal{O}_\Sigma^{\oplus n}$.

We have a short exact sequence of coherent sheaves on Σ

$$0 \longrightarrow E^* \xrightarrow{f_s^*} \mathcal{O}_\Sigma^{\oplus n} \longrightarrow \mathcal{K}^* \oplus \mathcal{T} =: \mathcal{Q} \longrightarrow 0\,, \tag{3.3}$$

where $\mathcal{T}$ is either a torsion sheaf supported on finitely many points of Σ, or $\mathcal{T} = 0$; in fact, $\mathcal{T}$ is isomorphic to the quotient sheaf $E/\mathrm{im}(f_s)$ (but there is no canonical isomorphism).

Since E^* is a subsheaf of a trivial vector bundle, it follows that the degree of E^* is never positive; hence we will require throughout that

$$d := \deg(E) = -\deg(E^*) \geq 0\,. \tag{3.4}$$

We now introduce an ample line bundle $\mathcal{L} \longrightarrow \Sigma$ over the surface where the vortices live. (For the purposes of the present section, this line bundle plays an auxiliary role, and its choice does not affect any of the results.) Since $\ell := \deg(\mathcal{L})$ is necessarily positive, there is an integer

$$\delta_E \in \mathbb{N} \tag{3.5}$$

such that, for all $\delta \geq \delta_E$,

$$H^1(\Sigma,\, E^* \otimes \mathcal{L}^{\otimes \delta}) \,=\, 0\,, \tag{3.6}$$

and the natural evaluation homomorphism

$$H^0(\Sigma,\, E^* \otimes \mathcal{L}^{\otimes \delta}) \otimes_{\mathbb{C}} \mathcal{O}_\Sigma \,\longrightarrow\, E^* \otimes \mathcal{L}^{\otimes \delta} \tag{3.7}$$

is surjective. The second condition means that the vector bundle $E^* \otimes \mathcal{L}^{\otimes \delta}$ is generated by its global holomorphic sections. We emphasize that at this stage δ_E depends on the holomorphic vector bundle E. The Riemann–Roch theorem yields

$$\dim H^0(\Sigma,\, E^* \otimes \mathcal{L}^{\otimes \delta}) - \dim H^1(\Sigma,\, E^* \otimes \mathcal{L}^{\otimes \delta}) \,=\, r\ell\delta - d + r(1-g) \tag{3.8}$$

and this determines the dimension of $H^0(\Sigma, E^* \otimes \mathcal{L}^{\otimes \delta})$ whenever $\delta \geq \delta_E$, by (3.6).

Suppose that an integer δ is fixed, satisfying $\delta \geq \delta_E$. Tensoring (3.3) with $\mathcal{L}^{\otimes \delta}$, we obtain the short exact sequence of coherent sheaves

$$0 \longrightarrow E^* \otimes \mathcal{L}^{\otimes \delta} \longrightarrow (\mathcal{L}^{\otimes \delta})^{\oplus n} \longrightarrow \mathcal{Q} \otimes \mathcal{L}^{\otimes \delta} \longrightarrow 0\,. \tag{3.9}$$

This will give rise to a long exact sequence of cohomology groups

$$0 \longrightarrow H^0(\Sigma,\, E^* \otimes \mathcal{L}^{\otimes \delta}) \longrightarrow H^0(\Sigma,\, (\mathcal{L}^{\otimes \delta})^{\oplus n}) \xrightarrow{\ q\ } H^0(\Sigma,\, \mathcal{Q} \otimes \mathcal{L}^{\otimes \delta}) \longrightarrow 0\,, \tag{3.10}$$

where the right-exactness follows from (3.6).

Consider the quotient map $Q : H^0(\Sigma,\, (\mathcal{L}^{\otimes \delta})^{\oplus n}) \longrightarrow H^0(\Sigma,\, \mathcal{Q} \otimes \mathcal{L}^{\otimes \delta})$ in (3.10). The subsheaf

$$E^* \otimes \mathcal{L}^{\otimes \delta} \,\subset\, (\mathcal{L}^{\otimes \delta})^{\oplus n}$$

in (3.9) can be reconstructed from Q, and from it the morphism f_s in (3.2), by a procedure that we will now describe.

Let

$$\widehat{\mathcal{K}} \,:=\, \ker Q$$

be the kernel of the quotient map, and let

$$\mathcal{S} \,\subset\, (\mathcal{L}^{\otimes \delta})^{\oplus n} \tag{3.11}$$

be the subsheaf generated by the sections lying in the subspace $\widehat{\mathcal{K}}$. From the exactness of the sequence (3.10) we know that $\widehat{\mathcal{K}}$ coincides with the subspace

$$H^0(\Sigma,\, E^* \otimes \mathcal{L}^{\otimes \delta}) \hookrightarrow H^0(\Sigma,\, (\mathcal{L}^{\otimes \delta})^{\oplus n})$$

determined by the section $s \in H^0(\Sigma, E^{\oplus n})_0$. Also, the holomorphic vector bundle $E^* \otimes \mathcal{L}^{\otimes \delta}$ is generated by its global sections (recall that the homomorphism in (3.7) is surjective). Consequently, the subsheaf $\mathcal{S}$ in (3.11) coincides with the subsheaf

$$E^* \otimes \mathcal{L}^{\otimes \delta} \,\subset\, (\mathcal{L}^{\otimes \delta})^{\oplus n}$$

in (3.9). In other words, we have reconstructed the subsheaf $E^* \otimes \mathcal{L}^{\otimes\delta}$ of $(\mathcal{L}^{\otimes\delta})^{\oplus n}$ from the quotient vector space $H^0(\Sigma,\, (\mathcal{L}^{\otimes\delta})^{\oplus n})/H^0(\Sigma,\, E^* \otimes \mathcal{L}^{\otimes\delta})$, or equivalently from the linear map Q in (3.10).

Let $E' \longrightarrow \Sigma$ be a holomorphic vector bundle, and let $s \in H^0(\Sigma,\, (E')^{\oplus n})_0$ (see Definition 3.1) be such that

$$H^1(\Sigma,\, (E')^* \otimes \mathcal{L}^{\otimes\delta}) \,=\, 0\,,$$

and also assume that $(E')^* \otimes \mathcal{L}^{\otimes\delta}$ is generated by its global sections. Let $\mathcal{Q}'$ be the quotient of $\mathcal{O}_\Sigma^{\oplus n}$ constructed from E' just as $\mathcal{Q}$ is constructed from E (see (3.3)). Therefore, $\mathcal{Q}' \otimes \mathcal{L}^{\otimes\delta}$ is a quotient of $(\mathcal{L}^{\otimes\delta})^{\oplus n}$. If the two quotients $H^0(\Sigma,\, \mathcal{Q} \otimes \mathcal{L}^{\otimes\delta})$ and $H^0(\Sigma,\, \mathcal{Q}' \otimes \mathcal{L}^{\otimes\delta})$ of $H^0(\Sigma,\, (\mathcal{L}^{\otimes\delta})^{\oplus n})$ coincide, then the subsheaf $E^* \otimes \mathcal{L}^{\otimes\delta}$ of $(\mathcal{L}^{\otimes\delta})^{\oplus n}$ (see (3.9)) coincides with the subsheaf $(E')^* \otimes \mathcal{L}^{\otimes\delta}$ constructed as in (3.9) using E'. Indeed, this follows from the above observation that we can reconstruct the subsheaf $E^* \otimes \mathcal{L}^{\otimes\delta}$ of $(\mathcal{L}^{\otimes\delta})^{\oplus n}$ from the quotient map $Q : H^0(\Sigma,\, (\mathcal{L}^{\otimes\delta})^{\oplus n}) \longrightarrow H^0(\Sigma,\, \mathcal{Q} \otimes \mathcal{L}^{\otimes\delta})$ in (3.10).

We put down the observations above in the form of what we will call a "reconstruction" lemma:

Lemma 3.2. *The quotient* $H^0(\Sigma,\, (\mathcal{L}^{\otimes\delta})^{\oplus n}) \longrightarrow H^0(\Sigma,\, \mathcal{Q} \otimes \mathcal{L}^{\otimes\delta})$ *in* (3.10) *uniquely determines the subsheaf*

$$E^* \otimes \mathcal{L}^{\otimes\delta} \,\subset\, (\mathcal{L}^{\otimes\delta})^{\oplus n}$$

in (3.9).

Remark 3.3. Consider the subsheaf $E^* \otimes \mathcal{L}^{\otimes\delta} \;\subset\; (\mathcal{L}^{\otimes\delta})^{\oplus n}$ in Lemma 3.2. Its dual $E \otimes (\mathcal{L}^*)^{\otimes\delta}$ is a quotient of $((\mathcal{L}^{\otimes\delta})^{\oplus n})^* \,=\, ((\mathcal{L}^*)^{\otimes\delta})^{\oplus n}$. Tensoring this quotient homomorphism

$$((\mathcal{L}^*)^{\otimes\delta})^{\oplus n} \,\longrightarrow\, E \otimes (\mathcal{L}^*)^{\otimes\delta}$$

with the identity homomorphism of $\mathcal{L}^{\otimes\delta}$, we get back the homomorphism

$$f_s \,:\, (\mathcal{O}_\Sigma)^{\oplus n} \,\longrightarrow\, E$$

used to construct the quotient in (3.10). So the quotient effectively determines the n-pair.

We will now show that, for fixed $n \geq r$, a suitably large integer $\delta > \delta_E$ in (3.5) can be uniformly found that depends only on the rank and the degree of the vector bundles E arising in stable n-pairs, such that (3.7) is surjective and (3.6) holds; so we can replace all the relevant δ_E by an integer $\delta_{n,r,d}$ depending only on topological quantities, and not on the holomorphic structure. This fact implies that, once integers n, r, d have been fixed, with $n \geq r$, the moduli space of stable n-pairs can be described as a Quot scheme of finite type $\mathcal{M}_\Sigma(n,r,d)$ parametrizing quotients q as in (3.10), and from which the pair (E, s) (and hence a vortex solution to (1.1) and (1.2)) can be reconstructed.

Proposition 3.4. *Fix a positive integer r, a nonnegative integer d and an integer $n \geq r$. Given an ample line bundle $\mathcal{L} \longrightarrow \Sigma$, there is an integer $\delta_{n,r,d}$ such that for any n-pair $(E\,,s)$ with $\mathrm{rk}(E) = r$, $\deg(E) = d$ and*

$$s \in H^0(\Sigma,\, E^{\oplus n})_0$$

(see Definition 3.1), and any integer $\delta \geq \delta_{n,r,d}$,

- *the homomorphism in* (3.7) *is surjective, and*
- (3.6) *holds.*

Proof. The strategy of the proof is to first show, using the idea of Quot scheme, that all such pairs of the given numerical type form a bounded family; then the proof is completed using upper semicontinuity for dimension of cohomology.

Take a pair $(E\,,s)$, where $E \longrightarrow \Sigma$ is a holomorphic vector bundle of rank r and degree d, and

$$s \in H^0(\Sigma,\, E^{\oplus n})_0\,.$$

The vector bundle E^* is a subsheaf of $\mathcal{O}_\Sigma^{\oplus n}$ of rank r and degree $-d$ (see (3.3)). Therefore, all possible pairs $(E^*\,, f_s^*)$ (see (3.3)) are parametrized by a projective scheme $\mathbb{T}$ over $\mathbb{C}$ of finite type [HL, p. 40, Theorem 2.2.4] (set S in [HL, Theorem 2.2.4] to be a point). Now from upper semicontinuity of dimension of H^1 we conclude that there is an integer k_0, that depends only on n, r and d, such that for all $(E\,,s)$ of the above type and all $\delta \geq k_0$,

$$H^1(\Sigma,\, E^* \otimes \mathcal{L}^{\otimes\delta}) = 0\,.$$

Take any point $x \in \Sigma$. Consider the short exact sequence of sheaves

$$0 \longrightarrow E^* \otimes \mathcal{L}^{\otimes\delta} \otimes \mathcal{O}_\Sigma(-x) \longrightarrow E^* \otimes \mathcal{L}^{\otimes\delta} \longrightarrow (E^* \otimes \mathcal{L}^{\otimes\delta})_x \longrightarrow 0\,.$$

Let

$$H^0(\Sigma,\, E^* \otimes \mathcal{L}^{\otimes\delta}) \longrightarrow (E^* \otimes \mathcal{L}^{\otimes\delta})_x \longrightarrow H^1(\Sigma,\, E^* \otimes \mathcal{L}^{\otimes\delta} \otimes \mathcal{O}_\Sigma(-x)) \tag{3.12}$$

be the corresponding long exact sequence in cohomology. From (3.12) we conclude that if

$$H^1(\Sigma,\, E^* \otimes \mathcal{L}^{\otimes\delta} \otimes \mathcal{O}_\Sigma(-x)) = 0\,, \tag{3.13}$$

then the homomorphism $H^0(\Sigma,\, E^* \otimes \mathcal{L}^{\otimes\delta}) \longrightarrow (E^* \otimes \mathcal{L}^{\otimes\delta})_x$ in (3.12) is surjective. Therefore, given $(E^*\,, f_s^*)$, if (3.13) holds for all $x \in \Sigma$, then the homomorphism in (3.7) is surjective.

Now all possible pairs $(E^*\,, f_s^*)$ (see (3.3)) are parametrized by a projective scheme over $\mathbb{C}$ (see above). From upper semicontinuity of dimension of H^1, we conclude again that there is an integer k_1 such that, for all $(E\,,s)$ of the type in the statement of the proposition, and all $\delta \geq k_1$, the homomorphism in (3.7) is surjective.

Consequently, the integer

$$\delta_{n,r,d} := \max\{k_0\,, k_1\}\,, \tag{3.14}$$

which depends only on r, d and n, has the property that for all $\delta \geq \delta_{n,r,d}$, and for any pair $(E\,,s)$ of the of the type in the statement of the proposition, the homomorphism in (3.7) is surjective, and (3.6) holds. This completes the proof of the proposition. □

Note that in Proposition 3.4 we assume the degree d to be nonnegative because of the inequality in (3.4).

4. Embedding in a Grassmannian

As in Section 3, fix a positive integer r, a nonnegative integer d and an integer $n \geq r$, specifying the topology of $E \longrightarrow \Sigma$ and the number of copies of E in a direct sum. For a given ample line bundle $\mathcal{L} \longrightarrow \Sigma$ of degree ℓ, fix also an integer $\delta \geq \delta_{n,r,d}$, where $\delta_{n,r,d}$ is as in Proposition 3.4, cf. (3.14). Notice that we can always set δ to be the minimal $\delta_{n,r,d}$ ensuring that both (3.6) is surjective and the vanishing in (3.7) holds, and in fact we will be doing so by default. A consequence of our previous discussion is that

$$\ell\delta \geq \frac{d}{r} + g - 1\,; \tag{4.1}$$

this follows from equations (3.8) and (3.6).

At this point, we shall introduce metric structures on the basic objects that we have been considering in the previous section. We equip Σ with a Kähler metric ω_Σ, and the ample line bundle $\mathcal{L} \longrightarrow \Sigma$ in Section 3 with a Hermitian structure $h_{\mathcal{L}}$. If the Kähler class $[\omega_\Sigma] \in H^2(\Sigma, \mathbb{R})$ is integral, which amounts to

$$\int_\Sigma \omega_\Sigma \,\in\, \mathbb{Z}\,,$$

it would be natural to require $(\mathcal{L}, h_{\mathcal{L}})$, together with its Chern connection $\nabla_{\mathcal{L}}$, to be a prequantization of the Kähler structure on Σ, in the sense that its curvature is proportional to the Kähler form as

$$F_{\nabla_{\mathcal{L}}} \,=\, 2\pi\sqrt{-1}\cdot \omega_\Sigma\,; \tag{4.2}$$

but for now we need not impose this condition. Consider the vector space

$$H^0(\Sigma,\, (\mathcal{L}^{\otimes\delta})^{\oplus n}) \,\cong\, H^0(\Sigma,\, \mathcal{L}^{\otimes\delta})^{\oplus n}\,.$$

The Hermitian structure $h_{\mathcal{L}}$ on $\mathcal{L}$ together with the Kähler form ω_Σ on Σ define an L^2 inner product on $H^0(\Sigma,\, (\mathcal{L}^{\otimes\delta})^{\oplus n})$.

Let

$$\mathrm{Gr} \,:=\, \mathrm{Gr}(H^0(\Sigma,\, (\mathcal{L}^{\otimes\delta})^{\oplus n})\,, r(\ell\delta - g + 1) - d) \tag{4.3}$$

be the Grassmannian of subspaces of $H^0(\Sigma,\, (\mathcal{L}^{\otimes\delta})^{\oplus n})$ of dimension $r(\ell\delta - g+1) - d$ (see (3.8) and (3.10)). The inner product on $H^0(\Sigma,\, (\mathcal{L}^{\otimes\delta})^{\oplus n})$ defines a Fubini–Study Kähler form on Gr. Indeed, for any subspace

$$H^0(\Sigma,\, (\mathcal{L}^{\otimes\delta})^{\oplus n}) \,\supset\, V \,\in\, \mathrm{Gr}\,,$$

we have

$$T_V\mathrm{Gr} \,=\, V^* \otimes (H^0(\Sigma,\, (\mathcal{L}^{\otimes\delta})^{\oplus n})/V)\,,$$

where $T_V\mathrm{Gr}$ is the holomorphic tangent space at the point V of Gr. The L^2 inner product we have on $H^0(\Sigma,\, (\mathcal{L}^{\otimes\delta})^{\oplus n})$ defined above induces inner products on both V and $H^0(\Sigma,\, (\mathcal{L}^{\otimes\delta})^{\oplus n})/V$. Therefore, we get an inner product on $T_V\mathrm{Gr}$. It is easy to see that the Hermitian structure on Gr constructed in this way is actually Kähler.

Another way of describing this Kähler structure is to consider the Fubini–Study metric on the projective space of lines in $\bigwedge^{r(\ell\delta-g+1)-d} H^0(\Sigma,\, (\mathcal{L}^{\otimes\delta})^{\oplus n})$

$$\mathbb{P}(\wedge^{r(\ell\delta-g+1)-d} H^0(\Sigma,\, (\mathcal{L}^{\otimes\delta})^{\oplus n}))$$

induced by the inner product on $H^0(\Sigma, \mathcal{L}^{\otimes\delta})$. The Plücker map [GH]

$$P \,:\, \mathrm{Gr} \,\longrightarrow\, \mathbb{P}(\bigwedge^{r(\ell\delta-g+1)-d} H^0(\Sigma,\, (\mathcal{L}^{\otimes\delta})^{\oplus n})), \tag{4.4}$$

defined by

$$\mathrm{Gr} \,\ni\, \mathrm{span}_{\mathbb{C}}\{s_1, \ldots, s_{r(\ell\delta-g+1)-d}\} \,\longmapsto\, s_1 \wedge \cdots \wedge s_{r(\ell\delta-g+1)-d},$$

embeds Gr as a complex submanifold of the target. The above Kähler structure on Gr coincides with the restriction of the Fubini–Study metric on the projective space to the image of P.

Let

$$\mathcal{M}_\Sigma \,:=\, \mathcal{M}_\Sigma(n, r, d) \tag{4.5}$$

be the moduli space of isomorphism classes of all n-pairs $(E\,, s)$, on Σ where the holomorphic bundle $E \,\longrightarrow\, \Sigma$ has rank r and degree d, and

$$s \,\in\, H^0(\Sigma,\, E^{\oplus n})_0\,.$$

We now claim that we have an embedding

$$\varphi \,:\, \mathcal{M}_\Sigma \,\longrightarrow\, \mathrm{Gr} \tag{4.6}$$

that sends any $(E\,, s)$ to the subspace $H^0(\Sigma,\, E^* \otimes \mathcal{L}^{\otimes\delta}) \,\subset\, H^0(\Sigma,\, (\mathcal{L}^{\otimes\delta})^{\oplus n})$ in (3.10), where Gr is defined in (4.3). Note that (3.6) and (3.8) together imply that $H^0(\Sigma,\, E^* \otimes \mathcal{L}^{\otimes\delta})$ has dimension $r(\ell\delta - g + 1) - d$, and this means that φ is well defined. The map φ is also injective from Lemma 3.2 and Remark 3.3. In this way, the moduli space $\mathcal{M}_\Sigma$ can be regarded as a closed subvariety of the Grassmannian Gr in (4.3).

One advantage of our description of the moduli space $\mathcal{M}_\Sigma$ is that one can address its smoothness in a straightforward way. Take any point $\underline{z} \,:=\, (E\,, s) \,\in\, \mathcal{M}_\Sigma$ of the moduli space. Let

$$0 \,\longrightarrow\, E^* \,\xrightarrow{f_s^*}\, \mathcal{O}_\Sigma^{\oplus n} \,\longrightarrow\, \mathcal{K}^* \oplus \mathcal{T} \,=:\, \mathcal{Q} \,\longrightarrow\, 0$$

be the short exact sequence constructed in (3.3) from the n-pair $(E\,, s)$. The tangent space to $\mathcal{M}_\Sigma$ at the point $\underline{z} \,:=\, (E\,, s)$ has the following description:

$$T_{\underline{z}}\mathcal{M}_\Sigma \,=\, H^0(\Sigma,\, \mathcal{H}om(E^*\,, \mathcal{Q})) \,=\, H^0(\Sigma,\, E \otimes \mathcal{Q})\,. \tag{4.7}$$

The obstruction to smoothness of $\mathcal{M}_\Sigma$ at $\underline{z}$ is given by

$$\mathrm{Ext}^1_{\mathcal{O}_\Sigma}(E^*\,,\mathcal{Q})\,,$$

where $\mathrm{Ext}^1_{\mathcal{O}_\Sigma}$ is the global Ext. Since E^* is a vector bundle,

$$\mathrm{Ext}^1_{\mathcal{O}_\Sigma}(E^*\,,\mathcal{Q}) \;=\; H^1(\Sigma,\, E^*\otimes\mathcal{Q}^*)\,. \tag{4.8}$$

In the local case, where $n = r := \mathrm{rank}(E)$, the quotients $\mathcal{Q}$ in (3.3) are torsion sheaves supported on finitely many points of Σ. In that case, $E^*\otimes\mathcal{Q}^*$ is a torsion sheaf, and hence

$$H^1(\Sigma,\, E\otimes\mathcal{Q}) \;=\; 0\,.$$

Therefore, from (4.8) we conclude $\mathrm{Ext}^1_{\mathcal{O}_\Sigma}(E^*\,,\mathcal{Q}) \;=\; 0$ if $n = r$, implying that the variety $\mathcal{M}_\Sigma$ is smooth if $n = r$.

From the description in terms of local n-pairs, one can compute the dimension of the moduli spaces by standard Riemann–Roch arguments, recovering results in [BDW2]. Assume that $d > r(g-1)$. Then we claim that

$$\dim\mathcal{M}_\Sigma \;=\; nd + r(r-n)(g-1)\,. \tag{4.9}$$

To see this, note that the dimension of the space of infinitesimal deformations of a simple vector bundle $E\longrightarrow\Sigma$ of rank r, which coincides with $\dim H^1(\Sigma,\,\mathrm{End}(E))$, is $r^2(g-1)+1$ by the Riemann–Roch theorem. Also, for a general vector bundle E of rank r and degree d with $d > r(g-1)$, the dimension of $H^0(X,\,E)$ is $d-r(g-1)$ again by Riemann–Roch and the fact that $H^1(X,\,E) \;=\; 0$. When $d \le r(g-1)$, there will not be such a simple formula because $H^1(\Sigma, E) \neq 0$ for general E. However, in the local case $r = n$, the formula (4.9) will remain valid, as we will see explicitly in Section 6 below.

Since the map φ in (4.6) embeds $\mathcal{M}_\Sigma$ in Gr as a complex submanifold, we can obtain Kähler structures on the moduli space $\mathcal{M}_\Sigma$ by restricting a Kähler forms on the Grassmann manifold Gr to it. In the following, we shall denote by ω_{Gr} the Kähler form on the moduli space $\mathcal{M}_\Sigma$ obtained by pulling back the Fubini–Study 2-form on Gr described above, using the embedding φ. In the next section, we will see when it will be possible to make ω_{Gr} cohomologous to the usual Kähler structure ω_{L^2} on the moduli space of vortices, in the abelian case where the Kähler class $[\omega_{L^2}]$ is known.

Although the Kähler form ω_{L^2} depends on both the metric on Σ and a Hermitian metric on the vector bundle $E\longrightarrow\Sigma$, there is a natural splitting $\omega_{L^2} = \omega_1+\omega_2$, where ω_1 is a closed $(1,1)$-form depending only on ω_Σ (see [MN] for the abelian case). A natural question to ask is how ω_1 is related to ω_{Gr} when the prequantization condition (4.2) is imposed. This is one issue that we plan to address in future work.

5. Abelian local vortices: $n = r = 1$

It is natural to ask whether the Kähler form $\omega_{\rm Gr}$ on the vortex moduli space $\mathcal{M}_\Sigma$ induced from the embedding φ into the Grassmannian manifold, as described in Section 4, is related to the L^2 Kähler structure inherited naturally from the gauge theory, which is of interest to physicists. We shall address this issue in the present section, but our discussion will be restricted to the case of abelian local vortices, where $n = 1$, $r = 1$. So throughout this section we will be assuming that

$$\mathcal{M}_\Sigma \,=\, \mathcal{M}_\Sigma(1,1,d)\,.$$

5.1. **Some background on the geometry of the abelian local case.** Let us briefly recall how the Kähler structures ω_{L^2} on $\mathcal{M}_\Sigma$ arise in the abelian local case. There are many alternative descriptions of the L^2 metrics of vortices, but here we will concentrate on a particularly insightful one given by García-Prada in [Ga], which uses infinite-dimensional symplectic geometry. The space of fields appearing as variables in the vortex equations (1.1)–(1.2) is $\mathcal{A}\times\mathcal{C}$, where $\mathcal{A}$ is the space of unitary connections on the line bundle $E \longrightarrow \Sigma$ and $\mathcal{C} = C^\infty(\Sigma, E)$ is the vector space of smooth sections of this bundle. Any two connections differ by a global real 1-form on Σ with values on the Lie algebra $\mathfrak{u}(1) \cong \sqrt{-1}\cdot\mathbb{R}$, so $\mathcal{A}$ is an affine space modelled on the vector space $\Omega^1(\Sigma,\mathbb{R})$. Thus in fact $\mathcal{A}\times\mathcal{C}$ is a complex manifold with complex structure induced from the one on Σ:

$$(\dot{A},\dot{\phi}) \,\longmapsto\, (*A\,,\sqrt{-1}\,\dot{\phi})\,. \tag{5.1}$$

The component of this map in the first factor is the Hodge star operator on Σ acting on 1-forms, which squares to $-{\rm id}_{\Omega^1(\Sigma)}$, whereas the component in the second factor is the complex structure on the fibers of the bundle $E \longrightarrow \Sigma$.

There is an action of the gauge group ${\rm Aut}_\Sigma(E) \cong C^\infty(\Sigma, {\rm U}(1))$ on fields $(A,\phi) \in \mathcal{A}\times\mathcal{C}$, namely

$$(A,\phi) \,\longmapsto\, (A - u^{-1}{\rm d}u, u\phi)\,, \tag{5.2}$$

where $u \in {\rm Aut}_\Sigma(E)$. This action turns out to be Hamiltonian with respect to a natural product symplectic form,

$$\omega_{\mathcal{A}} + \omega_{\mathcal{C}}\,, \tag{5.3}$$

defined on the space of fields. The factor denoted by $\omega_{\mathcal{A}}$ in (5.3) is the Atiyah–Bott structure [AB] on the space of connections $\mathcal{A}$, while $\omega_{\mathcal{C}}$ is the natural symplectic structure (of constant coefficients, hence closed) on $\mathcal{C}$ produced out of the Kähler structure on Σ and the Hermitian metric on E. The latter is usually simply called the L^2 structure (on $\mathcal{C}$), since it is associated to the metric

$$||\dot{\phi}||^2_{L^2} = \int_\Sigma (\dot{\phi},\dot{\phi})_{h_E}\omega_\Sigma$$

defined for all sections $\dot{\phi} \in C^\infty(\Sigma, E) \cong T_\phi\mathcal{C}$, for any $\phi \in \mathcal{C}$. The complex structure (5.1) on the space of fields $\mathcal{A} \times \mathcal{C}$ preserves (5.3), so one can regard this space as a Kähler manifold.

The first vortex equation (1.1) is invariant under the complex structure (5.1), so the infinite-dimensional submanifold $\mathcal{N}$ of solutions to this equation (pairs (A, ϕ) where ϕ is a holomorphic section for the holomorphic structure on $E \longrightarrow \Sigma$ associated to the connection A, cf. [DK]) has an induced Kähler structure, which is again preserved by the $\mathrm{Aut}_\Sigma(E)$-action (5.2). It turns out that the left-hand side of the second vortex equation (1.2) is a moment map for this action. So the moduli space of solutions of both (1.1) and (1.2), where the action of the group of gauge transformations is quotiented out, can be understood as the infinite-dimensional Meyer–Marsden–Weinstein quotient

$$\mathcal{M}_\Sigma = \mathcal{N} /\!\!/ \mathrm{Aut}_\Sigma(E)\,. \tag{5.4}$$

This receives a symplectic structure, denoted as ω_{L^2}, and which is usually referred to as the L^2 structure on the moduli space of vortices $\mathcal{M}_\Sigma$. In fact, this argument is formal, since we are dealing with an infinite-dimensional quotient, but the intuitive picture just given is confirmed by the analysis carried out in [Ga], which is itself quite insightful. The Kähler form ω_{L^2} satisfies the properties

$$p^*\omega_{L^2} = i^*(\omega_\mathcal{A} + \omega_\mathcal{C}), \qquad i : \mathcal{N} \hookrightarrow \mathcal{A} \times \mathcal{C}\,,$$

where p denotes the projection from $\mathcal{N}$ to the space of $\mathrm{Aut}_\Sigma(E)$-orbits.

Under the stability condition (2.2), Bradlow [Br1] and García-Prada [Ga] showed that the quotient $\mathcal{M}_\Sigma$ in (5.4) can be identified with the d-th symmetric power of Σ,

$$\mathcal{M}_\Sigma \cong \mathrm{Sym}^d(\Sigma) := \Sigma^d/\mathfrak{S}_d \tag{5.5}$$

as a complex manifold. This space parametrizes effective divisors of degree d, interpreted as portrayals of vortex locations on Σ. But the symplectic structure ω_{L^2} on $\mathcal{M}_\Sigma$ turns out to be much more difficult to describe explicitly.

When comparing ω_L^2 with ω_{Gr}, the most basic question to ask is whether the two are cohomologous (up to a scalar multiple, say) for any choice of the data. The answer to this question is trivially affirmative if $g = 0$, since then $\mathrm{Sym}^d(\Sigma) \cong \mathbb{P}^d$ and $H^2(\mathbb{P}^d, \mathbb{Z}) \cong \mathbb{Z}$, so the interesting setting for this question is $g \geq 1$. Then the cohomology ring of $\mathrm{Sym}^d(\Sigma)$ is more complicated; the intersection

$$H^{1,1}(\mathrm{Sym}^d(\Sigma), \mathbb{C}) \cap H^2(\mathrm{Sym}^d(\Sigma), \mathbb{Z}), \tag{5.6}$$

where the Kähler classes of the moduli space are contained, turns out to be a rank two lattice. The Kähler class $[\omega_{L^2}]$ has been computed as [MN, Ba2]

$$[\omega_{L^2}] = \left(\pi\tau\mathrm{Vol}(\Sigma) - \frac{4\pi^2 d}{e^2}\right)\eta + \frac{2\pi^2}{e^2}\sigma\,; \tag{5.7}$$

a description of the generators η and σ of (5.6) will be given in Section 5.3. It is remarkable that this formula involves so little detail on the geometrical data needed to set up the vortex equations and to define the L^2 metric. In the following, we shall be interested in calculating the Kähler class $[\omega_{\mathrm{Gr}}]$ and relating it with $[\omega_{L^2}]$. The result (5.7) has been used to compute the symplectic volume of the moduli space [MN] and the total scalar curvature [Ba2] — such quantities carry only cohomological information.

5.2. **Description of the embedding.** In the abelian local case, we can describe the embedding (4.6) constructed in Section 4 more explicitly. More precisely, we will be interested in characterizing the composition $P\circ\varphi$, where P is the Plücker embedding (4.4). Given the result (5.5), the map we are interested in is

$$P\circ\varphi\,:\,\mathrm{Sym}^d(\Sigma)\,\longrightarrow\,\mathbb{P}(\wedge^{\ell\delta-g-d+1}H^0(\Sigma,\,\mathcal{L}^{\otimes\delta}))\tag{5.8}$$

where P and φ are constructed in (4.4) and (4.6), respectively. We shall give a description the holomorphic line bundle on $\mathrm{Sym}^d(\Sigma)$ associated to this projective embedding.

Let p_1 (respectively, p_2) be the projection of $\mathrm{Sym}^d(\Sigma)\times\Sigma$ to $\mathrm{Sym}^d(\Sigma)$ (respectively, Σ). Let also

$$\Delta_0\,\subset\,\mathrm{Sym}^d(\Sigma)\times\Sigma$$

be the tautological divisor consisting of all points $(z\,,x)\,\in\,\mathrm{Sym}^d(\Sigma)\times\Sigma$ such that $x\,\in\,z$.

Consider the line bundle $p_2^*\mathcal{L}^{\otimes\delta}$ on $\mathrm{Sym}^d(\Sigma)\times\Sigma$, and the torsion sheaf defined by

$$\mathcal{B}\,:=\,p_2^*\mathcal{L}^{\otimes\delta}/(p_2^*\mathcal{L}^{\otimes\delta}\otimes\mathcal{O}_{\mathrm{Sym}^d(\Sigma)\times\Sigma}(-\Delta_0))\,\longrightarrow\,\mathrm{Sym}^d(\Sigma)\times\Sigma.$$

The support of $\mathcal{B}$ is Δ_0, which is finite over $\mathrm{Sym}^d(\Sigma)$ of degree d. Hence the direct image

$$p_{1*}\mathcal{B}\,\longrightarrow\,\mathrm{Sym}^d(\Sigma)$$

is a vector bundle of rank d. So $\bigwedge^d p_{1*}\mathcal{B}$ is a line bundle over $\mathrm{Sym}^d(\Sigma)$.

We have a canonical isomorphism of line bundles over $\mathrm{Sym}^d(\Sigma)$

$$\bigwedge^d p_{1*}\mathcal{B}\,=\,(P\circ\varphi)^*\mathcal{O}_{\mathbb{P}(\wedge^{\ell-g-d+1}H^0(\Sigma,\,\mathcal{L}^{\otimes\delta}))}(1)\,,\tag{5.9}$$

where $P\circ\varphi$ is the map in (5.8), and

$$\mathcal{O}_{\mathbb{P}(\wedge^{\ell-g-d+1}H^0(\Sigma,\,\mathcal{L}^{\otimes\delta}))}(1)\,\longrightarrow\,\mathbb{P}(\wedge^{\ell-g-d+1}H^0(\Sigma,\,\mathcal{L}^{\otimes\delta}))$$

is the tautological line bundle. This means that the embedding (5.8) is associated to the complete linear system corresponding to the holomorphic line bundle $\bigwedge^d p_{1*}\mathcal{B}\longrightarrow\mathrm{Sym}^d(\Sigma)$.

5.3. **Representability of the L^2 Kähler structure.** Our main goal in this section is to prove the following representability result:

Theorem 5.1. *Consider the embedding (5.8), constructed from an ample line bundle $\mathcal{L} \longrightarrow \Sigma$ of degree ℓ and an integer $\delta > \delta_{1,1,d}$, where $\delta_{1,1,d}$ is as in Proposition 3.4 and $d > 1$. Then the Fubini–Study metric on* $\mathrm{Sym}^d(\Sigma)$ *(obtained by pulling back the usual Fubini–Study metric using this map) is cohomologous to a multiple of the L^2-metric of vortices on the line bundle $E \longrightarrow \Sigma$ exactly when*

$$q := \frac{\tau e^2}{4\pi}\mathrm{Vol}(\Sigma) \in \mathbb{N} \tag{5.10}$$

and the integers ℓ, δ are chosen such that

$$\ell\delta = q + g - 1\,. \tag{5.11}$$

This result means that, at least in the abelian local case, the Kähler structure ω_{Gr} on $\mathcal{M}_\Sigma$ discussed in Section 4 provides a realization of the Kähler class of the L^2 geometry of vortices if (5.10) and (5.11) hold. Note that the condition (5.10) is rather natural from the point of view of geometric quantization, as it implies that the symplectic structure $\frac{e^2}{2\pi^2}\,\omega_{L^2}$ is (pre)quantizable in the sense of Weil:

$$\left[\frac{e^2}{2\pi^2}\,\omega_{L^2}\right] \in H^2(\mathcal{M}_\Sigma, \mathbb{Z}) \tag{5.12}$$

(From (5.7), it follows that the Weil quantization condition (5.12) is equivalent to $q \in \frac{1}{2}\mathbb{N}$.) It would be very striking if the full L^2 geometry were to be described by a Fubini–Study structure, but we will not attempt to address this question here. Even in the case $g = 0$, for which the representability of $[\omega_{L^2}]$ in the sense we are using is trivial, this question has not yet been settled rigorously.

To set the stage for the proof of Theorem 5.1, we introduce the following curves on $\mathrm{Sym}^d(\Sigma)$, regarded as the space of degree d effective divisors on Σ:

$$\begin{aligned} \Sigma_\emptyset &:= \{dx \mid x \in \Sigma\}, \\ \Sigma_p &:= \{p + (d-1)x \mid x \in \Sigma\}, \qquad \text{for} \quad p \in \Sigma. \end{aligned}$$

We shall denote their cohomology classes by

$$\Sigma_0 = [\Sigma_\emptyset] \qquad \text{and} \qquad \Sigma_1 = [\Sigma_p],$$

respectively. (Clearly, the cohomology class of Σ_p is independent of $p \in \Sigma$ because we are assuming that Σ is connected.) Let us also set

$$d_0 := \deg(P \circ \varphi(\Sigma_\emptyset)), \tag{5.13}$$

$$d_1 := \deg(P \circ \varphi(\Sigma_p)) \tag{5.14}$$

to be the degrees of the images of the curves above by the map (5.8), whose target is a complex projective space of dimension

$$N = \binom{\ell\delta - g + 1}{d} - 1\,.$$

We first claim that the integers d, g, d_0 and d_1 determine the cohomology class

$$[(P\circ\varphi)^*\omega_{\rm FS}] \in H^{1,1}({\rm Sym}^d(\Sigma),\mathbb{C}) \cap H^2({\rm Sym}^d(\Sigma),\mathbb{Z}).$$

To describe this, we start by recalling the basic result [M]

$$H^k({\rm Sym}^d(\Sigma),\mathbb{Z}) \;\cong\; H^k(\Sigma^d,\mathbb{Z})^{\mathfrak{S}_d}, \qquad k \,\in\, \mathbb{N}\,.$$

The intersection of cohomology groups in (5.6) is generated over $\mathbb{Z}$ by the two cohomology classes of degree two [MN]

$$\eta \;=\; \sum_{i=1}^{d} \beta_i \qquad \text{and} \qquad \sigma \;=\; \sum_{j=1}^{g} \sigma_j\,. \tag{5.15}$$

Here, the cohomology classes β_i come from the fundamental class $\beta \,\in\, H^2(\Sigma,\mathbb{Z})$; more precisely, $\beta_i \,=\, \pi_i^*\beta$, where $\pi_i \,:\, \Sigma^d \,\longrightarrow\, \Sigma$ denotes the projection to the i-th factor. Moreover, we denote

$$\sigma_j := \xi_j\xi_{j+g}, \qquad \text{where} \qquad \xi_j = \sum_{k=1}^{d}\alpha_{j,k}, \qquad 1\le j\le 2g, \tag{5.16}$$

and the $\alpha_{j,k}$ are classes of degree one which come from the middle cohomology of Σ, namely

$$\alpha_{j,k} = \pi_k^*\alpha_j\,.$$

In this expression, the α_j denote elements in a standard basis of $H^1(\Sigma,\mathbb{Z})$, satisfying [F]

$$\begin{aligned} \alpha_i\alpha_j &= 0 & i &\neq j\pm g,\\ \alpha_i\alpha_{i+g} &= -\alpha_{i+g}\alpha_i = \beta & 1 &\le i\le g. \end{aligned}$$

So we may write

$$(P\circ\varphi)^*[\omega_{\rm FS}] \;=\; C_\eta\eta + C_\sigma\sigma\,, \tag{5.17}$$

where η and σ are the generators in (5.15), so our task is to obtain the coefficients $C_\eta, C_\sigma \in \mathbb{Z}$ as functions of d, g, d_0 and d_1.

Lemma 5.2. *The duality pairing on* ${\rm Sym}^d(\Sigma)$ *satisfies:*

$$\langle\eta,\Sigma_j\rangle = d-j \quad \textit{and} \quad \langle\sigma,\Sigma_j\rangle = (d-j)^2 g \quad \textit{for} \quad j\in\{0,1\}.$$

Proof. The pairings for $j = 0$ can be reduced to computations in the cohomology ring of ${\rm Sym}^d(\Sigma)$, which has been given a presentation in [M, (6.3)]. In fact, the statement in reference [M] is not totally accurate — we refer the reader to Section 2 of [BT] for the corrected result. For our purposes, it will suffice to state that $H^*({\rm Sym}^d(\Sigma),\mathbb{Z})$ is generated by the classes η in (5.15) and ξ_j in (5.16), $j = 1,\dots,2g$, which supercommute according to the parity of their degrees; in particular, one has

$$\eta\sigma_j = \sigma_j\eta, \qquad j = 1,\dots,g,$$

where σ_j were defined in (5.16), since η and ξ_j commute. The extra relations among the generators can be expressed as follows: given three disjoint subsets

$$I_1, I_2, J \subset \{1, \ldots, g\}$$

and a nonnegative integer r satisfying [BT, (2.3)]

$$r \geq d - |I_1| - |I_2| - 2|J| + 1,$$

there is a nontrivial relation

$$\eta^r \prod_{i_1 \in I_1} \xi_{i_1} \prod_{i_2 \in I_2} \xi_{i_2+g} \prod_{j \in J} (\eta - \sigma_j) = 0. \tag{5.18}$$

For $d \geq 1$, we have the relation

$$\eta^{d-1}\sigma = g\eta^d. \tag{5.19}$$

This follows from summing the relations

$$\eta^{d-1}\sigma_j = \eta^d, \qquad j = 1, \ldots, g \tag{5.20}$$

over j. Notice that (5.20) can be obtained from (5.18) by taking $r = d-1$, $J = \{j\}$ and $I_1 = I_2 = \emptyset$.

Another relation contained in (5.18) is that, for $i \neq j$ and $d > 1$,

$$\eta^{d-2}\sigma_i\sigma_j = \eta^{d-1}(\sigma_i + \sigma_j) - \eta^d; \tag{5.21}$$

this one is obtained by setting $r = d-2$, $J = \{i, j\}$ and $I_1 = I_2 = \emptyset$. Since $\sigma_j^2 = 0$ from the anticommutativity of the ξ_j's (for each $j = 1, \ldots, g$), we also have that

$$\begin{aligned}
\eta^{d-2}\sigma^2 &= 2 \sum_{1 \leq i < j \leq g} \eta^{d-2}\sigma_i\sigma_j \\
&= 2 \sum_{1 \leq i < j \leq g} \eta^{d-1}(\sigma_i + \sigma_j) - g(g-1)\eta^d && (5.22)\\
&= g(g-1)\eta^d. && (5.23)
\end{aligned}$$

Step (5.22) made use of (5.21), whereas (5.23) used (5.20).

Another useful result by Macdonald [M, (15.4)] is that the Poincaré dual of the homology class Σ_0, for $d > 1$, is given by

$$\mathrm{PD}(\Sigma_0) = d(d + (g-1)(d-1))\eta^{d-1} - d(d-1)\eta^{d-2}\sigma. \tag{5.24}$$

This can be applied to calculate

$$\begin{aligned}
\langle \eta, \Sigma_0 \rangle &= d \int_{\mathrm{Sym}^d\Sigma} (d + (d-1)(g-1))\eta^d - (d-1)\eta^{d-1}\sigma \\
&= d \int_{\mathrm{Sym}^d\Sigma} (d + (d-1)(g-1) - (d-1)g)\eta^d && (5.25)\\
&= d \int_{\mathrm{Sym}^d\Sigma} \eta^d \\
&= d. && (5.26)
\end{aligned}$$

The second step (5.25) used (5.19), whereas the last step (5.26) follows from the fact that η^d is the fundamental class of $\mathrm{Sym}^d(\Sigma)$.

Using (5.24) once again, we can write

$$\begin{aligned} \langle \sigma, \Sigma_0 \rangle &= d \int_{\mathrm{Sym}^d\Sigma} (d + (d-1)(g-1))\sigma\eta^{d-1} - (d-1)\sigma\eta^{d-2}\sigma \\ &= d \int_{\mathrm{Sym}^d\Sigma} ((d + (d-1)(g-1))g - (d-1)g(g-1))\eta^d \\ &= d^2 g, \end{aligned} \tag{5.27}$$

where (5.27) is a consequence of (5.20) and (5.23).

Now consider the map $\iota : \Sigma \longrightarrow \Sigma_p$ given by

$$x \longmapsto p + (d-1)x \quad \in \quad \mathrm{Sym}^d(\Sigma)\,,$$

which is a biholomorphism. We have

$$\iota^*\eta = (d-1)\beta \tag{5.28}$$

and

$$\iota^*\xi_j = (d-1)\alpha_j, \qquad j = 1, \ldots, 2g,$$

which in turn implies

$$\iota^*(\xi_j\xi_{j+g}) = (d-1)^2\alpha_j\alpha_{j+g} = (d-1)^2\beta, \qquad j = 1, \ldots, g.$$

It follows that

$$\iota^*\sigma = (d-1)^2 g\beta\,. \tag{5.29}$$

So we can finally compute

$$\begin{aligned} \langle \eta, \Sigma_1 \rangle &= \int_{\Sigma_p} \eta \\ &= \int_{\Sigma} \iota^*\eta \\ &= (d-1) \int_{\Sigma} \beta \\ &= d-1 \end{aligned}$$

using (5.28), and likewise, from (5.29),

$$\langle \sigma\,, \Sigma_1 \rangle \;=\; (d-1)^2 g\,.$$

This completes the proof of the lemma. □

Since

$$\langle (P \circ \varphi)^*[\omega_{\mathrm{FS}}], \Sigma_j \rangle = d_j \qquad \text{for} \quad j = 0, 1,$$

the constants C_η and C_σ in (5.17) can be determined by solving a linear system whose coefficients are the four pairings in Lemma 5.2. The solution is

$$C_\eta = \frac{d^2 d_1 - (d-1)^2 d_0}{d(d-1)}, \tag{5.30}$$

$$C_\sigma = \frac{(d-1)d_0 - d d_1}{d(d-1)g}, \tag{5.31}$$

and this establishes our claim.

We want to compare the resulting Kähler class (5.17) with the Kähler class $[\omega_{L^2}]$ in (5.7) associated to the L^2 metric of vortices. The next task is to determine the degrees d_0 and d_1 defined in (5.13)–(5.14), in terms of the basic topological data.

Lemma 5.3. *$d_j = (d-j)(\ell\delta + (d-j-1)(g-1) - j)$ for $j \in \{0,1\}$.*

Proof. Let

$$\psi : \Sigma \longrightarrow \mathrm{Sym}^d(\Sigma) \tag{5.32}$$

be the morphism defined by $x \longmapsto dx$. Note that d_0 in (5.13) is the degree of

$$(P \circ \varphi \circ \psi)^* \mathcal{O}_{\mathbb{P}(\wedge^{\ell\delta - g - d + 1} H^0(\Sigma, \mathcal{L}^{\otimes\delta}))}(1)\,,$$

where $P \circ \varphi$ is the morphism in (5.8).

Let K_Σ be the holomorphic cotangent bundle of Σ.

Take any point $x \in \Sigma$. We have a natural filtration of coherent sheaves

$$\mathcal{L}^\delta \otimes \mathcal{O}_\Sigma(-dx) \subset \mathcal{L}^\delta \otimes \mathcal{O}_\Sigma((1-d)x) \subset \cdots \subset \mathcal{L}^\delta \otimes \mathcal{O}_\Sigma(-ix)$$
$$\subset \mathcal{L}^\delta \otimes \mathcal{O}_\Sigma((1-i)x) \subset \cdots \subset \mathcal{L}^\delta \otimes \mathcal{O}_\Sigma(-x) \subset \mathcal{L}^\delta\,.$$

For any $i \in [1\,,d]$, the quotient $\mathcal{L}^\delta \otimes \mathcal{O}_\Sigma((1-i)x)/\mathcal{L}^\delta \otimes \mathcal{O}_\Sigma(-ix)$ is the torsion sheaf $\mathcal{L}^\delta_x \otimes (K_\Sigma^{\otimes(i-1)})_x$ supported at x. Consequently, we have a canonical identification

$$(P \circ \varphi \circ \psi)^* \mathcal{O}_{\mathbb{P}(\wedge^{\ell\delta - g - d + 1} H^0(\Sigma, \mathcal{L}^{\otimes\delta}))}(1)_x = \mathcal{L}^{d\delta}_x \otimes (K_\Sigma^{\otimes d(d-1)/2})_x$$

(see (5.9)), where ψ is the map in (5.32). Moving x over Σ, this isomorphism produces an isomorphism of line bundles

$$(P \circ \varphi \circ \psi)^* \mathcal{O}_{\mathbb{P}(\wedge^{\ell\delta - g - d + 1} H^0(\Sigma, \mathcal{L}^{\otimes\delta}))}(1) = \mathcal{L}^{d\delta}_x \otimes (K_\Sigma^{\otimes d(d-1)/2})$$

Since $\deg(K_\Sigma) = 2(g-1)$, this immediately implies that $d_0 = d(\ell\delta + (g-1)(d-1))$.

Fix a point $p \in \Sigma$. Let

$$\psi_1 : \Sigma \longrightarrow \mathrm{Sym}^d(\Sigma) \tag{5.33}$$

be the morphism defined by $x \longmapsto p + (d-1)x$. Note that d_1 in (5.14) coincides with

$$\deg((P \circ \varphi \circ \psi_1)^* \mathcal{O}_{\mathbb{P}(\wedge^{\ell\delta - g - d + 1} H^0(\Sigma, \mathcal{L}^{\otimes\delta}))}(1))\,,$$

where $P \circ \varphi$ is the morphism in (5.8).

For notational convenience, the line bundle $\mathcal{L}^\delta \otimes \mathcal{O}_\Sigma(-p)$ will be denoted by ζ.

As before, take any point $x \in \Sigma$. We have a natural filtration of coherent sheaves

$$\zeta \otimes \mathcal{O}_\Sigma((1-d)x) \subset \zeta \otimes \mathcal{O}_\Sigma((2-d)x) \subset \cdots \subset \zeta \otimes \mathcal{O}_\Sigma(-x) \subset \zeta\,.$$

For any $i \in [1\,, d-1]$, the quotient $\zeta \otimes \mathcal{O}_\Sigma((1-i)x)/\mathcal{L}^\delta \otimes \mathcal{O}_\Sigma(-ix)$ is the torsion sheaf $\zeta_x \otimes (K_\Sigma^{\otimes(i-1)})_x$ supported at x. Consequently, we have a canonical identification

$$(P\circ\varphi\circ\psi_1)^*\mathcal{O}_{\mathbb{P}(\wedge^{\ell\delta-g-d+1}H^0(\Sigma,\,\mathcal{L}^{\otimes\delta}))}(1)_x \;=\; \zeta_x^{\otimes(d-1)} \otimes (K_\Sigma^{\otimes(d-1)(d-2)/2})_x \otimes \mathcal{L}_p^{\otimes\delta}\,,$$

(see (5.9)), where ψ_1 is the map in (5.33). Fixing an isomorphism of the line $\mathcal{L}_p^{\otimes\delta}$ with $\mathbb{C}$ (recall that p is fixed), and moving x over Σ, the above isomorphism gives an isomorphism of line bundles

$$(P\circ\varphi\circ\psi_1)^*\mathcal{O}_{\mathbb{P}(\wedge^{\ell\delta-g-d+1}H^0(\Sigma,\,\mathcal{L}^{\otimes\delta}))}(1) \;=\; \zeta_x^{\otimes(d-1)} \otimes (K_\Sigma^{\otimes(d-1)(d-2)/2})\,.$$

Since $\deg(\zeta) = \ell\delta-1$, this implies that $d_1 = (\ell\delta-1)(d-1)+(g-1)(d-1)(d-2)$. □

Proof of Theorem 5.1. Using Lemma 5.3 in (5.30) and (5.31), we find

$$\begin{aligned} C_\eta &= \ell\delta - d - g + 1,\\ C_\sigma &= 1, \end{aligned}$$

which are integers as expected. Comparing with the coefficients of η and σ in (5.7), the formula (5.11) for the quantity q in Theorem 5.1 immediately follows. The quantization condition (5.10) results from all the other terms in (5.11) being integers. □

Note that when the inequality (2.2) ensuring stability is saturated, in the situation

$$\tau \to \frac{4\pi d}{e^2 \mathrm{Vol}(\Sigma)} \tag{5.34}$$

which is called the limit of "dissolved" vortices by Manton and Romão in [MR], the quantization condition (5.10) is automatically satisfied with $q = d$. Then imposing the condition (5.11) implies that (4.1) also becomes an equality, which unfortunately makes the Grassmannian (4.3) collapse. The nontrivial situation closest to this collapse would be to consider

$$\tau = \frac{4\pi(d+1)}{e^2\mathrm{Vol}(\Sigma)} \quad \Rightarrow \quad q = d+1,$$

for which the Grassmannian (4.3) is a projective space; if the area of Σ is taken to be large, this value of τ will still be close to the critical value (5.34). In this context (provided $\delta_{1,1,d}$ does not turn out to be too large), the geometry of the Kähler structure ω_{Gr} we introduced in Section 4, assuming $\ell\delta = d+g$, should give an approximation of the L^2 geometry of the moduli spaces, as an extension of the work by Baptista and Manton [BM] in the case $g = 0$.

6. Nonabelian local vortices: $n = r > 1$

In this section we shall assume that $r = n$ and will give a description of the non-abelian situation $n > 1$ by means of Hecke modifications [FB, HL] on holomorphic vector bundles over Σ. So from now on we shall let

$$\mathcal{M}_\Sigma \,=\, \mathcal{M}_\Sigma(n,n,d)$$

denote the moduli space of local vortices (see (4.5)). We know that these can also be described as n–pairs if the condition (2.2) holds; these sections generically generate the vector bundle.

Take any $(E\,,s) \,\in\, \mathcal{M}_\Sigma$. Consider the homomorphism f_s in (3.2). Since the sections of E in s generate E generically, the quotient $E/f_s(\mathcal{O}_\Sigma^{\oplus n})$ is a torsion sheaf supported on finitely many points, and we have

$$\dim H^0(\Sigma,\, E/f_s(\mathcal{O}_\Sigma^{\oplus n})) \,=\, d\,.$$

Let

$$\Phi\,:\, \mathcal{M}_\Sigma \,\longrightarrow\, \mathrm{Sym}^d(\Sigma) \tag{6.1}$$

be the map to the symmetric product that sends each pair $(E\,,s)$ to the scheme-theoretic support [EH] of the torsion sheaf $E/f_s(\mathcal{O}_\Sigma^{\oplus n})$. To explain what this map Φ does, let m_x denote the dimension of the stalk of $E/f_s(\mathcal{O}_\Sigma^{\oplus n})$ at each point $x \,\in\, \Sigma$. Then Φ sends $(E\,,s)$ to $\sum_{x\in\Sigma} m_x\cdot x$.

The map Φ in (6.1) is clearly surjective. In what follows, we shall describe its fibers step by step, and obtain a description of the moduli space as a stratification by the type of the partitions of d associated to effective divisors of degree d.

6.1. The case of distinct points.

Let $\mathbb{P}^{r-1}$ be the projective space of hyperplanes in $\mathbb{C}^r$. Take d distinct points

$$x_1\,,\ldots\,,x_d\,\in\,\Sigma\,.$$

Let $\underline{x} \,\in\, \mathrm{Sym}^d(\Sigma)$ be the point defined by $\{x_1\,,\cdots\,,x_d\}$. We will show that the fiber of Φ over $\underline{x}$ is the Cartesian product $(\mathbb{P}^{r-1})^d$. This is a description of the generic fiber of the map Φ, and it coincides with the one in [Ba1].

Take any $(H_1\,,\cdots\,,H_d) \,\in\, (\mathbb{P}^{r-1})^d$. So each H_i is a hyperplane in $\mathbb{C}^r$. The fiber of the trivial vector bundle $(\mathcal{O}_\Sigma^{\oplus n})^* \,=\, \mathcal{O}_\Sigma^{\oplus n}$ over x_i is identified with $\mathbb{C}^r$. Thus the hyperplane $H_i \,\subset\, \mathbb{C}^r$ defines a hyperplane $\widetilde{H}_i$ in the fiber of $(\mathcal{O}_\Sigma^{\oplus n})^*$ over the point x_i. Let

$$\widetilde{q}\,:\, (\mathcal{O}_\Sigma^{\oplus n})^* \,\longrightarrow\, \bigoplus_{i=1}^d((\mathcal{O}_\Sigma^{\oplus n})^*)_{x_i}/\widetilde{H}_i \tag{6.2}$$

be the quotient map. The kernel of $\widetilde{q}$ will be denoted by $\widetilde{\mathcal{K}}$, and we have the following short exact sequence of sheaves on Σ:

$$0 \longrightarrow \widetilde{\mathcal{K}} \xrightarrow{h} (\mathcal{O}_\Sigma^{\oplus n})^* \xrightarrow{\widetilde{q}} \bigoplus_{i=1}^{d} ((\mathcal{O}_\Sigma^{\oplus n})^*)_{x_i}/\widetilde{H}_i \longrightarrow 0 .$$

Now consider the dual of the homomorphism h above,

$$h^* : \mathcal{O}_\Sigma^{\oplus n} \longrightarrow \widetilde{\mathcal{K}}^* .$$

It is easy to see that the pair $(\widetilde{\mathcal{K}}^*, h^*)$ defines a point in the fiber of Φ (see (6.1)) over the point $\underline{x}$, and that all the points in the fiber can be obtained by choosing the hyperplanes H_i suitably. This construction identifies the fiber of Φ over $\underline{x}$ with the Cartesian product $(\mathbb{P}^{r-1})^d$. Employing the usual terminology, we can say that we have constructed the bundle $E = \widetilde{\mathcal{K}}$ of an n-pair by performing d elementary Hecke modifications (one at each x_i) on the trivial bundle of rank $r = n$ over Σ, and the inclusion h yields the morphism $h^* = f_s$ in (3.2) which is equivalent to a holomorphic section $s \in H^0(\Sigma, E^{\oplus n})$.

6.2. **Case of multiplicity two.** Now take $d-1$ distinct points

$$x_1, \dots, x_{d-1} \in \Sigma .$$

Let $\underline{x} \in \mathrm{Sym}^d(\Sigma)$ be the point defined by $2x_1 + \sum_{j=2}^{d-1} x_j$. We will describe the fiber of Φ over $\underline{x}$.

Let H_1 be a hyperplane in $\mathbb{C}^r$. Let

$$q_1 : (\mathcal{O}_\Sigma^{\oplus n})^* \longrightarrow ((\mathcal{O}_\Sigma^{\oplus n})^*)_{x_1}/\widetilde{H}_1$$

be the quotient map, where, just as in (6.2), $\widetilde{H}_1$ is the hyperplane in the fiber of $(\mathcal{O}_\Sigma^{\oplus n})^*$ over x_1 given by H_1. Let $\mathcal{K}(H_1)$ denote the kernel of q_1. So we have a short exact sequence of sheaves on Σ

$$0 \longrightarrow \mathcal{K}(H_1) \longrightarrow (\mathcal{O}_\Sigma^{\oplus n})^* \longrightarrow ((\mathcal{O}_\Sigma^{\oplus n})^*)_{x_1}/\widetilde{H}_1 \longrightarrow 0 . \tag{6.3}$$

Consider the space $\mathcal{S}_2$ of all objects of the form

$$(H_1, H_2, \cdots, H_{d-1}; H^1),$$

where H_i, $1 \leq i \leq d-1$, is a hyperplane in $\mathbb{C}^r$, and H^1 is a hyperplane in the fiber over x_1 of the vector bundle $\mathcal{K}(H_1)$. There is a natural surjective map from this space $\mathcal{S}_2$ to the fiber of Φ over the point $\underline{x}$ of $\mathrm{Sym}^d(\Sigma)$. To construct this map, first note that it follows from (6.3) that for any point $x \in \Sigma$ different from x_1, the fibers of $\mathcal{K}(H_1)$ and $(\mathcal{O}_\Sigma^{\oplus n})^*$ over x are identified. Hence for any $2 \leq j \leq d-1$, the hyperplane H_j gives a hyperplane in the fiber of $\mathcal{K}(H_1)$ over the point x_j; this hyperplane in the fiber of $\mathcal{K}(H_1)$ will be denoted by $\widetilde{H}_j$. Let $\mathcal{K}$ be the holomorphic vector bundle over Σ that fits in the following short exact sequence of sheaves:

$$0 \longrightarrow \mathcal{K} \xrightarrow{h} \mathcal{K}(H_1) \longrightarrow (\mathcal{K}(H_1)_{x_1}/H^1) \oplus \bigoplus_{j=2}^{d-1} \mathcal{K}(H_1)_{x_j}/\widetilde{H}_j \longrightarrow 0 . \tag{6.4}$$

As before, let h^* denote the dual map to h in (i3) and $\mathcal{K}^*$ the dual sheaf to $\mathcal{K} = \ker h$. The pair $(\mathcal{K}^*\,, h^*)$ defines an element of the moduli space $\mathcal{M}_\Sigma$ that lies over $\underline{x}$ for the projection Φ. Moreover, all elements in the fiber over $\underline{x}$ arise in this way for some element of $\mathcal{S}_2$.

Sending any $(H_1\,, H_2\,, \cdots\,, H_{d-1}\,; H^1) \in \mathcal{S}_2$ to $(H_1\,, H_2\,, \cdots\,, H_{d-1})$ we see that $\mathcal{S}_2$ is a projective bundle over $(\mathbb{P}^{r-1})^{d-1}$ of relative dimension $r-1$. Therefore, we have the following lemma:

Lemma 6.1. *The fiber $\Phi^{-1}(\underline{x})$ admits a natural surjective map from $\mathcal{S}_2$. The variety $\mathcal{S}_2$ is a projective bundle over $(\mathbb{P}^{r-1})^{d-1}$ of relative dimension $r-1$.*

6.3. **Case of multiplicity $m > 2$.** Let m be an integer satisfying $2 < m \leq d$, and fix $d-m+1$ distinct points $x_1\,, x_2\,, \cdots\,, x_{d-m+1}$ of Σ. Let

$$\underline{x} \in \mathrm{Sym}^d(\Sigma)$$

be the point defined by $m \cdot x_1 + \sum_{j=2}^{d-m+1} x_j$.

Let H_1 be a hyperplane in $\mathbb{C}^r$. Construct $\mathcal{K}(H_1)$ as in (6.3). Let

$$H^1 \subset \mathcal{K}(H_1)_{x_1}$$

be a hyperplane in the fiber of the vector bundle $\mathcal{K}(H_1)$ over the point x_1. Let $\mathcal{K}(H^1)$ be the holomorphic vector bundle over Σ that fits in the following exact sequence of sheaves

$$0 \longrightarrow \mathcal{K}(H^1) \longrightarrow \mathcal{K}(H_1) \longrightarrow \mathcal{K}(H_1)_{x_1}/H^1 \longrightarrow 0\,.$$

Now fix a hyperplane

$$H^2 \subset \mathcal{K}(H^1)_{x_1}$$

in the fiber of $\mathcal{K}(H^1)$ over x_1. Let $\mathcal{K}(H^2)$ be the holomorphic vector bundle over Σ that fits in the following short exact sequence of sheaves

$$0 \longrightarrow \mathcal{K}(H^2) \longrightarrow \mathcal{K}(H^1) \longrightarrow \mathcal{K}(H^1)_{x_1}/H^2 \longrightarrow 0\,.$$

Inductively, after j steps as above, fix a hyperplane

$$H^{j+1} \subset \mathcal{K}(H^j)_{x_1}$$

and construct the vector bundle $\mathcal{K}(H^{j+1})$ that fits in the short exact sequence

$$0 \longrightarrow \mathcal{K}(H^{j+1}) \longrightarrow \mathcal{K}(H^j) \longrightarrow \mathcal{K}(H^j)_{x_1}/H^{j+1} \longrightarrow 0\,. \tag{6.5}$$

Consider the space $\mathcal{S}_m$ of all elements of the form

$$(H_1\,, H_2\,, \cdots\,, H_{d-m+1}\,; H^1\,, H^2\,, \cdots\,, H^{m-1})\,,$$

where H_i is a hyperplane in $\mathbb{C}^r$, while H^1 is a hyperplane in $\mathcal{K}(H_1)_{x_1}$, and each H^j is a hyperplane in the fiber over x_1 of the vector bundle $\mathcal{K}(H^{j-1})$. There is a natural map from $\mathcal{S}_m$ to the fiber of Φ over the point $\underline{x}$. To construct this map, first note that, from (6.5), it follows inductively that for any point $x \in \Sigma \setminus \{x_1\}$, the fiber of $\mathcal{K}(H^{j+1})$ over x is identified with the fiber of $(\mathcal{O}_\Sigma^{\oplus n})^*$ over x. Therefore, for $2 \leq i \leq d-m+1$, the hyperplane H_i in $\mathbb{C}^r$ defines a hyperplane in the fiber of

$\mathcal{K}(H^{m-1})$ over the point x_i; this hyperplane in the fiber $\mathcal{K}(H^{m-1})_{x_i}$ will be denoted by $\widetilde{H}_i$. Let $\mathcal{K}$ be the holomorphic vector bundle over Σ that fits in the following short exact sequence of sheaves:

$$0 \longrightarrow \mathcal{K} \xrightarrow{h} \mathcal{K}(H^{m-1}) \longrightarrow \bigoplus_{j=2}^{d-m+1} \mathcal{K}(H^{m-1})_{x_j}/\widetilde{H}_j \longrightarrow 0\,. \tag{6.6}$$

The pair $(\mathcal{K}^*\,, h^*)$ in (6.6) defines a point of the moduli space $\mathcal{M}_\Sigma$ that lies over $\underline{x}$. All points in the fiber over $\underline{x}$ arise in this way, for some element of $\mathcal{S}_m$.

Consider the $m-1$ maps

$$\mathcal{S}_m \longrightarrow \cdots \longrightarrow (\mathbb{P}^{r-1})^{d-m+1}$$

defined by

$$(H_1\,, H_2\,, \cdots\,, H_{d-m+1}\,; H^1\,, H^2\,, \cdots\,, H^{m-1}) \longmapsto$$

$$(H_1\,, H_2\,, \cdots\,, H_{d-m+1}\,; H^1\,, H^2\,, \cdots\,, H^{m-2}) \longmapsto \cdots \longmapsto (H_1\,, H_2\,, \cdots\,, H_{d-m+1})\,.$$

Each of these is a projective bundle of relative dimension $r-1$. Therefore, we have the following generalization of Lemma 6.1:

Lemma 6.2. *The fiber $\Phi^{-1}(\underline{x})$ admits a natural surjective map from $\mathcal{S}_m$. There is a chain of $m-1$ maps starting from $\mathcal{S}_m$ ending in $(\mathbb{P}^{r-1})^{d-m+1}$ such that each one is a projective bundle of relative dimension $r-1$.*

6.4. **The general case.** The general case is not harder to understand than the previous case.

Take any point $\underline{x} := \sum_{i=1}^a m_i \cdot x_i$ of $\mathrm{Sym}^d(\Sigma)$, where m_i are arbitrary positive integers addind up to d and $x_i \in \Sigma$, $i = 1, \ldots, a$. For each point x_i, fix data $(H_i\,, H_i^1\,, \cdots\,, H_i^{m_i-1})$, where H_i is a hyperplane in $\mathbb{C}^r$, and the H_i^j are hyperplanes in the fibers, over x_i, of vector bundles constructed inductively as in the previous case. From the set of such objects, there is a canonical surjective map to the fiber of Φ over $\underline{x}$ by repeating the argument above.

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