

GREEN FUNCTIONS, THE FINE TOPOLOGY AND RESTORING COVERINGS

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ABSTRACT. There are several equivalent ways to define continuous harmonic functions $H(K)$ on a compact set K in \mathbb{R}^n . One may let $H(K)$ be the uniform closures of all functions in $C(K)$ which are restrictions of harmonic functions on a neighborhood of K , or take $H(K)$ as the subspace of $C(K)$ consisting of functions which are finely harmonic on the fine interior of K . In [DG74] it was shown that these definitions are equivalent. Using a localization result of [BH78] one sees that a function $h \in H(K)$ if and only if it is continuous and finely harmonic on every fine connected component of the fine interior of K . Such collection of sets are usually called *restoring*.

Another equivalent definition of $H(K)$ was introduced in [P97] using the notion of Jensen measures which leads another restoring collection of sets. The main goal of this paper is to reconcile the results in [DG74] and [P97].

To study these spaces, two notions of Green functions have previously been introduced. One by [P97] as the limit of Green functions on domains D_j where the domains D_j are decreasing to K , and alternatively following [F72, F75] one has the fine Green function on the fine interior of K . Our Theorem 3.2 shows that these are equivalent notions.

In Section 4 a careful study of the set of Jensen measures on K , leads to an interesting extension result (Corollary 5.1) for superharmonic functions. This has a number of applications. In particular we show that the two restoring coverings are the same. We are also able to extend some results of [GL83] and [P97] to higher dimensions.

1. INTRODUCTION

There are several ways to define the spaces $(S(K))-H(K)$ of continuous (super)-harmonic functions on a compact set K in \mathbb{R}^n . Let $C(K)$ denote the space of all continuous real functions on K . The natural definition is to let $H(K)$ or $S(K)$ be the uniform closures of all functions in $C(K)$ which are restrictions of harmonic (resp. superharmonic) functions on a neighborhood of K . More fashionably, we can define $H(K)$ and $S(K)$ as the subspaces of $C(K)$ consisting of functions which are finely harmonic (resp. finely superharmonic) on the fine interior of K . The equivalence of these definitions was shown in [BH75] and [BH78].

Another definition was introduced in [P97] using the notion of Jensen measures. A measure μ supported by K is Jensen with barycenter $x \in K$ if for every open set V containing K and every superharmonic function u on V we have $u(x) \geq \mu(u)$. The set of such measures will be denoted by $\mathcal{J}_x(K)$. Then $H(K)$ is the subspace of

Date: December 20, 2010.

2010 Mathematics Subject Classification. Primary: 31B05; Secondary: 31C40.

Key words and phrases. Harmonic measure, Jensen measures, Subharmonic functions, Potential theory, Fine topology, Restoring coverings.

$C(K)$ consisting of functions h such that $h(x) = \mu(h)$ for all $\mu \in \mathcal{J}_x(K)$ and $x \in K$. It was shown in [P97] that this definition is equivalent to the definitions above.

Despite the existence of so many equivalent definitions it is still difficult to verify whether a function on a compact set is harmonic or superharmonic. In [DG74] it was shown that a function $h \in H(K)$ if and only if it is continuous and finely harmonic on the fine interior of K . A localization result from [BH78] implies that a function $h \in H(K)$ if and only if it is continuous and finely harmonic on every fine connected component of the fine interior of K . Such collection of sets are usually called *restoring*.

In its turn in [P97] another restoring collection of sets was introduced. For $x \in K$ let $I(x)$ be the set of all points $y \in K$ such that $\mu(V) > 0$ for every $\mu \in \mathcal{J}_x(K)$ and every open set V containing y . It was shown that the sets $I(x)$ form the restoring covering.

The main goal of this paper is to reconcile the results in [DG74] and [P97]. It required the understanding of a connection between fine topology and Jensen measures. For this we use the fact from [P97] that $I(x)$ is the closure of the set $Q(x)$ of all $y \in K$ such that $G_K(x, y) > 0$, where the Green function G_K on K is defined as the limit of Green functions on domains D_j decreasing to K .

Fuglede [F72, F75, F83] defined a Green function on K as the fine Green function on the fine interior $int_f(K)$ of K . We denote the fine Green function on a finely open set U by $G_U^f(x, y)$ (see [F75, F83, F99] for the definition, and Section 2 for some basic properties).

As the first step we show at Section 3 (Theorem 3.2) that these two notions of Green functions are constant multiples of each other. These leads to Proposition 3.3 which claims that the set $Q(x)$ is a fine connected component of $int_f(K)$.

To finish the reconciliation process in Section 4 we study closely the set $\mathcal{J}_x(K)$. The main result (Theorem 4.6) provides Corollary 4.7 claiming that $\mu \in \mathcal{J}_x(K)$ if and only if $\mu \in \mathcal{J}_x(I(x))$. This corollary proves to be quite useful. From it we are able to derive a number of applications in Section 5. In particular Corollary 5.1 an extension result for superharmonic functions shows that for every $f \in S(I(x))$ there is a $\hat{f} \in S(K)$ such that $\hat{f}|_{I(x)} = f$. Also following from Corollary 4.7 is the desired reconciliation of the restoring theorem of Poletsky [P97] and the [DG74] result, proved here as Theorem 5.2.

In 1983, Gamelin and Lyons have shown [GL83, Theorem 3.1] that for $K \subset \mathbb{R}^2$

$$H(K)^\perp = \bigoplus H(\bar{A}_j)^\perp$$

where A_j are the fine components (fine open, fine connected) of the fine interior of K . However their work follows from an estimate for harmonic measure of the radial projection of a set, proved by Beurling in his thesis, which has no analog in \mathbb{R}^n for $n > 2$. By using Theorem 5.2 we are now able to extend this result to higher dimensions in Corollary 5.3. As an application of this we are able to show, Proposition 5.5, that every Jensen set is Wermer, which was first proved by Poletsky in [P97] for $n = 2$.

We are especially grateful to Eugene Poletsky for his excellent guidance and support.

2. BASIC PROPERTIES

Let $U \subset \mathbb{R}^n$ be a domain. The set of harmonic and superharmonic functions on U are denoted by $H(U)$ and $S(U)$, respectively. The set of Jensen measures on U at $x \in U$, denoted $\mathcal{J}_x(U)$, is the collection of all positive Radon measures μ with support compactly contained in U such that $\mu(f) \leq f(x)$ for all $f \in S(U)$, where $\mu(f) = \int f d\mu$. The Jensen measures on K are given by

$$\mathcal{J}_x(K) = \bigcap_{K \subset U} \mathcal{J}_x(U), \quad x \in K$$

where the intersection is over all domains U containing K .

Poletsky [P97, Theorem 3.1] (see also [BH86, BH78, DG74, H85]) has shown a connection between the set $H(K)$ and the set of Jensen measures on K .

Theorem 2.1. *A function is in $H(K)$ if and only if it is continuous and satisfies the averaging property with respect to every Jensen measure on K , that is*

$$H(K) = \{h \in C(K) : h(x) = \mu(h), \text{ for all } \mu \in \mathcal{J}_x(K) \text{ and every } x \in K\}.$$

By analogy one can define the set of continuous superharmonic functions on K . The set of functions $S(K)$ is defined to be the uniform closure of the set of functions $f|_K$ where f is continuous and superharmonic in some neighborhood of K .

See [P97] for further properties of the space $H(K)$ and $\mathcal{J}_x(K)$. For our purpose it is sufficient to draw some connections between these sets and their counterparts as seen from a fine potential theoretic viewpoint.

The two books [B71, F72] are classical references on the fine topology and many books on potential theory contain chapters on the topic, e.g. [AG01, Chapter 7] and [H69, Chapter 10]. Furthermore the topic is generally subsumed in the more abstract potential theory involving balayage spaces, e.g. [BH86].

The *fine topology* on \mathbb{R}^n is the coarsest topology on \mathbb{R}^n such that all superharmonic functions are continuous in the extended sense of functions taking values in $[-\infty, \infty]$. When referring to a topological concept in the fine topology we will follow the standard policy of either using the words “fine” or “finely” prior to the topological concept or attaching the letter f to the associated symbol. For example, the fine boundary of K , $\partial_f K$, is the boundary of K in the fine topology. The fine topology is strictly finer than the Euclidean topology.

A set E is said to be *thin* at a point x_0 if x_0 is not a fine limit point of E , i.e. if there is a fine neighborhood U of x_0 such that $E \setminus \{x_0\}$ does not intersect U . An example of a thin set is given by the Lebesgue spine in \mathbb{R}^3 defined by

$$L = \{(x, y, z) : x > 0 \text{ and } y^2 + z^2 < \exp(-c/x)\}, \quad c > 0$$

which is thin at the origin.

Many of the key concepts of classical potential theory have analogous definitions in relation to the fine topology. Presently we will recall a few of them. Relative to a finely open set V in \mathbb{R}^n the *harmonic measure* $\delta_x^{\mathbb{C}V}$ is defined as the swept-out of the Dirac measure δ_x on the complement of V . A function u is said to be *finely hyperharmonic* on a finely open set U if it is lower finite, finely lower semicontinuous, and

$$-\infty < \delta_x^{\mathbb{C}V}(u) \leq u(x),$$

for all $x \in V$ and all relatively compact finely open sets V with fine closure contained in U . A finely hyperharmonic function u is called *finely superharmonic* if $u \not\equiv \infty$,

and a function h is said to be *finely harmonic* if h and $-h$ are finely hyperharmonic. We will need the concept of a fine Green function. See [F75, F83, F99] for the definitions and basic properties. It was shown [F75] that every bounded fine open set U admits a fine Green function which we shall denote by $G_U^f(x, y)$.

Following [G78] the Choquet boundary of K with respect to $S(K)$ is

$$Ch_{S(K)}K = \{x \in K : J_x(K) = \{\delta_x\}\}.$$

Many nice properties of the Choquet boundary are known. In particular, we will need the following characterization, see, for example, [BH86, VI.4.1] and [H85].

Lemma 2.2. *The Choquet boundary of K with respect to $S(K)$ is the fine boundary of K , i.e.*

$$Ch_{S(K)}K = \partial_f K.$$

Proof. Since the fine topology is strictly finer than the Euclidean topology, any point in the interior of K will also be in the fine interior of K , and any point of $\mathbb{R}^n \setminus K$ can be separated from K by an Euclidean (therefore fine) open set. Therefore the fine boundary of K is contained in ∂K . The result follows immediately from [P97, Theorem 3.3] or [BH86, Proposition 3.1] which states that $J_x(K) = \{\delta_x\}$ if and only if the complement of K is non-thin at x , i.e. x is a fine boundary point of K . \square

In particular,

Corollary 2.3. *If $J_x(K) \neq \{\delta_x\}$, then $x \in \text{int}_f K$.*

The set $\partial_f K$ is also called the stable boundary of K . In fact the lemma shows that $Ch_{S(K)}K$ is the finely regular boundary of the fine interior of K . For more details on finely regular boundary points and other related concepts, see [BH86, VII.5-7] and [H85].

The following lemma has been known since the book of Fuglede [F72, p. 147].

Lemma 2.4. *A fine open set U in \mathbb{R}^n has at most countably many fine open connected components.*

3. ON THE GREEN FUNCTION ASSOCIATED TO A COMPACT SET

We now proceed to study the Green function on K . Recall [D83, Theorem (b) 1.VII.6, p. 94] that if D is an open Greenian set in \mathbb{R}^n so that $\{D_j\}$ is a decreasing sequence of open sets converging to D , then the sequence $\{G_{D_j}(\cdot, y)\}$ of Green functions associated to $\{D_j\}$ is decreasing to $G_D(\cdot, y)$ for every $y \in D$. By analogy one can define a Green function on a compact set K as the limit of the sequence $\{G_{D_j}(\cdot, y)\}$ where $y \in K$ and $\{D_j\}$ is any decreasing sequence of open sets converging to K . In the article [P97] Poletsky defines a Green function on a compact set in this way.

Recall, [D83, p. 90], that for a regular open set D the associated Green function $G_D(\cdot, y)$ extends continuously as $\hat{G}_D(\cdot, y)$ to \mathbb{R}^n for any $y \in D$ where $\hat{G}_D(\cdot, y) = 0$ on $\complement D$, the complement of D , and this extension $\hat{G}_D(\cdot, y)$ is subharmonic on $\mathbb{R}^n \setminus \{y\}$.

In the following proposition we outline some of the basic properties of \hat{G}_K .

Proposition 3.1. *For all $y \in K$, the function $\hat{G}_K(\cdot, y) : \mathbb{R}^n \rightarrow [0, \infty]$ defined as $\hat{G}_K(\cdot, y) = \liminf_j \hat{G}_{D_j}(\cdot, y)$ has the following properties:*

- i. $\hat{G}_K(x, y) = 0$ when $x \in \mathbf{C}K := \mathbb{R}^n \setminus K$ and $y \in K$,
- ii. \hat{G}_K does not depend on the sequence $\{D_j\}$ chosen,
- iii. $\hat{G}_K \geq 0$ and $\hat{G}_K(y, y) = +\infty$ for all $y \in K$,
- iv. \hat{G}_K is symmetric, i.e. $\hat{G}_K(x, y) = \hat{G}_K(y, x)$, for all $x, y \in K$,
- v. $\hat{G}_K(\cdot, y)$ is super-averaging on K , i.e. $\hat{G}_K(x, y) \geq \int \hat{G}_K(\zeta, y) d\mu(\zeta)$ for all $\mu \in \mathcal{J}_x(K)$ with $x \in K$, and
- vi. $\hat{G}_K(\cdot, y)$ is subharmonic on $\mathbb{R}^n \setminus \{y\}$.

proof of i. This follows from the fact that $\hat{G}_{D_j}(x, y) = 0$ whenever $x \notin \overline{D_j}$. \square

proof of ii. If $D_1 \supset D_2$ then $\hat{G}_{D_1}(\cdot, y) \geq \hat{G}_{D_2}(\cdot, y)$ for any Greenian sets D_1 and D_2 . Alternatively we could have defined \hat{G}_K by

$$\hat{G}_K(\cdot, y) = \inf\{\hat{G}_D(\cdot, y) : D \supset K, D \text{ Greenian}\}, \quad y \in K. \quad \square$$

proof of iii. As $\hat{G}_D \geq 0$ and $\hat{G}_D(y, y) = +\infty$ for all $x, y \in D$ for any Greenian D . \square

proof of iv. Since $\hat{G}_D(x, y) = \hat{G}_D(y, x)$ for all $x, y \in D$ for any Greenian D . \square

proof of v. For any Greenian set D the function $\hat{G}_D(\cdot, y)$ is superharmonic on D . Then $\hat{G}_D(x, y) \geq \int \hat{G}_D(\zeta, y) d\mu(\zeta)$ for all $\mu \in \mathcal{J}_x(D)$ with $\zeta \in D$. If D_j is a decreasing sequence of domains converging to K , then $\hat{G}_{D_j}(\cdot, y)$ is decreasing to $\hat{G}_K(\cdot, y)$. Therefore by the Lebesgue Monotone Convergence Theorem $\hat{G}_K(x, y) \geq \int \hat{G}_K(\zeta, y) d\mu(\zeta)$ for all $\mu \in \cap_j \mathcal{J}_x(D_j) := \mathcal{J}_x(K)$ with $x \in K$. \square

proof of vi. Let $\{D_j\}$ be a decreasing sequence of regular domains converging to K . Then $\hat{G}_{D_j}(\cdot, y)$ is continuous, and so $\hat{G}_K(\cdot, y)$ must be upper semicontinuous. For any j and any $y \in D_j$, by [D83, p. 90] the extension $\hat{G}_{D_j}(\cdot, y)$ of Green function $\hat{G}_{D_j}(\cdot, y)$ by 0 is subharmonic on $\mathbb{R}^n \setminus \{y\}$. Therefore by the Lebesgue Monotone Convergence Theorem $\hat{G}_K(\cdot, y)$ is subaveraging on $\mathbb{R}^n \setminus \{y\}$ as it is the decreasing limit of a sequence of subharmonic functions. Since $\hat{G}_K(\cdot, y)$ is upper semicontinuous and subaveraging, $\hat{G}_K(\cdot, y)$ is subharmonic on $\mathbb{R}^n \setminus \{y\}$. \square

It was shown in [F75] that every bounded fine open set U admits a fine Green function which we shall denote by $G_U^f(x, y)$. The following result shows that for a compact set K the functions $\hat{G}_K(x, y)$ and $G_{int_f K}^f(x, y)$ are scalar multiples of each other.

Theorem 3.2. *For any compact set $K \subset \mathbb{R}^n$, $n \geq 2$, there is $c > 0$ such that $\hat{G}_K(x, y) = cG_{int_f K}^f(x, y)$ for any $y \in int_f K$.*

Proof. Fuglede has given a simple characterization of the fine Green function up to multiplication by a positive constant. Indeed, if a function $g: U \times U \rightarrow \mathbb{R}$ has the following properties

1. $g(\cdot, y)$ is a nonnegative finely superharmonic function on U ,
2. if v is finely subharmonic on U and $v \leq g(\cdot, y)$, then $v \leq 0$,
3. $g(\cdot, y)$ is finely harmonic on $U \setminus \{y\}$ for any $y \in U$, and
4. $g(y, y) = +\infty$

then $g(x, y) = cG_U^f(x, y)$ for some $c > 0$ for all $x, y \in U$.

Hence to prove the theorem we need only to check these properties. Firstly, we note that by Lemma 3.1 $\hat{G}_K(\cdot, y)$ is subharmonic (and thereby finely subharmonic) on $\mathbb{R}^n \setminus \{y\}$, which implies ([F72, Theorem 9.10]) fine continuity on $\mathbb{R}^n \setminus \{y\}$.

In fact, we will shall now see that $\hat{G}_K(\cdot, y)$ is finely continuous at y when $y \in \text{int}_f K$. Every bounded fine open set admits a fine Green function, cf. [F75, F99]. Let $G_{\text{int}_f K}^f$ denote the fine Green function corresponding to the bounded fine open set $\text{int}_f K$. Since $\text{int}_f K \subset D_j$ we have $G_{\text{int}_f K}^f(\cdot, y) \leq \hat{G}_{D_j}(\cdot, y)$. As $\hat{G}_K(\cdot, y)$ is the decreasing limit of $\hat{G}_{D_j}(\cdot, y)$ we have the inequalities

$$G_{\text{int}_f K}^f(\cdot, y) \leq \hat{G}_K(\cdot, y) \leq \hat{G}_{D_j}(\cdot, y),$$

for all $y \in \text{int}_f K$. Since $G_{\text{int}_f K}^f(\cdot, y)$ and $\hat{G}_{D_j}(\cdot, y)$ are finely continuous, $\hat{G}_K(\cdot, y)$ must be finely continuous at y as

$$\infty = \text{f-lim}_{x \rightarrow y} G_{\text{int}_f K}^f(x, y) \leq \text{f-lim}_{x \rightarrow y} \hat{G}_K(x, y) \leq \text{f-lim}_{x \rightarrow y} \hat{G}_{D_j}(x, y) = \infty.$$

Therefore $\hat{G}_K(\cdot, y)$ is finely continuous on \mathbb{R}^n when $y \in \text{int}_f K$.

Thus $\hat{G}_K(\cdot, y)$ is finely superharmonic on $\text{int}_f K$ as it is finely continuous and the decreasing limit of $\{\hat{G}_{D_j}(\cdot, y)\}$, a sequence of finely superharmonic functions on $\text{int}_f K$ and this implies that 1. holds.

Suppose that $\hat{G}(x_0, y) > 0$ for $x_0 \in \partial_f K$. Then there is a fine neighborhood V of x_0 such that $\hat{G}_K(x, y) > 0$ for all $x \in V$. By definition $x \in \partial_f K$ if and only if $\mathbb{C}K$ is non-thin at x . As $x_0 \in \partial_f K$, this means that $V \cap \mathbb{C}K \neq \emptyset$. However by Lemma 3.1, $\hat{G}_K(x, y) = 0$ for $x \in \mathbb{C}K$ and $y \in K$, a contradiction. Therefore $\hat{G}_K(x, y) = 0$ for all $x \in \partial_f K$ and $y \in \text{int}_f K$. So \hat{G}_K is a fine potential on $\text{int}_f K$ by the minimum principle [BH86, III.4.1] (see also [F72, Theorem 9.1]) and this implies 2.

We have seen above that $\hat{G}_K(\cdot, y)$ is finely superharmonic on $\text{int}_f K$. By Proposition 3.1.vi $\hat{G}_K(\cdot, y)$ is finely subharmonic on $\text{int}_f K \setminus \{y\}$. Therefore $\hat{G}_K(\cdot, y)$ is finely harmonic on $\text{int}_f K \setminus \{y\}$ and we checked 3.

The property 4. follows immediately from Proposition 3.1.iii and the theorem is proved. \square

Proposition 3.3. *The Green function $\hat{G}_K(x, y) > 0$ for $x, y \in K$ if and only if x and y are in the same fine connected component of $\text{int}_f K$.*

Proof. By the previous proposition $\hat{G}_K(\cdot, y)$ is finely superharmonic on $\text{int}_f K$. If $\hat{G}_K(x, y) = 0$, then by [F72, Theorem 12.6] for all ζ in the fine component of y we have $\hat{G}_K(\zeta, y) = 0$. Therefore $\hat{G}_K(\cdot, y) > 0$ on the fine component containing y .

Suppose that $\text{int}_f K$ has multiple components. Each component is fine open and therefore has its own Green function. We can define a function $g(x, y)$ on $\text{int}_f K$ by

$$g(x, y) = \begin{cases} G_{Q_x}^f(x, y), & y \in Q_x \\ 0, & y \in (\text{int}_f K) \setminus Q_x \end{cases}$$

where Q_x is the fine component containing x . Since fine subharmonicity and fine harmonicity are local properties, g satisfies the requirements mentioned in the proof

of Theorem 3.2 to be a positive multiple of the fine Green function on $\text{int}_f K$. Therefore $G_{\text{int}_f K}^f(x, y)$ is positive if and only if x and y are in the same fine component of $\text{int}_f K$. So $\hat{G}_K(x, y) = 0$ when x and y are in different fine connected components.

In the proof of the previous proposition we proved that $\hat{G}_K(x, y) = 0$ for $x \in \partial_f K$ and $y \in K \setminus \{x\}$. \square

In [P97] Poletsky introduced the sets

$$Q(x) = \{y \in K : \hat{G}_K(x, y) > 0\},$$

for every $x \in K$. The following corollary directly follows from Proposition 3.3 and characterizes these sets in terms of the fine topology.

Corollary 3.4. *For all $x \in \text{int}_f K$, the set $Q(x)$ is the fine connected component of $\text{int}_f K$ which contains x . Additionally the point $x \in K$ is in $\partial_f K$ if and only if $Q(x) = \{x\}$.*

4. JENSEN MEASURES

Some results from [P97] now follow from standard properties of the fine potential theory and the fine topology. For example [P97, Theorem 3.6 (2)] is the partitioning the set K into the fine connected components of $\text{int}_f K$ and singleton sets for peak points (i.e. the set $\partial_f K$) forms an equivalence relation, [P97, Theorem 3.6 (3)] is the fine minimum principle, and [P97, Theorem 3.6 (4)] is that fine connected components have positive measure. We can now extend/rephrase some results of [P97] and use them to obtain some new results.

Theorem 4.1. *For $x \in K$ and any $\varepsilon > 0$ there exists a $\mu \in \mathcal{J}_x(K)$ with $\mu(B(y, \varepsilon)) > 0$ if and only if the point y is in the (Euclidean) closure $\overline{Q(x)}$ of the fine component of x .*

Proof. In [P97] Poletsky defines $I(x)$ as the set of points $y \in K$ with the property that for any $\varepsilon > 0$ there exists a $\mu \in \mathcal{J}_x(K)$ with $\mu(B(y, \varepsilon)) > 0$ and in [P97, Theorem 3.6 (1)] proves that $I(x) = \overline{Q(x)}$. The result follows from Corollary 3.4. \square

The following corollary is an immediate consequence of the previous theorem.

Corollary 4.2. *Let K be a compact set in \mathbb{R}^n , $n \geq 2$. Then $\text{supp } \mu \subset \overline{Q(x)}$ for all $\mu \in \mathcal{J}_x(K)$.*

For use in the following proposition we recall the notion of a reduced function, see [AG01, Definition 5.3.1]. Fix a Greenian open set $\Omega \subset \mathbb{R}^n$. Let $U_+(\Omega)$ be the set of non-negative superharmonic functions on Ω . For $u \in U_+(\Omega)$ and $E \subset \Omega$, the reduced function of u relative to E in Ω is defined by

$$R_u^E(x) = \inf\{v(x) : v \in U_+(\Omega) \text{ and } v \geq u \text{ on } E\}, \quad x \in \Omega.$$

Also note that \hat{R}_u^E is the lower semicontinuous regularization of R_u^E .

Proposition 4.3. *Let U and V be disjoint fine open sets. Then $V \cap \overline{U}$ is a polar set.*

Proof. It suffices to prove this statement when U and V are bounded. Otherwise, we may consider intersections of these sets with increasing sequence of open balls.

Let Ω be any open Greenian set containing U and V . Since U is disjoint from V , U is thin at y for every $y \in V$. Then by [AG01, Theorem 7.3.5] there is a bounded continuous potential $u^\#$ on Ω with the property that $\hat{R}_{u^\#}^U(y) < u^\#(y)$ for all $y \in V \cap \bar{U}$. By construction $R_{u^\#}^U \geq u^\#$ and $R_{u^\#}^U(x) = \hat{R}_{u^\#}^U(x) = u^\#(x)$ for all $x \in U$. Therefore $V \cap \bar{U} \subset \{R_{u^\#}^U \neq \hat{R}_{u^\#}^U\}$, and by [AG01, Theorem 5.7.1] the set $\{R_{u^\#}^U \neq \hat{R}_{u^\#}^U\}$ is polar. \square

Corollary 4.4. *For a compact set $K \subset \mathbb{R}^n$, let $\{A_i\}$ be the collection of disjoint fine connected components of the fine interior of K . Then $\text{int}_f \bar{A}_i = A_i$ for all i .*

Proof. We will show that $\text{int}_f \bar{A}_i$ has only one fine component and so it must be A_i . Suppose that $\text{int}_f \bar{A}_i = A \cup V$ where A is the fine component containing A_i and V is fine open and disjoint from A . First we note that $A_i = A$ as $A_i \subset A \subset \text{int}_f K$ and A_i is a fine component of $\text{int}_f K$. Secondly, V is disjoint from A_i and contained in $\text{int}_f \bar{A}_i$, hence $V \subset \bar{A}_i \setminus A_i$. Therefore by Proposition 4.3, we have that V must be polar and cannot be fine open. \square

The following corollary tells us that the only the trivial Jensen measures can have support in the closure of two fine components. We use the notation $\mathcal{J}(K) := \cup_{x \in K} \mathcal{J}_x(K)$ to denote the collection of all Jensen measures on K .

Corollary 4.5. *Let $\{A_j\}$ be the fine connected components of the fine interior of K . Then*

$$\mathcal{J}(\bar{A}_i) \cap \mathcal{J}(\bar{A}_j) = \bigcup_{x \in \bar{A}_i \cap \bar{A}_j} \{\delta_x\},$$

where $i \neq j$.

Proof. Let $\mu \in \mathcal{J}(\bar{A}_i) \cap \mathcal{J}(\bar{A}_j)$ with $i \neq j$. Then there is an $x_i \in \bar{A}_i$ and $x_j \in \bar{A}_j$ so that $\mu \in \mathcal{J}_{x_i}(\bar{A}_i) \cap \mathcal{J}_{x_j}(\bar{A}_j)$. As the coordinate functions are harmonic, this implies that $x_i = x_j$. Let us call $x_0 := x_i = x_j \in \bar{A}_i \cap \bar{A}_j$. As A_i and A_j are disjoint, we have by Corollary 4.4 that x_0 must be in the fine boundary of either \bar{A}_i or \bar{A}_j . However the only way that x_0 can be in the fine boundary (see Lemma 2.2) is if $\mu = \delta_{x_0}$. \square

The following theorem gives sufficient condition on a subset E of K so that the Jensen measures on K with barycenter $x \in E$ belong to the Jensen measures on E .

Theorem 4.6. *Let $A \subset K \subset \mathbb{R}^n$, $n \geq 2$, with K compact with A and $\text{int}_f K \setminus A$ fine open, that is A is the union of fine connected components of $\text{int}_f K$. Suppose that $\text{supp } \mu \subset \bar{A}$ for all $\mu \in \mathcal{J}_x(K)$ and all $x \in A$ then $\mathcal{J}_x(K) \subset \mathcal{J}_x(\bar{A})$ for all $x \in A$.*

Proof. Suppose there exists $\mu \in \mathcal{J}_{x_0}(K)$, $f \in S(\bar{A})$ and $a > 0$, such that $\mu(f) > f(x_0) + a$. As $cf + c'$ is also in $S(\bar{A})$ for $c > 0$ and since the functions in $S(\bar{A})$ are uniform limits of continuous superharmonic functions defined in neighborhoods of \bar{A} , we may assume that $f \in C(G) \cap S(G)$ for some open set $G \supset \bar{A}$ with the properties $-1 < f(x_0) < \mu(f) < 0$ and $-1 < f < 0$. Let $a := \mu(f) - f(x_0) > 0$ and take G' open with $\bar{A} \subset G'$ and $\bar{G}' \subset G$.

Pick $\phi \in C(\mathbb{R}^n)$ with $\phi = 0$ on \overline{A} , $\phi = 1$ on $\mathbb{R}^n \setminus G'$ and $0 < \phi < 1$ on $G' \setminus \overline{A}$. By Edwards Theorem (see [CR97])

$$E\phi(y) = \inf\{f(y) : f \in S(K), f \geq \phi\} = \sup\{\nu(\phi) : \nu \in \mathcal{J}_y(K)\}.$$

By assumption $\text{supp}(\nu) \subset \overline{A}$ for all $\nu \in \mathcal{J}_y(K)$ and every $y \in A$. So $E\phi(y) = 0$ for every $y \in A$. Therefore there exists a $g \in S(K)$ with $0 \leq \phi \leq g \leq 1$ and $g(x_0) < \varepsilon < a/3$.

Actually we can say a little more. By Corollary 2.3, we know that $\mathcal{J}_y(K) \neq \{\delta_y\}$ if and only if $y \in \text{int}_f K$. This allows us to decompose \overline{A} into three sets; A , $\partial_1 A \subset \partial A$ where $\mathcal{J}_y(K) = \{\delta_y\}$ for $y \in \partial_1 A$, and $\partial_2 A = \overline{A} \setminus (A \cup \partial_1 A)$. Each point in $\partial_2 A$ belongs to $\text{int}_f K \setminus A$. Recall that by hypothesis $\text{int}_f K \setminus A$ is fine open. Therefore $\partial_2 A \subset \overline{A} \cap (\text{int}_f K \setminus A)$, which means that $\partial_2 A$ is polar by Proposition 4.3. Since $\partial_2 A$ is a polar set, we see that $\mu(\partial_2 A) = 0$.

Thus there exists C a compact neighborhood of x with $C \subset A \cup \partial_1 A$ so that $\mu(C) > 1 - \varepsilon$. As $E\phi|_{A \cup \partial_1 A} = 0$, trivially $E\phi|_C = 0$. For every $y \in C$ there are a continuous and superharmonic function $g_y \geq \phi$ in a neighborhood of K and an open neighborhood U_y of y with $g_y < \varepsilon$ on U_y . The sets U_y cover C , so by compactness we can pick up y_1, \dots, y_N so that $C \subset U_{y_1} \cup \dots \cup U_{y_N}$. Then $g = \min\{g_{y_1}, \dots, g_{y_N}\}$ has the property $g|_C < \varepsilon$ and $\mu(\{g > \varepsilon\}) < \varepsilon$.

Consider the function $f + g$. As $g \geq 0$ we have

$$\mu(f + g) = \mu(f) + \mu(g) > f(x_0) + g(x_0) + a - g(x_0) > (f(x_0) + g(x_0)) + a - \varepsilon.$$

As $\phi \leq g$, we have that $f + g \geq 0$ on $K \setminus \overline{G'}$. Note also that

$$f(x_0) + g(x_0) = \mu(f) - a + g(x_0) < -a + g(x_0) < -a + \varepsilon < 0.$$

So

$$h(y) = \begin{cases} 0, & K \setminus \overline{G'} \\ \min\{f + g, 0\}, & G' \cap K \end{cases}$$

is in $C(K)$, $h \equiv 0$ on $K \setminus G'$ and $h(x_0) = f(x_0) + g(x_0)$.

To see that h is in $S(K)$ we use a localization argument. Let \mathcal{V} be a covering of the fine interior of K by fine open sets such that $V \in \mathcal{V}$ has the property: if $V \cap G' \neq \emptyset$ then $V \subset G'$. If $V \subset G'$, then $h = \min\{f + g, 0\} \in S(V)$. If $V \cap G' = \emptyset$ then $h \equiv 0 \in S(V)$. Thus $h \in S(K, \text{int}_f K, \mathcal{V}) = S(K)$, by [BH78, Proposition 3.5].

Thus

$$\mu(h) = \int_{\{f+g < 0\}} (f + g) d\mu = \mu(f + g) - \int_{\{f+g \geq 0\}} (f + g) d\mu.$$

Now $\mu(f + g) > f(x_0) + g(x_0) + a - \varepsilon$ and

$$\int_{\{f+g \geq 0\}} (f + g) d\mu = \int_{\{f+g \geq 0\}} f d\mu + \int_{\{f+g \geq 0\}} g d\mu.$$

The first integral on the right is negative and because $g \geq 0$

$$\int_{\{f+g \geq 0\}} g d\mu \leq \int g d\mu.$$

But the last integral is equal to

$$\int_{\{g>\varepsilon\}} g d\mu + \int_{\{g\leq\varepsilon\}} g d\mu \leq \mu(\{g > \varepsilon\}) + \varepsilon.$$

Recall that $\mu(\{g > \varepsilon\}) < \varepsilon$. Thus

$$\mu(h) > f(x_0) + g(x_0) + a - 3\varepsilon > h(x_0).$$

However this contradicts that $\mu \in J_{x_0}(K)$ and $h \in S(K)$. Hence $\mathcal{J}_x(K) \subset \mathcal{J}_x(\overline{A})$ for all $x \in A$. Since the reverse inclusion is trivial the theorem is proved. \square

We also get the following useful restriction property of Jensen measures.

Corollary 4.7. *Let $K \subset \mathbb{R}^n$, $n \geq 2$, be a compact set. For all $x \in K$, we have*

$$\mathcal{J}_x(\overline{Q(x)}) = \mathcal{J}_x(K).$$

Proof. We will show $\mathcal{J}_x(\overline{Q(x)}) \subset \mathcal{J}_x(K)$ first. Consider $\mu \in \mathcal{J}_x(\overline{Q(x)})$ and $u \in S(K)$. Then $u|_{\overline{Q(x)}} \in S(\overline{Q(x)})$, so that $u(x) \geq \mu(u)$. Thus $\mu \in \mathcal{J}_x(K)$.

By Corollary 4.2, $\text{supp } \mu \subset \overline{Q(x)}$ for all $\mu \in \mathcal{J}_x(K)$, which by Theorem 4.6 means that $\mathcal{J}_x(K) \subset \mathcal{J}_x(\overline{Q(x)})$. \square

5. APPLICATIONS

An interesting corollary follows immediately from the proof of Theorem 4.6. For any cone of functions \mathcal{R} , we define the closure $\widetilde{\mathcal{R}}$ of \mathcal{R} as all continuous functions which can be represented as the infimum of functions from \mathcal{R} .

Corollary 5.1. *Let $K \subset \mathbb{R}^n$, $n \geq 2$, be a compact set with $\{A_j\}$ the fine connected components of the fine interior of K . Then*

$$S(\overline{A_j}) = \widetilde{S(K)}|_{\overline{A_j}}$$

for every component A_j .

Proof. It is clear that $S(K)|_{\overline{A_j}} \subset S(\overline{A_j})$. Consider any function $f \in S(\overline{A_j})$. By Edwards Theorem (see [CR97])

$$f(x) = \inf\{\phi(x) : \phi \in S(\overline{A_j}) \text{ and } \phi \geq f \text{ on } \overline{A_j}\} = \sup\{\mu(f) : \mu \in \mathcal{J}_x(\overline{A_j})\},$$

for all $x \in \overline{A_j}$. From Corollary 4.7 we have $\mathcal{J}_x(\overline{A_j}) = \mathcal{J}_x(K)$ when $x \in A_j$. Therefore we may apply Edwards Theorem again to see that

$$f(x) = \sup\{\mu(f) : \mu \in \mathcal{J}_x(K)\} = \inf\{\phi(x) : \phi \in S(K) \text{ and } \phi \geq f \text{ on } K\},$$

for $x \in A_j$. Thus $f \in \widetilde{S(K)}|_{\overline{A_j}}$. \square

The following theorem shows that the restoring covering of [P97] is given by the fine connected components of $\text{int}_f K$.

Theorem 5.2. *Let $K \subset \mathbb{R}^n$, $n \geq 2$, be a compact set with $\{A_j\}$ denoting the fine components (fine open, fine connected) of the fine interior of K . For any $f \in C(K)$, $f \in H(K)$ if and only if $f \in H(\overline{A_j})$ for all j .*

Proof. Recall that

$$H(K) = \{f \in C(K) : f(x) = \mu(f) \text{ for all } \mu \in \mathcal{J}_x(K) \text{ and every } x \in K\}$$

However if x is in A_j by Corollary 4.7 we have that $\mathcal{J}_x(\overline{A_j}) = \mathcal{J}_x(K)$ which implies the result. \square

As a corollary we may extend the [GL83] result to higher dimensions. Recall that for any compact set E , the set $H(E)^\perp$ is the set of Radon measures μ with $\text{supp}(\mu) \subset E$ such that $\mu(h) = 0$ for all $h \in H(E)$, and if $m(E)$ is any set of Radon measures with support in E the set ${}^\perp m(E)$ consists of all $f \in C(E)$ such that $\mu(f) = 0$ for all $\mu \in m(E)$.

Corollary 5.3. *For any $K \subset \mathbb{R}^n$, $n \geq 2$, compact*

$$H(K)^\perp = \bigoplus H(\overline{A_j})^\perp$$

where A_j are the fine components (fine open, fine connected) of the fine interior of K .

Proof. Consider any $\mu \in \bigoplus H(\overline{A_j})^\perp$ and $h \in H(K)$. Then $h|_{\overline{A_j}} \in H(\overline{A_j})$, so $\mu(h) = 0$. Thus $\bigoplus H(\overline{A_j})^\perp \subset H(K)^\perp$.

Conversely, suppose that $h \in C(K)$ and $h \in {}^\perp(\bigoplus H(\overline{A_j})^\perp)$. Then $h|_{\overline{A_j}} \in {}^\perp(H(\overline{A_j})^\perp) = H(\overline{A_j})$. The restoring property (Theorem 5.2) then implies that $h \in H(K)$. Therefore ${}^\perp(\bigoplus H(\overline{A_j})^\perp) \subset H(K)$ and so $H(K)^\perp \subset \bigoplus H(\overline{A_j})^\perp$. \square

Recall the following definitions of Poletsky [P97, Def 3.9, 3.15].

Definition 5.4. *A compact set $K \subset \mathbb{R}^n$, $n \geq 2$, is called Jensen if $K = \overline{Q(x)}$ for some $x \in K$, and Wermer if for all $x \in K$, either $\overline{Q(x)} = K$ or $\overline{Q(x)} = \{x\}$.*

It has been shown in [P97, Corollary 3.16] that every Jensen set is a Wermer set in the plane. We can now provide a proof of this in \mathbb{R}^n .

Proposition 5.5. *A Jensen set is Wermer.*

Proof. Suppose K is Jensen. Then $K = \overline{Q(x_0)}$ for some $x_0 \in K$. Every $y \in K$ is either a fine boundary point or in the fine interior. If y is in the fine boundary of K , then $Q(y) = \{y\}$ by Corollary 3.4.

We will show that $\text{int}_f K$ has only one fine component which must be $Q(x_0)$. Suppose that $\text{int}_f K = Q(x_0) \cup V$ where V is fine open and disjoint from $Q(x_0)$. Since $\overline{Q(x_0)} = K$, by Proposition 4.3, we have that V must be polar and cannot be fine open.

Thus for any $y \in \text{int}_f K$, we have $Q(y) = Q(x_0)$ and so $\overline{Q(y)} = K$. \square

The set $K = [0, 1] \subset \mathbb{R}^2$ provides a simple example of a Wermer set that is not Jensen. Every point is a fine boundary point, so $Q(x) = \{x\}$ for all $x \in K$. However there is no point $x_0 \in K$ such that $K = \overline{Q(x_0)}$. Proposition 5.5 can be interpreted as saying that if K is Wermer then either $H(K) = C(K)$ or K is Jensen.

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