ORDER-THEORETIC PROPERTIES OF BASES IN TOPOLOGICAL SPACES I

MENACHEM KOJMAN, DAVID MILOVICH, AND SANTI SPADARO

Abstract. We study some cardinal invariants of an order-theoretic fashion on products and box products of topological spaces. In particular, we concentrate on the Noetherian type (Nt), defined by Peregudov in the 1990s. Some highlights of our results include:

(1) There are spaces X and Y such that $Nt(X \times Y) < \min\{Nt(X), Nt(Y)\}$.
(2) In several classes of compact spaces, the Noetherian type is preserved by their square and their dense subspaces.
(3) The Noetherian type of some countably supported box products cannot be determined in ZFC. In particular, it is sensitive to square principles and some Chang Conjecture variants.
(4) PCF theory can be used to provide ZFC upper bounds to Noetherian type on countably supported box products. The underlying combinatorial notion is a weakening of Shelah's freeness.

1. Introduction

Van Douwen's Problem (see [20]) asks about the existence of a compact homogeneous space whose cellularity exceeds the continuum. We say that a homogeneous compactum is *exceptional* if it is not homeomorphic to a product of dyadic compacta and first-countable compacta. By Arhangel′skiĭ's Theorem first-countable compacta have size $\leq \mathfrak{c}$. Moreover dyadic compacta are ccc. So all non-exceptional homogeneous compacta have cellularity bounded by $\mathfrak{c}$. To the best of our knowledge there are essentially two examples of exceptional homogeneous compacta (see [26]).

We are interested in order-theoretic cardinal functions that, just like cellularity, have bounds on the class of all known homogeneous compacta. Here we study the behavior of these cardinal functions with regard to products and countably supported box-products, motivated in part by the structure theory of homogeneous compacta.

All the cardinal functions we are interested in are obtained from the classical ones by means of the following definition.

Definition 1.1. [27] Given a cardinal κ, define a poset to be κ^{op}*-like* if no element is below κ many elements.

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Definition 1.2. [29] The *Noetherian type of* X ($Nt(X)$) is defined as the least infinite cardinal κ such that X has a base which is κ^{op}-like with respect to inclusion. The π-*Noetherian type of* X ($\pi Nt(X)$) is defined as the least infinite cardinal κ such that X has a κ^{op}-like π-base. The *local Noetherian type at the point* x ($\chi Nt(x,X)$) is defined as the least infinite cardinal κ such that x has a κ^{op}-like local base. The *local Noetherian type of* X ($\chi Nt(X)$) is defined as $\chi Nt(X) = \sup\{\chi Nt(x,X) : x \in X\}$.

Spaces with Noetherian type ω (respectively, ω_1) were called *Noetherian* (respectively *weakly Noetherian*) by Peregudov and Šapirovskiĭ [30]. Spaces with countable Noetherian type were also studied under the name of *spaces with an Open in Finite (OIF) base* by Balogh, Bennett, Burke, Gruenhage, Lutzer and Mashburn in [6], by Bennett and Lutzer in [7] and by Bailey in [4], especially in the context of generalized metric spaces, metrization theorems and generalized ordered spaces.

Theorem 1.3. [29, 27] *Let* $X = \prod_{i\in I} X_i$. *Then:*

$$Nt(X) \le \sup_{i\in I} Nt(X_i)$$

$$\pi Nt(X) \le \sup_{i\in I} \pi Nt(X_i)$$

$$\chi Nt(X) \le \sup_{i\in I} \chi Nt(X_i)$$

All information about the Noetherian type of a space is lost in its large powers. This is a direct consequence of the following theorem of Malykhin.

Theorem 1.4. *Let* $X = \prod_{i\in I} X_i$ *where each* X_i *has a minimal open cover of size two (which is the case, for example, if* X *is* T_1 *and has more than one point). If* $\sup_{i\in I} w(X_i) \le |I|$, *then* $Nt(X) = \omega$.

In particular, $Nt(X^{w(X)}) = \omega$ for every T_1 space X.

Another easy, but nonetheless surprising consequence of the above theorem is the following.

Example 1.5. There are compact spaces X and Y such that $Nt(X \times Y) < Nt(X) \cdot Nt(Y)$.

Proof. Let κ be a regular infinite cardinal. Let $X = 2^\kappa$, with the usual topology and $Y = \kappa + 1$ with the order topology. By Theorem 1.4 we have $Nt(X \times Y) = \omega$. However, it is easy to see using the Pressing-down Lemma that $Nt(Y) = \kappa^+$. □

In view of the above example it is natural to ask:

Question 1.6. Is is true that for every (compact) space X then $Nt(X^2) = Nt(X)$?

Balogh, Bennett, Burke, Gruenhage, Lutzer and Mashburn similarly asked whether there exists X with $Nt(X^2) = \omega < Nt(X)$ (see [6], Question 1). We will offer some partial positive answers for the compact case, as well as an example of $Nt(X \times Y) < \min\{Nt(X), Nt(Y)\}$.

In the final section of our paper we will study Noetherian type in spaces where G_δ sets are open. We will give a Noetherian analogue of a classical bound of Juhász on the cellularity of the G_δ modification of a compact space. While Juhasz's was a ZFC theorem, we will need to assume (a weakening of) the GCH and another condition in our result. However, modulo large cardinals, we will show that ours is the sharpest possible such result. Finally, we will apply tools from PCF theory to give a detailed study of the Noetherian type of a certain product with the countably supported box topology. It will turn out that although its exact value is independendent of the axioms of set theory, still a lot can be said about it in ZFC.

2. Subsets of bases and the Noetherian type of compact squares and dense subspaces

The only approach we know towards proving that $Nt(X^2) = Nt(X)$ for a space X is based on the following lemma.

Lemma 2.1. *([28]) Let X be any space and $n \in \omega$. Then $Nt_{box}(X^n) = Nt(X)$.*

Where $Nt_{box}(X^n)$ is the minimum infinite cardinal κ such that X^n has a κ^{op}-like base consisting of boxes.

If we were able to prove that every base of X^n consisting of boxes contains a base which is $Nt(X^n)^{op}$-like, then $Nt(X) = Nt_{box}(X^n) \leq Nt(X^n)$, so we would be done because $Nt(X^n) \leq Nt(X)$ by Theorem 1.3. Unfortunately, this is not true. A counterexample is offered by the irrationals. The following theorem partially answers Question 2 from [28].

Theorem 2.2. *The Baire space ω^ω (homeomorphic to the space $\mathbb{P}$ of irrationals) has a base B that lacks an ω^{op}-like subcover (and hence contains no ω^{op}-like base).*

Proof. For each $s \in \omega^{<\omega}$ and $n \in \omega$, let $U_{s,n}$ be the clopen set of all $f \in \omega^\omega$ for which $s^\frown i \subseteq f$ for some $i \leq n$. Let $\mathcal{B}$ consist of the sets of the form $U_{s,n}$. This makes $\mathcal{B}$ a base of ω^ω. Now suppose that $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{A}$ is ω^{op}-like. For each $s \in \omega^{<\omega}$, there can be at most finitely many $U_{s,n} \in \mathcal{A}$. Set $t_0 = \varnothing$ and, given $k < \omega$ and $t_k \in \omega^{<\omega}$, choose $i_k \in \omega$ such that $i_k > n$ for all $U_{t_k,n} \in \mathcal{A}$. Set $t_{k+1} = t_k^\frown i_k$. Set $f = \bigcup_{k<\omega} t_k$. If $f \in U_{s,n}$ for some $U_{s,n} \in \mathcal{A}$, then $s = t_k$ for some k, which implies that $i_k \leq n$, in contradiction with how we constructed f. Therefore, $\bigcup \mathcal{A} \neq \omega^\omega$. □

Corollary 2.3. *If $X = \omega^\omega$, then, for all $\alpha \in [1, \omega_1)$, X^α has a base B consisting of boxes such that B lacks an $Nt(X^\alpha)^{op}$-like subcover.*

Proof. Let $p\colon \alpha \times \omega \leftrightarrow \omega$ and let $h\colon \omega^\omega \cong (\omega^\omega)^\alpha$ be given by $h(f)(i)(j) = f(p(i,j))$. Observe that the h-image of every $U_{s,n}$ from the proof of Theorem 2.2 is a box. Therefore, X^α has a base of boxes not containing an ω^{op}-like subcover. Finally, observe that $Nt(X^\alpha) = Nt(\omega^\omega) = \omega$ by Theorem 1.4. □

We actually rediscovered Theorem 2.2. The first reference we have to it is in [2], where it is credited to Konstantinov. (See also page 26 of [3].) Whether every base of a metric space contains an ω^{op}-like base is closely related to total metacompactness and total paracompactness.

Definition 2.4. A space X is *totally metacompact* (*totally paracompact*) if every base $\mathcal{B}$ of X has a point-finite (locally finite) subcover $\mathcal{A}$.

Compact implies totally paracompact implies totally metacompact; less obviously, totally metacompact does not imply totally paracompact: Balogh and Bennett [5] noticed that Example 1 of [17] is a counterexample. (That counterexample is a Moore space, but we do not know if there is a metrizable counterexample.)

On the other hand, Lelek [22] has shown that total metacompactness, total paracompactness, and the Menger property are equivalent in the context of separable metric spaces. The next theorem connects these covering properties with ω^{op}-like bases.

Definition 2.5.

- A family $\mathcal{F}$ of subsets of a space X is *open in finite*, or *OIF*, if every nonempty open set of X has at most finitely many supersets in $\mathcal{F}$.
- A space is *totally OIF* if every base has an OIF subcover.
- Let $bNt(X)$ denote the least $\kappa \geq \omega$ such that every base of X includes a κ^{op}-like base of X.

Theorem 2.6. *If X is a metric space, then $bNt(X) = \omega$ if and only if X is totally OIF.*

Proof. If $bNt(X) = \omega$, then every base contains an ω^{op}-like base, which is also an OIF subcover. Conversely, if $\mathcal{A}$ is a base of X and $\mathcal{B}_n$ is an OIF subcover of the elements of A with diameter $\leq 2^{-n}$, for all $n < \omega$, then $\bigcup_{n<\omega} \mathcal{B}_n$ is an ω^{op}-like base. □

Corollary 2.7. *If X is a totally metacompact metric space, then $bNt(X) = \omega$.*

Question 2.8. Is there a metric space that has some but not all of the three properties totally OIF, totally metacompact, and totally paracompact?

Corollary 2.9 (Lemma 2.9, [27])**.** *$bNt(X) = Nt(X) = \omega$ for all compact metrizable X.*

Question 2.10. Is there a compact space X having a base that does not contain an $Nt(X)^{op}$-like base? In other words, is $bNt(X) < Nt(X)$ possible for a compact X?

Many non-compact metric spaces X satisfy $bNt(X) = Nt(X)$ too. Every σ-locally compact metric space X is totally paracompact [10], so it satisfies $bNt(X) = Nt(X) = \omega$. (To be σ-locally compact is to be a countable union of closed subspaces that are each locally compact. It is not hard to show that a paracompact, locally σ-locally compact space is already σ-locally compact.) Indeed, every scattered metric space (even every C-scattered metric space) is totally paracompact (and σ-locally compact) [34].

Remark. $Nt(X) = \omega$ for all metrizable X. Moreover, it was noted by Bennett and Lutzer in [7] that, "it is easy to prove that any metric space, and indeed any metacompact Moore space, has an OIF base." Indeed, the proof would be an easy modification of the proof of Theorem 2.6: if $\langle \mathcal{D}_n \rangle_{n<\omega}$ is a devolopment, then, after choosing a point-finite refinement $\mathcal{R}_n$ of each $\mathcal{D}_n$, we obtain an OIF (and therefore ω^{op}-like) base: $\bigcup_{n<\omega} \mathcal{R}_n$.

Returning our focus from metric spaces back to compacta, we are going to prove that for several classes of compact spaces X, Question 1.6 has a positive answer: $Nt(X^2) = Nt(X)$. Theorem 2.12 below says the answer is yes for the wide class of spaces X satisfying $\chi(p, X) = w(X)$ for all $p \in X$. We will present some further partial answers to this question. In particular, it is consistent that the answer is yes for all homogeneous compacta.

Proposition 2.11. *If X is a space and $\mathcal{A}$ is a $(w(X)^+)^{\mathrm{op}}$-like base of X, then $|\mathcal{A}| \leq w(X)$.*

Proof. Seeking a contradiction, suppose that $|\mathcal{A}| > w(X)$. Let $\mathcal{B}$ be a base of X of size $w(X)$. Every element of $\mathcal{A}$ then contains an element of $\mathcal{B}$. Hence, some $U \in \mathcal{B}$ is contained in $w(X)^+$-many elements of $\mathcal{A}$. Clearly U contains some $V \in \mathcal{A}$, so $\mathcal{A}$ is not $(w(X)^+)^{\mathrm{op}}$-like. □

Theorem 2.12 (Lemma 3.20, [27])**.** *Suppose that X is a space with no isolated points and $\chi(p, X) = w(X)$ for all $p \in X$. Further suppose that $\kappa = \operatorname{cf}\kappa \leq \min\{\mathrm{Nt}(X), w(X)\}$ and X has a network consisting of at most $w(X)$-many κ-compact sets. Every base of X then contains a $\mathrm{Nt}(X)^{\mathrm{op}}$-like base of X.*

Remark. If X is T_3 and locally compact, then it is easily seen that X has a network consisting of at most $w(X)$-many compact sets.

The following two lemmas are easy modifications of Dow's Propositions 2.3 and 2.4 from [11].

Lemma 2.13. *Let X be a space with base $\mathcal{A}$; let $\omega < \operatorname{cf}\kappa = \kappa$, $\{X, \mathcal{A}, \kappa\} \subseteq M \prec H(\theta)$, and $\kappa \cap M \in \kappa + 1$. Set $B = \{p \in X : \operatorname{ord}(p, \mathcal{A}) < \kappa\}$. We then have $\{U \in \mathcal{A} : p \in U\} \subseteq M$ for every $p \in \overline{B \cap M}$.*

Proof. Suppose that $p \in \overline{B \cap M}$ and $p \in U \in \mathcal{A}$. Choose $q \in U \cap B \cap M$. Since $\kappa \cap M \in \kappa + 1$, we have

$$U \in \{V \in \mathcal{A} : q \in V\} \in [H(\theta)]^{<\kappa} \cap M \subseteq [M]^{<\kappa};$$

hence, $U \in M$. □

Remark. The conclusion of the above lemma immediately implies that $\overline{B \cap M} \subseteq B$ if $|M| < \kappa$ (but we do not use this fact).

Lemma 2.14. *Let X be a compact T_1 space with base $\mathcal{A}$ and let M be such that $X, \mathcal{A} \in M \prec H(\theta)$ and $\mathcal{A} \cap M$ includes a local base at every $p \in \overline{X \cap M}$. We then have $\overline{X \cap M} = X$; hence, $\mathcal{A} \cap M$ is a base of X.*

Proof. Seeking a contradiction, suppose that $q \in X \setminus \overline{X \cap M}$. Choose $\mathcal{B} \subseteq \mathcal{A} \cap M$ such that $q \notin \bigcup \mathcal{B} \supseteq \overline{X \cap M}$. Choose a finite $\mathcal{F} \subseteq \mathcal{B}$ such that $\bigcup \mathcal{F} \supseteq \overline{X \cap M}$. Since $\mathcal{F} \in M$, we have $X \subseteq \bigcup \mathcal{F}$ by elementarity, in contradiction with $q \notin \bigcup \mathcal{B}$. □

Theorem 2.15. *Let X be a compact T_1 space with base $\mathcal{A}$ and let κ be a regular uncountable cardinal. Set $B = \{p \in X : \operatorname{ord}(p, \mathcal{A}) < \kappa\}$. We then have $w(\overline{B}) < \kappa$.*

Proof. Choose M to be as in Lemma 2.13 and to have size less than κ. Applying Lemma 2.14 to the space $\overline{B}$ and its base $\mathcal{U} = \{U \cap \overline{B} : U \in \mathcal{A}\}$, we get a sufficiently small base $\mathcal{U} \cap M$ of $\overline{B}$. □

The following lemma improves upon Theorem 1 of Peregudov [29], which says that if X is a compactum, then $w(X) \leq \pi\chi(X) lNt(X)$, where $lNt(X)$ is the supremum of all cardinals strictly below $\mathrm{Nt}(X)$.

Lemma 2.16. *Let X be a compact space such that $w(X) \geq \kappa$ where κ is some regular uncountable cardinal. If X has a dense set of points of π-character $< \kappa$, then $Nt(X) > \kappa$.*

Proof. Let $\mathcal{B}$ be any base for X. By Theorem 2.15, there is an open set $U \subset X$ such that every point of U has order at least κ. Let $p \in U$ be a point of π-character less than κ, and $\mathcal{C} \subset \mathcal{B}$ be a set such that $|\mathcal{C}| = \kappa$ and $p \in \bigcap \mathcal{C}$. Since p has π-character less than κ, there is a nonempty open set that is in κ-many members of $\mathcal{C}$. So, $Nt(X) > \kappa$. $\square$

The above lemma fails if κ is allowed to be singular.

Example 2.17. For one example, if Y is the one-point compactification of $\bigoplus_{n<\omega} 2^{\omega_n}$, then $\pi\chi(p, Y) < \aleph_\omega = w(Y)$ for all $p \in Y$, yet $\mathrm{Nt}(Y) = \omega$ is witnessed by joining the canonical bases of 2^{ω_n} for $n < \omega$ with $\{Y \setminus \bigcup_{m<n} 2^{\omega_m} : n < \omega\}$.

Example 2.18. For another example, let $X = \prod_{n<\omega} A_{\aleph_n}$ where for all infinite cardinals κ, A_κ denotes the one-point compactification $D_\kappa \cup \{\infty\}$ of the discrete space D_κ with underlying set κ. Notice that $w(X) = w(A_{\aleph_\omega}) = \aleph_\omega$ and $\pi\chi(X) = \pi\chi(A_{\aleph_\omega}) = \omega$. Let us show that $\mathrm{Nt}(A_{\aleph_\omega}) = \aleph_{\omega+1}$, but $\mathrm{Nt}(X) = \aleph_\omega$.

First, let us show that actually $\mathrm{Nt}(A_\kappa) = \kappa^+$ for all uncountable κ. Let $\mathcal{U}$ be a base of A_κ. Set $F = \{\sigma \subseteq \kappa : A_\kappa \setminus \sigma \in \mathcal{U}\} \in [[\kappa]^{<\omega}]^\kappa$. Set $S = \{\lambda^+ : \omega \leq \lambda < \kappa\}$. For each $\mu \in S$, choose $I_\mu \in [F]^\mu$ such that I_μ is a Δ-system with root r_μ. Partition each I_μ into disjoint subsets J_μ and K_μ each of size μ. Observe that if

$$J = \bigcup_{\mu \in S} \left\{ \sigma \in J_\mu : \varnothing = (\sigma \setminus r_\mu) \cap \bigcup_{\nu \in \mu \cap S} K_\nu \right\},$$

then $\bigcup J$ has size κ but does not equal κ. Thus, $\bigcap_{\sigma \in J}(A_\kappa \setminus \sigma)$ includes an isolated point. Hence, $\mathcal{U}$ is not κ^{op}-like; hence, $\kappa^+ \leq \mathrm{Nt}(A_\kappa) \leq w(A_\kappa)^+ = \kappa^+$.

Second, by Theorem 1.3, $\mathrm{Nt}(X) \leq \sup_{n<\omega} \mathrm{Nt}(A_{\aleph_n}) = \aleph_\omega$. Finally, $\mathrm{Nt}(X) \geq \aleph_\omega$ by Lemma 2.16.

Example 2.19. Building on the previous example, let Z be the one-point compactification of $\bigoplus_{\alpha<\omega_1} A_{\aleph_\alpha}$. Observe that $w(Z) = w(A_{\aleph_{\omega_1}}) = \aleph_{\omega_1}$ and $\pi\chi(Z) = \pi\chi(A_{\aleph_{\omega_1}}) = \omega$. As argued above, $\mathrm{Nt}(A_{\aleph_{\omega_1}}) = \aleph_{\omega_1+1}$. However, we will show that $\mathrm{Nt}(Z) = \aleph_{\omega_1}$. First, by Lemma 2.16, $\mathrm{Nt}(Z) \geq \aleph_{\omega_1}$. Second, we can build an $\aleph_{\omega_1}^{\mathrm{op}}$-like base $\mathcal{C}$ of Z as follows. For each $\alpha < \omega_1$, let $\mathcal{A}_\alpha$ be (a copy of) a base of $A_{\aleph_\alpha}$ of size $\aleph_\alpha$. Set $\mathcal{B} = \{Z \setminus \bigcup_{\alpha \in \sigma} A_{\aleph_\alpha} : \sigma \in [\omega_1]^{<\omega}\}$. Set $\mathcal{C} = \mathcal{B} \cup \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$.

Theorem 2.20. *If X is a homogeneous compactum with regular weight, then every base of X contains an $\mathrm{Nt}(X)^{\mathrm{op}}$-like base.*

Proof. If $\chi(X) = w(X)$, then just apply Theorem 2.12. If $\chi(X) < w(X)$, then $\mathrm{Nt}(X) = w(X)^+$ by Lemma 2.16. So, if $\mathcal{A}$ is any base for X, then every base of size $w(X)$ contained in $\mathcal{A}$ would be $Nt(X)^{op}$-like. $\square$

We can exchange the above requirement that $w(X)$ be regular for a weak form of GCH.

Corollary 2.21. *Suppose that every limit cardinal is strong limit. For every homogeneous compactum X, every base of X then contains an $\mathrm{Nt}(X)^{\mathrm{op}}$-like base.*

Proof. By Arhangel′skiĭ's Theorem, $\chi(X) \leq w(X) \leq 2^{\chi(X)}$. If $\chi(X) < w(X)$, then $w(X)$ is a successor cardinal; apply Theorem 2.20. If $\chi(X) = w(X)$, then apply Theorem 2.12. $\square$

Corollary 2.22. *(GCH) Let X be a homogeneous compactum. Then $Nt(X^n) = Nt(X)$ for every $n \in \omega$.*

Geschke and Shelah [14] have shown that for every infinite cardinal $\kappa \leq \mathfrak{c}$, there is a first countable homogeneous compactum with weight κ. Therefore, it is consistent to have a homogeneous compactum X such that our above theorems do not determine whether $Nt(X^2) = Nt(X)$.

If we don't assume homogeneity, then we still have the following weak results.

Theorem 2.23 (Lemma 3.23, [27])**.** *Suppose that $\kappa = \operatorname{cf}\kappa > \omega$ and X is a space such that $\pi\chi(p, X) = w(X) \geq \kappa$ for all $p \in X$. Further suppose that X has a network consisting of at most $w(X)$-many κ-compact sets. Every base of X then contains a $w(X)^{\mathrm{op}}$-like base of X.*

Remark. If X is T_3 and locally compact, then it is easily seen that X has a network consisting of at most $w(X)$-many compact sets.

Theorem 2.24. *Suppose that κ is a regular cardinal and X is a locally κ-compact T_3 space such that $\mathrm{Nt}(X) \leq w(X) = \kappa$. Every base of X then contains a κ^{op}-like base of X.*

Proof. Let $\mathcal{A}$ be a base of X and let $\mathcal{B}$ be a κ^{op}-like base of X. We may assume that $|\mathcal{A}| = |\mathcal{B}| = \kappa$. Suppose that $\kappa = \omega$. The space X is then metrizable and σ-compact, so, as noted earlier for the wider class of σ-locally compact metric spaces, every base of X contains an ω^{op}-like base.

Suppose that $\kappa > \omega$. Let $\langle M_\alpha \rangle_{\alpha \leq \kappa}$ be a continuous elementary chain such that $\{M_\beta : \beta < \alpha\} \cup \{\mathcal{A}, \mathcal{B}\} \subseteq M_\alpha \prec H(\theta)$ and $|M_\alpha| < \kappa$ and $M_\alpha \cap \kappa \in \kappa$ for all $\alpha < \kappa$. The inclusion $\mathcal{A} \cup \mathcal{B} \subseteq M_\kappa$ follows immediately. For each $\alpha < \kappa$, let $\mathcal{U}_\alpha$ denote the set of all $U \in \mathcal{A} \cap M_{\alpha+1}$ for which U has a superset in $\mathcal{B} \setminus M_\alpha$. Set $\mathcal{U} = \bigcup_{\alpha<\kappa} \mathcal{U}_\alpha \subseteq \mathcal{A}$. First, let us show that $\mathcal{U}$ is κ^{op}-like. Suppose that $\alpha < \kappa$ and $\mathcal{U}_\alpha \ni U \subseteq V \in \mathcal{U}$. There then exist $\beta < \kappa$ and $B \in \mathcal{B} \setminus M_\beta$ such that $B \supseteq V \in M_{\beta+1}$. Hence, $U \subseteq B$; hence, $B \in \{W \in \mathcal{B} : U \subseteq W\} \in M_{\alpha+1} \cap [\mathcal{B}]^{<\kappa}$; hence, $B \in M_{\alpha+1}$; hence, $\beta \leq \alpha$; hence, $V \in M_{\alpha+1}$. Thus, $\mathcal{U}$ is κ^{op}-like.

Finally, let us show that $\mathcal{U}$ is a base of X. Suppose that $p \in B \in \mathcal{B}$ and $\overline{B}$ is κ-compact. It then suffices to find $U \in \mathcal{U}$ such that $p \in U \subseteq B$. Let β be the least $\alpha < \kappa$ such that there exists $A \in \mathcal{A} \cap M_{\alpha+1}$ satisfying $p \in A \subseteq \overline{A} \subseteq B$. Fix such an $A \in \mathcal{A} \cap M_{\beta+1}$. If $B \notin M_\beta$, then $A \in \mathcal{U}_\beta$ and $p \in A \subseteq B$. Hence, we may assume that $B \in M_\beta$. For each $q \in \overline{A}$, choose $\langle A_q, B_q \rangle \in \mathcal{A} \times \mathcal{B}$ such that $q \in A_q \subseteq B_q \subseteq \overline{B}_q \subseteq B$. There then exists $\sigma \in \left[\overline{A}\right]^{<\kappa}$ such that $\overline{A} \subseteq \bigcup_{q\in\sigma} A_q$. By elementarity, we may assume that $\langle\langle A_q, B_q \rangle\rangle_{q\in\sigma} \in M_{\beta+1}$; hence, $A_q, B_q \in M_{\beta+1}$ for all $q \in \sigma$. Choose $q \in \sigma$ such that $p \in A_q$. If $B_q \notin M_\beta$, then $A_q \in \mathcal{U}_\beta$ and $p \in A_q \subseteq B$. Hence, we may assume that $B_q \in M_\beta$; hence, we may choose $\alpha < \beta$ such that $B_q \in M_{\alpha+1}$. It follows that $B \in \{W \in \mathcal{B} : B_q \subseteq W\} \in M_{\alpha+1} \cap [\mathcal{B}]^{<\kappa}$; hence, $B \in M_{\alpha+1}$. For each $r \in \overline{B_q}$, choose $W_r \in \mathcal{A}$ such that $r \in W_r \subseteq \overline{W}_r \subseteq B$. There then exists $\tau \in \left[\overline{B}_q\right]^{<\kappa}$ such that $\overline{B}_q \subseteq \bigcup_{r\in\tau} W_r$. By elementarity, we may assume that $\langle W_r \rangle_{r\in\tau} \in M_{\alpha+1}$. Choose $r \in \tau$ such that $p \in W_r$. We then have $W_r \in \mathcal{A} \cap M_{\alpha+1}$ and $p \in W_r \subseteq \overline{W_r} \subseteq B$, in contradiction with the minimality of β. Thus, $\mathcal{U}$ is a base of X. □

Theorem 2.25. *Let X be a compact space such that $w(X)$ is a regular cardinal and X does not map onto $I^{w(X)}$. Then $Nt(X^n) = Nt(X)$ for every $n \in \omega$.*

Proof. By a well-known consequence of Šhapirovskiĭ's Theorem on maps onto Tychonoff cubes (see [18], 3.20) X has a dense set of points of π-character $< w(X)$.

But then also X^n has a dense set of points of π-character $< w(X)$. Therefore, by Lemma 2.16, we have $w(X) = w(X^n) < Nt(X^n)$. Let $\mathcal{B}$ be a base for X^n of size $w(X)$ consisting of boxes. Then $\mathcal{B}$ is trivially $Nt(X^n)^{op}$-like base and hence we are done. □

Corollary 2.26. *$Nt(X^n) = Nt(X)$ for every compact space such that $w(X)$ is a regular cardinal and at least one of the following conditions holds:*

(1) *X is hereditarily normal.*
(2) *$\beta\omega$ does not embed in X.*
(3) *$|X| < 2^{w(X)}$.*

Proof. The case of the third item follows readily from Theorem 2.25. In case X is like in the first or the second item then X cannot even map onto I^{ω_1} by the argument in the proof of 3.21 and 3.22 of [18]. □

We now proceed to show the strongest instance of the failure of productivity of Noetherian type that we know of so far. Recall that a partial order is called *directed* if any two elements have a common upper bound. A map between partial orders is called *Tukey* if it sends unbounded sets into unbounded sets and *convergent* if it maps cofinal sets into cofinal sets. Tukey and Schmidt proved (see [35], Proposition 1) that there is a Tukey map from P to Q if and only if there is a convergent map from Q to P.

Let κ be a regular cardinal such that $\kappa^\omega = \kappa$; order $[\kappa]^{<\omega}$ with respect to containment. Let S_0 and S_1 be two stationary subsets of κ with non-stationary intersection. Let D_i be the set of all countable compact subsets of S_i, ordered with respect to containment. Todorcevic [35] has proved that there is no Tukey map between $[\kappa]^{<\omega}$ and D_i but there is a Tukey map $T : [\kappa]^{<\omega} \to D_0 \times D_1$, where the ordering on the codomain is the product ordering. Note that since $D_0 \times D_1$ is directed and $[\kappa]^{<\omega}$ has no infinite unbounded sets, any injection of $D_0 \times D_1$ into $[\kappa]^{<\omega}$ is Tukey. Therefore, the map T can be chosen to be convergent.

Example 2.27. There are spaces X and Y such that $Nt(X\times Y) < \min\{Nt(X), Nt(Y)\}$.

Proof. For $i = 0, 1$, let X_i be the set $[\kappa]^{<\omega}$ topologized in such a way that a local base at the point $x \in X_i$ is $\{\langle x, E\rangle : E \in D_i\}$, where $\langle x, E\rangle = \{x \cup z : z \in [\kappa \setminus E]^{<\omega}\}$. We claim that we even have $\chi Nt(X_i) \geq \aleph_1$ for $i = 0, 1$. Indeed, let $\mathcal{B}$ be a local base at the point $x \in X_i$. Since $\kappa^\omega = \kappa$, we can assume that $|\mathcal{B}| = \kappa$. Moreover, we can assume that $\mathcal{B}$ is of the form $\{\langle x, E\rangle : E \in \mathcal{E}\}$ where $\mathcal{E} \subset D_i$ is cofinal. Now fix an injection $F : [\kappa]^{<\omega} \to \mathcal{E}$. By Todorcevic's result, we can find an unbounded set A such that $\{F(a) : a \in A\}$ is bounded by some E. Therefore, we have $\langle x, E\rangle \subset \langle x, F(a)\rangle$ for every $a \in A$, which shows that $\chi Nt(X) \geq \aleph_1$.

Now we claim that $Nt(X \times Y) = \omega$. Indeed, let $T : [\kappa]^{<\omega} \to D_0 \times D_1$ be a convergent Tukey map, and consider $\{\langle x, T(y)_0\rangle \times \langle z, T(y)_1\rangle : x, y, z \in [\kappa]^{<\omega}\}$. This set is a base because the range of T is cofinal. Suppose that $\langle x, T(y)_0\rangle \times \langle x', T(y)_1\rangle \subset \langle x_j, T(y_j)_0\rangle \times \langle x'_j, T(y_j)_1\rangle$ for every $j \in \omega$. Then for every $j \in \omega$ we have $x_j \subset x$ and $x'_j \subset x'$. So, we can assume that there exist z and z' such that $\langle x, T(y)_0\rangle \times \langle x', T(y)_1\rangle \subset \langle z, T(y_j)_0\rangle \times \langle z', T(y_j)_1\rangle$ for every $j \in \omega$. Then $T(y_j)_0 \subset T(y)_0 \cup x$ and $T(y_j)_1 \subset T(y)_1 \cup x'$, contradicting the fact that T is a Tukey map. □

Question 2.28. Do there exist compact spaces X and Y such that $Nt(X \times Y) < \min\{Nt(X), Nt(Y)\}$?

The methods of this section can be used to attack also Question 2 from [6], which in our terminology reads *does every dense subspace of a regular space of countable Noetherian type have countable Noetherian type*? This is because of the following theorem.

Theorem 2.29. *Let X be a regular space such that every base of X contains a $Nt(X)^{op}$ like base of X. Then $Nt(D) \leq Nt(X)$ for every $D \subset X$.*

Proof. Let $\mathcal{B}$ be a base consisting of regular open sets (that is, $Int(\overline{B}) = B$ for every $B \in \mathcal{B}$. Let $\mathcal{U} \subset \mathcal{B}$ be a $Nt(X)^{op}$-like. Let $\mathcal{V} = \{D \cap U : B \in \mathcal{U}\}$. Then $\mathcal{U}$ is a base for $\mathcal{B}$. To see that $\mathcal{U}$ is $Nt(X)^{op}$-like just note that $U \cap D \subset V \cap D$ implies that $U \subset V$ whenever U and V are regular open. □

Define $\delta Nt(X) = \sup\{Nt(D) : D \text{ is a dense subset of } X\}$. Note that we always have $\delta Nt(X) \geq Nt(X)$. It is well-known that doing the same procedure for cellularity doesn't give rise to a new cardinal function. In other words, the cellularity of a dense subspace is always equal to the cellularity of the whole space. However, the authors of [6] showed that this is not the case for Noetherian type, at least if one is willing to forego regularity.

Theorem 2.30. [6] *There is a Hausdorff space X such that $\delta Nt(X) > Nt(X)$.*

Corollary 2.31. *$\delta Nt(X) = Nt(X)$ whenever X is a compact space such that $w(X)$ has regular weight and one of the following conditions holds:*

1. *X is homogeneous.*
2. *X is hereditarily normal.*
3. *$\beta\omega$ does not embed in X.*
4. *$|X| < 2^{w(X)}$.*

So Corollary 2.31 provides partial answers to Question 2 from [6], which we now pose in a more general form.

Question 2.32. Is $\delta Nt(X) = Nt(X)$ for every regular space X?

Bailey [4] introduced a natural strengthening of countable Noetherian type which implies countable Noetherian type of every dense subspace.

3. Noetherian type, sparse families and square principles

The second author proved that if X is a compact dyadic homogeneous space, then $Nt(X) = \omega$ [27]. In his proof a generalization of a continuous elementary chain of countable elementary submodels is used to approximate X by compact metric spaces and then coherence and arguments similar to the proof of Theorem 2.24 are used to cook up an ω^{op}-like base for the whole space from ω^{op}-like bases for each of the approximations. In particular, the Noetherian type of compact groups is countable. A simple operation that destroys compactness on every infinite space is the G_δ modification. Let X_δ denote the space obtained from X by declaring all the G_δ sets to be open. It turns out that even the Noetherian type of such a simple space as $(2^{\aleph_\omega})_\delta$ depends on cardinal arithmetic. We were originally motivated to look at the Noetherian type of the countably supported topology on $2^{\aleph_\omega}$ by the following theorem.

Theorem 3.1. *[$(\forall\kappa)(\lambda < \kappa \Rightarrow \lambda^\omega \leq \kappa)$] Let X be a (countably) compact space such that $Nt(X)$ has uncountable cofinality. Then $Nt(X_\delta) \leq 2^{Nt(X)}$.*

Proof. Suppose that $Nt(X) = \kappa$ and let $\mathcal{B}$ be a κ^{op}-like base for X. Moreover, let $\mathcal{B}_\delta$ be the set of all countable intersections from $\mathcal{B}$. Clearly $\mathcal{B}_\delta$ is a base for X_δ. Now, suppose it's not $(2^\kappa)^{op}$-like. Then some $B \in \mathcal{B}_\delta$ is contained in every element of some family $\mathcal{F} = \{B_\alpha : \alpha < 2^\kappa\} \subset \mathcal{B}_\delta$ of distinct G_δ sets. Let $\mathcal{U} \subset \mathcal{B}$ be the set of all open sets that make up elements in $\mathcal{F}$. Then $|\mathcal{U}| \geq \kappa$, because if $|\mathcal{U}| < \kappa$ then $|\mathcal{F}| \leq |\mathcal{U}|^\omega \leq \kappa < 2^\kappa$. So take some enumeration $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$. Observe that every G_δ set in a regular space contains a closed G_δ, then let $G = \bigcap_{i \in \omega} \overline{G_i} \subset B$ be some closed G_δ set. Observe that $G \subset U_\alpha$ for every $\alpha < \kappa$, so use compactness to find for every $\alpha < \kappa$ an $n \in \omega$ such that $\bigcap_{i=1}^n G_i \subset U_\alpha$. Since κ has uncountable cofinality there has to be some $R \subset \kappa$ and $n \in \omega$ such that $|R| = \kappa$ and $\bigcap_{i=1}^n G_i \subset U_\alpha$ for every $\alpha \in R$. Let now $V \in \mathcal{B}$ be such that $V \subset \bigcap_{i=1}^n G_i$. Then κ^{op}-ness of $\mathcal{B}$ is contradicted. □

The set-theoretic assumption in Theorem 3.1 is essential, as the following example shows.

Example 3.2. A compact space X such that $\mathrm{cf}(Nt(X)) > \omega$, but $Nt(X_\delta) > 2^{Nt(X)}$ in a model where $(\aleph_\omega)^\omega = \aleph_{\omega+2}$.

Proof. Start from a model of ZFC+GCH+κ is a measurable cardinal of Mitchell order κ^{++}. Force with Gitik-Magidor forcing ([16], see also [15]). In a generic extension GCH will fail only at $\aleph_\omega$ where we have $2^{\aleph_\omega} = \aleph_\omega^\omega = \aleph_{\omega+2}$. Note that in a generic extension we must have $\mathrm{cov}(\aleph_\omega, \omega) = \aleph_{\omega+2} = 2^{\aleph_{\omega+1}}$. (See Definition 3.17.) Let now X be the one-point compactification of $\aleph_\omega$ with the discrete topology. Then $Nt(X) = \aleph_{\omega+1}$. (See Example 2.18.) We are now going to show that $Nt(X_\delta) = cov(\aleph_\omega, \omega)^+$ so that X will satisfy the statement of the example in a generic extension. Indeed, note that $Nt(X_\delta) \leq cov(\aleph_\omega, \omega)^+$ since $w(X_\delta) = cov(\aleph_\omega, \omega)$ and $Nt(X_\delta) \leq w(X_\delta)^+$. For the reverse inequality, let $\lambda = cov(\aleph_\omega, \omega)$ and $\mathcal{B}$ be any base for X, and suppose by contradiction that $Nt(X) \leq \lambda$. Let $\mathcal{C} = \{C \in [\aleph_\omega]^\omega : X \setminus C \in \mathcal{B}\}$. Enumerate $\mathcal{C} = \{C_\alpha : \alpha < \lambda\}$. Let γ be any ordinal less than $\aleph_\omega$. If we could find λ-many elements of $\mathcal{C}$ which miss γ, then the isolated point γ would have λ-many supersets in $\mathcal{B}$. Hence, we can assume that for every $\alpha < \aleph_1$ we can find $\beta_\alpha < \lambda$ such that $\alpha \in C_\gamma$ for every $\gamma \geq \beta_\alpha$. Let $\beta = \sup_{\alpha < \aleph_1} \beta_\alpha$. We have that $\beta < \lambda$ since λ is regular and $\lambda \geq \aleph_{\omega+1}$. But this implies $\{\alpha : \alpha < \aleph_1\} \subset C_{\beta+1}$, which contradicts the fact that $C_{\beta+1}$ is countable. Therefore, $Nt(X) \geq \lambda^+$ and we are done. □

The proof of Theorem 3.1 easily generalizes to show that if $\lambda^{\aleph_0} < \kappa$ for all $\lambda < \kappa$ and $\mathrm{cf}(\kappa) > \omega$, then $\mathrm{Nt}(X) \leq \kappa$ implies $\mathrm{Nt}(X_\delta) \leq \kappa$, without any assumption on $\mathrm{cf}(\mathrm{Nt}(X))$. In particular, $Nt(X_\delta) \leq \left(2^{Nt(X)}\right)^+$ and $Nt((2^{\aleph_\omega})_\delta) \leq \mathfrak{c}^+$. However, as we shall see in the next section, it is consistent (relative to large cardinals) that the upper bound $Nt((2^{\aleph_\omega})_\delta) \leq \mathfrak{c}^+$ cannot be improved, making the assumption $\mathrm{cf}(Nt(X)) > \omega$ essential to Theorem 3.1. In constrast, the cellularity of the G_δ modification of a compact space X is always bounded above by $2^{c(X)}$. (See [19].)

3.1. **Sparse families.** To see that the assumption about the cofinality of the Noetherian type is essential in Theorem 3.1 we need to introduce a new combinatorial object.

Definition 3.3.

(1) Let κ be a cardinal. A family of sets $\mathcal{F}$ is κ-small if $|\bigcup F| < \kappa$. Equivalently, there exists a set B with $|B| < \kappa$such that $\mathcal{F} \subseteq \mathcal{P}(B)$.
(2) A family of sets $\mathcal{F}$ is (μ, κ)-*sparse* if no $\mathcal{G} \subset \mathcal{F}$ with $|\mathcal{G}| \geq \mu$ is κ-small. In other words, $|\bigcup \mathcal{G}| \geq \kappa$ for every $\mathcal{G} \in [\mathcal{F}]^\mu$.
(3) A family $\mathcal{F}$ is called ν-*uniform* if each member of $\mathcal{F}$ is a set of cardinality ν.

Let us list a few basic properties of (μ, κ)-sparse families of sets.

(1) A (μ, κ)-sparse family $\mathcal{F}$ is (μ', κ')-sparse whenever $\mu' \geq \mu$ and $\kappa' \leq \kappa$.
(2) Every ν-uniform $\mathcal{F}$ is $((2^\nu)^+, \nu^+)$-sparse.
(3) If $\mu > |\mathcal{F}|$ then $\mathcal{F}$ is (μ, κ)-sparse for every cardinal κ (vacuously) and if $\kappa > |\bigcup \mathcal{F}|$ then $\mathcal{F}$ is not (μ, κ)-sparse for any μ.
(4) If κ is limit and $\mathcal{F}$ is (μ, θ)-sparse for every $\theta < \kappa$ then $\mathcal{F}$ is (μ, κ)-sparse.
(5) For every cardinal κ the class of cardinals μ for which $\mathcal{F}$ is *not* (μ, κ)-sparse is closed under limits of cofinality $< \operatorname{cf} \kappa$.
(6) If $\mathcal{F}$ is ν-uniform then the least μ for which $\mathcal{F}$ is (μ, ν^+)-sparse satisfies $\operatorname{cf} \mu > \nu$.

The first 4 items are obvious and (6) follows from (5). To prove (5) suppose $\langle \mu_i : i < \theta \rangle$ is an increasing sequence of ordinals with limit μ for some $\theta < \operatorname{cf} \kappa$ and that $\mathcal{F}$ is not (μ_i, κ)-sparse for each $i < \theta$. For each $i < \theta$ fix a κ-small $\mathcal{G}_i \subseteq \mathcal{F}$ of cardinality μ_i and let $\mathcal{G} = \bigcup_{i<\theta} \mathcal{G}_i$. Now $\mathcal{G}$ has cardinality μ and is κ-small because $\theta < \operatorname{cf} \kappa$.

Recall that $\operatorname{cov}(\theta, \kappa)$ is the least size of a collection $\mathcal{A} \subseteq [\theta]^{<\kappa}$ such that every $X \in [\theta]^{<\nu}$ is contained in some member of the collection.

Claim. *Suppose $\mathcal{F}$ is ν-uniform and (μ, ν^+)-sparse. Then $\mathcal{F}$ is (μ, κ)-sparse for every $\kappa \geq \nu^+$ such that for all $\nu < \rho < \kappa$ it holds that* $\operatorname{cov}(\rho, \nu) < \operatorname{cf} \mu$.

Proof. Suppose that, contrary to the claim, $|B| = \rho < \kappa$ and that $|\mathcal{P}(B) \cap \mathcal{F}| \geq \mu$. Fix a covering collection $\mathcal{B} \subseteq [B]^{<\nu}$ of cardinality $|\mathcal{B}| < \operatorname{cf} \mu$. It follows that some $Y \in \mathcal{B}$ contains μ members of $\mathcal{F}$ which, as $|Y| = \nu$, contradics (μ, ν)-sparseness. □

Claim. *Suppose that $\mathcal{F}$ is $\aleph_\alpha$-uniform for some infinite cardinal $\aleph_\alpha$, and μ is the least cardinal for which $\mathcal{F}$ is $(\mu, \aleph_{\alpha+1})$-sparse. Then $\mathcal{F}$ is $(\mu', \aleph_{\alpha+\beta})$-sparse for every $1 \leq \beta \leq \omega$ and $\mu' = \max\{\mu, \aleph_{\alpha+\beta}\}$.*

Proof. The case $\beta = \omega$ follows from the case $1 \leq \beta < \omega$, which we prove by induction on n.

Recall that $\operatorname{cf} \mu > \nu$. Given $n + 1 \geq 1$ let $\mu' = \max\{\mu, \aleph_{\alpha+n+1}\}$ and now also $\operatorname{cf} \mu' > \nu$. Assume, contrary to the claim, that there exists some set B or cardinality $\aleph_{\omega+n+1}$ such that $|\mathcal{P}(B) \cap \mathcal{F}| \geq \mu'$. As $\operatorname{cov}(\aleph_{\alpha+n+1}, \aleph_{\alpha+n}) = \aleph_{\alpha+n+1}$ we are done. □

Corollary 3.4. *If $\mathcal{F}$ is an $\aleph_0$-uniform family and there exists n such that $\mathcal{F}$ is $(\aleph_n, \aleph_1)$-sparse then $\mathcal{F}$ is $(\aleph_\alpha, \aleph_\alpha)$-sparse for all $n \leq \alpha \leq \omega$.*

Claim.
(1) *if $\{F_\alpha : \alpha < \lambda\} \subseteq [\aleph_\omega]^{\aleph_0}$ is a (μ, κ)-sparse family and $\{G_\alpha : \alpha < \lambda\}$ is any family which is cofinal in $([\aleph_\omega]^\omega, \subset)$ then $\{F_\alpha \cup G_\alpha : \alpha < \lambda\}$ is both (μ, κ)-sparse and cofinal.*
(2) *Let $\lambda = \operatorname{cf}([\aleph_\omega]^{\aleph_0}], \subseteq)$. If a cofinal family $\mathcal{F} \subseteq [\aleph_\omega]^{\aleph_0}$ is $(\aleph_n, \aleph_1)$-sparse for some $n \in \omega$ then it is $(\aleph_n, \aleph_n)$-sparse for all $m \geq n$ in ω.*

Proof. Item (1) is obvious and (2) folllows from (1). To prove (3) suppose that $\mathcal{F}$ is an $(\aleph_n, \aleph_1)$-sparse cofinal family and let $m \geq n$ be given. So $\mathcal{F}$ is $(\aleph_m, \aleph_1)$-sparse for all $m \geq n$. As $\mathrm{cov}(\aleph_m, \aleph_0) = \aleph_m$, it follows from Claim 3.1 that $\mathcal{F}$ is $(\aleph_m, \aleph_m)$-sparse. □

Our sparse families generalize Shelah's free families studied for example in [32] and in Magidor and Shelah's paper [24]. A family of sets which is κ-free satisfies that each of its κ subfamilies has an injective choice function; this condition of course implies (κ, κ)-sparseness.

Proof. Suppose $\mathcal{F}$ is ν-uniform and (ν^{+n}, κ)-sparse. As $\mathrm{cov}(\kappa^{+m}, \kappa, \nu^+, 2) = \kappa^{+m}$ for $m \geq n$, it follows bt Claim 3.1 that $\mathcal{F}$ □

3.2. **Sparse cofinal families and Noetherian type.** The following theorem links sparse cofinal families and the Noetherian type of the countably supported product topology on the Cantor Cube of weight $\aleph_\omega$.

Theorem 3.5. *Let $Y \subset (2^{\aleph_\omega})_\delta$ be a dense subset. Then $Nt(Y) \leq \kappa$ if and only if there is a $(\kappa, \aleph_1)$-sparse cofinal family in $([\aleph_\omega]^{\aleph_0}, \subseteq)$.*

Proof. Let $\mathcal{F}$ be a $(\kappa, \aleph_1)$-sparse cofinal family. Let $\mathcal{B} = \{[\sigma] \cap Y : \mathrm{dom}\, \sigma \in \mathcal{F}\}$. It is easy to see that $\mathcal{B}$ is a base for Y. To see that it is κ^{op}-like suppose by contradiction that there is a countable partial function $[\sigma]$ and a family of countable partial functions $\{\sigma_\alpha : \alpha < \kappa\}$ such that $[\sigma] \cap Y \subset [\sigma_\alpha] \cap Y$ for every $\alpha < \kappa$ and $[\sigma_\alpha] \neq [\sigma_\beta]$ whenever $\alpha \neq \beta$. By taking closures we see that $[\sigma] \subset [\sigma_\alpha]$ for every $\alpha < \kappa$. Note that when $\alpha \neq \beta$, $\mathrm{dom}\, \sigma_\alpha$ and $\mathrm{dom}\, \sigma_\beta$ are distinct or otherwise the corresponding basic open sets would be disjoint. Now $\mathrm{dom}\, \sigma_\alpha \subset \mathrm{dom}\, \sigma$ for every $\alpha < \kappa$, which contradicts $(\kappa, \aleph_1)$-sparseness of the family $\mathcal{F}$.

Viceversa, suppose that $Nt(Y) \leq \kappa$ and let $x \in Y$. Let $\mathcal{B}$ be a κ^{op}-like local base at x. We can assume that every element of $\mathcal{B}$ is the intersection of a box with Y. Note also that the set $\mathcal{B}'$ of all closures of elements of $\mathcal{B}$ is κ^{op}-like. Since $\mathcal{B}'$ is a κ^{op}-like local base at x in $(2^{\aleph_\omega})_\delta$ the set $\{\mathrm{dom}\, \sigma : [\sigma] \in \mathcal{B}'\}$ is a $(\kappa, \aleph_1)$-sparse cofinal family. □

Example 3.6. In Theorem 3.5, we cannot weaken density of Y to, for example, somewhere density, because we can embed a space $(2^{\aleph_\omega})_\delta \oplus X_\delta$ into $(2^{\aleph_\omega})_\delta$ such that X_δ is as in the proof of Example 3.2 and the embedded copy of $(2^{\aleph_\omega})_\delta$ is open in $(2^{\aleph_\omega})_\delta$. Indeed, that proof showed, in ZFC, that $Nt(X_\delta) = cov(\aleph_\omega, \omega)^+$, and we shall show in Lemma 3.20 that there is a $(cov(\aleph_\omega, \omega), \aleph_1)$-sparse cofinal family.

Recall that the notation $(\kappa, \lambda) \twoheadrightarrow (\alpha, \beta)$ abbreviates the statement that for every structure $M = (A, B, \dots)$ with countable signature, $|A| = \kappa$, and $|B| = \lambda$, there is an elementary substructure $N = (C, D, \dots) \prec M$ such that $|C| = \alpha$ and $|D| = \beta$. The statement $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ is known as *Chang's Conjecture for* $\aleph_\omega$. Chang's Conjecture for $\aleph_\omega$ is consistent with the GCH by [23]. It is easy to see that if Chang's Conjecture for $\aleph_\omega$ holds, then no family of countable subsets of $\aleph_\omega$ whose size is $> \aleph_\omega$ can be $(\aleph_1, \aleph_1)$-sparse. Therefore we have the following theorem due to Lajos Soukup.

Corollary 3.7. ([33]) *Assume Chang's Conjecture for $\aleph_\omega$. Then $Nt((2^{\aleph_\omega})_\delta) \geq \aleph_2$. If CH is also assumed, then $Nt((2^{\aleph_\omega})_\delta) = \aleph_2$.*

At the expense of breaking GCH, we prove the consistency of $\pi Nt((2^{\aleph_\omega})_\delta) \geq \aleph_2$ in the next theorem. We do not know if this inequality is consistent with GCH.

Definition 3.8. $\log(\kappa) = \min\{\lambda : 2^\lambda \geq \kappa\}$.

Lemma 3.9. *The space* $(2^{\aleph_\omega})_\delta$ *has a dense subspace of size* $(\log(\aleph_\omega))^\omega$.

Proof. Let $\kappa = \log(\aleph_\omega)$ and fix a subspace Z of $(2^\kappa)_\delta$ of size $\aleph_\omega$. Fix a base $\mathcal{B}$ for Z of size at most κ^ω. Let $\mathcal{F}$ be the set of all countable pairwise disjoint subsets of $\mathcal{B}$. Identify $(2^{\aleph_\omega})_\delta$ with $X = (2^Z)_\delta$ and let D be the set of all functions $f \in X$ for which, for some $F \in \mathcal{F}$, f is constant on each $y \in F$ and f is 0 outside of $\bigcup F$. Since countable subsets of $(2^\kappa)_\delta$ have disjoint open expansions, D is easily seen to be dense in X. □

Lemma 3.10. *If* $\mathbb{P}$ *is a forcing that preserves cardinals, then* $\mathbb{P}$ *cannot destroy Chang's Conjecture at* $\aleph_\omega$ *(or anywhere else).*

Proof. An equivalent formulation of Chang's Conjecture at $\aleph_\omega$ is that for all sufficiently large regular θ and all $A \in H(\theta)$, there exists $M \prec H(\theta)$ such that $A \in M$, $|M \cap \aleph_{\omega+1}| = \aleph_1$, and $|M \cap \aleph_\omega| = \aleph_0$.

Assume Chang's Conjecture at $\aleph_\omega$ and let G be a V-generic filter of $\mathbb{P}$. Choose θ large enough that $\mathbb{P}, \aleph_{\omega+1} \in H(\theta)$. In $V[G]$, let $A \in H(\theta)$. Back in V, let $\dot{A}$ be a $\mathbb{P}$-name for A and let $N \prec H(\theta)$ be such that $\mathbb{P}, \dot{A} \in N$, $|N \cap \aleph_{\omega+1}| = \aleph_1$, and $|N \cap \aleph_\omega| = \aleph_0$. Set $M = N[G] (= \{\tau_G : \tau \in V^{\mathbb{P}} \cap N\})$. Since $\mathbb{P} \in N$, we have $M \prec H(\theta)[G]$. Moreover, $M \cap \theta = N \cap \theta$ because forcings don't add ordinals. Hence, in $V[G]$ we have $M \prec H(\theta)$, $A \in M$, $|M \cap \aleph_{\omega+1}| = \aleph_1$, and $|M \cap \aleph_\omega| = \aleph_0$, as desired. (Note that the above argument generalizes to any Chang conjecture $(\kappa, \lambda) \twoheadrightarrow (\mu, {<}\nu)$.) □

Theorem 3.11. *It is consistent, relative to a 2-huge cardinal, that* $\pi\mathrm{Nt}((2^{\aleph_\omega})_\delta) = \aleph_2$.

Proof. Levinski, Magidor, and Shelah constructed a model of GCH and Chang's Conjecture at $\aleph_\omega$, assuming a 2-huge embedding. [23] (Actually, the embedding was only assumed to be slightly more than huge.) Starting from such a model, force with countably supported binary functions on $\aleph_{\omega+1}$. Passing to a generic extension, we still have CH and $\aleph_\omega^\omega = \aleph_{\omega+1}$, but $2^{\aleph_1} = \aleph_{\omega+1}$. By Lemma 3.10, we still have Chang's Conjecture at $\aleph_\omega$.

Let $\mathcal{A}$ be a π-base of $X = (2^{\aleph_\omega})_\delta$. By Fact 3.22, $|\mathcal{A}| \geq \aleph_{\omega+1}$. By Lemma 3.9, there exists $\mathcal{B} \in [\mathcal{A}]^{\aleph_{\omega+1}}$ such that $\bigcap \mathcal{B} \neq \emptyset$. Choose $f \in \bigcap \mathcal{B}$ and, for each $U \in \mathcal{B}$, choose $\sigma(U) \in [\aleph_\omega]^{\aleph_0}$ such that every $g \in X$ extending $f \upharpoonright \sigma(U)$ is in U. By Chang's Conjecture at $\aleph_\omega$, there exists $M \prec H(\theta)$ such that $\sigma \in M$, $|M \cap \mathcal{B}| = \aleph_1$, and $|M \cap \aleph_\omega| = \aleph_0$. Set $a = \bigcup \mathrm{ran}(\sigma \upharpoonright M)$, which is a subset of $M \cap \aleph_\omega$, and therefore countable. Every $g \in X$ extending $f \upharpoonright a$ also extends $f \upharpoonright \sigma(U)$ for all $U \in \mathcal{B} \cap M$. Hence, $\bigcap(\mathcal{B} \cap M)$ has nonempty interior. Thus, $\pi\mathrm{Nt}(X) \geq \aleph_2$. By CH, $\pi\mathrm{Nt}(X) \leq \aleph_2$. □

Question 3.12. Does GCH imply the following?

(⋆) Every cofinal family of countable binary functions on $\aleph_\omega$ contains a pairwise compatible subfamily of size $\aleph_{\omega+1}$.

If GCH implies ($\star$), then GCH and Chang's Conjecture at $\aleph_\omega$ together imply that $\pi\mathrm{Nt}((2^{\aleph_\omega})_\delta) = \aleph_2$. However, we do not know if ($\star$) is even consistent with $\aleph_\omega$ being a strong limit.

On a different note, let us remark that $\pi Nt(2^{\aleph_\omega}_\delta) = \aleph_1$ is consistent with Chang's Conjecture for $\aleph_\omega$.

Theorem 3.13. *There is a model of Chang's Conjecture for $\aleph_\omega$ where $\pi Nt((2^{\aleph_\omega})_\delta) = \aleph_1$.*

Proof. Assume GCH plus Chang's Conjecture (at $\aleph_\omega$) in the ground model and force with finite partial functions on $\aleph_{\omega+1}$. Then, in a generic extension, $\mathfrak{c} = \aleph_{\omega+1} = 2^{\aleph_\omega}$ and Chang's Conjecture still holds by Lemma 3.10. Moreover, $(2^{\aleph_\omega})_\delta$ is homeomorphic to $(((2^\omega)_\delta)^{\aleph_\omega})_\delta$, which in turn is homeomorphic to $(D(\aleph_{\omega+1})^{\aleph_\omega})_\delta$, where $D(\aleph_{\omega+1})$ denotes the discrete space of size $\aleph_{\omega+1}$. We now prove that in a generic extension $\pi Nt((D(\aleph_{\omega+1})^{\aleph_\omega})_\delta) = \aleph_1$. Indeed, let $\{\sigma_\alpha : \alpha < \aleph_{\omega+1}\}$ be a cofinal family of countable partial functions from $\aleph_\omega$ to $\aleph_{\omega+1}$. For every $\alpha < \aleph_{\omega+1}$, choose $\beta_\alpha \notin \mathrm{dom}(\sigma_\alpha)$. Define $\mathcal{F} = \{\sigma_\alpha \cup \langle \beta_\alpha, \alpha\rangle : \alpha < \aleph_{\omega+1}\}$, which is a cofinal family. Suppose by contradiction that $\langle \mathcal{F}, \supseteq\rangle$ is not ω_1^{op}-like. Then there is an uncountable set $A \subset \aleph_{\omega+1}$ and a countable partial function τ such that $\sigma_\alpha \cup \langle \beta_\alpha, \alpha\rangle \subset \tau$ for every $\alpha \in A$. If the β_αs are all distinct then τ has uncountable domain, while if there are distinct $\alpha, \gamma \in A$ such that $\beta_\alpha = \beta_\gamma$, then τ is not a partial function. ☐

Remark. The above proof shows that $2^\omega = \aleph_\omega^\omega$ implies $\pi\mathrm{Nt}((2^{\aleph_\omega})_\delta) = \aleph_1$ in ZFC.

Corollary 3.14. *There is a model of ZFC where $\pi Nt((2^{\aleph_\omega})_\delta) = \aleph_1 < \aleph_2 = Nt((2^{\aleph_\omega})_\delta)$*

Contrast this with $\pi w((2^{\aleph_\omega})_\delta) = w((2^{\aleph_\omega})_\delta)$ in every model of ZFC.

We would still be interested in examples showing the sharpness of Theorem 3.1 using milder set-theoretic assumptions.

Question 3.15. Is the existence of a compact space X such that $Nt(X_\delta) > 2^{Nt(X)}$ equiconsistent with ZFC?

The space $(2^{\aleph_\omega})_\delta$ does not answer Question 3.15 since its Noetherian type can consistently be $\aleph_1$. (See Theorem 3.21, (2).)

Question 3.16. Is there a characterization of the subspaces of $(2^{\aleph_\omega})_\delta$ whose Noetherian type can be determined in ZFC?

At first we conjectured that under Chang's Conjecture for $\aleph_\omega$ plus the GCH every $\aleph_{\omega+1}$-sized subset of $(2^{\aleph_\omega})_\delta$ would either have *large* Noetherian type or be discrete (note that the set of all characteristic functions of members of an $\aleph_{\omega+1}$-sized almost disjoint family of countable subsets of $\aleph_\omega$ is an $\aleph_{\omega+1}$-sized discrete set). But this conjecture is easily disproved by embedding into $(2^{\aleph_\omega})_\delta$ a copy of the sum of $\aleph_{\omega+1}$-many copies of the one-point Lindelöfication of a discrete set of size $\aleph_1$.

3.3. **Sparse Families and PCF theory.** To gain more insight on the order theory of bases in the countably supported box product topology, we need some concepts from PCF theory, which we now review for the reader's convenience. The proofs of Shelah's theorems quoted below can be found in [32] (see also [21] for an expository treatment).

Definition 3.17. Let $\operatorname{cov}(\aleph_\omega,\omega)$ be the cofinality of the partial order $([\aleph_\omega]^\omega,\subseteq)$. This number is the *tame factor* of a singular cardinal power.

In fact, it is easy to realize that $\aleph_\omega^\omega = Cov(\aleph_\omega,\omega)\cdot\mathfrak{c}$, and while the continuum has no bound in ZFC, the number $\operatorname{cov}(\aleph_\omega,\omega)$ does have a bound.

Theorem 3.18. *(Shelah)* $\operatorname{cov}(\aleph_\omega,\omega)<\aleph_{\omega_4}$.

PCF theory studies the *possible* cofinalities of products of small sets of regular cardinals modulo filters. Let $A=\{\aleph_n : n\in\omega\}$ and let U be a filter on A. We define the set $\prod A/U$ as the set of all equivalence classes of functions in $\prod A$ modulo the equivalence relation $=_U$ defined as $f =_U g$ if and only if $\{\aleph_n\in A : f(\aleph_n) = g(\aleph_n)\}\in U$, for $f,g\in\prod A$.

We define a partial order on $\prod A/U$ as follows: $f<_U g$ if and only if $\{\aleph_n\in A : f(\aleph_n)<g(\aleph_n)\}\in U$.

The *bounding number* $\mathfrak{b}(\prod A/U)$ is the least cardinality of an unbounded subset of $\prod A/U$ and it always regular when U is a proper filter. If $\mathfrak{b}(\prod A/U)=\operatorname{cf}(\prod A/U)$ then $\prod A/U$ is said to have *true cofinality* (denoted by tcf) and one can find a linearly ordered cofinal subset of it. Such a subset is called *a scale*.

Let $Pcf(A)=\{\operatorname{tcf}(\prod A/U) : U \text{ is a filter on } A\}$. An important theorem of PCF theory states that this set has a maximum.

Theorem 3.19. *(Shelah) If $A=\{\aleph_n : n\in\omega\}$, then $Pcf(A)$ is a set of regular cardinals with a maximum and* $\max Pcf(A)=Cov(\aleph_\omega,\omega)$.

The notion of a PCF scale allows us to give ZFC upper bounds on the Noetherian type of the countably supported topology. To prove that the upper bound can consistently drop to $\aleph_1$ we will need PCF scales with stronger properties whose existence is independent of ZFC.

Recall that a function $g\in On^\omega$ is said to be an *exact upper bound* for a $<^*$-increasing sequence $\{f_\alpha:\alpha<\lambda\}\subset On^\omega$ if $f_\alpha<^* g$ for every $\alpha<\lambda$ and whenever $g'<^* g$ there is $\beta<\lambda$ such that $g'<^* f_\beta$. A $<^*$-increasing sequence $\{f_\alpha:\alpha<\beta\}$ where $cf(\beta)=\delta>\aleph_0$ is called *flat* if there exists a $<$-increasing sequence $\{h_i : i<\delta\}\subset On^\omega$ so that for all $i<\delta$ there is $\alpha<\beta$ with $h_i<^* f_\alpha$ and for every $\alpha<\beta$ there is $i<\delta$ with $f_\alpha<^* h_i$.

A scale $\overline{f}=\{f_\alpha:\alpha<\lambda\}\subset On^\omega$ is called *good* if for every $\beta<\lambda$ such that $cf(\beta)>\aleph_0$ the sequence $\overline{f}\restriction\beta=\{f_\alpha:\alpha<\beta\}$ is flat and the function f_β is an exact upper bound for it. Chang's Conjecture for $\aleph_\omega$ negates the existence of good $\aleph_{\omega+1}$-scales (see Claim 4.3 of [12]).

Lemma 3.20.

1. *There is in ZFC a $(Cov(\aleph_\omega,\omega),\aleph_\omega)$-sparse family of size* $\operatorname{cov}(\aleph_\omega,\omega)$ *which is cofinal in $[\aleph_\omega]^\omega$.*
2. *(Shelah,* [32]*) If* $\operatorname{cov}(\aleph_\omega,\omega)=\aleph_{\omega+1}$ *and there exists a good scale of size* $\aleph_{\omega+1}$ *in $(\prod_{n\in\omega}\aleph_n,\le^*)$, then there is an $(\aleph_1,\aleph_1)$-sparse cofinal family which is cofinal in $[\aleph_\omega]^\omega$.*

Proof. For the proof of Item 1 let $\lambda=Cov(\aleph_\omega,\omega)$. By Theorem 3.19 there is a filter U and a scale $\overline{f}=\{f_\alpha:\alpha<\lambda\}$ in $\prod A/U$. Clearly, U contains no finite sets.

We claim that $\{\operatorname{ran} f_\alpha : \alpha<\lambda\}$ is a $(\lambda,\aleph_\omega)$-sparse family in $[\aleph_\omega]^\omega$. Indeed, suppose that $S\subset\lambda$ is of size λ such that $F=\bigcup_{\alpha\in S}\operatorname{ran} f_\alpha$ has cardinality $<\aleph_\omega$. Let g be the function defined by letting $g(n)=\sup(\bigcup S\cap\omega_n)$. As $|\bigcup S|<\aleph_\omega$ there is some m_0 such that for all $n \geq m_0$ it holds that $g(n) < \omega_n$. As $\overline{f}$ is a scale in $\prod A/U$ there is some $\alpha < \lambda$ such that for all $\alpha \leq \beta < \lambda$ it holds that $g <_U f_\beta$. Since $|S| = \lambda$, there must be some $\beta \in S$ such that $g <_U f_\beta$. The set $\{n : g(n) < f_\beta(n)\}$ belongs to U and is therefore infinite. Fix, then, $n > m_0$ such that $g(n) < f_\beta(n)$. This demonstrates that $f_\beta(n) \notin \bigcup\{\operatorname{ran} f_\beta : \beta \in S\}$ — a contradiction.

For the proof of Item 2 let $\overline{f} = \{f_\alpha : \alpha < \aleph_{\omega+1}\}$ be a good scale in $(\prod_{n\in\omega} \aleph_n, \leq^*)$. We claim that $\{\operatorname{ran} f_\alpha : \alpha < \aleph_{\omega+1}\}$ is an $(\aleph_1, \aleph_1)$-sparse family. Indeed, let $\{f_{\alpha_i} : i < \omega_1\} \subset \overline{f}$, where $\{\alpha_i : i < \omega_1\}$ is an increasing sequence of ordinals. Then $\gamma = \sup_{i<\omega_1} \alpha_i$ has cofinality $\aleph_1$ and hence the sequence $\overline{f} \restriction \gamma$ is flat. Suppose that $\{h_\beta : \beta < \omega_1\}$ is a $<$-increasing sequence witnessing flatness. For every $\beta < \omega_1$ find $i(\beta)$ such that $h_\beta <^* f_{\alpha_{i(\beta)}}$. By thinning out we may assume that $h_\beta <^* f_{\alpha_{i(\beta)}} <^* h_{\beta+1}$. Since ω_1 has uncountable cofinality, by further thinning out and reenumerating we may assume that, for a fixed $m \in \omega$ and for every $\beta < \omega_1$, we have $h_\beta(m) < f_{\alpha_{i(\beta)}}(m) < h_{\beta+1}(m)$. For every $\beta < \gamma < \omega_1$ we have $f_{\alpha_{i(\beta)}}(m) < h_{\beta+1}(m) \leq h_\gamma(m) < f_{\alpha_{i(\gamma)}}(m)$. Therefore, $\bigcup_{i<\omega_1} \operatorname{ran} f_{\alpha_i}$ is uncountable.

Shelah's original proof actually provided a stronger results. □

We can now provide some upper bounds on $Nt((2^{\aleph_\omega})_\delta)$. The ZFC upper bound will later be improved by a finer analysis of the relationship between sparseness and PCF scales.

Theorem 3.21.

(1) $Nt((2^{\aleph_\omega})_\delta) \leq Cov(\aleph_\omega, \omega)$.
(2) *If* $\operatorname{cov}(\aleph_\omega, \omega) = \aleph_{\omega+1}$ *and there exists a good scale of size* $\aleph_{\omega+1}$ *then* $Nt((2^{\aleph_\omega})_\delta) = \aleph_1$.

Proof. Let $\mathcal{F}$ be a $(\kappa, \aleph_1)$-sparse cofinal family in $([\aleph_\omega]^\omega, \subseteq)$ where $\kappa = Cov(\aleph_\omega, \omega)$ in Item 1 and $\kappa = \aleph_1$ in Item 2. Apply Theorem 3.5 (and note that trivially $Nt((2^{\aleph_\omega})_\delta) \geq \aleph_1$). □

Remarkably, the cardinal functions weight and π-weight don't enjoy ZFC bounds on $(2^{\aleph_\omega})_\delta$. For completeness we include a proof of the following probably well-known fact.

Fact 3.22. $w((2^{\aleph_\omega})_\delta) = \pi w((2^{\aleph_\omega})_\delta) = (\aleph_\omega)^{\aleph_0}$.

Proof. Let $\mathcal{F}$ be the set of all partial countable functions from $\aleph_\omega$ to $\{0,1\}$. The cofinality of $(\mathcal{F}, \subseteq)$ is clearly the π-weight of $(2^{\aleph_\omega})_\delta$, and is at least $\max\{Cov(\aleph_\omega, \omega), \mathfrak{c}\} = (\aleph_\omega)^{\aleph_0}$. Clearly, $\pi w((2^{\aleph_\omega})_\delta) \leq w((2^{\aleph_\omega})_\delta) \leq (\aleph_\omega)^{\aleph_0}$, so we are done. □

(In contrast, it is equally easy to prove that $\chi((2^{\aleph_\omega})_\delta)$ and $\pi\chi((2^{\aleph_\omega})_\delta)$ both equal $\operatorname{cov}(\aleph_\omega, \omega)$.)

Corollary 3.23. *If* $\square_{\aleph_\omega}$ *and* $\operatorname{cov}(\aleph_\omega, \omega) = \aleph_{\omega+1}$ *hold then* $Nt((2^{\aleph_\omega})_\delta) = \aleph_1$.

Proof. By Theorem 4 of [9], $\square_{\aleph_\omega}$ implies that there is a good (actually, "very good") scale of length $\aleph_{\omega+1}$ on $(\prod_{n\in A} \aleph_n, \leq^*)$ for some infinite $A \subseteq \omega$. The proof of Lemma 3.20 can be trivially modified to accomodate the restriction of the index set to A. □

Let us also note three reasons why $Nt((2^{\aleph_\omega})_\delta) = \aleph_1$ is consistent with large cardinals. First, it is standard that we can add a $\square_{\aleph_\omega}$-sequence (and force GCH at $\aleph_\omega$) with a mild forcing (*i.e.*, a forcing smaller than any large cardinal). Second, we can directly produce an $(\aleph_1, \aleph_1)$-sparse cofinal family with a mild forcing. Assume that $\mathfrak{c} < \aleph_\omega$ in the ground model (or force it) and let $\mathbb{P} = [[\aleph_\omega]^\omega]^\omega$ with $q \leq p$ iff $q \supseteq p$ and $y \not\subseteq x$ for all $x \in p$ and $y \in q \setminus p$. If G is a V-generic filter of $\mathbb{P}$, then $\mathcal{F} = \bigcup G$ is cofinal in $([\aleph_\omega]^\omega)^V$. Since $\mathbb{P}$ is countably closed and has the $\mathfrak{c}^+$-cc, $([\aleph_\omega]^\omega)^{V[G]} = ([\aleph_\omega]^\omega)^V$, so $\mathcal{F}$ is actually cofinal in the $[\aleph_\omega]^\omega$ of $V[G]$. Therefore, for every $x \in [\aleph_\omega]^\omega$ we can find y, p with $x \subseteq y \in p \in G$, which implies $\mathcal{F} \cap \mathcal{P}(x) \subseteq p$. Thus, $\mathcal{F}$ is $(\aleph_1, \aleph_1)$-sparse.

Third, the combinatorial principle *Very Weak Square* of Foreman and Magidor [12] implies that a continuous scale contains a club set of functions such that every function indexed by an ordinal of cofinality ω_1 is a flat point ([12], Claim 4.4). So if we restrict ourselves to that club set of points, using the same argument of Lemma 3.20, (2) we get an $(\aleph_1, \aleph_1)$-sparse cofinal family of countable subsets of $\aleph_\omega$. Now, by Theorem 2.5 of [12], if κ is supercompact in a model M of GCH, there is a generic extension of M in which cardinals and cofinalities are preserved, Very Weak Square holds at the successor of every singular cardinal, and κ remains supercompact. This third reason has advantage of easily generalizing Theorem 3.21 about $\mathrm{Nt}((2^{\aleph_\omega})_\delta)$ to, for example, the global consistency of $\mathrm{Nt}((2^{\aleph_{\alpha+\omega}})_\delta) = \aleph_1$ for all ordinals α, even in the presence of large cardinals.

3.4. **Steps towards a tight bound.** We now improve the bound from Theorem 3.21, (1). The proof uses the main idea in Shelah's proof of the existence of a stationary set $S \subseteq S^\lambda_\kappa$ in Shelah's ideal $I[\lambda]$ for regular κ and λ satisfying $\kappa^{++} < \lambda$. (For an exposition, see [21].)

Theorem 3.24. *There exists a cofinal family $\mathcal{F} \subseteq [\aleph_\omega]^{\aleph_0}$ which is $(\aleph_\alpha, \aleph_\alpha)$-sparse for every $4 \leq \alpha \leq \omega$.*

Corollary 3.25. *The Noetherian type of $(2^{\aleph_\omega})_\delta$ is at most $\aleph_4$.*

Proof of Theorem. It suffices to prove the existence of a family $\mathcal{F} \subseteq [\aleph_\omega]^{\aleph_0}$ of cardinality $\mathrm{cov}(\aleph_\omega, \omega)$ which is $(\aleph_4, \aleph_1)$-sparse by Claim 3.1. By Claim 3.1 the proof below is needed only when the continuum is larger than $\aleph_3$, but we do not make this assumption.

Let Ω be a sufficiently large regular cardinal and let $\langle H(\Omega), \in, \prec, \ldots\rangle$ be the structure of all sets of hereditary cardinality smaller than Ω with a well ordering and perhaps some countably many constants. For example, for every regular $\kappa, \lambda < \Omega$ satisfying $\kappa^+ < \lambda$, the structure $\langle H(\Omega), \in, \prec, \ldots\rangle$ contains a constant for a club guessing sequence of the form $\overline{C} = \langle c_\delta : \delta \in S^\lambda_\kappa\rangle$. Such a sequence shall be called "canonical".

Denote $\lambda = \max \mathrm{pcf}(\{\aleph_n : n \in \omega\}) = \mathrm{cov}(\aleph_\omega, \omega)$ and recall that λ is regular. Let I be an ideal such that $\mathrm{tcf}(\prod_{n<\omega} \aleph_n / I) = \mathrm{cov}(\aleph_\omega, \omega)$. In $\langle H(\Omega), \in, \prec, \ldots\rangle$ there is a canonical λ-scale $\langle f_\alpha : \alpha < \lambda\rangle \subseteq \langle \prod_n \omega_n, <_I\rangle$.

Fix a continuous sequence $\overline{M} := \langle M_i : i < \lambda\rangle$ of elementary submodels of $\langle H(\Omega), \in, \prec, \ldots\rangle$ satisfying the following for all $i < \lambda$.

- $i + 1 \subseteq M_i$ and $||M_i|| < \lambda$
- $\overline{M} \restriction i \in M_{i+1}$

Let $E \subseteq \lambda$ be the club set of points $i < \lambda$ for which $M_i \cap \lambda = i$. The sequence $\langle f_i : i \in E\rangle$ is a λ-scale. Finally, set $\mathcal{F} = \{\mathrm{ran}\, f_i : i \in E\}$. To prove that $\mathcal{F}$ is $(\aleph_4, \aleph_1)$-sparse let $A \in [E]^{\aleph_4}$ be given of order-type ω_4, and we shall find some $B \in [A]^{\aleph_1}$ such that $|\bigcup_{j \in B} \mathrm{ran}\, f_j| = \aleph_1$.

Fix an increasing and continuous chain $\overline{N} = \langle N_\zeta : \zeta \le \omega_3 \rangle$ of submodels of $\langle H(\Omega), \in, \prec, \ldots \rangle$ satisfying the following for all $\zeta \le \omega_3$.

- $||N_\zeta|| = \aleph_3$
- $\{\overline{M}, A, E\} \cup \omega_3 \subseteq N_0$
- $\overline{N} \restriction (\zeta + 1) \in N_{\zeta+1}$

Let $h(\zeta) = \sup(N_\zeta \cap A)$ for all $\zeta \le \omega_3$. As A has order-type ω_4 and N_ζ has cardinality ω_3, $h(\zeta) < \sup A$ for all $\zeta \le \omega_3$. Also, as $A, E \in N_\zeta$, it follows that $h(\zeta) \in E$ and is a limit of A for every $\zeta \le \omega_3$.

For $\zeta \le \omega_3$ let $j(\zeta) = \min\{A \setminus h(\zeta)\}$. So $j(\zeta) \ge h(\zeta)$ and by elementarity, $j(\zeta) < h(\zeta + 1)$ for all $\zeta < \omega_3$.

In the model $M_{h(\omega_3)+1}$ there exists some canonical function $g : \omega_{n+2} \to h(\omega_3)$ which is increasing and continuous and has range cofinal in $h(\omega_3)$.

Let $C \subseteq \omega_3$ be the club set of points $\zeta < \omega_3$ which satisfy $h(\zeta) = g(\zeta)$, and let $\delta \in S^{\omega_3}_{\omega_1}$ be such that $c_\delta \subseteq C$ (where c_δ is the δ-th element in the canonical club guessing sequence $\overline{C} = \langle c_\delta : \delta \in S^{\omega_3}_{\omega_1} \rangle$).

Let $B = \{j(\xi) : \xi \in c_\delta\}$. As $\mathrm{otp}(c)_\delta = \omega_1$ and $\zeta \mapsto j(\zeta)$ is order-preserving, $B \in [A]^{\aleph_1}$.

We shall show that $\bigcup_{j \in B} \operatorname{ran} f_j = \bigcup_{\xi \in c_\delta} \operatorname{ran} f_{j(\xi)}$ has cardinality $\aleph_1$ by proving that for every $\rho \in c_\delta$ the union $\bigcup_{\xi \in c_\delta \cap \rho} \operatorname{ran} f_{j(\xi)}$ does not contain the full union $\bigcup_{\xi \in c_\delta} \operatorname{ran} f_{j(\xi)}$.

Given $\rho \in c_\delta$, let $t = \sup\{f_{j(\xi)} : \xi \in c_\delta \cap \rho\}$. As the sequence $\langle j(\xi) : \xi \in c_\delta \cap \rho \rangle$ belongs to $N_{\rho+1}$, also $t \in N_{\rho+1}$.

Since $c_\delta \subset C$, we have $h(\xi) = g(\xi)$ for all $\xi \in c_\delta \cap \rho$. Since $g, c_\delta, \rho \in M_{h(\omega_3)+1}$ and $g(\xi) = h(\xi)$ for all $\xi \in c_\delta$, the set $\{g(\xi) : \xi \in c_\delta \cap \rho\} = \{h(\xi) : \xi \in c_\delta \cap \rho\}$ belongs to $M_{h(\omega_3)+1}$. So also the pointwise supremum function $t = \sup\{f_{j(\xi)} : \xi \in c_\delta \cap \rho\}$ belongs also to $M_{h(\omega_3)+1}$ and hence $t <_I f_{\sup(M_{i+1} \cap \lambda)}$.

There are ω_4 points in A above $h(\omega + 3) + 1$. Let $j(*)$ be the least point in A strictly above $h(\omega_3)$. Find some $i(*) > j(*)$ in E such that $i(*) < \sup A$. The pair $j(*)$ and $i(*)$ witness the truth of the sentence: "there exists $i(*) \in (E \cap \sup A)$ and some $j(*) \in A \cap M_{i(*)}$ such that $t <_I f_{j(*)}$".

As $t, A, \overline{f}, \overline{M} \in N_{\rho+1}$, by elementarity of $N_{\rho+1}$ we can find such $i(*)$ in $N_{\rho+1}$. Consequently, there exists some $j(**) \in A$ below $f(\min\{c_\delta \setminus (\rho+1)\})$. In particular, $t <_I f_{j(\min(c_\delta \setminus (\rho+1))}$, hence $\bigcup_{\zeta \in c_\delta \cap \rho} \operatorname{ran} f_{j(\zeta)}$ does not contain $f_{j(\min(c_\delta \setminus (\rho+1))}$.

□

So at this point the Noetherian type of $(2^{\aleph_\omega})_\delta$ can be any cardinal between $\aleph_1$ and $\aleph_4$.

Question 3.26. Is it consistent that $Nt((2^{\aleph_\omega})_\delta) > \aleph_2$?

(Since $Nt((2^{\aleph_\omega})_\delta) \le \mathfrak{c}^+$, this question is only interesting in the context of $\neg CH$.)

Question 3.26 is related to approachability. Given a sequence $\langle C_i : i < \lambda \rangle$ where $C_i \subseteq i$ is unbounded, and, for club many i, $\mathrm{otp}(C_i) = \mathrm{cf}(i)$, an ordinal $i < \aleph_{\omega+1}$ is approachable with respect to $\overline{C}$ if $\{C_i \cap j : j < i\} \subseteq \{C_j : j < i\}$. As argued by Foreman and Magidor in the proof of Claim 4.4 of [12], for every $\overline{C}$ as above and every continuous scale $\langle f_i \rangle_{i<\lambda}$ of a reduced product $\prod_{n<\omega} \aleph_n / U$, there is a club $D \subseteq \lambda$ such that if $i \in D$ is approachable with respect to $\overline{C}$, then $\overline{f}$ is flat at i. Therefore, if we could find a club E and $\overline{C}$ to which every $\alpha \in E \cap S^{Cov(\aleph_\omega, \omega)}_{\omega_2}$ is approachable, then we could deduce $\mathrm{Nt}((2^{\aleph_\omega})_\delta) \le \aleph_2$, arguing as in the proof of Lemma 3.20, (2). (Foreman and Magidor asked a related question, whether ZFC+GCH implies a version of Very Weak Square for $S^{\aleph_{\omega+1}}_{\omega_2}$.)

Sharon and Viale [31] have shown that MM implies that club many points in $S^{\aleph_{\omega+1}}_{>\omega_1}$ are approachable. Now, MM implies that $\mathrm{cov}(\aleph_\omega, \omega) = \aleph_{\omega+1}$ because MM implies that $\mathfrak{c} = \aleph_2$ and $\aleph_\omega^\omega = \aleph_{\omega+1}$ (see [13], Theorem 10 and Corollary 11). Thus, MM implies $\mathrm{Nt}((2^{\aleph_\omega})_\delta) \leq \aleph_2$.

Question 3.27. Does MM imply that $Nt((2^{\aleph_\omega})_\delta) = \aleph_2$?

A yes answer would reduce the consistency strength thus far required to break $Nt((2^{\aleph_\omega})_\delta) = \aleph_1$, for the consistency of Martin's Maximum has been proved relative to a superstrong cardinal [13]. Mild evidence for a positive answer is provided by Menachem Magidor's result that MM negates the existence of good scales (see [8], Theorem 17.1, for a proof).

Finally, we remark that the consistency of Chang conjecture's variant $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_2, \aleph_1)$ would imply the consistency of $Nt((2^{\aleph_\omega})_\delta) > \aleph_2$. Indeed, it implies that every cofinal family of countable subsets of $\aleph_\omega$ contains $\aleph_2$-many members whose union U has size $\aleph_1$. But then, by the pigeonhole principle, $\aleph_2$-many of them would have to be contained in an initial segment of U (according to some ordering of type ω_1). Thus, by Theorem 3.5, we would have $Nt((2^{\aleph_\omega})_\delta) > \aleph_2$. However, the consistency of this version of Chang's Conjecture alone is an open problem [31].

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School of Mathematics, Institute for Advanced Study, Einstein Drive, Princeton, NJ, 08540 USA
E-mail address: kojman@ias.edu

Department of Engineering, Mathematics and Physics, Texas A&M International University, 5201 University Blvd, Laredo, TX, 78041 USA
E-mail address: david.milovich@tamiu.edu

Department of Mathematics, Ben Gurion University of the Negev, P.O.B. 653, Be'er Sheva, 84105 Israel
E-mail address: santi@cs.bgu.ac.il