SYMMETRIC OPERATORS WITH REAL DEFECT SUBSPACES OF THE MAXIMAL DIMENSION. APPLICATIONS TO DIFFERENTIAL OPERATORS

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Abstract. Let $\mathfrak{H}$ be a Hilbert space and let A be a simple symmetric operator in $\mathfrak{H}$ with equal deficiency indices $d := n_\pm(A) < \infty$. We show that if, for all λ in an open interval $I \subset \mathbb{R}$, the dimension of defect subspaces $\mathfrak{N}_\lambda(A) (= \mathrm{Ker}(A^* - \lambda))$ coincides with d, then every self-adjoint extension $\widetilde{A} \supset A$ has no continuous spectrum in I and the point spectrum of $\widetilde{A}$ is nowhere dense in I. Application of this statement to differential operators makes it possible to generalize the known results by Weidmann to the case of an ordinary differential expression with both singular endpoints and arbitrary equal deficiency indices of the minimal operator. Moreover, we show in the paper, that an old conjecture by Hartman and Wintner on the spectrum of a self-adjoint Sturm - Liouville operator is not valid.

1. Introduction

Let $\mathfrak{H}$ be a Hilbert space, let A be a simple symmetric densely defined operator in $\mathfrak{H}$ with equal and finite deficiency indices $d = n_\pm(A) < \infty$ and let $\mathfrak{N}_z(A) = \mathrm{Ker}(A^* - z)$, $z \in \mathbb{C}$ be a defect subspace of A. As is known [15] $\dim \mathfrak{N}_\lambda(A) \leq d$ for all $\lambda \in \mathbb{R}$ and $\dim \mathfrak{N}_\lambda(A) = d$ if the range of $A - \lambda$ is closed, i.e., if λ belongs to the set $\hat{\rho}(A)$ of all regular type points of A (note that $\mathrm{Ker}(A - \lambda) = \{0\}$, since the operator A is simple). Moreover, if $I = (\mu_1, \mu_2)$ is an interval such that $I \subset \hat{\rho}(A)$, then for any self-adjoint extension $\widetilde{A} \supset A$ the spectrum $\sigma(\widetilde{A})$ in I consists of isolated eigenvalues of $\widetilde{A}$ with finite multiplicity (the discrete spectrum $\sigma_d(\widetilde{A})$). In this connection it seems to be rather interesting to find out if the situation is the same for the weaker condition

$$\dim \mathfrak{N}_\lambda(A) = d, \quad \lambda \in I. \tag{1.1}$$

It turns out that the answer is negative. More precisely, we show in the paper (see Proposition 3.6) that for any interval I there is an operator A such that (1.1) is satisfied and for any (equivalently for some) self-adjoint extension $\widetilde{A} \supset A$ the set of all points $\lambda \in I$ belonging to the essential spectrum $\sigma_e(\widetilde{A}) (= \sigma(\widetilde{A}) \setminus \sigma_d(\widetilde{A}))$ is infinite. At the same time the spectrum of such an extension $\widetilde{A}$ is "small" enough. Namely, in the main theorem of the paper we prove that under the condition (1.1) the following statement (s) is valid for any self-adjoint extension $\widetilde{A} \supset A$:

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(s) the set $\sigma(\widetilde{A}) \cap I$ is nowhere dense in I and coincides with the closure of the set $\sigma_p(\widetilde{A}) \cap I$, where $\sigma_p(\widetilde{A})$ is the set of all eigenvalues of $\widetilde{A}$ (the point spectrum).

Our considerations are substantially inspired by the book [18] and the recent paper [17] in Journal of Funct. Anal., where similar results were obtained for differential operators. Namely, let L_0 be the minimal symmetric operator generated by a formally self-adjoint differential expression $l[y]$ of an even order $2n$ on an interval $(a,b), -\infty \leq a < b \leq \infty$ (see (4.1)). For the operator L_0 satisfying (1.1) the validity of the statement (s) for any extension $\widetilde{A} = \widetilde{A}^* \supset L_0$ was proved by Weidmann [18] under the assumptions, that a is a regular endpoint for the expression $l[y]$ and L_0 has minimal deficiency indices $d(= n_\pm(L_0)) = n$. Moreover, it was shown in [17] that in the case of the regular endpoint a and an arbitrary defects d the statement (s) holds for some self-adjoint extension $\widetilde{A} \supset L_0$ defined by separated boundary conditions.

In the present paper we generalize the Weidmann's result to the case of arbitrary (regular or singular) endpoints a and b and arbitrary equal deficiency indices $d = n_\pm(L_0)$. More precisely, let L_{a0} and L_{b0} be minimal operators for the expression $l[y]$ on intervals (a,c) and (c,b) respectively (with some $c \in (a,b)$), let $n_+(L_{a0}) = n_-(L_{a0}) =: d_a$, $n_+(L_{b0}) = n_-(L_{b0}) =: d_b$ and let for some interval $I = (\mu_1, \mu_1) \subset \mathbb{R}$

$$\dim \mathfrak{N}_\lambda(L_{a0}) = d_a, \quad \dim \mathfrak{N}_\lambda(L_{b0}) = d_b, \quad \lambda \in I.$$

We show in Theorem 4.1 that under such assumptions the statement (s) holds for any self-adjoint extension $\widetilde{A} \supset L_0$.

In the paper [9] Hartman and Wintner suggested that for the second order, i.e. Sturm - Liouville, operator L_0 on the semiaxis $[0,\infty)$ with

$$\dim \mathfrak{N}_\lambda(L_0) = 1 (= d), \quad \lambda \in I = (\mu_1, \mu_2). \tag{1.2}$$

the statement (s) can be strengthened to "the spectrum of any self-adjoint extension $\widetilde{A} \supset L_0$ is discrete in I" (similar conjecture for the operator L_0 of an arbitrary order $2n$ is contained in [18, 17]). We prove in the paper, that this conjecture is not valid. More precisely we show that for any finite interval $I = (\mu_1, \mu_2)$ there exists a Sturm - Liouville operator L_0 such that (1.2) holds and for any self-adjoint extension $\widetilde{A} \supset L_0$ the set $\sigma_e(\widetilde{A}) \cap I$ is infinite (see Proposition 4.3).

In conclusion note that our approach is based on the concepts of a boundary triplet for A^* and the corresponding abstract Weyl function, which has become a convenient tool in the extension theory of symmetric operators and its applications (see [8, 3, 13, 5, 14] and references therein). Such an approach enabled us to obtain the above results without complicated construction of the self-adjoint extension $\widetilde{A} \supset L_0$ with the desired properties of the spectrum $\sigma(\widetilde{A})$ (cf. [17]).

2. Preliminaries

In the sequel we use the following notations: $\mathfrak{H}$, $\mathcal{H}$ denote separable Hilbert spaces; $[\mathcal{H}_1, \mathcal{H}_2]$ is the set of all bounded linear operators defined on $\mathcal{H}_1$ with values in $\mathcal{H}_2$; $[\mathcal{H}] := [\mathcal{H}, \mathcal{H}]$; $\mathbb{C}_+ \, (\mathbb{C}_-)$ is the upper (lower) half-plain of the complex plain.

Moreover, for a (not necessarily bounded) operator T from $\mathcal{H}_1$ to $\mathcal{H}_2$ we denote by $\mathcal{D}(T)$, $\mathcal{R}(T)$ and $\mathrm{Ker}T$ the domain, range and the kernel of T respectively.

For a closed operator T in $\mathfrak{H}$ we denote by $\hat{\rho}(T) = \{\lambda \in \mathbb{C} : \mathrm{Ker}(T-\lambda) = \{0\},\ \overline{\mathcal{R}(T-\lambda)} = \mathcal{R}(T-\lambda)\}$ and $\rho(T) = \{\lambda \in \hat{\rho}(T) : \mathcal{R}(T-\lambda) = \mathfrak{H}\}$ the set of regular type points and the resolvent set of T respectively.

Let $\mathcal{H}$ be a finite dimensional Hilbert space. Recall that a holomorphic operator function $\Phi(\cdot) : \mathbb{C}_+ \cup \mathbb{C}_- \to [\mathcal{H}]$ is called a Nevanlinna function (and is referred to the class $R[\mathcal{H}]$) if $\mathrm{Im}\, z \cdot \mathrm{Im}\Phi(z) \geq 0$ and $\Phi^*(z) = \Phi(\overline{z}),\ z \in \mathbb{C}_+ \cup \mathbb{C}_-$. According to [10, 2] a function $\Phi(\cdot) : \mathbb{C}_+ \cup \mathbb{C}_- \to [\mathcal{H}]$ belongs to the class $R[\mathcal{H}]$ if and only if it admits the integral representation

$$\Phi(z) = C_0 + z\,C_1 + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) dF_\Phi(t), \tag{2.1}$$

where $C_0, C_1 \in [\mathcal{H}],\ C_0 = C_0^*,\ C_1 \geq 0$ and $F_\Phi(\cdot) : \mathcal{B}_b \to [\mathcal{H}]$ is an operator valued measure defined on the ring $\mathcal{B}_b$ of all bounded Borel sets in $\mathbb{R}$ and such that

$$\int_{\mathbb{R}} \frac{1}{t^2+1} dF_\Phi(t) \in [\mathcal{H}]. \tag{2.2}$$

The operator valued measure $F_\Phi(\cdot)$ in (2.1) is called the spectral measure of the function $\Phi(\cdot) \in R[\mathcal{H}]$.

The following lemma is well known.

Lemma 2.1. *Let $\Phi(\cdot) \in R[\mathcal{H}]$ and let $F(\cdot) = F_\Phi(\cdot)$ be the corresponding spectral measure. Then for each $\lambda \in \mathbb{R}$ the following relations are equivalent:*

$$\begin{aligned} &(i) \quad \lim_{y\to 0} 1/y\ \mathit{Im}\,(\Phi(\lambda+iy)h, h) < \infty, \quad h \in \mathcal{H};\\ &(ii) \quad \int_{\mathbb{R}} \frac{d(F(t)h,h)}{(t-\lambda)^2} < \infty, \quad h \in \mathcal{H}; \end{aligned} \tag{2.3}$$

If the relation (i) (or, equivalently, (ii)) is satisfied, then there exists the limit

$$M(\lambda + i0) := \lim_{y\to 0} M(\lambda + iy)$$

and $Im\, M(\lambda + i0) = 0$.

Let A be a closed densely defined symmetric operator in $\mathfrak{H}$ and let A^* be the adjoint operator. For each $z \in \mathbb{C}$ denote by

$$\mathfrak{N}_z(A) := \mathrm{Ker}(A^* - z)(= \mathfrak{H} \ominus \mathcal{R}(A - \overline{z}))$$

the defect subspace of A and let $n_\pm(A) = \dim \mathfrak{N}_z(A)\ (z \in \mathbb{C}_\pm)$ be the deficiency indices of A.

Definition 2.2. [8] A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ consisting of an auxiliary Hilbert space $\mathcal{H}$ and linear mappings $\Gamma_j : \mathcal{D}(A^*) \to \mathcal{H},\ j \in \{0,1\}$ is called a boundary triplet for A^* if the mapping $\Gamma = (\Gamma_0\ \ \Gamma_1)^\top : \mathcal{D}(A^*) \to \mathcal{H} \oplus \mathcal{H}$ is surjective and the following abstract Green's identity holds

$$(A^*f, g) - (f, A^*g) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g), \quad f, g \in \mathcal{D}(A^*).$$

The following Proposition was proved in [4].

Proposition 2.3. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Then $n_+(A) = n_-(A) = \dim \mathcal{H}$ and the equalities*

$$\mathcal{D}(A_0) = Ker\Gamma_0 = \{f \in \mathcal{D}(A^*) : \Gamma_0 f = 0\}, \qquad A_0 = A^* \upharpoonright \mathcal{D}(A_0) \tag{2.4}$$

define a self-adjoint extension $A_0 \supset A$.

Conversely, let A be a symmetric operator in $\mathfrak{H}$ with $n_+(A) = n_-(A)$ and let $\widetilde{A}$ be a self-adjoint extension of A. Then there exists a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* such that $\widetilde{A} = A_0 (= A^* \upharpoonright Ker\Gamma_0)$.

It turns out that for any $z \in \rho(A_0)$ the operator $\Gamma_0 \upharpoonright \mathfrak{N}_z(A)$ isomorphically maps $\mathfrak{N}_z(A)$ onto $\mathcal{H}$. This enables one to introduce the following definition.

Definition 2.4. [3] The operator function $M(\cdot) : \rho(A_0) \to [\mathcal{H}]$ defined by

$$\Gamma_1 \upharpoonright \mathfrak{N}_z(A) = M(z)\Gamma_0 \upharpoonright \mathfrak{N}_z(A), \quad z \in \rho(A_0)$$

is called the Weyl function corresponding to the boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$.

As was shown in [3] the Weyl function $M(\cdot)$ belongs to the class $R[\mathcal{H}]$ and $0 \in \rho(\mathrm{Im} M(z))$, $z \in \mathbb{C}_+ \cup \mathbb{C}_-$.

3. Symmetric operators with real defect subspaces of the maximal dimension

In the sequel we denote by A a simple symmetric densely defined operator in $\mathfrak{H}$ with equal deficiency indices $d = n_\pm(A) < \infty$. Since the operator A is simple, it follows that $\mathrm{Ker}(A - \lambda) = \{0\}$ and, consequently, $\dim \mathfrak{N}_\lambda(A) \leq d$ for all $\lambda \in \mathbb{R}$. We denote by $\widetilde{\rho}(A)$ the set of all $\lambda \in \mathbb{R}$ such that $\dim \mathfrak{N}_\lambda(A) = d$.

Proposition 3.1. *Assume that A is a simple symmetric operator in $\mathfrak{H}$ with $d = n_\pm(A) < \infty$, $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^*, A_0 is the self-adjoint extension* (2.4) *and $M(\cdot)$ is the corresponding Weyl function. Then a real point λ belongs to $\widetilde{\rho}(A)$ and $Ker(A_0 - \lambda) = \{0\}$ if and only if*

$$\lim_{y \to 0} 1/y \, Im(M(\lambda + iy)h, h) < \infty, \quad h \in \mathcal{H}. \tag{3.1}$$

Proof. For a point $\lambda \in \mathbb{R}$ denote by $\mathcal{H}_\lambda$ the subspace in $\mathcal{H}$ given by $\mathcal{H}_\lambda = \Gamma_0 \mathfrak{N}_\lambda(A)$. It follows from (2.4) that

$$\mathrm{Ker}(\Gamma_0 \upharpoonright \mathfrak{N}_\lambda(A)) = \mathcal{D}(A_0) \cap \mathfrak{N}_\lambda(A) = \mathrm{Ker}(A_0 - \lambda)$$

and, consequently,

$$\dim \mathfrak{N}_\lambda(A) = \dim \mathrm{Ker}(A_0 - \lambda) + \dim \mathcal{H}_\lambda. \tag{3.2}$$

Since $\dim \mathfrak{N}_\lambda(A) \leq d$, the equality (3.2) yields the equivalences

$$(\mathrm{Ker}(A_0 - \lambda) = \{0\} \text{ and } \dim \mathfrak{N}_\lambda(A) = d) \iff \dim \mathcal{H}_\lambda = d \iff \mathcal{H}_\lambda = \mathcal{H}. \tag{3.3}$$

Moreover according to [13] for any $h \in \mathcal{H}$ the following equivalence holds

$$h \in \mathcal{H}_\lambda \iff \lim_{y \to 0} 1/y \, \mathrm{Im}(M(\lambda + iy)h, h) < \infty. \tag{3.4}$$

In view of (3.4) the equality $\mathcal{H}_\lambda = \mathcal{H}$ is equivalent to the condition (3.1), which together with (3.3) gives the desired statement. $\square$

Remark 3.2. It is easily seen that for all $\lambda \in \mathbb{R}$ there exists a self-adjoint extension $A_0 \supset A$ with $\mathrm{Ker}(A_0 - \lambda) = \{0\}$. Moreover by Proposition 2.3 there exists a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* such that A_0 is defined by (2.4). This implies that Proposition 3.1 provides actually the criterium (in terms of the Weyl function) for a point $\lambda \in \mathbb{R}$ belongs to $\widetilde{\rho}(A)$.

Lemma 3.3. *Assume that* $\dim \mathcal{H} < \infty$ *and* $F(\cdot) : \mathcal{B}_b \to [\mathcal{H}]$ *is an operator valued measure satisfying* (2.2) *and the relation*

$$\lim_{[\alpha,\beta)\to\mathbb{R}} (F([\alpha,\beta))h, h) = \infty, \quad h \in \mathcal{H}.$$

Moreover, let $L_2(F, \mathcal{H})$ *be the Hilbert space of vector functions* $f(\cdot) : \mathbb{R} \to \mathcal{H}$ *such that*

$$||f||^2_{L_2(F,\mathcal{H})} := \int_{\mathbb{R}} (dF(t)f(t), f(t)) < \infty$$

(see [11, 6]*) and let* $\widetilde{A}_F$ *be the self-adjoint multiplication operator in* $L_2(F, \mathcal{H})$ *given by*

$$\mathcal{D}(\widetilde{A}_F) = \{f \in L_2(F, \mathcal{H}) : tf(t) \in L_2(F, \mathcal{H})\}, \quad (\widetilde{A}_F f)(t) = tf(t). \tag{3.5}$$

Then: 1) the equalities

$$\mathcal{D}(A_F) = \left\{ f \in \mathcal{D}(\widetilde{A}_F) : \int_{\mathbb{R}} dF(t)f(t) = 0 \right\}, \quad (A_F f)(t) = tf(t) \tag{3.6}$$

define a simple symmetric densely defined operator in $L_2(F, \mathcal{H})$ *such that* $n_\pm(A_F) = \dim \mathcal{H}$ *and* $A_F \subset \widetilde{A}_F$*;*

2) for each point $\lambda \in \mathbb{R}$ *with* $F(\{\lambda\}) = 0 (\Leftrightarrow Ker(\widetilde{A}_F - \lambda) = \{0\})$ *the inclusion* $\lambda \in \widetilde{\rho}(A_F)$ *is equivalent to the relation* (2.3).

Proof. The statement 1) was proved in [5].

2) Let the function $M(\cdot) \in R[\mathcal{H}]$ be given by (2.1) with $C_1 = 0$ and $F_M(\cdot) = F(\cdot)$. Then according to [5] there exists a boundary triplet $\Pi_0 = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A_F^* such that $A_0 (= A^* \upharpoonright \mathrm{Ker}\Gamma_0) = \widetilde{A}_F$ and the corresponding Weyl function coincides with $M(\cdot)$. Applying now Proposition 3.1 to the triplet Π_0 and taking Lemma 2.1 into account one obtains the desired statement. $\square$

For a given operator A and an interval $I = (\mu_1, \mu_2)$, $-\infty \le \mu_1 < \mu_2 \le \infty$, we denote by $\widetilde{\rho}_I(A) = \widetilde{\rho}(A) \cap I$ the set of all points $\lambda \in I$ with $\dim \mathfrak{N}_\lambda(A) = d (= n_\pm(A))$ and let $\hat{\rho}_I(A) = \hat{\rho}(A) \cap I$ be the set of all regular type points of A belonging to I. Since $\dim \mathfrak{N}_\lambda(A) = d$ for all $\lambda \in \hat{\rho}_I(A)$, the inclusion $\hat{\rho}_I(A) \subset \widetilde{\rho}_I(A)$ is valid. Moreover, the set $\widetilde{\rho}_I(A) \setminus \hat{\rho}_I(A)$ consists of all points $\lambda \in I$ such that $\dim \mathfrak{N}_\lambda(A) = d$ and the range $\mathcal{R}(A - \lambda)$ is not closed.

As is known [16] the spectrum $\sigma(T)$ of a self-adjoint operator T admits the representation

$$\sigma(T) = \overline{\sigma_p(T)} \cup \sigma_c(T), \quad \sigma_c(T) = \sigma_{ac}(T) \cup \sigma_{sc}(T), \tag{3.7}$$

where $\sigma_p(T) = \{\lambda \in \mathbb{R} : \mathrm{Ker}(T-\lambda) \neq \{0\}\}$ is the point spectrum and $\sigma_c(T)$, $\sigma_{ac}(T)$ and $\sigma_{sc}(T)$ are continuous, absolutely continuous and singular continuous parts of $\sigma(T)$ respectively. Recall that the continuous spectrum $\sigma_c(T)$ is defined as the spectrum of the self-adjoint operator $T_c = T \upharpoonright \mathfrak{H}_c$, where $\mathfrak{H}_c := \mathfrak{H} \ominus \mathrm{span}\{Ker(T-\lambda) : \lambda \in \sigma_p(T)\}$ is the subspace reducing the operator T.

Another basic partition of the spectrum is in terms of the discrete spectrum $\sigma_d(T)$ and the essential spectrum $\sigma_e(T)$. Namely, $\sigma_d(T)$ is the set of all isolated eigenvalues of T with finite multiplicity and $\sigma_e(T) = \sigma(T) \setminus \sigma_d(T)$. It is clear that $\sigma_c(T) \subset \sigma_e(T)$. Moreover, the following lemma is well known.

Lemma 3.4. *Let A be a simple symmetric operator with $d = n_\pm(A) < \infty$. Then all self-adjoint extensions $\widetilde{A} \supset A$ have the same essential spectrum*

$$\sigma_e(\widetilde{A}) = \mathbb{R} \setminus \hat{\rho}(A). \tag{3.8}$$

Recall also that a set $X \subset (\mu_1, \mu_2)$ is called nowhere dense in (μ_1, μ_2) if for any interval $(\mu_1', \mu_2') \subset (\mu_1, \mu_2)$ there exists an interval $(\mu_1'', \mu_2'') \subset (\mu_1', \mu_2')$ such that $X \cap (\mu_1'', \mu_2'') = \emptyset$.

Now we are ready to prove the main theorem of the paper.

Theorem 3.5. *Assume that A is a simple symmetric densely defined operator in $\mathfrak{H}$ with equal deficiency indices $d = n_\pm(A) < \infty$ and $I = (\mu_1, \mu_2)$, $-\infty \le \mu_1 < \mu_2 \le \infty$, is an interval such that the set $I \setminus \widetilde{\rho}_I(A)$ is at most countable. Then:*

1) for each self-adjoint extension $\widetilde{A} \supset A$ the intersection $\sigma_c(\widetilde{A}) \cap I$ is empty and the set $\sigma(\widetilde{A}) \cap I$ is nowhere dense in I;

2) the set $I \setminus \hat{\rho}_I(A)$ is nowhere dense in I.

Proof. 1) Let $\widetilde{A}$ be a self-adjoint extension of A and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* with $\widetilde{A} = A_0 (= A^* \upharpoonright \mathrm{Ker}\Gamma_0)$ (such a triplet exists in view of Proposition 2.3). Moreover, let $M(\cdot) \in R[\mathcal{H}]$ be the corresponding Weyl function.

Next assume that

$$X_p = \{\lambda_k\} (= \sigma_p(\widetilde{A}) \cap I)$$

is the (at most countable) set of all eigenvalues of $\widetilde{A}$ belonging to I and let $X_1 := \widetilde{\rho}_I(A) \setminus X_p$, $X_2 := (I \setminus \widetilde{\rho}_I(A)) \setminus X_p$, so that

$$I = X_p \cup X_1 \cup X_2, \quad X_p \cap X_1 = X_p \cap X_2 = X_1 \cap X_2 = \emptyset. \tag{3.9}$$

Then $X_p \cup X_2$ is an at most countable subset in I and by Proposition 3.1 the Weyl function $M(\cdot)$ satisfies the relation (3.1) for all $\lambda \in X_1$. This and Lemma 2.1 yield the following statement (s_1) :

(s_1) there exists a subset $X_1 \subset I$ such that: (i) $I \setminus X_1$ is an at most countable set; (ii) for all $\lambda \in X_1$ the limit $M(\lambda + i0) := \lim\limits_{y \to 0} M(\lambda + iy)$ exists and $\mathrm{Im}\, M(\lambda + i0) = 0$.

According to [1, Theorem 4.3] the statement (s_1) implies that $\sigma_{sc}(\widetilde{A}) \cap I = \emptyset$, $\sigma_{ac}(\widetilde{A}) \cap I = \emptyset$ and, consequently,

$$\sigma_c(\widetilde{A}) \cap I = \emptyset. \tag{3.10}$$

Next assume that $E(\cdot)$ and $F(\cdot) = F_M(\cdot)$ are spectral measures of the operator $\widetilde{A}(= A_0)$ and the Weyl function $M(\cdot)$ respectively. According to [1, Lemma 3.2] the measures $E(\cdot)$ and $F(\cdot)$ are equivalent. Moreover, by (3.10)

$$\sigma(\widetilde{A}) \cap I = \overline{X}_p, \tag{3.11}$$

which implies that the measure $E(\cdot)$ is discrete on I and hence so is the measure $F(\cdot)$. Combining this statement with Proposition 3.1 and Lemma 2.1 one obtains

$$\int_{\mathbb{R}} \frac{d(F(t)h,h)}{(t-\lambda)^2} = \sum_k (F_k h,h)/(\lambda_k-\lambda)^2 < \infty, \quad \lambda \in X_1, \;\; h \in \mathcal{H}, \tag{3.12}$$

where $F_k = F(\{\lambda_k\}) \in [\mathcal{H}]$. Let $\{e_j\}_1^d$ be an orthonormal basis in $\mathcal{H}$ and let $c_k = \sum\limits_j (F_k e_j, e_j)$. Since $F_k \neq 0$ and $F_k \geq 0$, it follows that $c_k > 0$ and the relation (3.12) yields

$$\sum_k c_k/(\lambda_k-\lambda)^2 < \infty, \quad \lambda \in X_1. \tag{3.13}$$

Thus the following statement (s_2) is proved:

(s_2) there exists a decomposition (3.9) of the interval I and a sequence of positive numbers $\{c_k\}$ such that $X_p = \{\lambda_k\}$ and X_2 are at most countable sets and for all $\lambda \in X_1$ the relation (3.13) holds.

By using the statement (s_2) one can prove in the same way as it was done [18, Theorem11.7] that the set X_p is nowhere dense in I. This and the equality (3.11) imply that the set $\sigma(\widetilde{A}) \cap I$ is nowhere dense in I as well.

The statement 2) follows from the obvious inclusion $(I \setminus \hat{\rho}_I(A)) \subset \sigma(\widetilde{A}) \cap I$ and the statement 1). □

It turns out that the relation $\sigma_c(\widetilde{A}) \cap I = \emptyset$ in the statement 1) of Theorem 3.5 can not be replaced with the stronger one $\sigma_e(\widetilde{A}) \cap I = \emptyset$. More precisely the following proposition holds.

Proposition 3.6. *For any interval $I = (\mu_1, \mu_2)$, $-\infty \leq \mu_1 < \mu_2 \leq \infty$, and for any $d \in \mathbb{N}$ there exist a Hilbert space $\mathfrak{H}$ and a simple symmetric operator A in $\mathfrak{H}$ such that $n_\pm(A) = d$, $\widetilde{\rho}_I(A) = I$ and for any self-adjoint extension $\widetilde{A} \supset A$ the interval I contains infinitely many points of $\sigma_e(\widetilde{A})$. In view of* (3.8) *the last statement implies that the set $\widetilde{\rho}_I(A) \setminus \hat{\rho}_I(A)$ is infinite.*

Proof. First assume that $d = 1$ and consider the following two alternative cases:

(i) $\mu_2 < \infty$, i.e., the interval I is bounded from above.

Let $\{\lambda_k\}_0^\infty$ be a strictly increasing sequence of the points $\lambda_k \in I$ such that $\lim\limits_{k\to\infty} \lambda_k = \mu_2$ and let $\{\lambda_{jk}\}_{j,k=1}^\infty$ be a sequence of the points $\lambda_{jk} \in (\lambda_{k-1}, \lambda_k)$ such that $\lambda_{jk} < \lambda_{j+1,k}$, $\lambda_k - \lambda_{jk} < 1$ and $\lim\limits_{j\to\infty} \lambda_{jk} = \lambda_k$, $j,k \in \mathbb{N}$. Consider also two sequences of positive numbers $\{s_k\}_1^\infty$ and $\{u_{jk}\}_{j,k=1}^\infty$ such that

$$\sum_{k=1}^\infty s_k < \infty \quad \text{and} \quad \sum_{j=1}^\infty u_{jk} = s_k$$

and let $F_{jk} = u_{jk}(\lambda_k - \lambda_{jk})^2$. Since $F_{jk} \le u_{jk}$, it follows that

$$\sum_k \sum_j F_{jk} \le \sum_k \sum_j u_{jk} = \sum_k s_k < \infty.$$

This enables one to introduce the scalar discrete measure $F'(\cdot)$ on Borel sets $B \subset I$ by

$$F'(\{\lambda_{jk}\}) = F_{jk}, \quad F'(B) = \sum_{\lambda_{jk}\in B} F_{jk}. \tag{3.14}$$

Assume also that $F''(\cdot)$ is a scalar measure on bounded Borel sets $B \subset \mathbb{R}$ such that $F''(\mathbb{R}\setminus I) = \infty$ and $\int\limits_{\mathbb{R}\setminus I} (t^2+1)^{-1} dF''(t) < \infty$ (for example one can take as $F''(\cdot)$ the standard Lebesgue measure on the line). Then the equality

$$F(B) = F'(B \cap I) + F''(B \cap (\mathbb{R}\setminus I)) \tag{3.15}$$

defines the scalar measure $F(\cdot)$ on bounded Borel sets $B \subset \mathbb{R}$ such that $F(\mathbb{R}) = \infty$ and

$$\int_{\mathbb{R}} \frac{dF(t)}{t^2+1} < \infty. \tag{3.16}$$

Next we show that

$$\int_{\mathbb{R}} \frac{dF(t)}{(t-\lambda_k)^2} < \infty \tag{3.17}$$

for each point λ_k. Let (α_k, β_k) be an interval such that $\lambda_{k-1} < \alpha_k < \lambda_{1,k}$ and $\lambda_k < \beta_k < \lambda_{1,k+1}$. Then

$$\int\limits_{[\alpha_k,\beta_k)} \frac{dF(t)}{(t-\lambda_k)^2} = \sum_{j=1}^\infty \frac{F_{jk}}{(\lambda_{jk}-\lambda_k)^2} = \sum_{j=1}^\infty u_{jk} = s_k < \infty \tag{3.18}$$

and in view of (3.16) one has

$$\int\limits_{\mathbb{R}\setminus[\alpha_k,\beta_k)} \frac{dF(t)}{(t-\lambda_k)^2} < \infty. \tag{3.19}$$

Combining (3.18) and (3.19) we arrive at (3.17).

(ii) $\mu_2 = \infty$, i.e., the interval I is unbounded from above.

Assume also without loss of generality that $\mu_1 \le 1$. In this case we put $\lambda_k = k,\ \lambda_{jk} = k + \frac{1}{j+1},\ k, j \in \mathbb{N}$ and let $\{F_j\}_1^\infty$ be a sequence of positive numbers such that $\sum\limits_{j=1}^{\infty}(j+1)^2F_j<\infty$. Then $s:=\sum\limits_{j=1}^{\infty}F_j<\infty$ and the equalities

$$F(\{\lambda_{jk}\})=F_j,\quad F(B)=\sum_{\lambda_{jk}\in B}F_j. \tag{3.20}$$

define the scalar measure $F(\cdot)$ on bounded Borel sets $B\subset\mathbb{R}$. For this measure we have

$$F([k,k+1))=\sum_{j=1}^{\infty}F_j=s,\quad k\subset\mathbb{N}$$

and, consequently, $F(\mathbb{R})=\infty$. Moreover,

$$\int\limits_{(-\infty,1)}\frac{dF(t)}{t^2+1}=0\ \text{ and }\int\limits_{[k,k+1)}\frac{dF(t)}{t^2+1}=\sum_{j=1}^{\infty}\frac{F_j}{\left(k+\frac{1}{j+1}\right)^2+1}\leq\sum_{j=1}^{\infty}\frac{F_j}{k^2}=\frac{1}{k^2}s$$

for all $k\in\mathbb{N}$, which implies that

$$\int\limits_{\mathbb{R}}\frac{dF(t)}{t^2+1}=\sum_{k=1}^{\infty}\int\limits_{[k,k+1)}\frac{dF(t)}{t^2+1}\leq s\sum_{k=1}^{\infty}\frac{1}{k^2}<\infty.$$

Hence the measure $F(\cdot)$ satisfies the relation (3.16). Next for a given $\lambda_k=k$ consider the interval (α_k,β_k) such that $(k-1)+\frac{1}{2}<\alpha_k<k$ and $k+\frac{1}{2}<\beta_k<k+1$. Then

$$\int\limits_{[\alpha_k,\beta_k)}\frac{dF(t)}{(t-\lambda_k)^2}=\sum_{j=1}^{\infty}\frac{F_j}{\left[\left(k+\frac{1}{j+1}\right)-k\right]^2}=\sum_{j=1}^{\infty}(j+1)^2F_j<\infty$$

and by (3.16) the inequality (3.19) is also valid. This gives the relation (3.17) for the measure (3.20).

Thus in both cases (i) and (ii) we constructed the countable infinite subsets $Y_1=\{\lambda_{jk}\}$ and $Y_2=\{\lambda_k\}$ of I and the discrete measure $F(\cdot)$ with the following properties: 1) Y_1 consists of isolated points and Y_2 is the set of all limit points of Y_1 belonging to I; 2) the measure $F(\cdot)$ is concentrated on Y_1, $F(\mathbb{R})=\infty$ and the relations (3.16) and (3.17) are satisfied. This properties imply that the multiplication operator $\widetilde{A}_F$ in $L_2(F)$ (see (3.5)) satisfies the equality

$$\sigma_e(\widetilde{A}_F)\cap I=Y_2(=\{\lambda_k\}). \tag{3.21}$$

Next assume that $A_F\subset\widetilde{A}_F$ is a simple symmetric operator in $L_2(F)$ given by (3.6). Then $n_\pm(A_F)=1$ and in view of (3.21) and Lemma 3.4 the set $\sigma_e(\widetilde{A})\cap I(=\{\lambda_k\})$ is infinite for any self-adjoint extension $\widetilde{A}\supset A_F$. Moreover, by (3.17) and Lemma 3.3, 2) $Y_2\subset\widetilde{\rho}_I(A_F)$ and the equality (3.8) gives

$$I\setminus Y_2=\hat{\rho}_I(A_F)\subset\widetilde{\rho}_I(A_F).$$

This implies that $I=\widetilde{\rho}_I(A_F)$ and hence A_F is a desired operator.

In the case of an arbitrary $d \in \mathbb{N}$ we put $A = \bigoplus_{k=1}^{d} A_F$, where A_F is the constructed above simple symmetric operator with $n_{\pm}(A_F) = 1$. It is clear that the operator A has the required properties. $\square$

4. Differential operators

In this section we apply the obtained results to differential operators generated by the formally self-adjoint differential expression

$$l[y] = \sum_{k=1}^{n} (-1)^k ((p_{n-k} y^{(k)})^{(k)} - \tfrac{i}{2}[(q_{n-k}^* y^{(k)})^{(k-1)} + (q_{n-k} y^{(k-1)})^{(k)}]) + p_n y \quad (4.1)$$

of an even order $2n$. The coefficients $p_k(\cdot)$ and $q_k(\cdot)$ of this expression are defined on an interval (a,b), $-\infty \leq a < b \leq \infty$, take on values in $[\mathbb{C}^m]$ and possess the following properties:

(a) p_k, q_k are measurable on (a,b);

(b) $p_k(t) = p_k^*(t)$ $(k = 0 \div n)$ and $0 \in \rho(p_0(t))$ almost everywhere on (a,b);

(c) the operator functions p_k $(k = 2 \div n)$, q_k $(k = 1 \div (n-1))$, p_0^{-1}, $q_0^* p_0^{-1}$ and $\frac{1}{4} q_0^* p_0^{-1} q_0 - p_1$ are locally integrable on (a,b).

The expression (4.1) is called regular at a, if $a > -\infty$ and the assumptions on the coefficients are satisfied in $[a,b)$ instead of (a,b). The regularity of (4.1) at b is defined correspondingly.

Next assume that $y^{[k]}(\cdot)$, $k = 0 \div 2n$ are the quasi-derivatives of a function $y(\cdot) : (a,b) \to \mathbb{C}^m$ [15, 12] and let $\mathcal{D}(l)$ be the set of all functions $y(\cdot)$ such that the quasi-derivatives $y^{[k]}(\cdot)$, $k = 0 \div (2n-1)$ are absolutely continuous in (a,b). Then for each function $y \in \mathcal{D}(l)$ the equality $l[y] = y^{[2n]}$ is valid.

For a given interval $(\alpha, \beta) \subset \mathbb{R}$ denote by $L_2(\alpha,\beta)$ the Hilbert space of all measurable functions $f(\cdot) : (\alpha,\beta) \to \mathbb{C}^m$ such that $\int_{\alpha}^{\beta} ||f(t)||^2 \, dt < \infty$. As is known [15, 18] the expression (4.1) generates the maximal operator L in $L_2(a,b)$ defined on the domain $\mathcal{D}(L) := \{y \in \mathcal{D}(l) \cap L_2(a,b) : l[y] \in L_2(a,b)\}$ by $Ly = l[y]$, $y \in \mathcal{D}(L)$. Moreover, the minimal operator L_0 is defined by $L_0 = \overline{L_0'}$, where L_0' is a restriction of L onto the linear manifold of all functions $y \in \mathcal{D}(l)$ with compact support. It is known [15, 18] that L_0 is a densely defined symmetric operator in $L_2(a,b)$ and $L_0^* = L$.

For a given point $c \in (a,b)$ denote by $l_a[y]$ and $l_b[y]$ the restrictions of the expression $l[y]$ onto the intervals (a,c) and (c,b) respectively and let L_{a0} (L_{b0}) be the minimal operator in $L_2(a,c)$ (resp. $L_2(c,b)$) generated by $l_a[y]$ (resp. $l_b[y]$). It is clear that for each $\lambda \in \mathbb{C}$ the defect subspace $\mathfrak{N}_\lambda(L_{a0})$ $(\mathfrak{N}_\lambda(L_{b0}))$ is the set of all solutions of the equation

$$l[y] - \lambda y = 0, \quad (4.2)$$

which lie in $L_2(a,c)$ (resp. $L_2(c,b)$). Therefore the defect number $n_+(L_{a0})$ $(n_+(L_{b0}))$ can be defined as the number of linearly independent solutions of the equation (4.2) with $\lambda = i$ belonging to $L_2(a,c)$ (resp. $L_2(c,b)$). Similarly one defines (with $\lambda = -i$ in place of $\lambda = i$) the defect numbers $n_-(L_{a0})$ and $n_-(L_{b0})$.

If the operators L_{a0} and L_{b0} have equal deficiency indices

$$n_+(L_{a0}) = n_-(L_{a0}) =: d_a, \quad n_+(L_{b0}) = n_-(L_{b0}) =: d_b, \tag{4.3}$$

then $n\,m \le d_a \le 2n\,m, \quad n\,m \le d_b \le 2n\,m$ and the operator L_0 also has equal deficiency indices

$$n_+(L_0) = n_-(L_0) = d_a + d_b - 2n\,m.$$

In this connection note that the relations (4.3) hold if $m = 1$ (the scalar case) and in formula (4.1) $q_k = 0$. Observe also that all the above definitions and assertions do not depend on the choice of the point $c \in (a,b)$.

Application of Theorem 3.5 to the minimal differential operator L_0 gives the following result.

Theorem 4.1. *Let the operators L_{a0} and L_{b0} have equal deficiency indices* (4.3) *and let $I = (\mu_1, \mu_2)$, $-\infty \le \mu_1 < \mu_2 \le \infty$, be an interval such that for some (equivalently, for all) $c \in (a,b)$ and for all $\lambda \in I$, besides an at most countable set $X \subset I$, the equation* (4.2) *has d_a linearly independent solutions belonging to $L_2(a,c)$ and d_b linearly independent solutions which lie in $L_2(c,b)$. Then for any self-adjoint extension $\widetilde{A} \supset L_0$ the statement 1) of Theorem 3.5 holds.*

Proof. Since the expressions $l_a[y]$ and $l_b[y]$ are regular at c, it follows that the corresponding minimal operators L_{a0} and L_{b0} are simple (see for instance [7]). Hence the symmetric operator $\hat{L}_0 := L_{a0} \oplus L_{b0}$ in $L_2(a,b)$ is also simple and in view of the equality $\mathfrak{N}_\lambda(\hat{L}_0) = \mathfrak{N}_\lambda(L_{a0}) \oplus \mathfrak{N}_\lambda(L_{b0})$ one has

$$\dim \mathfrak{N}_\lambda(\hat{L}_0) = \dim \mathfrak{N}_\lambda(L_{a0}) + \dim \mathfrak{N}_\lambda(L_{b0}), \quad \lambda \in \mathbb{C}.$$

Therefore by (4.3) $n_\pm(\hat{L}_0) = d_a + d_b$ and

$$\dim \mathfrak{N}_\lambda(\hat{L}_0) = d_a + d_b (= n_\pm(\hat{L}_0)), \quad \lambda \in I \setminus X,$$

which implies that $\widetilde{\rho}_I(\hat{L}_0) = I \setminus X$. Moreover, $\hat{L}_0 \subset L_0$ and consequently $\hat{L}_0 \subset \widetilde{A}$ for any self-adjoint extension $\widetilde{A} \supset L_0$. Now it remains to apply Theorem 3.5 to $\hat{L}_0$ and any self-adjoint extension $\widetilde{A} \supset L_0 (\supset \hat{L}_0)$. $\square$

The following corollary is immediate from Theorems 4.1 and 3.5.

Corollary 4.2. *Let the expression* (4.1) *be regular at a and let the corresponding minimal operator L_0 has equal deficiency indices $d = n_\pm(L_0)$. Moreover, let $I = (\mu_1, \mu_2)$, $-\infty \le \mu_1 < \mu_2 \le \infty$, be an interval such that Eq.* (4.2) *has d linearly independent solutions which lie in $L_2(a,b)$ for all $\lambda \in I$ besides an at most countable set $X \subset I$. Then:*

1)for any self-adjoint extension $\widetilde{A} \supset L_0$ the statement 1) of Theorem 3.5 holds;

2) the set of all points $\lambda \in I$ such that $\overline{\mathcal{R}(L_0 - \lambda)} \ne \mathcal{R}(L_0 - \lambda)$ is nowhere dense in I.

The particular case of (4.1) is the scalar Sturm - Liouville expression

$$l[y] = -y'' + p(t)y, \quad t \in (0, \infty), \tag{4.4}$$

where $p(\cdot) : (0,\infty) \to \mathbb{C}$ is a scalar function such that $p(t) = \overline{p(t)}$ and $p(t) \in L_1(0,c)$ for every $c \in (0,\infty)$ (this means that the expression (4.4) is regular at 0). Let the minimal operator L_0 of the expression (4.4) has minimal deficiency indices $n_\pm(L_0) = 1$ (the limit point case). For a given $\theta \in \mathbb{R}$ consider the boundary value problem defined by the equation (4.2) and the boundary condition

$$y'(0) - \theta y(0) = 0. \tag{4.5}$$

Assume that $\varphi(t,\lambda)$ is the solution of (4.2) with the initial data $\varphi(0,\lambda) = 1,\ \varphi'(0,\lambda) = \theta$ and let $\widetilde{A}_\theta$ be a self-adjoint extension of L_0 with the domain

$$\mathcal{D}(\widetilde{A}_\theta) = \{y \in \mathcal{D}(L) : y'(0) = \theta y(0)\}.$$

As is known [15] the scalar measure $F(\cdot) : \mathcal{B}_b \to \mathbb{R}$ is called a spectral measure of the boundary problem (4.2), (4.5) if the relation (the Fourier transform)

$$L_2(0,\infty) \ni f \to (Vf)(\lambda) = \int_0^\infty \varphi(t,\lambda) f(t)\, dt \, \in L_2(F) \tag{4.6}$$

defines the unitary operator $V : L_2(0,\infty) \to L_2(F)(= L_2(F,\mathbb{C}))$ such that the operators $\widetilde{A}_\theta$ and $\widetilde{A}_F$ (see (3.5)) are unitary equivalent by means of V.

If $F(\cdot)$ is a spectral measure of the problem (4.2), (4.5), then the unitary operator V (4.6) gives the unitary equivalence between the minimal operator L_0 and the symmetric operator A_F defined by (3.6). Observe also that $F(\cdot)$ is the spectral measure of the Titchmarsh–Weyl function $m(\cdot)$ of the boundary problem (4.2), (4.5)[15] and hence it satisfies the relation (2.2).

In the following proposition we show that the conjecture by Hartman and Wintner on the spectrum of a self-adjoint Sturm - Liouville operator is false (for more details see Introduction).

Proposition 4.3. *For any finite interval $I = (\mu_1, \mu_2)$, $-\infty < \mu_1 < \mu_2 < \infty$, there exists a Sturm–Liouville expression* (4.4) *such that the deficiency indices of the minimal operator L_0 are $d = n_\pm(L_0) = 1$ and the following statements hold:*

1) for all $\lambda \in I$ Eq. (4.2) *has the unique solution which lies in $L_2(0,\infty)$;*

2) for any self-adjoint extension $\widetilde{A} \supset L_0$ the interval I contains infinitely many points of the essential spectrum $\sigma_e(\widetilde{A})$.

Proof. Let $I = (\mu_1, \mu_2)$ be a finite interval, let $\{\lambda_k\}$. $\{\lambda_{jk}\}$ and $\{F_{jk}\}$ be the same as in the proof of Proposition 3.6 (case (i)) and let $F'(B)$ be the measure on Borel sets $B \subset I$ defined by (3.14). According to [15, ch. 8. 26.3] the measure $F'(\cdot)$ can be extended to the measure $F(\cdot)$ on bounded Borel sets $B \subset \mathbb{R}$ with the following property: there exists the Sturm–Liouville expression (4.4) and a real θ such that the corresponding minimal operator L_0 has the deficiency indices $d = n_\pm(L_0) = 1$ and $F(\cdot)$ is the spectral measure of the boundary problem (4.2), (4.5).

Next assume that A_F and $\widetilde{A}_F$ are the operators (3.5) and (3.6) respectively. Then repeating the same reasonings as in the proof of Proposition 3.6 one obtains that $\widetilde{\rho}_I(A_F) = I$ and for any self-adjoint extension $\widetilde{A} \supset A_F$ the set $\sigma_e(\widetilde{A}) \cap I$ is infinite. Since the minimal operator L_0 is unitary equivalent to A_F (by means of the Fourier transform (4.6)), the operator L_0 has the same properties as A_F. This implies the desired statements 1) and 2). □

References

[1] J.F. Brasche, M.M. Malamud, H. Neidhart, Weyl function and spectral properties of self-adjoint extensions, Integral Equations Operator Theory 43(3) (2002) 264–289.

[2] M.S. Brodskii, Triangular and Jordan Representations of Linear operators, Nauka, Moscow, 1968.

[3] V.A. Derkach, M.M. Malamud, Generalized resolvents and the boundary value problems for Hermitian operators with gaps, J. Funct. Anal. 95 (1991) 1–95.

[4] V.A. Derkach, M.M. Malamud, Characteristic functions of almost solvable extensions of Hermitian operators, Ukr. Mat. Zh. 44(4) (1992) 435–459.

[5] V.A. Derkach, M.M. Malamud, The extension theory of Hermitian operators and the moment problem, J. Math. Sciences 73(2) (1995) 141–242.

[6] N. Dunford, J.T. Schwartz, Linear operators. Part2. Spectral theory, Interscience Publishers, New York-London, 1963.

[7] R.C. Gilbert, Simplicity of linear ordinary differential operators, J. Differential Equations 11(1972), 672–681.

[8] V.I. Gorbachuk, M.L. Gorbachuk, Boundary Problems for Differential-operator Equations, Kluver Acad. Publ., Dordrecht-Boston-London, 1991. (Russian edition: Naukova Dumka, Kiev, 1984).

[9] Ph. Hartman, A. Wintner, A separation theorem for continuous spectra, Amer. J. Math. 71 (1949) 650–662.

[10] I.S. Kac, M.G. Krein, *R*-functions – analytic functions mapping the upper halfplane into itself, Supplement to the Russian edition of F.V. Atkinson , Discrete and Continuous Boundary Problems, Mir, Moscow, 1968 (English translation: Amer. Math. Soc. Transl. Ser. 2, 103 (1974) 1–18).

[11] I.S. Kats, On Hilbert spaces generated by monotone Hermitian matrix-functions, Khar'kov. Gos. Univ. Uchen. Zap. 34 (1950) 95-113=Zap.Mat.Otdel.Fiz.-Mat. Fak. i Khar'kov. Mat. Obshch. 22(4) (1950) 95–113.

[12] V.I. Kogan, F.S. Rofe-Beketov, On square-integrable solutions of symmetric systems of differential equations of arbitrary order, Proc. Roy. Soc. Edinburgh (74A) (1974), 5–40.

[13] M.M. Malamud, On a formula for the generalized resolvents of a non-densely defined Hermitian operator, Ukr. Math. Zh. 44(12) (1992) 1658–1688.

[14] V.I.Mogilevskii, Boundary triplets and Titchmarsh - Weyl functions of differential operators with arbitrary deficiency indices, Methods Funct. Anal. Topology 15(3) (2009) 280–300.

[15] M.A. Naimark, Linear Differential Operators, vol. 1 and 2, Harrap, London, 1967. (Russian edition: Nauka, Moscow, 1969).

[16] M. Reed, B. Simon, Methods of Modern Mathematical Physics, Acad. Press, New York-London, 1972.

[17] J. Sun, A. Wang, A. Zettl, Continuous spectrum and square-integrable solutions of differential operators with intermediate deficiency index,J. Funct. Anal. 255 (2008) 3229–3248.

[18] J. Weidmann, Spectral Theory of Ordinary Differential Operators, Lecture Notes in Math., vol. 1958, Springer-Verlag, Berlin, 1987.

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