Alternated Hochschild Cohomology

Pierre Lecomte Valentin Ovsienko

Abstract

In this paper we construct a graded Lie algebra on the space of cochains on a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ that are skew-symmetric on the subspace V_1. The Lie bracket is obtained from the classical Gerstenhaber bracket by (partial) skew-symmetrization; the coboundary operator is a skew-symmetrized version of the Hochschild differential. We show that an order-one element m satisfying the zero-square condition $[m, m] = 0$ defines an algebraic structure called "Lie antialgebra" in [17]. The cohomology (and deformation) theory of these algebras is then defined. We present two examples of non-trivial cohomology classes which are similar to the celebrated Gelfand-Fuchs and Godbillon-Vey classes.

Key Words: Hochschild cohomology, graded Lie algebra, Lie antialgebra.

1 Introduction

Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$-graded vector space. We consider the space of parity preserving multilinear maps on V, that are skew-symmetric on the subspace V_1

$$\varphi : (V_0 \otimes \cdots \otimes V_0) \otimes (V_1 \wedge \cdots \wedge V_1) \to V. \tag{1.1}$$

We will define a natural structure of graded Lie algebra on this space and develop an adequate cohomology theory.

The graded Lie algebra on the space of all multilinear maps on a (multi-graded) vector space V is the classic Gerstenhaber algebra [4, 5]. The graded Lie algebra on the space of skew-symmetric maps $V \wedge \cdots \wedge V \to V$ is another classic graded Lie algebra called the Nijenhuis-Richardson algebra [14, 15]. A natural homomorphism between the Gerstenhaber algebra and that of Nijenhuis-Richardson is given by the skewsymmetrization, see [7].

In this article, we introduce a graded Lie algebra defined by the Gerstenhaber bracket skewsymmetrized only in a part of variables. In this sense, our graded Lie algebra is a kind of "intermediate form" between the Gerstenhaber and Nijenhuis-Richardson algebras.

The related cohomology is defined in a usual way. We consider a *parity preserving* bilinear map $m : V \times V \to V$ of the form (1.1), understood as an *odd* element (of degree 1) in our graded Lie algebra. We assume that m satisfies the condition

$$[m, m] = 0. \tag{1.2}$$

This defines a coboundary operator

$$\delta = \mathrm{ad}_m \tag{1.3}$$

and the corresponding cochain complex. Note that the condition $\delta^2 = 0$ is an immediate consequence of (1.2) and the graded Jacobi identity. We then calculate the explicit combinatorial formula of the differential, it turns out to be an amazing "interpolation" between the Hochschild and Chevalley-Eilenberg differentials.

Let us stress on the fact that most of the known algebraic structures, such as associative or Lie algebras, Lie bialgebras, Poisson structures, etc. can be represented in terms of an order-one element m of a graded Lie algebra (usually the algebra of derivations of an associative algebra of tensors) that satisfies the condition (1.2). This general idea goes back to Gerstenhaber and Nijenhuis-Richardson and became a powerful tool for producing new (or better understanding of the known) algebraic structures, see [8] as an example of such approach.

It turns out quite remarkably, that a bilinear map m, symmetric on V_0 and skew-symmetric on V_1, satisfying the condition (1.2) is precisely the structure of *Lie antialgebra* introduced in [17] and further studied in [9]. This class of algebras is a particular class of Jordan superalgebras closely related to the Kaplansky superalgebras defined in [12]. Lie antialgebras appeared in symplectic geometry, see [17]. Deducing this algebraic structure directly from the Gerstenhaber algebra explains its cohomologic nature.

In this paper, we define cohomology of Lie antialgebras. We pay a special attention to lower degree cohomology spaces and explain their algebraic sense. In particular we show that the second cohomology space classifies extensions of Lie antialgebras already considered in [17], while the first cohomology space classifies extensions of modules. In the end of the paper, we present two examples of non-trivial cohomology classes generalizes two celebrated cohomology classes of infinite-dimensional Lie algebras. One of them is analog of the Gelfand-Fuchs class and the second one is analog of the Godbillon-Vey class.

The notion of deformation is also provided by the equation (1.2). Considering a family m_t satisfying $[m_t, m_t] = 0$ and such that $m_0 = m$, one immediately obtains the Maurer-Cartan equation for the variational part $\widetilde{m} = m_t - m$:

$$\delta\widetilde{m} = \tfrac{1}{2}\,[\widetilde{m}, \widetilde{m}] \tag{1.4}$$

crucial in the deformation theory. We hope to develop this approach in a subsequent work.

2 The graded Lie algebra

In this section, we introduce our main object, the graded Lie algebra on the space of maps (1.1). We start with a brief description of the most classical Gersternhaber algebra. We then discuss in some details the case of $\mathbb{Z}_2$-graded vector space and finally define our graded Lie algebra by a skew-symmetrization of the Gersternhaber bracket.

2.1 The Gerstenhaber algebra

Given a vector space V, consider the space $M(V)$ of all multilinear maps $\varphi : V \otimes \cdots \otimes V \to V$. The standard $\mathbb{Z}_{\geq -1}$-grading on $M(V)$ is given by

$$M(V) = \bigoplus_{k \geq -1} M^k(V),$$

where $M^k(V)$ is the space of $(k+1)$-linear maps.

The *Gerstenhaber product* of two elements $\varphi \in M^k(V)$ and $\varphi' \in M^{k'}(V)$ usually denoted by, $j_\varphi \, \varphi' \in M^{k+k'}(V)$ is given by

$$(j_\varphi \, \varphi') \, (x_0, \ldots, x_{k+k'}) = \sum_{i=0}^{k'} (-1)^{i\,k} \, \varphi' \, (x_0, \ldots, \varphi(x_i, \ldots, x_{i+k}), \ldots, x_{k+k'}) \, . \tag{2.1}$$

The classical result of Gerstenhaber [4] states that graded bracket

$$\left[\varphi, \varphi'\right] = j_\varphi \varphi' - (-1)^{kk'} \, j_{\varphi'} \varphi \tag{2.2}$$

equips $M(V)$ with a structure of graded Lie algebra. In particular, it satisfies the graded Jacobi identity

$$(-1)^{k_1 k_3} \, [\varphi_1, \, [\varphi_2, \varphi_3]] + (-1)^{k_1 k_2} \, [\varphi_2, \, [\varphi_3, \varphi_1]] + (-1)^{k_2 k_3} \, [\varphi_3, \, [\varphi_1, \varphi_2]] = 0, \tag{2.3}$$

where $\varphi_i \in M^{k_i}(V)$ for $i = 1, 2, 3$.

The most conceptual way to prove this statement consists in a simple observation [16] that the Gerstenhaber algebra is nothing but the algebra of derivations of the associative tensor algebra TV^*. Indeed, one obviously has $M^k(V) \cong (V^*)^{\otimes (k+1)} \otimes V$, as a vector space. On the other hand, a derivation $D \in \mathrm{Der}\,(TV^*)$ is uniquely defined (via the Leibniz rule) by its restriction to the first-order component V^* of the algebra TV^*, namely,

$$D|_{V^*} : V^* \to TV^*.$$

The derivation D is therefore identified with an element of $TV^* \otimes V$. One thus obtains the isomorphism

$$(M(V), \, [\, , \,]) \cong \mathrm{Der}\,(TV^*) \tag{2.4}$$

as vector spaces. The bracket (2.2) precisely the (graded) commutator in $\mathrm{Der}\,(TV^*)$ so that the above isomorphism is, indeed, an isomorphism of graded Lie algebras.

2.2 $\mathbb{Z}_2$-graded case

It is easy to generalize the above definitions in the case of $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$. To make this paper self-content, we give here all the necessary details.

The space $M^k(V)$ has the following decomposition

$$M^k(V) = \bigoplus_{\ell+m=k+1} M^{\ell,m}(V),$$

where $M^{\ell,m}(V)$ is the space of all (ℓ,m)-linear maps $\varphi : V_0^{\otimes \ell} \otimes V_1^{\otimes m} \to V$.

Every element $\varphi \in M^{\ell,m}(V)$ is a sum of two homogeneous components: $\varphi = \varphi_0 + \varphi_1$ with values in V_0 and V_1, respectively. The *parity* of a homogeneous $\varphi_i \in M^{\ell,m}(V)$ is defined by

$$p(\varphi_i) := \ell + i + 1 \qquad (\mathrm{mod}\, 2),$$

where $i = 0$ or 1.

Remark 2.1. Note that the parity in this definition is somewhat "inverted". That was already the case for the usual Gerstenhaber algebra, cf. Section 2.1. In particular, an even bilinear map, $m : V_i \times V_j \to V_{i+j}$, is of parity 1 and is understood as an *odd* element of $M(V)$. Furthermore, the space V is viewed as a subspace of $M(V)$, and parity of $v \in V_i$ is $p(v) = i + 1$.

The operations (2.1) and (2.2) are now defined via the standard sing rule. Note that this is a particular case of the multi-graded Gerstenhaber algebra defined in [7]. In order to fix the notation, we will give the formula for the Gerstanhaber product denoting by x elements of V_0 and by y elements of V_1. For an (ℓ,m)-linear map φ_0 with values in V_0 and an (ℓ',m')-linear map φ', one obtains an $(\ell+\ell'-1, m+m')$-linear map

$$\begin{gathered}(j_{\varphi_0}\,\varphi')\,(x_1,\ldots,x_{\ell+\ell'-1};\, y_1,\ldots,y_{m+m'}) = \\ \sum_{i=0}^{\ell'} (-1)^{i\,p(\varphi_0)}\,\varphi'\,(x_1,\ldots,\varphi_0(x_{i+1},\ldots,x_{i+\ell};\, y_1,\ldots,y_m),\ldots,x_{\ell+\ell'-1};\, y_{m+1},\ldots,y_{m+m'})\,.\end{gathered} \tag{2.5}$$

Similarly, for a map φ_1 with values in V_1, one has

$$\begin{gathered}(j_{\varphi_1}\,\varphi')\,(x_1,\ldots,x_{\ell+\ell'};\, y_1,\ldots,y_{m+m'-1}) = \\ (-1)^{\ell'\,p(\varphi_1)} \sum_{i=0}^{m'} \varphi'\,(x_1,\ldots,x_{\ell'};\, y_1,\ldots,\varphi_1(x_{\ell'+1},\ldots,x_{\ell+\ell'};\, y_i,\ldots,y_{i+m}),\ldots,y_{m+m'-1})\,.\end{gathered} \tag{2.6}$$

The commutator (2.2) with k_1 and k_2 replaced by $p(\varphi_1)$ and $p(\varphi_2)$, respectively, is again a graded Lie algebra structure.

Observe the lack of sign depending upon i in (2.6) is due to the fact that the $y \in V_1$ are of even parity in $M(V)$.

2.3 The Nijenhuis-Richardson algebra

The Nijenhuis-Richardson algebra, $A(V)$, is the graded Lie algebra defined on the space of skew-symmetric multilinear maps $V \wedge \cdots \wedge V \to V$. The space $A(V)$ is precisely the space of derivations $\mathrm{Der}(\Lambda V^*)$, the graded Lie bracket on $A(V)$ is defined and similar to the Gersternhaber bracket. This algebra is called the Nijenhuis-Richardson algebra, it is related to the Chevalley-Eilenberg cohomology of Lie (super)algebras, see [14]. We omit the explicit formulæ.

The definition of the Nijenhuis-Richardson algebra holds in the $\mathbb{Z}_2$-graded case.

Remark 2.2. If $V = V_0 \oplus V_1$, then multilinear maps $V \wedge \cdots \wedge V \to V$ are skew-symmetric in the graded sense, in particular φ is symmetric on V_1. The multilinear maps (1.1) that we consider in this paper are skew-symmetric on V_1.

It was proved in [7] that the natural homomorphism of graded Lie algebras $M(V) \to A(V)$ is given by skew-symmetrization.

2.4 Alternated Gerstenhaber algebra

In this section, we give our main construction. The vector space V we consider is $\mathbb{Z}_2$-graded.

Definition 2.3. A map $\varphi \in M^{\ell,m}(V)$ is called *parity preserving* if

$$\begin{cases} \varphi_0 = 0, & \text{if } m \text{ is odd}, \\ \varphi_1 = 0, & \text{if } m \text{ is even}, \end{cases}$$

where φ_0 is with values in V_0 and φ_1 is with values in V_1.

In the sequel, we restrict our considerations to the parity preserving cochains.

We denote by $\mathfrak{al}(V)$ the space of parity preserving multilinear maps of the form (1.1). Of course, $\mathfrak{al}(V)$ is a subspace of $M(V)$. There is a natural projection $\mathrm{Alt} : M(V) \to \mathfrak{al}(V)$ defined by skew-symmetrization in y-variables:

$$(\mathrm{Alt}\,\varphi)\,(x_1, \ldots, x_\ell;\, y_1, \ldots, y_m) = \frac{1}{m!} \sum_{\sigma \in S_m} \mathrm{sign}(\sigma)\, \varphi\left(x_1, \ldots, x_\ell;\, y_{\sigma(1)}, \ldots, y_{\sigma(m)}\right). \tag{2.7}$$

We define a bilinear skew-symmetric operation on $\mathfrak{al}(V)$ by projection of the Gerstenhaber bracket (2.2):

$$[\varphi, \varphi']_{\mathfrak{al}} := \mathrm{Alt}\,[\varphi, \varphi']. \tag{2.8}$$

Our first main result is the following.

Theorem 1. *The space $\mathfrak{al}(V)$ equipped with the bracket (2.8) is a graded Lie algebra.*

Proof. It will suffice to check that the Gerstenhaber bracket commutes with the map Alt, that is

$$\mathrm{Alt}\,[\varphi,\varphi'] = \mathrm{Alt}\,[\mathrm{Alt}\,\varphi, \mathrm{Alt}\,\varphi'], \tag{2.9}$$

for all $\varphi, \varphi' \in M(V)$. Indeed, the Jacobi identity for $[.,.]_{\mathfrak{al}}$ will then follow from the Jacobi identity for the Gerstenhaber bracket.

The formula (2.9) is obviously satisfied whenever both φ and φ' are skew-symmetric in y-variables since Alt is a projection. It remains to prove this formula when one of them belongs to the kernel of Alt.

Consider the alternated Gerstenhaber product (2.5-2.6). It is obvious that if φ_0 or φ' belongs to ker(Alt), then $j_{\varphi_0}\varphi'$ also belongs to ker(Alt). It is also clear that $j_{\varphi_1}\varphi' \in \ker(\mathrm{Alt})$ when $\varphi_1 \in \ker(\mathrm{Alt})$. It remains to show that this is also guaranteed when $\varphi' \in \ker(\mathrm{Alt})$. Indeed, due to the parity preserving property, the transposition

$$(\varphi_1(x_{\ell'+1},\ldots,x_{\ell+\ell'};\, y_i,\ldots,y_{i+m}), y_j) \leftrightarrow (y_j, \varphi_1(x_{\ell'+1},\ldots,x_{\ell+\ell'};\, y_i,\ldots,y_{i+m}))$$

is an odd permutation of $\{y_i,\ldots,y_{i+m},y_j\}$. Therefore $\mathrm{Alt}(j_{\varphi_1}\varphi')$ is the skew-symmetrization of

$$\mathrm{Alt}(\varphi')(x_1,\ldots,x_\ell;\varphi_1(x_{\ell'+1},\ldots,x_{\ell+\ell'};y_1,\ldots,y_m),y_{m+1},\ldots,y_{m+m'-1})$$

with respect to the y's. Hence it vanishes. $\square$

Note that the parity preserving condition is essential, identity (2.9) fails without it.

3 Cohomology

In this section, we define and calculate explicitly the cochain operator associated to an arbitrary element $m \in \mathfrak{al}^1(V)$ satisfying the condition (1.2). This is the cohomology theory we are interested in. We start with a brief account on the classic Hochschild cohomology defined within the context of Gerstenhaber algebra, as well as the Chevalley-Eilenberg cohomology of Lie (super)algebras related to the Nijenhuis-Richardson algebra.

3.1 Gerstenhaber algebra and Hochschild cohomology

Let us recall the most classical case of Hochschild cohomology in the purely even case. A bilinear map $m : V \times V \to V$ satisfies the condition (1.2) if and only if m is an associative product on V. The linear map $\delta_H := \mathrm{ad}_m$ from $M^k(V)$ to $M^{k+1}(V)$ is as follows

$$\begin{aligned}(\delta_H\,\varphi)(x_0,\ldots,x_{k+1}) &= m(x_0,\varphi(x_1,\ldots,x_{k+1})) \\ &- \sum_{i=0}^{k}(-1)^i\,\varphi(x_0,\ldots,m(x_i,x_{i+1}),\ldots,x_{k+1}) \\ &+ (-1)^k\,m(\varphi(x_0,\ldots,x_k),x_{k+1}),\end{aligned} \tag{3.1}$$

for an arbitrary $\varphi \in M^k(V)$. This map coincides with the classic Hochschild differential; the corresponding cohomology is the classic Hochschild cohomology of the associative algebra (V, m), see [11] for more details.

3.2 Nijenhuis-Richardson algebra and cohomology of Lie superalgebras

A skew-symmetric bilinear map $m \in A^1(V)$ satisfies the condition (1.2) if and only if $\mathfrak{g} = (V, m)$ is a Lie (super)algebra. One now uses the notation

$$[x_1, x_2] := m(x_1, x_2),$$

for all $x_1, x_2 \in V$. The map (1.3) coincides with the Chevalley-Eilenberg differential:

$$\begin{aligned} (\delta_{CE}\,\varphi)(x_0, \ldots, x_{k+1}) &= \sum_{i=0}^{k+1} (-1)^i \, [x_i, \varphi(x_0, \ldots, \widehat{x_i}, \ldots, x_{k+1})] \\ &+ \sum_{0 \le i < j \le k+1} (-1)^{i+j} \, \varphi([x_i, x_j], x_0, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_{k+1}), \end{aligned} \tag{3.2}$$

for every $\varphi \in A^k(V)$.

3.3 Combinatorial formula for the differential on $\mathfrak{al}(V)$

Consider now the graded Lie algebra $\mathfrak{al}(V)$ defined in Section 2.4. Let $m \in \mathfrak{al}^1(V)$ be a parity preserving bilinear map satisfying the condition $[m, m] = 0$. In this section, we calculate the combinatorial expression of the differential $\delta = \mathrm{ad}_m$.

Of course, the operator δ increases the degree of the cochains. We will use the notation δ^k for the restriction of δ to $\mathfrak{al}^k(V)$, viz

$$\delta^k : \mathfrak{al}^k(V) \to \mathfrak{al}^{k+1}(V).$$

Since m preserves the parity, it is of the form:

$$m : V_0 \times V_0 \to V_0, \qquad m : V_0 \times V_1 \to V_1, \qquad m : V_1 \times V_1 \to V_0.$$

It follows that the operator δ^k has three terms:

$$\delta^k = \delta^k_{1,0} + \delta^k_{0,1} + \delta^k_{-1,2}, \tag{3.3}$$

where $\delta^k_{i,j} : \mathfrak{al}^{p,q}(V) \to \mathfrak{al}^{p+i,q+j}(V)$, for $p + q = k$.

The following statement follows from Theorem 1.

Corollary 3.1. *The operator $\delta^k = \delta^k_{1,0} + \delta^k_{0,1} + \delta^k_{-1,2}$ defined by:*

(i) if $q = 0$, then $\delta_{1,0}$ is the standard Hochschild differential, if $q \neq 0$, then

$$\begin{aligned}(\delta^k_{1,0}\varphi)(x_0, \dots, x_p;\, y_0, \dots, y_{q-1}) = \\ & m\,(x_0,\, \varphi(x_1, \dots, x_p;\, y_0, \dots, y_{q-1})) \\ & -\sum_{i=0}^{p-1}(-1)^i\, \varphi(x_0, \dots, x_{i-1}, m(x_i, x_{i+1}), x_{i+2}, \dots, x_p;\, y_0, \dots, y_{q-1}) \\ & -\frac{1}{q}\sum_{j=0}^{q-1}(-1)^{p+j}\, \varphi(x_0, \dots, x_{p-1};\, m(x_p, y_j), y_0, \dots, \widehat{y_j}, \dots, y_{q-1});\end{aligned} \tag{3.4}$$

(ii) $\delta^k_{0,1}$ is given by

$$\begin{aligned}(\delta^k_{0,1}\varphi)(x_0, \dots, x_{p-1};\, y_0, \dots, y_q) = \\ & \frac{1}{q+1}\sum_{j=0}^{q}(-1)^{p+j}\, m\,(\varphi(x_0, \dots, x_{p-1};\, y_0, \dots, \widehat{y_j}, \dots, y_q),\, y_j)\end{aligned} \tag{3.5}$$

(iii) $\delta^k_{-1,2}$ is given by

$$\begin{aligned}(\delta^k_{-1,2}\,\varphi)(x_0, \dots, x_{p-2};\, y_0, \dots, y_{q+1}) = \\ & \frac{2}{(q+1)(q+2)}\sum_{i<j}(-1)^{p+i+j}\, \varphi(x_0, \dots, x_{p-2}, m(y_i, y_j);\, y_0, \dots, \widehat{y_i}, \dots, \widehat{y_j}, \dots, y_{q+1})\end{aligned} \tag{3.6}$$

is a coboundary operator, that is, $\delta^{k+1} \circ \delta^k = 0$.

Proof. This is just the formula (3.1) skew-symmetrized in y-variables, i.e. $\delta = \mathrm{ad}_m$ in the graded Lie algebra $\mathfrak{al}(V)$. $\square$

Note that the operator $\delta_{1,0}$ is very close to the Hochschild differential, while $\delta^k_{-1,2}$ is that of Chevalley-Eilenberg.

Proposition 3.2. *The relation $\delta^{k+1} \circ \delta^k = 0$ is equivalent to the following system:*

$$\begin{aligned}\delta^{k+1}_{1,0} \circ \delta^k_{1,0} &= 0 \\ \delta^{k+1}_{1,0} \circ \delta^k_{0,1} + \delta^{k+1}_{0,1} \circ \delta^k_{1,0} &= 0 \\ 2\,\delta^{k+1}_{0,1} \circ \delta^k_{0,1} + \delta^{k+1}_{1,0} \circ \delta^k_{-1,2} + \delta^{k+1}_{-1,2} \circ \delta^k_{1,0} &= 0 \\ \delta^{k+1}_{-1,2} \circ \delta^k_{-1,2} &= 0.\end{aligned} \tag{3.7}$$

Proof. The above equations obviously represent linearly independent terms in the equation $\delta^{k+1} \circ \delta^k = 0$. $\square$

3.4 Cohomology with coefficients in an arbitrary module

So far, the cohomology we considered was with coefficients in the algebra itself. However, this particular case actually contains the most general one. The definition of a module is universal for all classes of algebras.

Definition 3.3. Let (V, m) be an algebra (of an arbitrary type: associative, Lie, anti-Lie, etc.). Assume that the vector space V is a direct sun of two subspaces $V = V' \oplus W$. If the space V' is closed with respect to the bilinear map m, in other words,

$$m : V' \times V' \to V',$$

and, in addition,

$$m : V' \times W \to W \qquad \text{and} \qquad m : W \times W \to 0,$$

then the space W is called a *module* over the algebra (V', m)

The space of multilinear maps from V to V contains the subspace, $C(V', W)$, of multilinear maps from V' to W. This subspace is obviously stable under the differential $\delta = \mathrm{ad}_m$. The corresponding complex

$$\delta : C^k(V', W) \to C^{k+1}(V', W)$$

defines the cohomology of the algebra (V', m) with coefficients in the module W.

4 Lie antialgebras

The notion of Lie antialgebra was introduced in [17] in the context of symplectic geometry (see also [3] for the first example) and was further studied in [13] and [9]. It was shown in [9] that Lie antialgebras is a particular case of Jordan superalgebras. The definition of a Lie antialgebra is almost identical to that of a Kaplansky Jordan superalgebra, see [12] (except that a Lie antialgebra is not necessarily half-unital). Lie antialgebras are closely related to Lie superalgebras. More precisely, there is a Lie superalgebra $\mathfrak{g}_{\mathfrak{a}}$ canonically associated to every Lie antialgebra $\mathfrak{a}$.

In this section, we show that the structure of Lie antialgebra can be understood as the algebraic structure defined by an element $m \in \mathfrak{al}^1(V)$ satisfying the condition $[m, m] = 0$.

4.1 Definition and examples

We give two equivalent definitions, each of them has its advantages.

Definition 4.1. A Lie antialgebra $(\mathfrak{a}, \cdot)$ is a commutative $\mathbb{Z}_2$-graded algebra: $\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1$ and $\mathfrak{a}_i \cdot \mathfrak{a}_j \subset \mathfrak{a}_{i+j}$, so that for all homogeneous elements $a, b \in \mathfrak{a}$ one has

$$a \cdot b \;=\; (-1)^{\bar{a}\bar{b}}\, b \cdot a \tag{4.1}$$

where $\bar{a}$ is the degree of a, satisfying the third-order identities:

$$x_1 \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot x_3, \tag{4.2}$$

$$x_1 \cdot (x_2 \cdot y) = \tfrac{1}{2}\,(x_1 \cdot x_2) \cdot y, \tag{4.3}$$

$$x \cdot (y_1 \cdot y_2) = (x \cdot y_1) \cdot y_2 \ + \ y_1 \cdot \, (x \cdot y_2)\,, \tag{4.4}$$

$$y_1 \cdot (y_2 \cdot y_3) \ + \ y_2 \cdot (y_3 \cdot y_1) \ + \ y_3 \cdot (y_1 \cdot y_2) = 0, \tag{4.5}$$

for all $x_i \in \mathfrak{a}_0$ and $y_i \in \mathfrak{a}_1$. In particular, $\mathfrak{a}_0$ is a commutative associative subalgebra.

An equivalent definition is as follows. A $\mathbb{Z}_2$-graded commutative algebra $\mathfrak{a}$ is a Lie antialgebra if and only if the following three conditions are satisfied.

1. The subalgebra $\mathfrak{a}_0 \subset \mathfrak{a}$ is associative.
2. For all $x_1, x_2 \in \mathfrak{a}_0$, the operators of left multiplication commute: $x_1 \cdot (x_2 \cdot a) = x_2 \cdot (x_1 \cdot a)$.
3. For every $y \in \mathfrak{a}_1$, the operator of right multiplication by y is an *odd* derivation of $\mathfrak{a}$, i.e., one has
$$(a \cdot b) \cdot y = (a \cdot y) \cdot b + (-1)^{\bar{a}}\, a \cdot \, (b \cdot y)\,. \tag{4.6}$$

Example 4.2. The simplest example of a Lie antialgebra is 3-dimensional algebra known as *tiny Kaplansky Superalgebra* and denoted by K_3. This algebra has the basis $\{\varepsilon; a, b\}$, where ε is even and a, b are odd, satisfying the relations

$$\begin{aligned} &\varepsilon \cdot \varepsilon = \varepsilon \\ &\varepsilon \cdot a = \tfrac{1}{2}\, a, \quad \varepsilon \cdot b = \tfrac{1}{2}\, b, \\ &a \cdot b = \tfrac{1}{2}\, \varepsilon. \end{aligned} \tag{4.7}$$

It is simple, i.e., it contains no non-trivial ideal.

The corresponding algebra of derivations is the simple Lie superalgebra osp(1|2). Let us mention that the algebra K_3 plays an important rôle in the study of some exceptional Jordan algebras, cf. [1].

Example 4.3. The main example of (an infinite-dimensional) Lie antialgebra is the *conformal Lie antialgebra* $\mathcal{AK}(1)$. This is a simple infinite-dimensional Lie antialgebra with the basis $\left\{\varepsilon_n,\ n \in \mathbb{Z}; \quad a_i,\ i \in \mathbb{Z} + \frac{1}{2}\right\}$, where ε_n are even and a_i are odd and satisfy the following relations

$$\begin{aligned} \varepsilon_n \cdot \varepsilon_m &= \varepsilon_{n+m}, \\ \varepsilon_n \cdot a_i &= \tfrac{1}{2}\, a_{n+i}, \\ a_i \cdot a_j &= \tfrac{1}{2}\, (j - i)\, \varepsilon_{i+j}. \end{aligned} \tag{4.8}$$

The algebra $\mathcal{AK}(1)$ is closely related to the well-known Neveu-Schwarz conformal Lie superalgebra, $\mathcal{K}(1)$, namely $\mathcal{K}(1) = \mathrm{Der}(\mathcal{AK}(1))$, see [17]. Note that $\mathcal{AK}(1)$ contains infinitely many copies of K_3, among which we quote the one with basis

$$\left\{\varepsilon_0; a_{-\frac{1}{2}}, a_{\frac{1}{2}}\right\}.$$

A very similar algebra, called the full derivation algebra, was considered in [12]. This algebra is also defined by the formulæ (4.8), but the odd generators a_i are indexed by integer i's.

Example 4.4. Our next example is the simple infinite-dimensional Lie antialgebra $\mathcal{M}^1$ which is a "truncated version" of $\mathcal{AK}(1)$. The algebra $\mathcal{M}^1$ is the algebra of formal series in the elements of the basis:

$$\left\{\alpha_n,\, n \geq 0, \quad a_i,\, i \geq -\tfrac{1}{2}\right\},$$

subject to the relations (4.8).

We understand this algebra as analog of the Lie algebra, W_1, of formal vector fields on the real line. Note that W_1 plays an important role in algebra and in topology (see, i.g., [2]).

Further examples of finite-dimensional Lie antialgebras can be found in [17]. An interesting series of simple infinite-dimensional examples is constructed in [16].

4.2 Lie antialgebra and the zero-square condition

Let us show that the notion of Lie antialgebra is related to graded Lie algebra $\mathfrak{al}(V)$.

Given a Lie antialgebra $\mathfrak{a}$, in order to use the notation of Section 2.4, we denote by V the ambient vector space, i.e., $V \cong \mathfrak{a}$. Define the bilinear map $m : V \times V \to V$ as follows:

$$m(x_1, x_2) := \tfrac{1}{2}\, x_1 \cdot x_2\,, \qquad m(x, y) := x \cdot y\,, \qquad m(y_1, y_2) := y_1 \cdot y_2\,, \tag{4.9}$$

where $x_1, x_2 \in V_0 (= \mathfrak{a}_0)$ and $y_1, y_2 \in V_1 (= \mathfrak{a}_1)$ and where $\cdot$ stands for the product in $\mathfrak{a}$.

Proposition 4.5. *The operation m satisfies the condition $[m, m] = 0$ in the graded Lie algebra $\mathfrak{al}(V)$ if and only if $\mathfrak{a}$ is a Lie antialgebra.*

Proof. The condition $[m, m] = 0$ reads:

$$\begin{array}{rcl}
m\left(m(x_1, x_2), x_3\right) - m\left(x_1, m(x_2, x_3)\right) & = & 0, \\
m\left(m(x_1, x_2), y\right) - m\left(x_1, m(x_2, y)\right) & = & 0, \\
\frac{1}{2}\, m\left(m(x_1, y_1), y_2\right) - \frac{1}{2}\, m\left(m(x_1, y_2), y_1\right) - m\left(x_1, m(y_1, y_2)\right) & = & 0, \\
m\left(m(y_1, y_2), y_3\right) + \text{cycle} & = & 0.
\end{array}$$

Note that this is just the associativity skew-symmetrized with respect to the y-variables, where $y \in V_1$. Using the definition (4.9), one immediately checks that the above condition is indeed equivalent to the identities (4.2)–(4.5). $\square$

The structure of Lie antialgebra is therefore equivalent to the zero-square condition in the graded Lie algebra $\mathfrak{al}(V)$. It follows that the cohomology developed in Section 3.3 is the cohomology of Lie antialgebras. Let us give a detailed description of cohomology of a Lie antialgebra with coefficients in an arbitrary module.

4.3 Modules over Lie antialgebras

The notion of module over a Lie antialgebra fits into the general Definition 3.3. Given a Lie antialgebra $\mathfrak{a}$ and an $\mathfrak{a}$-module $\mathcal{B}$, the space $\mathfrak{a} \oplus \mathcal{B}$ is equipped with a Lie antialgebra structure. More precisely, for $a \in \mathfrak{a}$ and $b \in \mathcal{B}$, one has

$$(a, b) \cdot (a', b') = \Big(a \cdot a',\, \rho_a b' + (-1)^{\bar{a'}\bar{b}} \rho_{a'} b\Big)\,,$$

where $\rho : \mathfrak{a} \to \mathrm{End}(\mathcal{B})$ is the linear map that defines the $\mathfrak{a}$-action on $\mathcal{B}$. The Lie antialgebra structure on the space $\mathfrak{a} \oplus \mathcal{B}$ is called a *semi-direct product* and is denoted by $\mathfrak{a} \ltimes \mathcal{B}$.

Proposition 4.6. *Given an $\mathfrak{a}$-module $\mathcal{B}$, the dual space $\mathcal{B}^*$ is naturally an $\mathfrak{a}$-module the $\mathfrak{a}$-action being given by*

$$\langle \rho^*_a u,\, b\rangle := (-1)^{\bar{u}\bar{a}} \langle u, \rho_a b\rangle \tag{4.10}$$

for all $u \in \mathcal{B}^*$ and $b \in \mathcal{B}$.

Proof. Straightforward. □

In particular, the space $\mathfrak{a}^*$ dual to a given Lie antialgebra $\mathfrak{a}$ is an $\mathfrak{a}$-module, as the Lie antialgebra $\mathfrak{a}$ itself is obviously an $\mathfrak{a}$-module with $\rho_a b = a \cdot b$.

4.4 Cohomology with coefficients in a module

Given a Lie antialgebra $\mathfrak{a}$ and an $\mathfrak{a}$-module $\mathcal{B}$, we define the space, $C^{p,q}(\mathfrak{a}; \mathcal{B})$, of parity preserving multi-linear (skew-symmetric on $\mathfrak{a}_1$) maps

$$\varphi : \underbrace{(\mathfrak{a}_1 \otimes \cdots \otimes \mathfrak{a}_1)}_{p} \otimes \underbrace{(\mathfrak{a}_0 \wedge \cdots \wedge \mathfrak{a}_0)}_{q} \to \mathcal{B}. \tag{4.11}$$

We also consider the following space:

$$C^k(\mathfrak{a}; \mathcal{B}) = \bigoplus_{q+p=k} C^{(q,p)}(\mathfrak{a}; \mathcal{B})$$

that we call the space of k-cochains. One obviously has $C^k(\mathfrak{a}; \mathcal{B}) \subset \mathfrak{al}(\mathfrak{a} \ltimes \mathcal{B})$.

The coboundary map δ to the space of cochains $C^{p,q}(\mathfrak{a};\mathcal{B})$ is defined by formulae (3.4)-(3.6) of Theorem 3.1, where m is as in (4.9). We define the cohomology of a Lie antialgebra $\mathfrak{a}$ with coefficients in an $\mathfrak{a}$-module $\mathcal{B}$ in a usual way:

$$H^k(\mathfrak{a};\mathcal{B}) = \ker(\delta^k)/\mathrm{im}(\delta^{k-1}).$$

The space $\ker(\delta^k)$ is called the space of k-cocycles and the space $\mathrm{im}(\delta^{k-1})$ is called the space of k-coboundaries.

4.5 The case of trivial coefficients

In the case, where $\mathcal{B} = \mathbb{R}$ (or $\mathbb{C}$) is a trivial module, the coboundary map (3.3) becomes simpler. The operator $\delta_{0,1}$ vanishes identically, while the system (3.7) reads:

$${\delta_{1,0}}^2 = 0, \qquad {\delta_{-1,2}}^2 = 0, \qquad [\delta_{1,0}, \delta_{-1,2}] = 0.$$

One therefore obtains a structure of bicomplex with two commuting differentials, $\delta_{1,0}$ and $\delta_{-1,2}$. We denote by $H^k(\mathfrak{a})$ the k-th cohomology space with trivial coefficients of a Lie antialgebra $\mathfrak{a}$.

Example 4.7. Consider the Kaplansky superalgebra K_3. In this case, the cohomology with trivial coefficients are quite easy to calculate. The result is as follows:

$$H^0(K_3) = \mathbb{R}\,(\text{or}\,\mathbb{C}), \qquad H^k(K_3) = 0, \quad k > 0.$$

We omit the explicit computation.

5 Algebraic interpretation of lower degree cohomology

Cohomology spaces of lower degree have algebraic meaning quite similar to that in the usual case of Lie algebras. In this section, we use the notation $a \cdot b$ for the action $\rho_a b$ of $a \in \mathfrak{a}$ on an element $b \in \mathcal{B}$, thinking of $\mathcal{B}$ as an ideal in $\mathfrak{a} \ltimes \mathcal{B}$.

5.1 The cohomology of degree zero

The space $H^0(\mathfrak{a};\mathcal{B})$ is the space of elements $b \in \mathcal{B}_0$ satisfying the condition

$$b \cdot y = 0 \tag{5.1}$$

for all $y \in \mathfrak{a}_1$. We call such elements $\mathfrak{a}$-*invariants* of the module $\mathcal{B}$.

Indeed, a 0-cocycle is an element of $C^0(\mathfrak{a};\mathcal{B})$, that is, an element of $\mathcal{B}$. The parity preserving condition for the cochains means that b is even. The equation $\delta b = 0$ then has two independent terms: $\delta_{10} b = 0$ and $\delta_{01} b = 0$. The first equation reads $x \cdot b - b \cdot x = 0$ for $x \in \mathfrak{a}_0$ and is satisfied identically. The equation $(\delta_{01} b)(y) = 0$ is precisely (5.1).

5.2 The first cohomology space $H^1(\mathfrak{a};\mathcal{B})$

An even derivation of $\mathfrak{a}$ with values in the $\mathfrak{a}$-module $\mathcal{B}$ is a parity preserving linear map $c:\mathfrak{a}\to\mathcal{B}$ such that

$$c(a\cdot a') = c(a)\cdot a' + a\cdot c(a')$$

where $\cdot$ stays both for the product in $\mathfrak{a}$ and for the $\mathfrak{a}$-action on $\mathcal{B}$.

Given an even element $b\in\mathcal{B}_0$, one immediately associates a derivation c_b of $\mathfrak{a}$ by the formula

$$c_b(x)=0, \qquad c_b(y)=b\cdot y,$$

where $x\in\mathfrak{a}_0$ and $y\in\mathfrak{a}_1$. We call such derivations *inner*.

Proposition 5.1. *The first cohomology space $H^1(\mathfrak{a};\mathcal{B})$ is the space of even outer derivations of $\mathfrak{a}$ with values in $\mathcal{B}$.*

Proof. Let us first show that the space $Z^1(\mathfrak{a};\mathcal{B})$ of 1-cocycles on $\mathfrak{a}$ with coefficients in an $\mathfrak{a}$-module $\mathcal{B}$ is the space of even derivations. A parity preserving linear map $c:\mathfrak{a}\to\mathcal{B}$ is a sum $c=c_{00}+c_{11}$, where $c_{00}:\mathfrak{a}_0\to\mathcal{B}_0$ and $c_{11}:\mathfrak{a}_1\to\mathcal{B}_1$. The condition $\delta\, c=0$ reads:

$$\delta_{10}\, c_{00}=0, \qquad \delta_{01}\, c_{00}+\delta_{10}\, c_{11}=0, \qquad \delta_{-12}\, c_{00}+\delta_{01}\, c_{11}=0.$$

These equations are equivalent to $c(a\cdot a')=c(a)\cdot a'+a\cdot c(a')$. For instance, one easily obtains:

$$\delta_{01}\, c_{00}(x,y)=c_{00}(x)\cdot y, \qquad \delta_{10}\, c_{11}(x,y)=x\cdot c_{11}(y)-c_{11}(x\cdot y).$$

The second equation is thus equivalent to $c_{00}(x)\cdot y+x\cdot c_{11}(y)-c_{11}(x\cdot y)=0$ and similar for the other two equations.

The space of coboundaries is precisely the space of inner derivations. □

5.3 Second cohomology space $H^2(\mathfrak{a},\mathcal{B})$ and abelian extensions

An exact sequence of homomorphisms of Lie antialgebras

$$0 \longrightarrow \mathcal{B} \longrightarrow \widetilde{\mathfrak{a}} \longrightarrow \mathfrak{a} \longrightarrow 0, \tag{5.2}$$

where $\mathcal{B}$ is a trivial algebra, is called an abelian extension of the Lie antialgebra $\mathfrak{a}$ with coefficients in $\mathcal{B}$.

As a vector space, $\widetilde{\mathfrak{a}}=\mathfrak{a}\oplus\mathcal{B}$ and the subspace $\mathcal{B}$ is obviously an $\mathfrak{a}$-module, the Lie antialgebra structure on $\widetilde{\mathfrak{a}}$ being given by

$$(a,b)\cdot(a',b') = \Big(a\cdot a',\, a\cdot b' + (-1)^{\bar{a'}\bar{b}}\, a'\cdot b + c(a,a')\Big)\,, \tag{5.3}$$

where $a,a'\in\mathfrak{a}$ and $b,b'\in\mathcal{B}$ and where $c:\mathfrak{a}\times\mathfrak{a}\to\mathcal{B}$ is a bilinear map preserving the parity.

An extension (5.2) is called trivial if the Lie antialgebra $\widetilde{\mathfrak{a}}$ is isomorphic to the semi-direct sum $\mathfrak{a} \ltimes \mathcal{B}$, i.e., to the extension (5.3) with $c = 0$, and the isomorphism is of the form

$$(a, b) \mapsto (a, b + L(a)), \tag{5.4}$$

where $L : \mathfrak{a} \to \mathcal{B}$ is a Linear map. If such an isomorphism does not exist, then the extension is called *non-trivial*.

The following statement shows that second cohomology space $H^2(\mathfrak{a}, \mathcal{B})$ classifies non-trivial extensions of $\mathfrak{a}$ with coefficients in $\mathcal{B}$.

Proposition 5.2. *(i) The formula (5.3) defines a Lie antialgebra structure if and only if the map c is a 2-cocycle.*

(ii) Two extensions $\widetilde{\mathfrak{a}}$ and $\widetilde{\mathfrak{a}}'$ of the same Lie antialgebra $\mathfrak{a}$ by a module $\mathcal{B}$ are isomorphic if and only if the corresponding cocycles c_1 and c_2 define the same cohomology class in $H^2(\mathfrak{a}, \mathcal{B})$.

Proof. Part (i). The axioms (4.2-4.5) applied to formula (5.3) read

$$\begin{array}{rcl} x_1 \cdot c(x_2, x_3) - c(x_1 \cdot x_2,\, x_3) + c(x_1,\, x_2 \cdot x_3) - c(x_1, x_2) \cdot x_3 & = & 0, \\[4pt] c(x_1, x_2) \cdot y + c(x_1 \cdot x_2,\, y) - 2\, x_1 \cdot c(x_2, y) - 2\, c(x_1,\, x_2 \cdot y) & = & 0, \\[4pt] c(x,\, y_1 \cdot y_2) + x \cdot c(y_1, y_2) - c(x, y_1) \cdot y_2 & & \\ -c(x \cdot\, y_1,\, y_2) - c(y_1,\, x \cdot y_2) - y_1 \cdot c(x, y_2) & = & 0, \\[4pt] y_1 \cdot c(y_2, y_3) + c(y_1,\, y_2 \cdot y_3) + (\text{cycle}) & = & 0. \end{array} \tag{5.5}$$

Substituting m defined by (4.9), one obtains precisely the condition $\delta\, c = 0$, where δ is as in Theorem 3.1.

Part (ii). Assume that an extension (5.2) is trivial and there exists an isomorphism (5.4). One then readily checks that this is equivalent to $c = \delta\, L$. □

An extension (5.2) of a Lie antialgebra is called a central extension, if the module $\mathcal{B}$ is trivial, i.e., if the $\mathfrak{a}$-action on $\mathcal{B}$ is identically zero. In this case, $\mathcal{B}$ belongs to the center of $\widetilde{\mathfrak{a}}$. Central extensions with $\mathcal{B} = \mathbb{K}$ are classified by the second cohomology with trivial coefficients that we denote by $H^2(\mathfrak{a})$.

6 Two remarkable cocycles

In this section, we give examples of two non-trivial cohomology classes of the infinite-dimensional Lie antialgebras $\mathcal{AK}(1)$ and $\mathcal{M}^1$, see Examples 4.3 and 4.4. These cohomology classes are analogues of the famous Gelfand-Fuchs and Godbillon-Vey classes.

6.1 The Gelfand-Fuchs cocycle

Recall that the classical Gelfand-Fuchs cocycle is a 2-cocycle with trivial coefficients on the Lie algebra of vector fields on the circle. This cocycle defines the unique central extension of the Lie algebra of vector fields called the Virasoro algebra. This algebra plays an important rôle in geometry and mathematical physics, see [2, 6] and references therein. In the $\mathbb{Z}_2$-graded case, the graded version of the Gelfand-Fuchs cocycle defined the Neveu-Schwarz and Ramond conformal superalgebras.

In this section, we recall the definition of the Gelfand-Fuchs cocycle on the conformal Lie superalgebra $\mathcal{K}(1)$ and rewrite it in a form of a 1-cocycle with coefficients in the dual space $\mathcal{K}(1)^*$. This "dualized" version of the Gelfand-Fuchs cocycle will be of a particular interest for our purpose.

The conformal Lie superalgebra, $\mathcal{K}(1)$, is spanned by the basis

$$\left\{\ell_n,\ n\in\mathbb{Z};\qquad \xi_i,\ i\in\mathbb{Z}+\tfrac{1}{2}\right\}$$

with the following commutation relations

$$\begin{array}{rcl}
[\ell_n,\ell_m] &=& (m-n)\,\ell_{n+m},\\
[\ell_n,\xi_i] &=& \left(i-\frac{n}{2}\right)\xi_{n+i},\\
[\xi_i,\xi_j] &=& 2\,\ell_{i+j}.
\end{array}\tag{6.1}$$

The Lie subalgebra generated by ℓ_i is the Lie algebra of (polynomial) vector fields on S^1.

The second cohomology space of $\mathcal{K}(1)$ with trivial coefficients, $H^2(\mathcal{K}(1))$, is one-dimensional and is generated by the 2-cocycle

$$\begin{array}{rcl}
c_{GF}(\ell_n,\ell_m) &=& \left(n^3-n\right)\delta_{n+m,0}\\
c_{GF}(\xi_i,\xi_j) &=& \left(-4\,i^2+1\right)\delta_{i+j,0}\\
c_{GF}(\ell_n,\xi_i) &=& 0,
\end{array}\tag{6.2}$$

that we call the Gelfand-Fuchs cocycle. It defines a (unique) central extension of $\mathcal{K}(1)$. The even part of c_{GF} defines the Virasoro algebra.

Remark 6.1. The conformal Lie superalgebra $\mathcal{K}(1)$ contains a subalgebra spanned by the elements

$$\left\{\ell_{-1},\,\ell_0,\,\ell_1;\,\xi_{-\frac{1}{2}},\,\xi_{\frac{1}{2}}\right\},$$

isomorphic to the Lie superalgebra $\mathrm{osp}(1|2)$. The cocycle (6.2) can be characterized as *the unique* 2-cocycle on $\mathcal{K}(1)$ vanishing on $\mathrm{osp}(1|2)$.

6.2 The dual Gelfand-Fuchs cocycle

It is a general fact that a 2-cocycle on a Lie superalgebra with trivial coefficients corresponds to a 1-cocycle with coefficients in the dual space.

Definition 6.2. Given a Lie algebra $\mathfrak{g}$ and a 2-cocycle $c : \mathfrak{g} \wedge \mathfrak{g} \to \mathbb{K}$, the formula

$$\langle C(X),\, Y\rangle := c(X, Y),$$

for all $X, Y \in \mathfrak{g}$, defines a 1-cocycle $C : \mathfrak{g} \to \mathfrak{g}^*$ with values in the dual space $\mathfrak{g}^*$.

The "dual Gelfand-Fuchs cocycle" then reads:

$$C_{GF}(\ell_n) = (n^3 - n)\, \ell^*_{-n}, \qquad C_{GF}(\xi_i) = (-4\, i^2 + 1)\, \xi^*_{-i}, \tag{6.3}$$

where $\{\ell^*_n,\, \xi^*_i\}$ is the dual basis, i.e.,

$$\langle \ell^*_n, \ell_m\rangle = \delta_{n,m}, \qquad \langle \xi^*_i, \xi_j\rangle = \delta_{i,j}, \qquad \langle \ell^*_n, \xi_j\rangle = \langle \xi^*_i, \ell_m\rangle = 0.$$

This is a non-trivial 1-cocycle on $\mathcal{K}(1)$ with values in $\mathcal{K}(1)^*$.

6.3 A non-trivial cocycle on $\mathcal{AK}(1)$

The conformal Lie antialgebra $\mathcal{AK}(1)$ has no non-trivial central extension, see [17], therefore, there is no analog of the classical Gelfand-Fuchs cocycle (6.2). However, there exists an analog of the dual cocycle (6.3). We denote by $\{\varepsilon^*_n, a^*_i\}$ the of $\mathcal{AK}(1)^*$ dual to $\{\varepsilon_n, a_i\}$.

Theorem 2. *The linear map $\gamma : \mathcal{AK}(1) \to \mathcal{AK}(1)^*$ given by*

$$\gamma(\varepsilon_n) = -n\, \varepsilon^*_{-n}, \qquad \gamma(a_i) = \left(i^2 - \tfrac{1}{4}\right) a^*_{-i}, \tag{6.4}$$

is a non-trivial 1-cocycle on $\mathcal{AK}(1)$.

Proof. Let us check that the map (6.4) is, indeed, a 1-cocycle. The action of $\mathcal{AK}(1)$ on $\mathcal{AK}(1)^*$ can be easily calculated according to the formula (4.10). The result is as follows:

$$\begin{array}{rcl} \varepsilon_n \cdot \varepsilon^*_m &=& \varepsilon^*_{m-n}, \\ \varepsilon_n \cdot a^*_i &=& \frac{1}{2}\, a^*_{i-n} \\ a_i \cdot \varepsilon^*_m &=& \left(\frac{m}{2} - i\right) a^*_{m-i}, \\ a_i \cdot a^*_i &=& -\frac{1}{2}\, \varepsilon^*_{i-j}. \end{array}$$

Consider the following Ansatz:

$$\gamma(\varepsilon_n) = t(n)\, \varepsilon^*_{-n}, \qquad \gamma(a_i) = s(i)\, a^*_{-i}.$$

The 1-cocycle condition then leads to: $\gamma(\varepsilon_n \cdot \varepsilon_m) = \varepsilon_n \cdot \gamma(e_m) + \varepsilon_m \cdot \gamma(e_n)$, so that

$$t(n+m) = t(n) + t(m).$$

The next two conditions are: $\frac{1}{2}\gamma(\varepsilon_n \cdot a_i) = \varepsilon_n \cdot \gamma(a_i) + \gamma(\varepsilon_n) \cdot a_i$ and $\gamma(a_i \cdot a_j) = a_i \cdot \gamma(a_j) - a_j \gamma(a_i)$. They give the same equation:

$$s(i) - s(j) = i^2 - j^2.$$

The map (6.4) obviously satisfies both equations, so that this is, indeed, a 1-cocycle.

We have already seen in Section 5.2 that every coboundary vanishes on the even part of a Lie antialgebra. It follows that the 1-cocycle (6.4) is non-trivial. □

We conjecture that the space $H^1(\mathcal{AK}(1); \mathcal{AK}(1)^*)$ is one-dimensional and, thus generated by the 1-cocycle (6.4). We think that the cocycle (6.4) is characterized by the property that it vanishes on the subalgebra $K_3 \subset \mathcal{AK}(1)$ spanned by $\varepsilon_0, a_{-\frac{1}{2}}, a_{\frac{1}{2}}$.

Remark 6.3. The 1-cocycle (6.4) is skew-symmetric on the even part $\mathcal{AK}(1)_0$ and symmetric on the odd part $\mathcal{AK}(1)_1$. This is the reason why it cannot be understood as a 2-cocycle on $\mathcal{AK}(1)$ with trivial coefficients. This phenomenon is quite general for the Lie antialgebras: approximately a half of the statements that hold in the Lie algebra setting remains true.

6.4 A cocycle on $\mathcal{M}^1$ and the dual Godbillon-Vey cocycle

The Godbillon-Vey cocycle is a 3-cocycle on the Lie algebra of polynomial vector fields on the line. The simplest way to define this cocycle is as follows. Consider the Lie algebra W_1 with basis $\{\ell_n,\ n \geq -1\}$ and the relations $[\ell_n, \ell_m] = (m-n)\,\ell_{n+m}$. Then

$$c_{GV} = \ell^*_{-1} \wedge \ell^*_0 \wedge \ell^*_1$$

is a non-trivial 3-cocycle on W_1 (with trivial coefficients). Similarly to Section 6.2, one can "dualize" the above 3-cocycle and obtain a 2-cocycle with coefficients in W_1^*.

Let us consider the Lie antialgebra $\mathcal{M}^1$, see Example 4.4.

Theorem 3. *The bilinear map $\eta : \mathcal{M}^1 \otimes \mathcal{M}^1 \to (\mathcal{M}^1)^*$ given by*

$$\eta = a^*_{-\frac{1}{2}} \wedge a^*_{\frac{1}{2}} \otimes \varepsilon_0, \tag{6.5}$$

is a non-trivial 2-cocycle on $\mathcal{M}^1$.

Proof. First, one can easily check that

$$\eta = \lambda\, a^*_{-\frac{1}{2}} \wedge a^*_{\frac{1}{2}} \otimes \varepsilon_0 + \mu \left(\varepsilon^*_0 \wedge \varepsilon^*_0 \otimes \varepsilon_0 - \tfrac{1}{2}\, \varepsilon^*_0 \wedge a^*_{-\frac{1}{2}} \otimes a_{\frac{1}{2}} + \tfrac{1}{2}\, \varepsilon^*_0 \wedge a^*_{\frac{1}{2}} \otimes a_{-\frac{1}{2}} \right)$$

is a 2-cocycle for all λ, μ.

Second, for any coboundary $\delta\zeta$, where $\zeta : \mathcal{M}^1 \to (\mathcal{M}^1)^*$ is a linear map, one has

$$(\delta\zeta)(\varepsilon_0, \varepsilon_0) = 2\,(\delta\zeta)(a_{-\frac{1}{2}}, a_{\frac{1}{2}}),$$

so that $\lambda = \frac{1}{2}\mu$ if η is a coboundary. $\square$

We conjecture that the space $H^2(\mathcal{M}^1; (\mathcal{M}^1)^*)$ is one-dimensional. We also hope that the cocycle (6.5) has a topological meaning and can be associated to a characteristic class, similarly to the classical Godbillon-Vey cocycle, see [2].

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Pierre Lecomte,
Département de mathématiques,
Université de Liège,
Grande Traverse 12 (B37),
B-4000 Liège,
Belgique;
plecomte@ulg.ac.be

Valentin Ovsienko,
CNRS,
Institut Camille Jordan,
Université Lyon 1,
Villeurbanne Cedex, F-69622,
France;
ovsienko@math.univ-lyon1.fr