From constructive field theory to fractional stochastic calculus. (I) An introduction: rough path theory and perturbative heuristics.

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Let $B = (B_1(t), \ldots, B_d(t))$ be a d-dimensional fractional Brownian motion with Hurst index $\alpha \le 1/4$, or more generally a Gaussian process whose paths have the same local regularity. Defining properly iterated integrals of B is a difficult task because of the low Hölder regularity index of its paths. Yet rough path theory shows it is the key to the construction of a stochastic calculus with respect to B, or to solving differential equations driven by B.

We intend to show in a forthcoming series of papers how to desingularize iterated integrals by a weak singular non-Gaussian perturbation of the Gaussian measure defined by a limit in law procedure. Convergence is proved by using "standard" tools of constructive field theory, in particular cluster expansions and renormalization. These powerful tools allow optimal estimates of the moments and call for an extension of the Gaussian tools such as for instance the Malliavin calculus.

This first paper aims to be both a presentation of the basics of rough path theory to physicists, and of perturbative field theory to probabilists; it is only heuristic, in particular because the desingularization of iterated integrals is really a *non-perturbative* effect. It is also meant to be a general motivating introduction to the subject, with some insights into quantum field theory and stochastic calculus. The interested reader should read in a second time the companion article [48] (or a preliminary version [47]) for the constructive proofs.

Keywords: fractional Brownian motion, stochastic integrals, rough paths, constructive field theory, Feynman diagrams, renormalization, cluster expansion.

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Contents

0 Introduction

A major achievement of the probabilistic school since the middle of the 20th century is the study of diffusion equations, in connection with Brownian motion or more generally Markov processes – and also with partial differential equations, through the Feynman-Kac formula – with many applications in physics and chemistry [64]. One of the main tools is stochastic calculus with respect to semi-martingales M. An adapted integral such as $\int_s^t X(u)dM(u)$ may be understood as a limit in some sense to be defined. Classically one uses piecewise linear interpolations, $\sum_{s\le t_1<\ldots<t_N\le t} X(t_i)(M(t_{i+1})-M(t_i))$ or $\sum_{s\le t_1<\ldots<t_N\le t} \frac{X(t_i)+X(t_{i+1})}{2}(M(t_{i+1})-M(t_i))$; these approximations define in the limit $N\to\infty$ the Itô, resp. Stratonovich integral. The latter one is actually obtained e.g. if $M=W$ is Brownian motion and $X(t)=f(W_t)$ with f smooth as the limit $\lim_{\varepsilon\to 0}\int_s^t f(W_\varepsilon(u))dW_\varepsilon(u)$ for any smooth approximation $(W_\varepsilon)_{\varepsilon>0}$ of W converging a.s. to W (see [69], or [35] p. 169). The Stratonovich integral $\int_s^t X(u)d^{Strato}M(u)$ has an advantage over the Itô integral in that it agrees with the fundamental theorem of calculus, namely, $F(M(t))=F(M(s))+\int_s^t F'(M(u))d^{Strato}M(u)$.

The semi-martingale approach fails altogether when considering stochastic processes with lower regularity. Brownian motion, and more generally semi-martingales (up to time reparametrization), are $(1/2)^-$-Hölder, i.e. α-Hölder for any $\alpha<1/2$ [1]. Processes with α-Hölder paths, where $\alpha\ll 1/2$,

[1] Recall that a continuous path $X:[0,T]\to\mathbb{R}$ is α-Hölder, $\alpha\in(0,1)$, if $\sup_{s,t\in[0,T]}\frac{|X_t-X_s|}{|t-s|^\alpha}<\infty$.

are maybe less common in nature but still deserve interest. Among these, the family of multifractional Gaussian processes is perhaps the most widely studied [52], but one may also cite diffusions on fractals [32], sub- or superdiffusions in porous media [26, 37] and the fascinating multi-fractal random measures/walks in connection with turbulence and two-dimensional Liouville quantum gravity [9, 13]. Many models in hydrodynamics take as input a space-time noise which is often chosen colored in space [34]. In this respect, let us mention in particular the Kraichnan model for passive advection of scalars, for which anomalous correlation exponents [36, 14, 6] may be expanded in α for $\alpha \to 0$.

We concentrate in this article on *multiscale Gaussian processes* (the terminology is ours) with *scaling dimension* or more or less equivalently *Hölder regularity* $\alpha \in (0, 1/2)$, the best-known example of which being *fractional Brownian motion* (fBm for short) with Hurst index α, $B^{\alpha}(t)$ or simply $B(t)$ [2]. We consider more precisely a *two-dimensional fBm*, $B(t) = (B_1(t), B_2(t))$, with independent, identically distributed components [3]. The covariance kernel $\mathbb{E}B_i(s)B_j(t) = \frac{1}{2}\delta_{i,j}(|s|^{2\alpha} + |t|^{2\alpha} - |t-s|^{2\alpha})$ is that of an *integrated colored noise* in the physical terminology [4]. It is a process with *long-range, negative correlations*, which is quite unusual from a statistical physics point of view; but the emphasis here is on the short-distance (or ultra-violet) behaviour, not on the long-distance one.

The simplest non-trivial stochastic integral is then

$$\mathcal{A}(s,t) := \int_s^t dB_1(t_1) \int_s^{t_1} dB_2(t_2) = \int_s^t (B_2(u) - B_2(s)) dB_1(u), \qquad (0.1)$$

a twice iterated integral, where $B = (B_1(t), B_2(t))$ is a two-component fBm with independent, identically distributed components. Since

$$\int_s^t dB_1(t_1) \int_s^{t_1} dB_2(t_2) + \int_s^t dB_2(t_2) \int_s^{t_2} dB_1(t_1) = (B_1(t) - B_1(s))(B_2(t) - B_2(s)),$$

one is mainly interested in the antisymmetrized quantity (measuring a *signed*

[2] It is (up to a constant) the unique self-similar Gaussian process with stationary increments. The last property implies that its derivative is a (distribution-valued) *stationary field*.

[3] The one-dimensional case is very different and much simpler, and has been treated in [27].

[4] In other words (informally at least) $\mathbb{E}B_i'(s)B_j'(t) \sim -c_\alpha |t-s|^{2\alpha-2}$ instead of $\delta(t-s)$.

area, as follows from the Green-Riemann formula),

$$\begin{aligned}\mathcal{LA}(s,t) &:= \int_s^t dB_1(t_1)\int_s^{t_1} dB_2(t_2) - \int_s^t dB_2(t_2)\int_s^{t_2} dB_1(t_1)\\ &= \int_s^t (B_2(u)-B_2(s))dB_1(u) - (B_1(u)-B_1(s))dB_2(u),\end{aligned} \tag{0.2}$$

called *Lévy area*. The corresponding Stratonovich integral, obtained as a limit either by linear interpolation or by more refined Gaussian approximations [11, 51, 58, 59], has been shown to *diverge* as soon as $\alpha \le 1/4$.

This seemingly no-go theorem, although clear and derived by straightforward computations that we reproduce in short in section 1, appears to be a puzzle when put in front of the results of *rough path theory* [43, 44, 28, 39, 40, 20]. The essential idea conveyed by this theory – we shall make this precise in section 2 – is that a path $\Gamma : \mathbb{R} \to \mathbb{R}^d$ with Hölder regularity index $\alpha \in (0,1)$ must be seen as the *projection* onto the d first components of some "essentially arbitrary" *rough path over* Γ, denoted by

$$\mathbf{\Gamma} : \mathbb{R}^2 \ni (s,t) \mapsto \mathbf{\Gamma}_{ts} := (\mathbf{\Gamma}^1_{ts}, \ldots, \mathbf{\Gamma}^N_{ts}) \in \mathbb{R}^d \times \mathbb{R}^{d^2} \times \ldots \times \mathbb{R}^{d^N}, \tag{0.3}$$

$N = \lfloor 1/\alpha \rfloor$ [5], which may be interpreted as iterated integrals of Γ in a limiting sense, namely, $\lim_{\varepsilon\to 0}\int_s^t d\Gamma^{\varepsilon}_{i_1}(t_1) = \Gamma_{i_1}(t) - \Gamma_{i_1}(s)$ for the d first components and

$$\lim_{\varepsilon\to 0}\int_s^t d\Gamma^{\varepsilon}_{i_1}(t_1)\int_s^{t_1} d\Gamma^{\varepsilon}_{i_2}(t_2), \ldots, \lim_{\varepsilon\to 0}\int_s^t d\Gamma^{\varepsilon}_{i_1}(t_1)\ldots\int_s^{t_{N-1}} d\Gamma^{\varepsilon}_{i_n}(t_n) \tag{0.4}$$

for the remaining ones, for some smooth family of approximations $(\Gamma^{\varepsilon})_{\varepsilon>0}$ of Γ. The limit must be understood in a Hölder norm sense, as explained in section 2. In other words, there exist infinitely many different families of approximations of B leading to as many different definitions of its iterated integrals! Alas, Gaussian approximations are unfortunately seemingly unable to produce such a definition for fBm with Hurst index $\alpha \le 1/4$.

Our project in this series of papers is to define an explicit rough path over fBm with arbitrary Hurst index, or more generally *multiscale Gaussian fields* (see a companion article [48] and [47] for a preliminary version) by an explicit, probabilistically meaningful limiting procedure, thus solving at

[5] where $\lfloor\ .\ \rfloor$ stands for the integer part of its argument.

last the problem of constructing a full-fledged, Stratonovich-like integration with respect to fBm.

Let us explain our strategy for $1/6 < \alpha < 1/4$. Roughly speaking, our rough path is obtained by making $B = (B(1), B(2))$ interact through a weak but singular quartic, non-local interaction, which plays the rôle of a squared kinetic momentum, or *bending energy*, and makes its Lévy area – and at the same time the iterated integrals of higher order – *finite*. Following the common use of quantum field theory, this is implemented by multiplying (probabilists would say: *penalizing*) the Gaussian measure by the exponential weight $e^{-\frac{1}{2}c'_\alpha \int\int \mathcal{L}_{int}(\phi_1,\phi_2)(t_1,t_2)|t_1-t_2|^{-4\alpha} dt_1 dt_2}$ [6], with

$$\mathcal{L}_{int}(\phi_1,\phi_2)(t_1,t_2) = \lambda^2 \left\{ (\partial\mathcal{A}^+)(t_1)(\partial\mathcal{A}^+)(t_2) + (\partial\mathcal{A}^-)(t_1)(\partial\mathcal{A}^-)(t_2) \right\}, \tag{0.5}$$

where: λ (the coupling parameter) is a small, positive constant; ϕ_1, ϕ_2 are the (infra-red divergent) stationary fields associated to B_1, B_2, with covariance kernel as in eq. (1.4), and similarly, $\mathcal{A}^\pm$ are stationary *left- and right-turning fields*, built out of ϕ_1, ϕ_2 and representing the singular part of the Lévy area (see section 1 for details). As usual in quantum field theory, one considers first the truncated measure obtained by an "ultra-violet cut-off" and on a finite "volume" (or finite horizon, in the probabilistic terminology) $V = [-T, T]$, i.e. one multiplies the Fourier transforms of the fields ϕ_1, ϕ_2 by some cut-off function with compact support in $[-M^\rho, M^\rho]$ (for some fixed constant $M > 1$) and integrates over V; see Definition 3.1 for the precise procedure. Then $\partial\mathcal{A}^\pm$ are replaced by the *truncated quantities* $(\partial\mathcal{A}^\pm)^{\to\rho}$ built out of the *truncated fields* $\phi^{\to\rho}$. The truncated interacting Lagrangian reads

$$\begin{aligned} &\frac{1}{2}c'_\alpha \int\int_{V\times V} |t_1-t_2|^{-4\alpha} \mathcal{L}^{\to\rho}_{int}(\phi_1,\phi_2)(t_1,t_2) dt_1 dt_2 + \int_V \mathcal{L}^{\to\rho}_{bdry} \\ &\quad := \frac{1}{2}c'_\alpha \lambda^2 \int\int_{V\times V} |t_1-t_2|^{-4\alpha} \left\{ (\partial\mathcal{A}^+)^{\to\rho}(t_1)(\partial\mathcal{A}^+)^{\to\rho}(t_2) \right. \\ &\qquad \left. + (\partial\mathcal{A}^-)^{\to\rho}(t_1)(\partial\mathcal{A}^-)^{\to\rho}(t_2) \right\} dt_1 dt_2 + \int_V \mathcal{L}^{\to\rho}_{bdry}, \end{aligned} \tag{0.6}$$

where $\mathcal{L}^{\to\rho}_{bdry}$ is some singular "Fourier boundary term" multiplied by an evanescent factor $M^{-\kappa\rho}$ ($\kappa > 0$), which cures unwanted difficulties due to

[6] The unessential constant c'_α is fixed e.g. by demanding that the Fourier transform of the kernel $c'_\alpha|t_1-t_2|^{-4\alpha}$ is the function $|\xi|^{4\alpha-1}$.

the ultra-violet cut-off [7]. When ρ and V are finite, the underlying Gaussian fields are smooth, which ensures the existence of the penalized measure. The assertion is that *the penalized measures converge weakly when* $\rho, |V| \to \infty$ *to some well-defined, unique measure*, while the *truncated iterated integrals* themselves converge in law to a *rough path over* B.

Note that the statistical weight is maximal when $\partial\mathcal{A}^+ = \partial\mathcal{A}^- = 0$, i.e. for sample paths which are "essentially" straight lines. Another way to motivate this interaction (following an image due to A. Lejay) is to understand that the divergence of the Lévy area is due to the accumulation in a small region of space of small loops [39]; the statistical weight is unfavorable to such an accumulation. On the other hand, the law of the quantities in the first-order Gaussian chaos, characterized by the n-point functions

$$\begin{aligned}&\langle B_{i_1}(x_1)\dots B_{i_n}(x_n)\rangle_\lambda\\ &\quad = \frac{1}{Z}\mathbb{E}\left[B_{i_1}(x_1)\dots B_{i_n}(x_n)e^{-\frac{1}{2}c'_\alpha \int\int \mathcal{L}_{int}(\phi_1,\phi_2)(t_1,t_2)|t_1-t_2|^{-4\alpha}dt_1dt_2}\right],\end{aligned} \tag{0.7}$$

$i_1,\dots,i_n = 1,2$, where

$$Z := \mathbb{E}\left[e^{-\frac{1}{2}c'_\alpha \int\int \mathcal{L}_{int}(\phi_1,\phi_2)(t_1,t_2)|t_1-t_2|^{-4\alpha}dt_1dt_2}\right] \tag{0.8}$$

is a normalization constant playing the rôle of a *partition function*, is insensitive to the interaction [8]. Thus we have built a rough path *over fBm*. This conveys the idea that the paths have been straightened by removing *in average* small bubbles of scale $M^{-\rho}$. In doing so, the paths of the limiting process when $\rho \to \infty$ are indistinguishable from those of B, *but* higher-order integrals have been corrected so as to become finite.

Starting from the above field-theoretic description, the proof of finiteness and Hölder regularity of the Lévy area for $\lambda > 0$ small enough follows, despite some specific features, the broad scheme of constructive field theory, see e.g. the monographies [1, 49, 56]. Constructive field theory is a program originally advocated in the sixties by A. S. Wightman [66], the aim of which was to give explicit examples of field theories with a non-trivial interaction ; see Glimm and Jaffe's book [23] for an introduction and references therein

[7] The exact form of $\mathcal{L}^{\to\rho}_{bdry}$ requires detailed constructive explanations and will not be required here. It is to be found in the companion article [48].

[8] In the two preceding equations, $\mathbb{E}\left[\cdot\, e^{-\frac{1}{2}c'_\alpha \int\int \mathcal{L}_{int}(\phi_1,\phi_2)(t_1,t_2)|t_1-t_2|^{-4\alpha}dt_1dt_2}\right]$ stands for the limit of $\mathbb{E}\left[\cdot\, e^{-\frac{1}{2}c'_\alpha \int\int_{V\times V} \mathcal{L}^{\to\rho}_{int}(\phi_1,\phi_2)(t_1,t_2)|t_1-t_2|^{-4\alpha}dt_1dt_2+\int_V \mathcal{L}^{\to\rho}_{bdry}}\right]$ when $\rho, |V| \to \infty$ as we explained above.

for an extensive bibliography. Let us give a short guide to the history of the subject.

The first contribution was made in 1965 by E. Nelson who introduced a scale analysis [50] to control the divergence of a model whose only divergence comes from Wick ordering. J. Glimm and A. Jaffe introduced the phase space analysis [24] for models having a finite number of divergent graphs. The cluster expansion was devised by J. Glimm, A. Jaffe and T. Spencer [25] to control infinite volume limits.

The Roman team [5] realized that the above phase space analysis was in some sense a continuous space version of the block-spin expansion, first written by Kadanoff for the Ising model, and then made into a major tool both in high-energy and statistical physics by K. G. Wilson through the introduction of the concept of *renormalization group* [67, 68]. The multi-scale expansion was devised in the eighties to provide a rigorous version of Wilson's renormalization group e.g. including the flow of the effective parameters: see [22] for the block-spin approach, and [17] for the continuous space *multi-scale cluster expansion*.

For some fermionic theories a simpler version of these constructions is available, due to the fact that (contrary to the bosonic case) the series expansion in terms of the effective coupling constants is *convergent* [42, 4]. The multi-scale cluster expansion has also allowed to study models with a singularity around a surface, like the so-called *jellium model* of interacting, non-relativistic fermions [18, 12], modelling the generation of Cooper pairs, in connection with the famous BCS (Bardeen-Cooper-Schrieffer) theory of supraconductivity [19].

In this work we use the multi-scale cluster expansion developed in [17] more than twenty years ago which seems to us the most appropriate for these probabilistic models; it reduces to the minimum the use of abstract combinatorial identities and algebra, to the benefit of a very intuitive and visual (though sometimes heavy) tree expansion.

The main theorem may be stated as follows. As a rule, we denote in this article by $\mathbb{E}[...]$ the Gaussian expectation and by $\langle ... \rangle_{\lambda,V,\rho}$ the expectation with respect to the λ-weighted interaction measure with scale ρ ultraviolet cut-off restricted to a compact interval V, so that in particular $\mathbb{E}[...] = \langle ... \rangle_{0,\infty}$.

Theorem 0.1 *Assume* $\alpha \in (\frac{1}{6}, \frac{1}{4})$. *Consider for* $\lambda > 0$ *small enough the* *family of probability measures (also called:* $(\phi, \partial\phi, \sigma)$-model*)*

$$\mathbb{P}_{\lambda,V,\rho}(\phi_1, \phi_2) = e^{-\frac{1}{2}c'_\alpha \int\int dt_1 dt_2 |t_1-t_2|^{-4\alpha} \mathcal{L}_{int}^{\to\rho}(\phi_1,\phi_2)(t_1,t_2) - \int \mathcal{L}_{bdry}^{\to\rho}} d\mu^{\to\rho}(\phi_1) d\mu^{\to\rho}(\phi_2), \tag{0.9}$$

where $d\mu^{\to\rho}(\phi_i) = d\mu(\phi_i^{\to\rho})$ *is a Gaussian measure obtained by an ultraviolet cut-off at Fourier momentum* $|\xi| \approx M^\rho$ *(*$M > 1$*), see Definition 3.1. Then* $(\mathbb{P}_{\lambda,V,\rho})_{V,\rho}$ *converges in law when* $|V|, \rho \to \infty$ *to some measure* $\mathbb{P}_\lambda$*, and the associated iterated integrals*

$$\int_s^t d\phi_{i_1}^{\to\rho}(t_1) \int_s^{t_1} d\phi_{i_2}^{\to\rho}(t_2), \ldots, \int_s^t d\phi_{i_1}^{\to\rho}(t_1) \int_s^{t_1} d\phi_{i_2}^{\to\rho}(t_2) \ldots \int_s^{t_{n-1}} d\phi_{i_n}^{\to\rho}(t_n), \ldots$$

converge in law to a rough path over B*.*

The result is not difficult to understand *heuristically*, at least for quantum field theory experts, if one resorts to the non-rigorous perturbation theory (see sections 3 and 4). First, by a Hubbard-Stratonovich transformation (a functional Fourier transform), one replaces the non-local interaction $\mathcal{L}(\phi_1, \phi_2)(t_1, t_2)|t_1 - t_2|^{-4\alpha}$ with a local interaction $\mathcal{L}(\phi_1, \phi_2, \sigma)(t)$ depending on a two-component exchange particle field $\sigma = (\sigma_+(t), \sigma_-(t))$. Then a *Schwinger-Dyson identity* (a functional integration by parts) relates the moments of $\mathcal{A}$ to those of σ. Simple power-counting arguments show that a connected $2n$-point function of σ alone is superficially divergent if and only if $1 - 4n\alpha \geq 0$. Thus, restricting to $\alpha > 1/8$, one only needs to renormalize the two-point function. Since the renormalized propagator of σ is screened by a positive, infinite mass term, the theory is free once one has integrated out the σ-field, hence one retrieves the underlying Gaussian theory (ϕ_1, ϕ_2). The Schwinger-Dyson identity then shows that the two-point functions of $\mathcal{A}$ have been made finite. Finally, simple arguments (not developed here) yield the convergence of higher-order iterated integrals in the interacting theory provided $\alpha > 1/6$.

Whereas these heuristic arguments are not difficult to follow in principle, they do not constitute at all a proof. Theorem 0.1 is proved in the companion article [48] by following – as explained above – the general scheme of constructive field theory. Although the constructive method is really a multi-scale refinement of the previous arguments, explaining it precisely is actually a formidable task, which is in general very much model-dependent, whereas perturbative renormalization always follows more or less the same lines; briefly said, the difference lies in the difference between a formal power series expansion and an analytic proof of convergence for a given quantity. This task we perform at long length and in great generality in the companion article, with the view of making constructive arguments into classical mathematical tools which probabilists may eventually reemploy.

Here is an outline of the article.

We begin in Section 1 by recalling classical arguments (due to the second author) explaining the divergence of the Lévy area for $\alpha \leq 1/4$, which is the starting point for all the story [58]; Fourier normal ordering [62, 61] – an indispensable tool for the sequel – is introduced there. Section 2 is a brief introduction into rough path theory, mainly for non-experts. Subsections 2.1 and 2.2 are standard and may be skipped by experts, whereas subsection 2.3 – a brief summary of the previous contributions of the second author to the subject – gives the context in which this series of papers arose.

The heart of the article is Section 3 and Section 4. Our problem is recast into a quantum field theoretic language in section 3; we take the opportunity to explain the basis of quantum field theory and renormalization at the same time. The interaction term is introduced at this point, where it comes out naturally. Finally, section 4 is dedicated to a heuristic perturbative "proof" of the convergence of the Lévy area of the interacting process, and serves also in some sense as an introduction to the companion paper [48].

1 A Fourier analysis of the Lévy area

The quantity we want to define in the case of fractional Brownian motion is the following.

Definition 1.1 (Lévy area) *The Lévy area of a two-dimensional path $\Gamma : \mathbb{R} \to \mathbb{R}^2$ between s and t is the area between the straight line connecting $(\Gamma_1(s), \Gamma_2(s))$ to $(\Gamma_1(t), \Gamma_2(t))$ and the curve $\{(\Gamma_1(u), \Gamma_2(u)); s \leq u \leq t\}$. It is given by the following antisymmetric quantity,*

$$\mathcal{LA}_\Gamma(s,t) := \int_s^t d\Gamma_1(t_1) \int_s^{t_1} d\Gamma_2(t_2) - \int_s^t d\Gamma_2(t_2) \int_s^{t_2} d\Gamma_1(t_1). \tag{1.1}$$

The purpose of this section is to show by using Fourier analysis why the Lévy area of fBm diverges when $\alpha \leq 1/4$. This is hopefully understandable to physicists, and also profitable to probabilists who are aware of other proofs of this fact, originally proved in [11], because Fourier analysis is essential in the analysis of Feynman graphs which shall be needed in section 4. We follow here the computations made in [61] or [60].

Definition 1.2 (Harmonizable representation of fBm) *Let $W(\xi),\xi\in\mathbb{R}$ be a complex Brownian motion* [9] *such that $W(-\xi)=-\overline{W(\xi)}$, and*

$$B_t := (2\pi c_\alpha)^{-\frac{1}{2}}\int_{-\infty}^{+\infty}\frac{e^{\mathrm{i}t\xi}-1}{\mathrm{i}\xi}|\xi|^{\frac{1}{2}-\alpha}dW(\xi),\quad t\in\mathbb{R}. \tag{1.2}$$

The field $B_t, t\in\mathbb{R}$ is called *fractional Brownian motion* [10] . Its paths are almost surely α^- Hölder, i.e. $(\alpha-\varepsilon)$-Hölder for every $\varepsilon>0$. It has dependent but identically distributed (or in other words, stationary) increments B_t-B_s. In order to gain translation invariance, we shall rather use the closely related *stationary process*

$$\phi(t) := \int_{-\infty}^{+\infty}\frac{e^{\mathrm{i}t\xi}}{\mathrm{i}\xi}|\xi|^{\frac{1}{2}-\alpha}dW(\xi),\quad t\in\mathbb{R} \tag{1.3}$$

– with covariance

$$\langle\phi(x)\phi(y)\rangle = \int e^{\mathrm{i}\xi(x-y)}\frac{1}{|\xi|^{1+2\alpha}}d\xi \tag{1.4}$$

– which is *infrared divergent*, i.e. *divergent around* $\xi=0$. However, the increments $\phi(t)-\phi(s)=B_t-B_s$ are well-defined for any $(s,t)\in\mathbb{R}^2$.

In order to understand the analytic properties of the Lévy area of fBm, we shall resort to a Fourier transform. One obtains, using the harmonizable representation of fBm,

$$\begin{aligned}\mathcal{A}(s,t) &:= \int_s^t dB_1(t_1)\int_s^{t_1}dB_2(t_2)\\ &= \frac{1}{2\pi c_\alpha}\int\frac{dW_1(\xi_1)dW_2(\xi_2)}{|\xi_1|^{\alpha-1/2}|\xi_2|^{\alpha-1/2}}\int_s^t dt_1\int_s^{t_1}dt_2\ \cdot\ e^{\mathrm{i}(t_1\xi_1+t_2\xi_2)}\end{aligned} \tag{1.5}$$

The Lévy area $\mathcal{LA}(s,t):=\mathcal{LA}_B(s,t)$ is obtained from this twice iterated integral by antisymmetrization. Note that $\mathcal{LA}(s,t)$ is homogeneous of degree 2α in $|t-s|$ since $B(ct)-B(cs)$, $c>0$ has same law as $c^\alpha(B(t)-B(s))$ by self-similarity.

Expanding the right-hand side yields an expression which is not homogeneous in ξ. Hence it is preferable to define instead the following stationary

[9] Formally, $\langle W'(\xi_1)W'(\xi_2)\rangle=0$ and $\langle W'(\xi_1)\overline{W'(\xi_2)}\rangle=\delta(\xi_1-\xi_2)$ if $\xi_1,\xi_2>0$.

[10] The constant c_α is conventionally chosen so that $\mathbb{E}(B_t-B_s)^2=|t-s|^{2\alpha}$.

quantity called *skeleton integral*, which depends only on *one* variable,

$$\begin{aligned}
\mathcal{A}(t) &:= \int^t dB_1(t_1) \int^{t_1} dB_2(t_2) \\
&= \frac{1}{2\pi c_\alpha} \int \frac{dW_1(\xi_1) dW_2(\xi_2)}{|\xi_1|^{\alpha-1/2} |\xi_2|^{\alpha-1/2}} \int^t dt_1 \int^{t_1} dt_2 \ \cdot \ e^{\mathrm{i}(t_1\xi_1 + t_2\xi_2)} \\
&= \frac{1}{2\pi c_\alpha} \int \frac{dW_1(\xi_1) dW_2(\xi_2)}{|\xi_1|^{\alpha-1/2} |\xi_2|^{\alpha-1/2}} \cdot \frac{e^{\mathrm{i}t(\xi_1+\xi_2)}}{[\mathrm{i}(\xi_1+\xi_2)][\mathrm{i}\xi_2]}, \qquad (1.6)
\end{aligned}$$

where by definition $\int^t e^{\mathrm{i}u\xi} du = \frac{e^{\mathrm{i}t\xi}}{\mathrm{i}\xi}$. From $\mathcal{A}(t)$ and the one-dimensional skeleton integral

$$\phi_i(t) = (2\pi c_\alpha)^{-\frac{1}{2}} \int^t dB_i(u) = \int \frac{dW_i(\xi)}{|\xi|^{\alpha-1/2}} \ \cdot \ \frac{e^{\mathrm{i}t\xi}}{\mathrm{i}\xi}, \qquad (1.7)$$

which is the above-defined infra-red divergent stationary process associated to B, one easily retrieves $\mathcal{A}(s,t)$ since

$$\begin{aligned}
\mathcal{A}(s,t) &= \int_s^t dB_1(t_1) \left(\int^{t_1} dB_2(t_2) - \int^s dB_2(t_2) \right) \\
&= \mathcal{A}(t) - \mathcal{A}(s) + \mathcal{A}_\partial(s,t), \qquad (1.8)
\end{aligned}$$

where $(2\pi c_\alpha)^{\frac{1}{2}} \mathcal{A}_\partial(s,t) := (B_1(t) - B_1(s))\phi_2(s)$ (called *boundary term*) is a *product of first-order integrals.*

One may easily estimate these quantities in each sector $|\xi_1| \gtrless |\xi_2|$. In practice, it turns out that estimates are easiest to get *after* a permutation of the integrals (applying Fubini's theorem) such that (for twice or multiple iterated integrals equally well) *innermost (or rightmost) integrals bear highest Fourier frequencies*; this is the essence of *Fourier normal ordering* [62, 16, 63]. This gives a somewhat different decomposition with respect to (1.8) since $\int_s^t dB_1(t_1) \int_s^{t_1} dB_2(t_2)$ is rewritten as $-\int_s^t dB_2(t_2) \int_t^{t_2} dB_1(t_1)$ in the "negative" sector $|\xi_1| > |\xi_2|$. After some elementary computations, one gets the following.

Lemma 1.3 *Let*

$$\begin{aligned}
\mathcal{A}^+(t) &:= 2\pi c_\alpha \int^t dt_1 \int^{t_1} dt_2 \mathcal{F}^{-1} \left((\xi_1, \xi_2) \mapsto \mathbf{1}_{|\xi_1|<|\xi_2|} (\mathcal{F}B_1')(\xi_1)(\mathcal{F}B_2')(\xi_2) \right) (t_1, t_2) \\
&= \int_{|\xi_1|<|\xi_2|} \frac{dW_1(\xi_1) dW_2(\xi_2)}{|\xi_1|^{\alpha-1/2} |\xi_2|^{\alpha-1/2}} \ \cdot \ \frac{e^{\mathrm{i}t(\xi_1+\xi_2)}}{[\mathrm{i}(\xi_1+\xi_2)][\mathrm{i}\xi_2]} \qquad (1.9)
\end{aligned}$$

and

$$\begin{aligned}\mathcal{A}^-(t) &:= 2\pi c_\alpha \int^t dt_2 \int^{t_2} dt_1 \mathcal{F}^{-1}\left((\xi_1,\xi_2)\mapsto \mathbf{1}_{|\xi_2|<|\xi_1|}(\mathcal{F}B_1')(\xi_1)(\mathcal{F}B_2')(\xi_2)\right)(t_1,t_2) \\ &= \int_{|\xi_2|<|\xi_1|} \frac{dW_1(\xi_1)dW_2(\xi_2)}{|\xi_1|^{\alpha-1/2}|\xi_2|^{\alpha-1/2}} \cdot \frac{e^{\mathrm{i}t(\xi_1+\xi_2)}}{[\mathrm{i}(\xi_1+\xi_2)][\mathrm{i}\xi_1]}. \end{aligned} \tag{1.10}$$

Then

$$\mathcal{A}(s,t) = \frac{1}{2\pi c_\alpha}\left\{ (\mathcal{A}^+(t)-\mathcal{A}^+(s)) - (\mathcal{A}^-(t)-\mathcal{A}^-(s)) + (\mathcal{A}^+_\partial(s,t)-\mathcal{A}^-_\partial(s,t))\right\}, \tag{1.11}$$

the boundary term $\mathcal{A}^+_\partial - \mathcal{A}^-_\partial$ *being given by*

$$\begin{aligned}\mathcal{A}^+_\partial(s,t)-\mathcal{A}^-_\partial(s,t) = &\left\{ -\int_{|\xi_1|<|\xi_2|} \frac{(e^{\mathrm{i}t\xi_1}-e^{\mathrm{i}s\xi_1})e^{\mathrm{i}s\xi_2}}{[\mathrm{i}\xi_1][\mathrm{i}\xi_2]} \right. \\ &\left. + \int_{|\xi_2|<|\xi_1|} \frac{(e^{\mathrm{i}t\xi_2}-e^{\mathrm{i}s\xi_2})e^{\mathrm{i}t\xi_1}}{[\mathrm{i}\xi_1][\mathrm{i}\xi_2]} \right\} \cdot \frac{dW_1(\xi_1)dW_2(\xi_2)}{|\xi_1|^{\alpha-1/2}|\xi_2|^{\alpha-1/2}}. \end{aligned} \tag{1.12}$$

Two lines of computations show immediately that

$$\begin{aligned}\mathrm{Var}\mathcal{A}^\pm_\partial(s,t) &\lesssim \int |e^{\mathrm{i}t\xi}-e^{\mathrm{i}s\xi}|^2|\xi|^{-1-4\alpha}d\xi \\ &\lesssim \int_{|\xi|>\frac{1}{|t-s|}} \frac{d\xi}{|\xi|^{1+4\alpha}} + \int_{|\xi|<\frac{1}{|t-s|}} \frac{|t-s|^2|\xi|^2}{|\xi|^{1+4\alpha}}d\xi \\ &\lesssim |t-s|^{4\alpha}, \end{aligned} \tag{1.13}$$

so that (essentially by the Kolmogorov-Centsov lemma, see section 2) the Hölder regularity indices of B_1 and B_2 add in the case of the boundary term, to produce a quantity which is $2\alpha^-$-Hölder. (Note that the artificial infrared divergence at $\xi_1=0$ disappears when Taylor expanding $e^{\mathrm{i}t\xi_1}-e^{\mathrm{i}s\xi_1}$). On the other hand, letting $\xi := \xi_1+\xi_2$ and introducing an ultra-violet cut-off at $|\xi_2| = \Lambda \gg 1$, one may see for instance $\mathcal{A}^+(t)$ as an inverse random Fourier transform of the integral $\xi \mapsto \int^\Lambda_{|\xi-\xi_2|<|\xi_2|} \frac{dW_2(\xi_2)}{\xi_2} \frac{1}{|\xi-\xi_2|^{\alpha-1/2}|\xi_2|^{\alpha-1/2}}$, whose variance diverges like $\int^\Lambda \frac{d\xi_2}{\xi_2^{4\alpha}} = O(\Lambda^{1-4\alpha})$ or $O(\ln\Lambda)$ in the ultra-violet limit $\Lambda\to\infty$ as soon as $\alpha \le 1/4$. Note that the ultraviolet divergence is in the region $|\xi_1|,|\xi_2| \gg |\xi|$.

It is apparent that the central rôle in this decomposition is played by the Fourier projection operator $D(\mathbf{1}_{|\xi_1|<|\xi_2|}) = \mathcal{F}^{-1}\left(\mathbf{1}_{|\xi_1|<|\xi_2|} \cdot \mathcal{F}(\ .\ )\right)$. Since

$\mathcal{A}_\partial^\pm$ are obtained by Fourier projecting $(B_1(t)-B_1(s))\phi_2(s)$, or $(B_2(t)-B_2(s))\phi_1(t)$, which are perfectly well-defined products of continuous fields [11], it was clear from the onset that these would be regular terms. Hence singularities come only from the *one-time quantity* $\mathcal{A}^\pm(t)$, which *does not* split into a product of first-order integrals, and that we shall call the *singular part of the Lévy area.*

2 An introduction to rough paths

2.1 General issues

Let $\Gamma = (\Gamma_1(t),\ldots,\Gamma_d(t))$ be a smooth path with d components. As explained in the Introduction, the Lévy area of Γ, $\mathbf{\Gamma}^2_{ts}(i,j) := \int_s^t d\Gamma_i(t_1) \int_s^{t_1} d\Gamma_j(t_2)$ is the simplest non-trivial iterated integral of Γ. The interest for iterated integrals of Γ comes from the study of two closely related problems in the case when Γ is not regular any more.

1. **Integration along an irregular path.**

Assume one wants to define the integral of the (say, smooth) one-form $f := \sum_{j=1}^d f_j(x)dx^j$ along the path Γ, namely, the quantity $\int_s^t f d\Gamma := \sum_j \int_s^t f_j(\Gamma(u))d\Gamma_j(u)$. Since Γ is not differentiable, $d\Gamma_j(u)$ may not be understood as $\frac{d\Gamma_j}{du}\cdot du$, and the very meaning of this quantity is unclear. Unfortunately, the Riemann-type sum $\sum_j \sum_{i=0}^{n-1} f_j(\Gamma(t_i))(\Gamma_j(t_{i+1})-\Gamma_j(t_i))$, with $s=t_0<\ldots<t_i = s+\frac{i}{n}(t-s)<\ldots<t_n=t$, may be shown to diverge in general as soon as $\alpha \le \frac{1}{2}$ [12].

A Taylor expansion to order N of the integrand yields (coming back to the case of a *regular path*) the improved Riemann-type sum

$$\sum_{i=0}^{n-1}\sum_{p=1}^{N}\sum_{j_1,\ldots,j_p=1}^{d} \frac{\partial^{p-1} f_{j_p}}{\partial x_{j_1}\ldots\partial x_{j_{p-1}}}(\Gamma(t_i))\mathbf{\Gamma}^p_{t_{i+1},t_i}(j_1,\ldots,j_p), \tag{2.1}$$

where

$$\mathbf{\Gamma}^p_{t_{i+1},t_i}(j_1,\ldots,j_p) = \int_s^t d\Gamma_{j_1}(t_1)\ldots\int_s^{t_{p-1}} d\Gamma_{j_p}(t_p) \tag{2.2}$$

[11] apart from the spurious infra-red divergence (see above)

[12] From a naive bound by the 1-variation of the path, $\sum_j\sum_i |\Gamma_j(t_{i+1})-\Gamma_j(t_i)| = O(n^{1-\alpha})$, one would come to the erroneous conclusion that the Riemann-type sums diverge when $\alpha<1$. The so-called *Young theory of integration* (see e.g. [39]) lowers the barrier to $\alpha=\frac{1}{2}$ by taking into account the Hölder regularity of the integrand $f(\Gamma(t))$.

is a p-th order iterated integral. The problem is, if Γ is *irregular*, iterated integrals of Γ are a priori ill-defined for the same reasons as before.

2. **Solutions of differential equations driven along an irregular path**

Consider the differential equation

$$dy_t = \sum_{i=1}^{d} V_j(y(t)) d\Gamma_j(t). \tag{2.3}$$

The following series gives a formal solution,

$$y_t = y_s + \sum_{N=1}^{\infty} \sum_{1 \le i_1, \ldots, i_N \le d} [V_{i_1} \cdots V_{i_N} \cdot \mathrm{Id}](Y_s) \ \cdot \ \mathbf{\Gamma}^{ts}(i_1, \ldots, i_N), \tag{2.4}$$

with $\mathbf{\Gamma}$ as in eq. (2.2). Solutions are usually computed by using some iterated numerical scheme. For instance, the Euler scheme of rank N gives the solution to (2.3) as the limit when $n \to \infty$ of the compound mapping,

$$\Phi(\mathbf{X}_{t,t_{n-1}}; \cdots \Phi(\mathbf{X}_{t_2,t_1}; \Phi(\mathbf{X}^{t_1,s}; y_s) \cdots), \tag{2.5}$$

where $\Phi(\mathbf{\Gamma}_{ts}; y_s)$ is the series (2.4) truncated to order N. If one takes for Γ an α-Hölder path, one stumbles again into the same problem of defining $\mathbf{\Gamma}_{ts} = (\mathbf{\Gamma}^1_{ts}, \ldots, \mathbf{\Gamma}^N_{ts})$.

In both cases, the hope is that, if one finds some (non necessarily unique!) way of defining iterated integrals of Γ with the correct regularity properties, then the refined Riemann-type sums (2.1) or Euler scheme (2.5) converge when the mesh $\frac{t-s}{n}$ goes to 0. Rough path theory shows this is possible [13] provided one chooses $N \ge \lfloor 1/\alpha \rfloor$ – here we choose $N = \lfloor 1/\alpha \rfloor$ minimal – and

$$\mathbf{\Gamma}_{ts} = \left(\mathbf{\Gamma}^1_{ts}(i_1)_{1 \le i_1 \le d}, \ldots, \mathbf{\Gamma}^N_{ts}(i_1, \ldots, i_N)_{1 \le i_1, \ldots, i_N \le d} \right) : \mathbb{R}^2 \to \mathbb{R}^d \times \mathbb{R}^{d^2} \times \ldots \times \mathbb{R}^{d^N} \tag{2.6}$$

is a rough path with Hölder regularity index α in the following sense:

Definition 2.1 (rough path) *An α-Hölder continuous rough path over Γ is a functional $\mathbf{\Gamma}^n_{ts}(i_1, \ldots, i_n)$, $n \le \lfloor N := 1/\alpha \rfloor$, $i_1, \ldots, i_n \in \{1, \ldots, d\}$, such that $\mathbf{\Gamma}_{ts}(i) = \Gamma_t(i) - \Gamma_s(i)$ are the increments of Γ, and the following 3 properties are satisfied:*

[13] Furthermore, the limit is α-Hölder and satisfies nice continuity properties with respect to the path Γ.

(i) (Hölder continuity) $\mathbf{\Gamma}^n_{ts}(i_1,\ldots,i_n)$ *is* $n\alpha$*-Hölder continuous as a function of two variables, namely,* $\sup_{s,t\in\mathbb{R}} \frac{|\mathbf{\Gamma}^n_{ts}(i_1,\ldots,i_n)|}{|t-s|^\alpha} < \infty$.

(ii) (Chen property)

$$\mathbf{\Gamma}^n_{ts}(i_1,\ldots,i_n) = \mathbf{\Gamma}^n_{tu}(i_1,\ldots,i_n) + \mathbf{\Gamma}^n_{us}(i_1,\ldots,i_n) + \sum_{n_1+n_2=n} \mathbf{\Gamma}^{n_1}_{tu}(i_1,\ldots,i_{n_1})\mathbf{\Gamma}^{n_2}_{us}(i_{n_1+1},\ldots,i_n); \tag{2.7}$$

(iii) (shuffle property)

$$\mathbf{\Gamma}^{n_1}_{ts}(i_1,\ldots,i_{n_1})\mathbf{\Gamma}^{n_2}_{ts}(j_1,\ldots,j_{n_2}) = \sum_{\boldsymbol{k}\in Sh(\boldsymbol{i},\boldsymbol{j})} \mathbf{\Gamma}^{n_1+n_2}_{ts}(k_1,\ldots,k_{n_1+n_2}), \tag{2.8}$$

where $Sh(\boldsymbol{i},\boldsymbol{j})$ *– the set of shuffles of the words* $\boldsymbol{i}$ *and* $\boldsymbol{j}$ *– is the subset of permutations of the union of the lists* $\boldsymbol{i},\boldsymbol{j}$ *leaving unchanged the order of the sublists* $\boldsymbol{i}$ *and* $\boldsymbol{j}$*. For instance,* $\mathbf{\Gamma}^2_{ts}(i_1,i_2)\mathbf{\Gamma}^1_{ts}(j_1) = \mathbf{\Gamma}^3_{ts}(i_1,i_2,j_1) + \mathbf{\Gamma}^3_{ts}(i_1,j_1,i_2) + \mathbf{\Gamma}^3_{ts}(j_1,i_1,i_2)$.

A formal *rough path over* Γ *is a functional satisfying all the above properties* except *Hölder continuity (i).*

In a random setting, the Hölder continuity estimates (i) are generally proved as a consequence of moment estimates such as $\mathbb{E}|\mathbf{\Gamma}^n_{ts}|^{2p} \le C_p|t-s|^{2pn\alpha}$, $p \ge 1, n = 1,\ldots,N$. This may be seen as a consequence of the well-known Kolmogorov-Centsov lemma stating that (for a measurable process random Γ)

$$\left(\mathbb{E}[|\Gamma(t)-\Gamma(s)|^{2p}] \le C|t-s|^{1+2p\alpha}\right) \Rightarrow \left(\forall \alpha^- < \alpha, \mathbb{E}\left[\left(\sup_{s,t\in[0,T]} \frac{|\Gamma(t)-\Gamma(s)|}{|t-s|^{\alpha^-}}\right)^{2p}\right] < \infty\right) \tag{2.9}$$

or more precisely of an extension (or a variant) of these estimates adaptated to functions of two variables (such as $(s,t)\mapsto \mathbf{\Gamma}^n_{ts}$) due to Garsia, Rodemich and Rumsey [21].

In particular, if Γ is smooth, then its natural iterated integrals $\int_s^t d\Gamma_{i_1}(t_1)\ldots\int_s^{t_{n-1}} d\Gamma_{i_n}(t_n)$ satisfy properties (ii) and (iii).

However, it is not clear a priori *in what sense* abstract data as in Definition 2.1 should represent iterated integrals in the usual sense.

2.2 Geometric approach

The answer to this question comes from a reinterpretation of rough paths in terms of group theory and geometric structures. We generally refer to the book by P. Friz and N. Victoir [20] for this paragraph. Consider the signature $\mathbf{\Gamma}_{ts} = (\mathbf{\Gamma}^1_{ts}, \mathbf{\Gamma}^2_{ts}, \ldots)$ of a smooth path Γ as

$$\mathbf{\Gamma}(s,t) := 1 + \sum_{i_1} \mathbf{\Gamma}^1_{ts}(i_1) X^{i_1} + \sum_{i_1,i_2} \mathbf{\Gamma}^2_{ts}(i_1,i_2) X^{i_1} \otimes X^{i_2} + \ldots, \qquad (2.10)$$

sitting inside the tensor algebra $T\mathbb{R}^d = \oplus_{n\ge 0} T^n \mathbb{R}^d$, with $X^1, \ldots, X^d$ generating a basis of $\mathbb{R}^d \simeq \mathbb{T}^1\mathbb{R}^d$. Note that the Chen property is trivially equivalent to the property $\mathbf{\Gamma}(s,t) = \mathbf{\Gamma}(s,u) \otimes \mathbf{\Gamma}(u,t)$, implying that $\mathbf{\Gamma}(s,t) = \mathbf{\Gamma}(0,s)^{\otimes -1} \otimes \mathbf{\Gamma}(0,t)$ is a *multiplicative increment*. In the particular case when $\Gamma(t) = tV$, $V \in \mathbb{R}^d$ is a straight line, $\mathbf{\Gamma}(0,t) = \exp t \sum_{i=1}^d V_i X^i$ belongs to $\exp T^1\mathbb{R}^d$. Easy arguments due to Chow show then that $t \mapsto \mathbf{\Gamma}(0,t)$ is a G-valued path, where $\mathfrak{g} = Lie(G)$ is the *free Lie algebra in d generators*, generated as a vector space by the successive commutators $X^{i_1}, [X^{i_1}, X^{i_2}], [X^{i_1}, [X^{i_2}, X^{i_3}]], \ldots$ In rough path theory, one quotients out by $\oplus_{n\ge N+1} T^n\mathbb{R}^d$. Then the quotient Lie algebra $\mathfrak{g}_N$ is the *free N-step nilpotent Lie algebra in d generators*, and $G_N = \exp \mathfrak{g}_N$ is a *Carnot group*. When $d = N = 2$, $\mathfrak{g}_2 \simeq \langle X, Y, Z := [X,Y]\rangle$ is isomorphic to the Heisenberg algebra, and the defect of additivity of the Lévy area $\mathcal{LA}_\Gamma(s,t) = \int_s^t d\Gamma_1(t_1) \int_s^{t_1} d\Gamma_2(t_2) - \int_s^t d\Gamma_2(t_2) \int_s^{t_2} d\Gamma_1(t_1)$, measured by the difference

$$\begin{aligned} \mathcal{LA}_\Gamma(s,t) - \mathcal{LA}_\Gamma(s,u) - \mathcal{LA}_\Gamma(u,t) = \\ (\Gamma_1(t) - \Gamma_1(u))(\Gamma_2(u) - \Gamma_2(s)) - (\Gamma_2(t) - \Gamma_2(u))(\Gamma_1(u) - \Gamma_1(s)), \end{aligned} \qquad (2.11)$$

is encoded into the non-commutativity of the product in the Heisenberg group, given by (in the exponential coordinates) $(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + y_1, x_2 + y_2, z_1 + z_2 + \frac{1}{2}(x_1 y_2 - x_2 y_1))$.

Carnot groups are naturally equipped by homogeneous norms coming from the sub-Riemannian Carnot-Carathéodory metric induced by horizontal geodesics, i.e.minimizing curves with tangent vectors in the Euclidean space $T^1\mathbb{R}^d$. Then an α-Hölder rough path over an α-Hölder path Γ is simply an α-Hölder G_N-valued path (in geometric terms, an α-Hölder section of the principal bundle $\mathbb{R} \times G_N$) which projects onto Γ.

One has the following two fundamental results.

Proposition 2.2 *(see Lyons [43], Lyons-Victoir [45], Friz-Victoir [20])*
Let $0 < \alpha^- < \alpha < 1$.

1. *(Existence theorem) There exists a (highly non-unique) α^--rough path over any α-Hölder path. In geometric terms, one may lift an α-Hölder section of the quotient bundle $\mathbb{R}\times\left(G_N/\exp\oplus_{n=2}^{N}T^n\mathbb{R}^d\right)\simeq\mathbb{R}\times T^1\mathbb{R}^d\simeq\mathbb{R}\times\mathbb{R}^d$ into an α^--Hölder section of the principal bundle $\mathbb{R}\times G_N$.*

2. *(Approximation theorem) Every α-Hölder rough path over Γ is the limit in α^--Hölder norm of the corresponding stack of natural iterated integrals over some smooth approximation family $\Gamma^\varepsilon,\varepsilon\to 0$ of Γ.*

The approximation theorem is essential in that it reduces differential equations driven by α-Hölder paths (through a limiting procedure which is often very subtle) to ordinary differential equations. Estimates for solutions in a deterministic setting are given in full details in the book by P. Friz and N. Victoir (see [20], Chap. 10).

This general approach is however insufficient for many purposes. Drawbacks are of two types:

– the arguments leading to the existence and approximation theorems are abstract, the first theorem relying on the axiom of choice (due to the arbitrariness of the lift), and the second one on an interpolation by sub-Riemannian geodesics which are notoriously complicated objects;

– in a random setting, this approach produces in principle deterministic, pathwise estimates, which moreover do not depend on the choice of rough path. Even in combination with probabilistic tools such as the Malliavin calculus, despite beautiful achivements in the case $\alpha>1/4$ (such as global existence of solutions for bounded potentials [20], existence of a density [10], ergodicity [31]),...) generalizing results known in the case of diffusion equations, it does not permit – in the case of stochastic differential equations driven by fBm for instance – to produce anything really better than a local existence theorem for solutions beyond the barrier $\alpha=1/4$.

Let us mention briefly *en passant* another related approach due to M. Gubinelli [28] and called *algebraic integration*. Without being too precise, it states the existence of a class of Γ-*controlled paths* – stable under functional transformations and under integration along Γ, and to which solutions of differential equations driven by Γ belong – whose increments are of the form

$$z_t-z_s=\sum_{n=1}^{N}\sum_{i_1,\ldots,i_n}\zeta_s^n(i_1,\ldots,i_n)\mathbf{\Gamma}_{ts}^n(i_1,\ldots,i_n) \tag{2.12}$$

for some functions $\zeta^n(i_1,\ldots,i_n)$, up to a remainder ρ_{ts} such that $\rho_{ts}=O(|t-s|^{1+\varepsilon}$, with $\varepsilon>0$. The right-hand side of (2.12) – viewed as a function of t – is a linear combination of the components of the rough path Γ, while the remainder is sufficiently regular so that conventional estimates apply. This essentially avoids the use of smooth approximations and requires only the knowledge of the quantities $\mathbf{\Gamma}^n_{ts}$, $n \le N$.

2.3 Fourier normal ordering

In contrast with this geometric approach, the point of view developed by the second author is that a rough path over an irregular path Γ is something "essentially arbitrary", and that one should rather look for *explicitly* constructed rough paths with "good" properties, which allow better estimates than the general ones.

Let us summarize very roughly the results obtained so far in the following Proposition:

Proposition 2.3 *(see [62, 16, 60])*

1. *A rough path is uniquely determined by an algorithm called* Fourier normal ordering algorithm *from its* tree data*, which are generalized Fourier normal ordered skeleton integrals on domains indexed by trees. As a consequence, any* arbitrary set of tree data *produces a* formal rough path *(see Definition 2.1).*

2. *Tree data yielding* Hölder-continuous rough paths *by Fourier normal ordering may be obtained by various, explicit regularization schemes applied to Fourier normal ordered tree skeleton integrals, using multiscale methods and inspired by the renormalization of Feynman graphs. In particular, one may construct rough paths $\mathbf{B} = (\mathbf{B}^1_{ts}, \dots, \mathbf{B}^N_{ts})$ over fBm such that $\mathbf{B}^j_{ts}$ is in the j-th chaos of fBm* [14].

Fourier normal ordering consists as in section 2 in (1) cutting iterated integrals like $I^{ts}_{\Gamma}(1, \dots, n) := \int_s^t d\Gamma_1(t_1) \int_s^{t_1} d\Gamma_2(t_2) \dots \int_s^{t_{n-1}} d\Gamma_n(t_n)$ into $n!$ pieces by applying the Fourier projection operators $\mathcal{P}^\sigma := D(\mathbf{1}_{|\xi_{\sigma(1)}|<\dots<|\xi_{\sigma(n)}|})$, where σ ranges in the group of permutations of $\{1, \dots, n\}$; (2) rewriting each piece $\mathcal{P}^\sigma I^{ts}_\Gamma(1, \dots, n)$ as a Fourier normal ordered integral over the inverse image of the simplex $\{t > t_1 > \dots > t_n > s\}$ by σ by using Fubini's theorem. The inverse image of the simplex decomposes as a union of elementary

[14] i.e. may be written as a j-linear integral expression in terms of B.

domains indexed by trees [15].

Thus the rôle of Fourier normal ordering is twofold: (1) it allows a general algebraic (combinatorial) classification of (formal) rough paths; (2) it induces a correct addition of the Hölder regularity indices of the tree data when recombining them by the Fourier normal ordering algorithm. We have seen an example of this when we estimated the variance of the boundary terms $\mathcal{A}^{\pm}_{\partial}$ in section 1.

The rough paths described in the above Proposition, in the case of fBm, say, are not obtained by an explicit limiting procedure; yet they suggest very strongly that the construction of rough paths is closely related to renormalization in quantum field theory. The purpose of the present series of articles is to give a probabilistic construction coming *directly* from quantum field theory. We actually conjecture that (some of) the rough paths of the above Proposition may be obtained by some limiting procedure from the construction of the next sections.

3 Definition of the interaction

We recall that $\int_s^t dB_1(t_1) \int_s^{t_1} dB_2(t_2)$ represents the area between the straight line connecting $(B_1(s), B_2(s))$ to $(B_1(t), B_2(t))$ and the curve. If the curve turns right, resp. left, then the Lévy area increases, resp. decreases. We have seen that $\mathcal{A}^{\pm}$ represents in some sense the *singular part* of the Lévy area.

It is conceivable that B_1, B_2 or ϕ_1, ϕ_2 represent the idealized, strongly self-correlated motion in $\mathbb{R}^2$ of a particle, which – although rotation-invariant – may not (probably as a consequence of a mechanical or electromagnetic rigidity due to the macroscopic dimension of the particle, or any other similar phenomenon) turn absolutely freely. A natural quantum field theoretic description of this rigidity phenomenon is to add an interaction Lagrangian of the form $\mathcal{L}_{int} = (\partial \mathcal{A}^{\pm})^2$. The fundamental intuition here is that the field B is in some sense a *mesoscopic field*, while $\mathcal{A}^{\pm}$ depends on *microscopic details of the theory*.

This is explained in great accuracy in [40], in a mathematical language. A. Lejay shows how a path Γ may be modified by inserting microscopic

[15] Given a rooted tree with n vertices indexed by $1, \ldots, n$, one integrates over the domain with coordinates $t_1, \ldots, t_n \in [s,t]$ such that $t_i < t_j$ whenever the vertex i is above the vertex j. When the tree is simply a trunk tree with no branching, one gets a usual iterated integral of order n.

bubbles all along, resulting in the limit in a path which is indistinguishable from the original one, while the Lévy area has been corrected by an *arbitrary* amount. Let us give a very simple example. Take for Γ a straight line $\Gamma(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}$, and insert (somewhat artificially) microscopic bubbles of size $\varepsilon = M^{-\alpha\rho}$ (covered in a time $O(M^{-\rho})$) at times which are multiples of $M^{-\rho}$. Then the resulting path Γ^{ε} has a Lévy area of order $M^{\rho} \cdot (M^{-\alpha\rho})^2 \to_{\rho\to\infty} \infty$, while $\Gamma^{\varepsilon} \to \Gamma$ in α^--Hölder norm whenever $\alpha^- < \alpha$ since $\frac{|(\Gamma^{\varepsilon}(t)-\Gamma^{\varepsilon}(s))-(\Gamma(t)-\Gamma(s))|}{|t-s|^{\alpha^-}} = O(M^{-(\alpha-\alpha^-)\rho}) \to_{\rho\to\infty} 0$. The inverse process of *removing* microscopic bubbles of a given path so as to make its Lévy area finite is of course much more hazardous, and looks a little bit like an "inverse Joule expansion" (i.e. like putting back all the molecules of a gas into the left compartment of a container after removing the wall which separated it from the right compartment, a statistical physicist's nightmare, sometimes called "Maxwell's devil").

Summarizing the above discussion, one must search for an interaction which cures the ultra-violet divergences of the microscopic scale, without modifying the theory at mesoscopic scale. This is where quantum field theory comes into play. The interested reader may refer to several excellent treatises on the subject (see e.g. [53] or [38]). It is impossible to give here a self-contained introduction to this theory which is one of the main foundations of the modern physics of both high-energy particles and condensed matter. Let us however explain in an informal way the most essential concepts, and introduce some useful terminology, in order to fill in the gap between probability theory and physics. We have tried to make the next two definitions as precise and as general as possible. In our case the space-time dimension D is simply one.

Definition 3.1 (ultra-violet cut-off) *1. Let $M > 1$ be a constant, and $\chi^0 : \mathbb{R}^D \to \mathbb{R}$, resp. χ^1 a non-negative, compactly supported function such that $\chi^0 \equiv 1$ in a neighbourhood of 0, resp. $\chi^1 \equiv 0$ in a neighbourhood of 0 and $\chi^1 \equiv 1$ in a neighbourhood of the hypersquare $\sup_{j=1,\ldots,D} |\xi_j| = 1$. These two functions may be chosen such that $(\chi^0, (\chi^j)_{j\geq 1})$, with $\chi^j := \chi^1(M^{-j}\cdot)$, define a partition of unity, i.e. $\chi^0 + \sum_{j\geq 1} \chi^j \equiv 1$. Let $\rho \in \mathbb{N}$. Then the* ultra-violet cut-off at scale ρ *of a function $f : \mathbb{R}^D \to \mathbb{R}^d$ is $f^{\to\rho} := \mathcal{F}^{-1}\left(\xi \mapsto \left[\sum_{j=0}^{\rho} \chi^j(\xi)\right] \mathcal{F}f(\xi)\right)$, where $\mathcal{F}$ is the Fourier transformation. Roughly speaking, the ultra-violet cut-off cuts away Fourier components of* momentum ξ *such that $|\xi| > M^{\rho}$.*

2. *Let $C_\phi(x,y) := C_\phi(x-y)$ be the covariance of a stationary Gaussian field $\phi : \mathbb{R}^D \to \mathbb{R}$. Then ϕ has same law as the series of independent Gaussian fields $\sum_{j\geq 0} \phi^j$, where ϕ^j has covariance kernel $C^j_\phi := \mathcal{F}^{-1}\left(\xi \mapsto \chi^j(\xi)\mathcal{F}C_\phi(\xi)\right)$. The* ultra-violet cut-off at scale *ρ of the Gaussian field ϕ is then $\phi^{\to\rho} := \sum_{j=0}^{\to\rho} \phi^j$, with covariance $C^{\to\rho}_\phi := \sum_{j=0}^{\rho} C^j_\phi$.*

Note that (at least for a good choice of the functions χ^0, χ^1) the Fourier transform of ϕ^j is supported on the union of two dyadic slices, $M^{j-1} < |\xi| < M^{j+1}$. In principle one may extend this decomposition to *negative scale indices* j, so that the limit $j \to -\infty$ describes the correlations at large distances. In our model however – and this makes it very different with respect to classical models in statistical physics, see comments below – it is only the transition from the microscopic scale ρ to the mesoscopic scale which is non-trivial, and one may essentially restrict to positive indices j.

Definition 3.2 (interacting fields) *Let $\phi : \mathbb{R}^D \to \mathbb{R}^d$ be a vector-valued Gaussian process on $\mathbb{R}^D$, $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_q)$ a set of real parameters, and $P_1, \ldots, P_q$ ($q \geq 1$) homogeneous polynomials on $\mathbb{R}^d \times (\mathbb{R}^d)^D$. Then the* interacting theory with interaction Lagrangian $\mathcal{L}_{int}(\phi)(x) = \sum_{p=1}^{q} \lambda_p P_p(\phi(x); \nabla\phi(x))$ *is (provided it exists!) the weak limit $\mathbb{P}(d\phi)$ of the* penalized measures

$$\mathbb{P}_{\boldsymbol{\lambda},V,\rho}(d\phi) := \frac{1}{Z_{V,\rho}} e^{-\int_V \mathcal{L}_{int}(\phi^{\to\rho})(x)dx} d\mu^{\to\rho}(\phi\big|_V), \tag{3.1}$$

when the volume $|V|$ and the ultra-violet scale ρ go to infinity, where: $V \subset \mathbb{R}^D$ is compact; $d\mu^{\to\rho}(\phi\big|_V)$ is the Gaussian measure corresponding to the cut-off field $\phi^{\to\rho}$ restricted to the finite volume V; $Z_{V,\rho}$ is a normalization constant called partition function *by reference to Gibbs measures.*

In general, ϕ is stationary, which accounts for the *finite volume cut-off* V, and $\int_V \mathcal{L}_{int}(\phi^{\to\rho})(x)dx$ diverges when $\rho \to \infty$, which accounts for the *ultra-violet cut-off at scale ρ*. The parameters $\lambda_1, \ldots, \lambda_q$ are called *bare coupling constants*. Usually the inverse of the covariance kernel of ϕ is a differential operator of the form $C_\phi^{-1} = \lambda_\nabla \nabla^2 + m^2$, where m is called the *mass*. (In the case of our model, C_ϕ^{-1} contains a *fractional* derivative operator instead, but the present discussion remains valid). Formally (forgetting about the cut-offs) $d\mu(\phi)$ gives the trajectories a weight proportional to the Onsager-Machlup functional $e^{-\frac{1}{2}((\lambda_\nabla\nabla^2+m^2)\phi,\phi)}$, so the parameters λ_∇ and m^2 play a rôle similar to the coupling constants $\lambda_1, \ldots, \lambda_q$, and the sum of the *interaction Lagrangian* and of the *Onsager-Machlup functional* is called simply the *Lagrangian*.

In general also, ϕ is self-similar (or at least asymptotically self-similar at short distances), so the term in the Lagrangian $P_p(\phi(x), \nabla\phi(x))dx$ has a certain *degree of homogeneity* with respect to a change of scale $x \mapsto ax$ or equivalently $\xi \mapsto a^{-1}\xi$ after a Fourier transform, which gives the main behaviour at large momenta ξ – or equivalently at short distances – of the correlations (or so-called n-point correlation functions) $\langle \phi_{i_1}(x_1)\ldots\phi_{i_n}(x_n)\rangle_{V,\rho} := \int \phi_{i_1}(x_1)\ldots\phi_{i_n}(x_n)\mathbb{P}_{\boldsymbol{\lambda},V,\rho}(d\phi)$.

Here we take a high-energy physics point of view. Then the *bare scale* is ρ; in other words, one uses a cut-off at short distances of order $M^{-\rho} \to_{\rho\to\infty} 0$, and wants to understand the behaviour of the correlations at macroscopic distances [16] . In principle, the theory is hopelessly divergent in the limit $\rho \to \infty$ if this degree of homogeneity is *negative* (the so-called *non-renormalizable case*). On the contrary, expanding the exponential $e^{-\int P_p(\phi(x),\nabla\phi(x))dx}$ into a series leads to only a finite number of diverging terms (called diverging *Feynman diagrams*) if the degree of homogeneity is *positive* (the so-called *super-renormalizable case*). When this degree of homogeneity is zero (the so-called *just renormalizable case*, often the most interesting one in practice) closer inspection is needed. In all cases, for a large variety of models, one obtains by iterated integration with respect to highest Fourier scales (i.e. with respect to the field components $\phi^\rho, \phi^{\rho-1}, \ldots, \phi^{j+1}$) an effective theory at scale j which may be described in terms of the same Lagrangian *but* with so-called *renormalized parameters*, by opposition to the *bare parameters*, $\lambda_p \rightsquigarrow \lambda_p^j$ or $\lambda_\Delta \rightsquigarrow \lambda_\nabla^j$, $m^2 \rightsquigarrow (m^2)^j$. One obtains in general a flow for the parameters, i.e. equations of the type $(\lambda_\nabla^j, (m^2)^j; (\lambda_p^j), p' = 1, \ldots, q) := F(\lambda_\nabla^{j+1}, (m^2)^{j+1}; (\lambda_{p'}^{j+1}), p' = 1, \ldots, q)$. Solving this flow down to small values of j is then the main task of renormalization. An interesting case is when one may show that the contribution of the renormalized vertex $\lambda_p P_p(\phi, \nabla\phi)$ goes to zero at distances which are large with respect to the bare scale; then the theory is said to be *asymptotically free at large distances*. The best-known examples of this behaviour are maybe the weakly avoiding path or the ϕ^4-theory, both in dimension $D = 4$; see [46, 22, 17] for rigorous results [17]. Our model is original for it combines in some sense features of models

[16] In statistical physics, the size of the lattice usually gives an explicit cut-off, so one may take $\rho = 0$.

[17] On the other hand, in high-energy physics, the main example in this respect is that of asymptotic freedom at *short distances* (or equivalently *high energy*) of quarks [30, 54], so exactly the opposite point of view with respect to the one we adopt here.

of both high-energy physics and statistical physics: namely, the bare scale is $O(M^{-\rho})$, *but* the theory is asymptotically free at large distances. Letting $\rho \to \infty$, the interaction disappears *at all finite scales*, hence one retrieves in the end a Gaussian theory, in which, however, the singular part of the Lévy area has been cancelled.

Perturbative methods are by far the most common in physics, because they are accessible to non-experts. They rely on an asymptotic analysis of the quantities obtained by expanding into power series in the coupling constants the exponential weight $e^{-\int \mathcal{L}_{int}(\phi)(x)dx}$. These are conventionally represented as *Feynman graphs* (we shall show some of these later on for our model). Unfortunately, in all interesting cases, the series diverges by and large because of huge combinatorial factors, hence perturbative theory has only a *heuristic status*. Constructive methods, on the other hand (when they work!), are based on particularly clever *finite Taylor expansions*, scale after scale, and produce *converging series* (but not power series!); in other terms, they are rigorous. However, the technical apparatus needed to explain constructive field theory is much more sophisticated.

Let us now return to the discussion of our model after this long parenthesis. In order to keep track of the degree of homogeneity of the fields – and to obtain eventually the expected Hölder regularity indices for iterated integrals – we need here a *just renormalizable* theory (or, in other terms, an integrated interaction which is homogeneous of degree 0). Since $(\partial A^{\pm})^2$ is homogeneous of degree $(4\alpha - 2)$ in time, one shall use in fact a *non-local* interaction lagrangian, $\frac{1}{2}c'_\alpha \int\int |t_1 - t_2|^{-4\alpha} \mathcal{L}_{int}(\phi_1, \phi_2)(t_1, t_2) dt_1 dt_2$, where

$$\mathcal{L}_{int}(\phi_1, \phi_2)(t_1, t_2) = \lambda^2 \left\{ \partial\mathcal{A}^+(t_1)\partial\mathcal{A}^+(t_2) + \partial\mathcal{A}^-(t_1)\partial\mathcal{A}^-(t_2) \right\}, \qquad (3.2)$$

which is *positive* for $\alpha < 1/4$ since the kernel $|t_1 - t_2|^{-4\alpha}$ is locally integrable and positive definite. Thus the Gaussian measure is *penalized* by the singular exponential weight $e^{-\frac{c'_\alpha}{2}\int\int \mathcal{L}_{int}(\phi_1,\phi_2)(t_1,t_2)|t_1-t_2|^{-4\alpha}dt_1dt_2}$. Equivalently, using the so-called Hubbard-Stratonovich transformation [18], we introduce two independent exchange particle fields $\sigma_\pm = \sigma_\pm(t)$ with covariance kernel $C_{\sigma_\pm}(s,t) = C_{\sigma_\pm}(t-s) = \mathbb{E}\sigma_\pm(s)\sigma_\pm(t) = c'_\alpha|s-t|^{-4\alpha}$ and rewrite (letting $d\mu(\phi)$, resp. $d\mu(\sigma)$ be the Gaussian measure associated to ϕ, resp. $\sigma = (\sigma_+, \sigma_-)$) the partition function $Z := Z(\lambda)$,

$$Z := \int e^{-\frac{c'_\alpha}{2}\int\int_{\mathbb{R}^2} |t_1-t_2|^{-4\alpha}\mathcal{L}_{int}(\phi_1,\phi_2)(t_1,t_2)dt_1dt_2} d\mu(\phi) \qquad (3.3)$$

[18] which is nothing else but an infinite-dimensional extension of the Fourier transform $\mathbb{E}[e^{i\lambda X}] = e^{-\sigma^2\lambda^2/2}$ for a random variable $X \sim \mathcal{N}(0,\sigma^2)$

as

$$Z := \int e^{-\int_{\mathbb{R}} \mathcal{L}_{int}(\phi_1,\phi_2,\sigma)(t)dt} d\mu(\phi)d\mu(\sigma), \tag{3.4}$$

where

$$\mathcal{L}_{int}(\phi_1, \phi_2, \sigma)(t) = \mathrm{i}\lambda \left(\partial A^+(t)\sigma_+(t) - \partial \mathcal{A}^-(t)\sigma_-(t)\right). \tag{3.5}$$

All of this is ill-defined mathematically since (1) σ is a distribution-valued process and $\partial A^{\pm}$ is not defined *at all* when $\alpha \leq 1/4$; (2) one integrates over $\mathbb{R}$ a translation-invariant quantity (note that ϕ_1, ϕ_2, σ are all *stationary fields*).

4 Heuristic perturbative proof of convergence

Let us now explain the basics of perturbative quantum field theory, and show how it suggests (at least heuristically) the assertions of Theorem 0.1. The general idea is to expand formally the exponential of the Lagrangian in order to compute polynomial moments, $\frac{1}{Z}\mathbb{E}\left[\psi_1(x_1)\ldots\psi_n(x_n)e^{-\int \mathcal{L}_{int}(\phi_1,\phi_2,\sigma)(t)dt}\right]$, also called *n-point functions* and denoted by $\langle\psi_1(x_1)\ldots\psi_n(x_n)\rangle_\lambda$, $\psi_i = \phi_1, \phi_2, \sigma_+$ or σ_-, as $\frac{1}{Z}\sum_{n\geq 0}\frac{(-1)^n}{n!}\mathbb{E}\left[\psi_1(x_1)\ldots\psi_n(x_n)\left(\int \mathcal{L}_{int}(\ .\ ;t)dt\right)^n\right]$. We do not bother too much about the volume and ultra-violet cut-off here, and write $\langle\ \cdot\ \rangle_\lambda$ instead of $\langle\ \cdot\ \rangle_{\lambda,V,\rho}$. Recall first the following classical combinatorial facts. A good reference for perturbative expansions in quantum field theory is e.g. [38].

Proposition 4.1 *1. (Wick's formula) Let $X = (X_1, \ldots, X_{2n})$ be a (centered) Gaussian vector. Then*

$$\mathbb{E}[X_1 \ldots X_{2n}] = \sum_{(i_1 i_2)\ldots(i_{2n-1}i_{2n})} \mathbb{E}[X_{i_1}X_{i_2}]\ldots\mathbb{E}[X_{i_{2n-1}}X_{i_{2n}}], \tag{4.1}$$

where the indices range over all pairings of the indices $1, \ldots, 2n$. Each term in the sum is represented as a graph with $2n$ points connected two by two.

2. (connected moments) Let $\langle\ \cdot\ \rangle := \frac{\mathbb{E}[\ \cdot\ e^{\Phi(X)}]}{\mathbb{E}[e^{\Phi(X)}]}$ be a penalized measure, where $X = (X_1, X_2, \ldots)$ is a (centered) Gaussian vector, and $\Phi(X)$ is a polynomial in $X_1, X_2, \ldots$. Then the connected expectation $\langle X_1 \ldots X_n\rangle_c$ (c for connected) is (formally at least) the sum of all connected graphs obtained by (i) expanding the exponential; (ii) applying Wick's formula and drawing links between the paired points; (iii) identifying all points coming from the same vertex, *i.e. from the same monomial in $\Phi(X)$ descended from the exponential.*

Connected expectations exclude in particular vacuum contributions, i.e. terms of the form $\mathbb{E}[e^{\Phi(X)}]\mathbb{E}[X_1 \ldots X_n] = Z\mathbb{E}[X_1 \ldots X_n]$. Discarding these contributions can be shown to provide automatically the normalizing factor $\frac{1}{Z}$. Then usual expectations $\langle X_1 \ldots X_n \rangle$ are obtained by taking all possible splittings of $\{1, \ldots, n\}$ into disjoint subsets $I_1 \uplus \ldots \uplus I_p$ and summing over the products of connected expectations $\sum_p \sum_{I_1,\ldots,I_p} \langle \prod_{i\in I_1} X_i\rangle_c \ldots \langle \prod_{i\in I_p} X_i\rangle_c$. In practice the last operation is trivial for two-point functions $\langle X_{i_1} X_{i_2}\rangle$ if by parity (which is often the case in quantum field theory) the one-point functions $\langle X_i\rangle_c$ vanish, so that $\langle X_{i_1} X_{i_2}\rangle = \langle X_{i_1} X_{i_2}\rangle_c$.

Let us return to our model. Using a straightforward extension of the above Proposition, one may represent $\langle \psi_1(x_1)\ldots\psi_n(x_n)\rangle_\lambda$, $\psi = \phi$ or σ as a sum over Feynman diagrams, $\sum_\Gamma A(\Gamma)$, where Γ ranges over a set of diagrams with n external legs, and $A(\Gamma) \in \mathbb{R}$ is the evaluation of the corresponding diagram (see examples below); connected expectations will then be obtained as a sum over connected Feynman diagrams. More precisely, one obtains formally a (diverging) power series in λ, $\sum_{n\geq 0} \lambda^n \sum_{\Gamma_n} A(\Gamma_n)$, where Γ_n ranges over the set of Feynman diagrams with n vertices. The Gaussian integration by parts formula [19] yields a so-called *Schwinger-Dyson identity*,

$$\begin{aligned}
\langle \partial\mathcal{A}^\pm(x)\partial\mathcal{A}^\pm(y)\rangle_\lambda &= -\frac{1}{\lambda^2 Z(\lambda)}\mathbb{E}\left[\frac{\delta}{\delta\sigma_+(y)}\frac{\delta}{\delta\sigma_+(x)}e^{-\int \mathcal{L}_{int}(\phi_1,\phi_2,\sigma_+)(t)dt}\right] \\
&= -\frac{1}{\lambda^2 Z(\lambda)}\mathbb{E}\left[(C_{\sigma_+}^{-1}\sigma_+)(y)\frac{\delta}{\delta\sigma_+(x)}e^{-\int \mathcal{L}_{int}(\phi_1,\phi_2,\sigma_+)(t)dt}\right] \\
&= -\frac{1}{\lambda^2}\left[-C_{\sigma_+}^{-1}(x,y) + \langle (C_{\sigma_+}^{-1}\sigma_+)(x)(C_{\sigma_+}^{-1}\sigma_+)(y)\rangle_\lambda\right],
\end{aligned} \tag{4.2}$$

with Fourier transform

$$\langle |\mathcal{F}(\partial\mathcal{A}^\pm)(\xi)|^2\rangle_\lambda = \frac{1}{\lambda^2}|\xi|^{1-4\alpha}\left[1 - |\xi|^{1-4\alpha}\langle |(\mathcal{F}\sigma_+)(\xi)|^2\rangle_\lambda\right]. \tag{4.3}$$

By parity, $\langle |\mathcal{F}(\partial\mathcal{A}^\pm)(\xi)|^2\rangle_\lambda$ is a power series in λ^2.

Introduce an ultra-violet cut-off at scale ρ as in Definition 3.1. For the simplicity of the exposition we shall actually use a brute-force ultra-violet cut-off at momentum M^ρ, i.e. cut off all Fourier components with

[19] an infinite-dimensional extension of the well-known formula for Gaussian vectors, $\mathbb{E}\left[\partial_{X_i} F(X_1,\ldots,X_n)\right] = \sum_j C^{-1}(i,j)\mathbb{E}\left[X_j F(X_1,\ldots,X_n)\right]$ if C is the covariance matrix of $(X_1,\ldots,X_n)$.

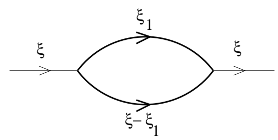

Figure 1: Bubble diagram with 2 vertices. By momentum conservation $\xi = \xi_1 + \xi_2$, which leaves out one free internal momentum.

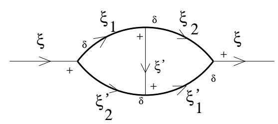

Figure 2: More complicated bubble diagram with 4 vertices. By momentum conservation $\xi = \xi_1 + \xi_2' = \xi_1' + \xi_2$ and $\xi_1 = \xi' + \xi_2$, which leaves out two independent internal momenta.

momentum $|\xi| > M^\rho$. After Fourier transformation, $\int \mathcal{L}_{int}(\cdot;t)dt$ becomes $\mathrm{i}\lambda \int_{|\xi_1|<|\xi_2|} d\xi_1 d\xi_2 d\xi \delta_0(\xi_1+\xi_2+\xi)\mathcal{F}\sigma_+(\xi)\mathcal{F}(\partial\phi_1)(\xi_1)\mathcal{F}\phi_2(\xi_2)$, minus a similar term involving σ_-. The square of this expression contributes the following term of order $O(\lambda^2)$ to $\langle|\mathcal{F}\sigma_+(\xi)|^2\rangle_\lambda$,

$$(-\mathrm{i}\lambda)^2 \int_{|\xi_1|<|\xi-\xi_1|}^{M^\rho} d\xi_1 \left\{ \left(\mathbb{E}[|\mathcal{F}\sigma_+(\xi)|^2]\right)^2 \ \mathbb{E}[|\mathcal{F}(\partial\phi_1)(\xi_1)|^2] \ \mathbb{E}[|\mathcal{F}\phi_2(\xi-\xi_1)|^2] \right\}$$
$$= -\lambda^2 |\xi|^{8\alpha-2} \int_{|\xi_1|<|\xi-\xi_1|}^{M^\rho} d\xi_1 |\xi_1|^{1-2\alpha} |\xi-\xi_1|^{-1-2\alpha} \sim_{\rho\to\infty} -K\lambda^2|\xi|^{8\alpha-2}(M^\rho)^{1-4\alpha}. \tag{4.4}$$

This is the evaluation of the *Feynman diagram represented* in Fig. 1, according to the following rules.

Definition 4.2 (Feynman rules) *A Feynman diagram in our theory is made up of (1)* bold lines *of type* $i = 1, 2$*, with momenta* $\xi_i, \xi_i', \ldots$ *evaluated as* $\mathbb{E}|\mathcal{F}\phi_i(\xi_i)|^2 = \frac{1}{|\xi_i|^{1+2\alpha}}$*; (2)* plain lines *of type* $\pm$*, with momenta* $\xi, \xi', \ldots$*, evaluated as* $\mathbb{E}|\mathcal{F}\sigma_\pm(\xi)|^2 = \frac{1}{|\xi|^{1-4\alpha}}$*; (3) vertices where two plain lines – one of each type – and a bold line meet, with a momentum conservation rule,* $\xi = \pm\xi_1 \pm \xi_2$ *(depending on the orientation of the lines). The definition of the interaction implies the presence of a further derivation – represented by the symbol* ∂ *on the Feynman diagram – on the* ϕ_1*-, resp.* ϕ_2*-field, and a momentum scale restriction* $|\xi_1| < |\xi_2|$*, resp.* $|\xi_1| > |\xi_2|$*, at vertices involving a* σ_+*-, resp.* σ_-*-field. The derivation translates into a multiplication by* $\mathrm{i}\xi_1$*, resp.* $\mathrm{i}\xi_2$ *when evaluating the diagram.*

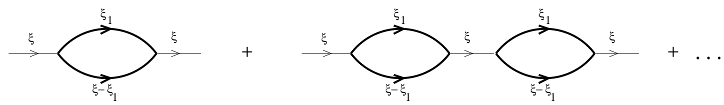

Figure 3: First three terms of the bubble series. The renormalized covariance of the σ-field is equal to the sum of the series.

It is sometimes useful to consider the evaluation of the corresponding *amputated Feynman diagram*, from which the contribution of the external legs has been removed. Here for instance, the evaluation of the amputated Feynman diagram associated to Fig. 1 is $(|\xi|^{1-4\alpha})^2$ times the previous expressions, hence is equivalent to the ξ-independent expression $-K\lambda^2(M^\rho)^{1-4\alpha}$ when $\rho \to \infty$. It is a diverging *negative* quantity. (Using the Fourier truncation of Definition 3.1 only changes the constant K.) However, resumming *formally* the bubble series as in Fig. 3 yields, starting from the right-hand side of eq. (4.3),

$$\begin{aligned}
&\frac{1}{\lambda^2}|\xi|^{1-4\alpha}\left[1-\sum_{n\geq 0}(-1)^n\left(\frac{1}{|\xi|^{1-4\alpha}}\cdot K\lambda^2(M^\rho)^{1-4\alpha}\right)\right] \\
&\quad = \frac{1}{\lambda^2}|\xi|^{1-4\alpha}\quad\cdot\quad\frac{K\lambda^2(M^\rho/|\xi|)^{1-4\alpha}}{1+K\lambda^2(M^\rho/|\xi|)^{1-4\alpha}} \\
&\quad \to_{\rho\to\infty}\ \frac{1}{\lambda^2}|\xi|^{1-4\alpha}.
\end{aligned} \tag{4.5}$$

On the other hand (see Fig. 3), the bare σ-covariance $\frac{1}{|\xi|^{1-4\alpha}}$ has been replaced with the renormalized covariance

$$\frac{1}{|\xi|^{1-4\alpha}}\cdot\frac{1}{1+K\lambda^2(M^\rho/|\xi|)^{1-4\alpha}}=\frac{1}{|\xi|^{1-4\alpha}+K\lambda^2(M^\rho)^{1-4\alpha}}, \tag{4.6}$$

which vanishes in the limit $\rho\to\infty$. The essential reason for this is of course that the oscillating signs $(-1)^n$ in the bubble series evaluation – due to the fact that the interaction Lagrangian $\mathcal{L}_{int}(\phi_1,\phi_2,\sigma)$ is purely *imaginary* – result by summing in a huge, virtually infinite denominator. Taking into account the possible insertion of σ_--lines between σ_+-lines amounts to a simple change of the constant K. In physical terms, the interaction in $\frac{1}{|\xi|^{1-4\alpha}}$ has been *screened* by a *huge mass term* $K\lambda^2 M^{\rho(1-4\alpha)} \to_{\rho\to\infty} +\infty$ (see section 3 for the definition of the mass). More complicated diagrams contributing to $\langle|(\mathcal{F}\sigma_+)(\xi)|^2\rangle_\lambda$, and involving internal σ-lines as in Fig. 2 also vanish when $\rho\to\infty$. Thus there remains simply:

$$\langle |\mathcal{F}\mathcal{A}^{\pm}(\xi)|^2| \rangle_\lambda = \frac{1}{\lambda^2}|\xi|^{-1-4\alpha}. \tag{4.7}$$

Hence $\mathbb{E}|\mathcal{A}^{\pm}(t)-\mathcal{A}^{\pm}(s)|^2 \lesssim \frac{1}{\lambda^2}|t-s|^{4\alpha}$, as in eq. (1.13).

As for the mixed term $\langle \partial\mathcal{A}^{\pm}(x)\partial\mathcal{A}^{\mp}(y)\rangle_\lambda$, its Fourier transform is given by $\frac{1}{\lambda^2}|\xi|^{1-4\alpha}\left[-\frac{1}{1+K''\lambda^2(\Lambda/|\xi|)^{1-4\alpha}}\right]$, where $K'' < K$ due to the constraints on the scales for bubbles of mixed type with one σ_+- and one σ_--leg, which vanishes in the limit $\rho\to\infty$ (note the disappearance of the factor 1 compared to eq. (4.5), due to the fact that $\mathbb{E}\sigma^+(x)\sigma^-(y)=0$). Thus the covariance of the two-component σ-field has been renormalized to $\frac{1}{|\xi|^{1-4\alpha}\mathrm{Id}+m^\rho}$, where m^ρ is a two-by-two positive "mass" matrix with eigenvalues $\approx \lambda^2 M^{\rho(1-4\alpha)}$.

Using eq. (1.11), one obtains:

$$\begin{aligned}
(2\pi c_\alpha)^2\langle \mathcal{A}(s,t)^2\rangle_\lambda &= \langle \left|\mathcal{A}^+(t)-\mathcal{A}^+(s)\right|^2\rangle_\lambda + \langle \left|\mathcal{A}^-(t)-\mathcal{A}^-(s)\right|^2\rangle_\lambda \\
&\qquad + \mathbb{E}\left|\mathcal{A}^+_\partial(s,t)-\mathcal{A}^-_\partial(s,t)\right|^2 \\
&= \frac{4}{\lambda^2}\int (1-\cos(t-s)\xi)|\xi|^{-1-4\alpha}d\xi + \mathbb{E}\left|\mathcal{A}^+_\partial(s,t)-\mathcal{A}^-_\partial(s,t)\right|^2 \\
&= (\frac{4}{\lambda^2}K_1+K_2)|t-s|^{4\alpha} \qquad (4.8)
\end{aligned}$$

for some constants K_1, K_2.

Let us now consider briefly other correlations. For a general discussion we need the following easy power-counting lemma:

Lemma 4.3 (power-counting rules) *Let Γ be a Feyman diagram with N_σ external σ-lines, N_ϕ external ϕ-lines, and $N_{\partial\phi}$ external $\partial\phi$-lines. Then the overall degree of homogeneity (in powers of ξ) of the evaluation of the corresponding amputated diagram – also called:* overall degree of divergence *– is $1-2\alpha N_\sigma+\alpha N_\phi+(\alpha-1)N_{\partial\phi}$.*

Proof. Let: I_σ, resp. I_ϕ, be the number of *internal* lines of type σ, resp. ϕ or $\partial\phi$; $I = I_\sigma + I_\phi$ be the total number of internal lines; and $L = I - V + 1$ be the number of loops, equal to the number of independent momenta (one per internal line, minus one per vertex due to momentum conservation, plus one due to overall momentum conservation). Since one σ- and two ϕ-lines meet at each vertex, one also has the relations $2I_\sigma + N_\sigma = V$, and $2I_\phi + N_\phi + N_{\partial\phi} = 2V$. Now the amputated diagram is homogeneous to $|\xi|^{-(1-4\alpha)I_\sigma-(1+2\alpha)I_\phi+L+V-N_{\partial\phi}}$ (counting one derivative per vertex, and minus one derivative per *external* $\partial\phi$-leg which is not taken into account in the evaluation). Putting all these relations together yields the result. $\square$

If a diagram is overall divergent, i.e. if its overall degree of divergence is positive, then the diagram diverges (except if by chance the coefficient of the term of highest degree in ξ vanishes). On the other hand, the fact that a diagram is overall convergent (i.e. its overall degree of divergence is negative) does not imply that it is convergent, since it may contain overall divergent *sub*-diagrams. One must hence study the behaviour of all possible diagrams, with arbitrary external leg structure.

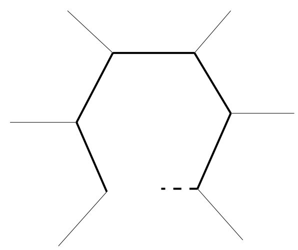

Figure 4: Higher connected moments of the Lévy area.

The above simple power-counting argument shows that the overall degree of divergence of a connected diagram with $2n$ external σ-legs is $1-4n\alpha$. For $n \geq 2$, this is $\leq 1-8\alpha < 0$ since $\alpha > \frac{1}{8}$ by hypothesis, so such diagrams are overall convergent. By the above arguments, there remain only the connected diagrams in the limit $\Lambda \to \infty$, see Fig. 4, whose evaluation is independent of λ.

General considerations following from the multi-scale expansions (one may refer to [65] for a good, accessible presentation, or to [60] for an application to the Gaussian renormalization of iterated integrals evoked in subsection 2.3) show that it is enough to consider the behaviour of diagrams whose internal legs have higher (or even: much higher) momentum scale than external legs, the so-called *dangerous diagrams*. Then the momentum scale constraint on the vertices coming from Fourier normal ordering implies that the external legs of dangerous diagrams may be either of type σ or of type $\partial\phi$, but not of type ϕ. Consider now any diagram whose external structure contains external $\partial\phi$-legs. By parity it has at least two such external legs, and the previous power-counting rules show that such a diagram is always overall convergent.

Finally, the law of the field ϕ is left unchanged by the interaction. Namely, all non-trivial diagrams contributing e.g. to $\langle\phi_1(x)\phi_2(x)\rangle_\lambda$ involve internal σ-lines which (as previously "shown") vanish in the limit $\rho \to \infty$.

On the whole, this is the content of Theorem 0.1.

The art of constructive field theory is to make the previous speculations rigorous. It relies on the following considerations, corresponding to the weak points (not to say flaws!) in the above arguments:

1. While going from eq. (4.4) to (4.5), we have replaced the amputated bubble diagram evaluation by its asymptotics when $\rho \to \infty$, namely, $-K\lambda^2 M^{\rho(1-4\alpha)}$, which is simply equal to its *evaluation at zero external momentum* ξ, also called *local part.* Thus we have actually not resummed the whole bubble series, but only the corresponding local parts, and observed that this was equivalent to adding a mass term of the form $K'\lambda^2 M^{\rho(1-4\alpha)} \int |\sigma(x)|^2 dx$ to the Lagrangian.

2. The bubble series is really a terribly *diverging geometric series.* Renormalization must actually be performed scale by scale. Considering only bubble diagrams with momentum in the dyadic slice $M^{\rho-1} < |\xi| < M^\rho$ leads on the other hand to a converging geometric series for λ small enough since the term between parentheses in eq. (4.5), $K\lambda^2 \left(\frac{M^\rho}{|\xi|}\right)^{1-4\alpha}$, is then < 1. This is equivalent to integrating out the highest field components (σ^ρ, ϕ^ρ), as explained in section 3. One obtains thus a running mass coefficient m^ρ of order $\lambda^2 M^{(1-4\alpha)\rho}$. The procedure must then be iterated by going down the scales step by step. Since renormalization *reduces* the covariance of the σ-field, the bound on λ ensuring convergence does not become worse and worse after each step.

3. We neglected more complicated bubble diagrams as in Fig. 2. Although these have the same order as the simple bubble diagram of Fig. 1, as follows from the above power-counting rules, taking into consideration *all possible* bubble diagrams lead to a terribly diverging power series in λ due to the rapidly increasing number of such diagrams in terms of the number of vertices, with a coefficient roughly of order $n!$ in front of λ^n. This divergence is actually due to the accumulation of vertices in a small region of space of size $O(M^{-j})$, where j is the momentum scale under consideration. *Multi-scale cluster expansions* in constructive field theory, by considering only *partial* series expansions, avoid this dangerous accumulation process.

4. By splitting each vertex $\int \mathcal{L}_{int}^{\to\rho}(\cdot; x)dx$ into its different scales, there may appear fields $\phi_1^{j_1}, \phi_2^{j_2}, \sigma^j$ with different scales $j_1 \neq j_2 \neq j$. Tak-

ing this into account in a coherent way in the previous partial series expansions lead to complicated combinatorial expressions encoded by so-called *polymers*, which are the main object in use in constructive field theory.

5. In the previous vertex splitting, the field with lowest momentum scale (j_1, j_2 or j, depending on the case) is called *low-momentum field*. Even though the cluster expansion in each momentum scale prevents an accumulation of vertices in the same region of space, the compound effect of *all* cluster expansions at *all* scales produces unavoidably accumulations of fields with very low momentum in very large regions of space, which is a dangerous problem called *domination problem*. This accounts for the addition of the extra boundary term $\mathcal{L}_{bdry}^{\to\rho}$ in the interaction Lagrangian. Writing out this term and explaining its precise form would however take us too far away.

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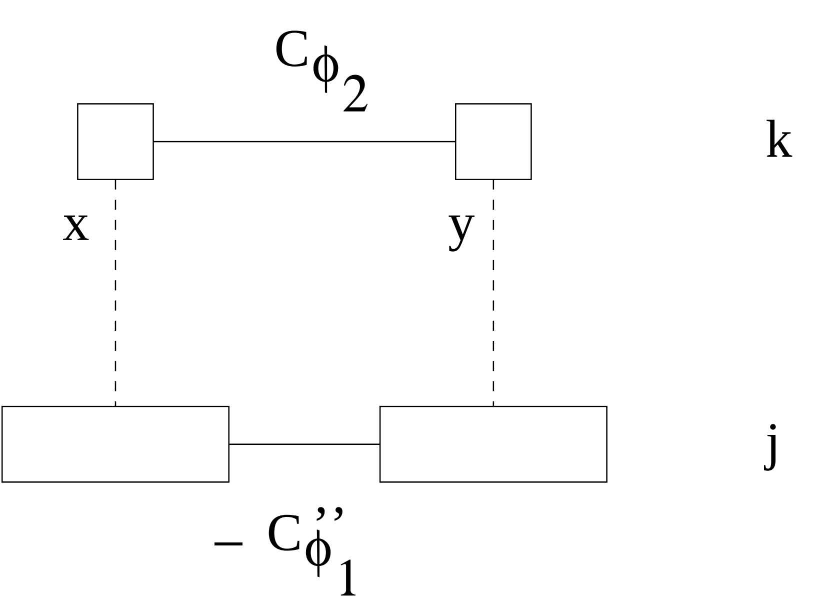

C_{ϕ_2}
k
x
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$- C_{\phi_1}''$