A note on Gaussian curvature of harmonic functions

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Abstract

It is proved that the fundamental group of the space of harmonic polynomials of degree $n(n \geq 2)$, with the same Gaussian curvature is not trivial. Furthermore, we give an example of topologically nonequivalent conjugate harmonic functions having the same Gaussian curvature.

1 Topological equivalency of harmonic functions

Let $w = u(x, y)$ be (at least) a C^2 harmonic function of real variables (x, y), defined on a region Ω of $\mathbb{R}^2$. It is well known that its Gaussian curvature, denoted by $k(x, y)$, can be given by $k(x, y) = (u_{xx}u_{yy} - u_{xy}^2)/(1 + u_x^2 + u_y^2)$. Set $z = x + iy$ and rewrite $w = u(x, y)$ as follows:

$$u(x, y) = \operatorname{Re} f(z) = \frac{1}{2}(f(z) + \overline{f(z)}).$$

We follow the next notations on holomorphic function $f(z)$ of complex variable z:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\,\frac{\partial}{\partial y}\right),\ \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\,\frac{\partial}{\partial y}\right),$$

and the next rules

$$\frac{\partial f}{\partial z} = f^{'}(z),\ \frac{\partial f}{\partial \overline{z}} = 0,\ \frac{\partial \overline{f}}{\partial z} = 0, \frac{\partial \overline{f}}{\partial \overline{z}} = \overline{f}^{\prime}(z).$$

It is easily seen that the Gaussian curvature of the graph obtained from $w = u(x, y)$ can be write as follows:

$$K(x, y) = \frac{|f^{''}(z)|^2}{-(1 + |f^{'}(z)|^2)^2}. \tag{1}$$

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We recall that a critical point of a smooth function $w = g(x_1, x_2)$ is called nondegenerate, if the determinant of Hessian matrix $\left(\dfrac{\partial^2 g(x_1, x_2)}{\partial x_i \partial x_j}\right)$ dose not vanish. Otherwise, it is called degenerate critical point. The following fact is about curvature function.

Lemma 1.1. *The Gaussian curvature of the graph obtained from a harmonic function $w = u(x, y)$, defined on a region Ω is nonpositive, and vanishes only in isolated points which are degenerate critical points of w.*

Proof. Let $f(x + \mathrm{i}\, y)$ be a holomorphic function, and $w = u(x, y)$ its real part. The Gaussian curvature of the graph of $w = u(x, y)$ vanishes when $f''(x+\mathrm{i}\, y) = 0$. Since $f(x, y)$ is holomorphic, it follows that $f''(x+\mathrm{i}\, y)$ is holomorphic therefore, its zeros are isolated. Moreover, critical points of $f(x+\mathrm{i}\, y)$ and $w = u(x, y)$ coincide. The zeros of $f''(x+\mathrm{i}\, y) = 0$ are critical points of $f(x+\mathrm{i}\, y)$ and hence critical points of $w = u(x, y)$. From the equation $K(x, y) = \dfrac{u_{xx}u_{yy} - u_{xy}^2}{(1 + u_x^2 + u_y^2)^2} = 0$, it follows $u_{xx}u_{yy} - u_{xy}^2 = 0$, if and only if when critical points are degenerate. □

Remark 1.1. *Suppose $w = u(x, y)$ is a harmonic function of degree $n(n > 2)$. Then it is obvious that the cardinality of the set Γ, of points at which the Gaussian curvature of the graph of $w = u(x, y)$ are zero, is not greater than $n - 2$. By theorem Gauss-Lucas (cf. [1]) points of Γ lie within the convex hull Δ, containing the zeros of holomorphic function $f(x + \mathrm{i}\, y) = u(x, y) + \mathrm{i}\, v(x, y)$. If the multiplicity of the zeros of $f(x + \mathrm{i}\, y)$ is not greater than two, then the points of Γ lie inside the convex hall $\delta \subset \Delta$, containing the the zeros of $f'(x + \mathrm{i}\, y)$.*

Remark 1.2. *It is immediate from the Equation 1 that if $f = u+\mathrm{i}\, v$ is a holomorphic function, then the Gaussian curvature of the graphs of u,v and f are the same, see [2].*

Recall that two smooth functions $w = q(x, y)$ and $v = r(x, y)$, are defined on a region Ω of $\mathbb{R}^2$, are called topologically equivalence if there exists homomorphism $\phi : \Omega \to \Omega$ and $\psi : \mathbb{R} \to \mathbb{R}$ such that $q(x, y) \circ k = v(x, y) \circ l$.

Proposition 1.1. *There exist topological nonequivalent conjugate harmonic functions having the same Gaussian curvature.*

Proof. It is well known that nondegenerate critical points are preserved by topological equivalency of smooth functions. Consider conjugate harmonic functions $u(x, y) = x^3 - 3xy^2 - 3x$ and $v(x, y) = -y^3 + 3x^2y - 3y$. Each of them have two nondegenerate critical points,$(\pm 1, 0)$.

But for the function $u(x,y)$ these critical points lie in different level lines, and for the function $v(x,y)$ critical points lie on one level line. Consequently, they are not topologically equivalent, but in light of Remark 1.2, their Gaussian curvature are the same. $\square$

Note that harmonic polynomials of different degree are never topologically equivalent; for each n there exists finite number of topologically nonequivalent harmonic polynomials $P^n(x,y)$ of degree n.

2 Gaussian Curvature of harmonic polynomials

Lemma 2.1. *The Gaussian curvature of the graph of harmonic polynomials of different degrees are always different.*

Proof. Let $Q(z)$ and $P(z)$ be complex polynomials of degree n and m resp., and let $u(x,y)$ and $v(x,y)$ be their real parts, resp. By assumption $m \neq n$. Functions $|P(z)|$ and $|Q(z)|$ attend to $|z^n|$ and $|z^m|$, resp., when z goes to infinity. Employing this fact for $K(x,y) = \dfrac{|f''(z)|^2}{-(1+|f'(z)|^2)^2}$, we obtain that

$$\frac{|P''(z)|^2}{-(1+|P'(z)|^2)^2} \neq \frac{|Q''(z)|^2}{-(1+|Q'(z)|^2)^2}.$$

Thus, the Gaussian curvature of polynomials $u(x,y)$ and $v(x,y)$ are different. $\square$

Theorem 2.1. *Suppose $P(z)$ and $Q(x)$ are complex polynomials. Then* $\dfrac{|P''(z)|^2}{-(1+|P'(z)|^2)^2}$ *and* $\dfrac{|Q''(z)|^2}{-(1+|Q'(z)|^2)^2}$ *coincide if and only if $P(z) = \alpha Q(z) + \beta$, where β is constant and $|\alpha| = 1$.*

Proof. **Sufficiency:** If $P(z) = \alpha Q(z) + \beta$, then from the Formula 1 it follows the conclusion of the theorem.

Necessity: We only need to show that $\dfrac{|P''(z)|}{1+|P'(z)|^2}$ and $\dfrac{|Q''(z)|}{1+|Q'(z)|^2}$ are equal. Put $P'(z) := n(z)$, obviously $n(z)$ is polynomial. Assume $\gamma(t)$ lies in a region where $n(z)$ is defined, with the initial point z_0 and the end point z, consider the integral

$$\int_{\gamma(t)} \frac{|n'(z)||d(z)|}{1+|n(z)|^2}.$$

we have

$$\int_{\gamma(t)} \frac{|n'(z)||d(z)|}{1+|n(z)|^2} = \int_{\gamma(t)} \frac{d(|n(z)|)}{1+|n(z)|^2} = \arctan|n(z)| + const.$$

Similarly, set $Q^{'}(z) = m(z)$ and perform the same arguments, we obtain

$$\int_{\gamma(t)} \frac{|m^{'}(z)||d(z)|}{1+|m(z)|^2} = \int_{\gamma(t)} \frac{d(|m(z)|)}{1+|m(z)|^2} = \arctan|m(z)| + const.$$

Thus, $\arctan|n(z)| = \arctan|m(z)| + \beta$, where β is constant, and therefore, $|n(z)| = |m(z)| + \beta$. Now we can easily show that $P(z) = \alpha Q(z) + \beta$, where $|\alpha| = 1$. $\square$

Remark 2.1. *Theorem 2.1 is not valid for any holomorphic functions; see examples in [2].*

Let $w = P(x, y)$ and $v = Q(x, y)$ be conjugate harmonic polynomials, if the parameter t varies in the unit circle in the complex plane. Then 1-parameter family of polynomials $\cos(t)u(x, y) - \sin(t)v(x, y)$ forms a loop in the space of harmonic polynomials having the same Gaussian curvature. It is easily seen that the functions $\pm P(x, y)$ and $\pm Q(x, y)$ lie in the loop. Hence, from Theorem 2.1 and Lemma 1.1 follows the next fact:

Proposition 2.1. *The fundamental group of the space of harmonic polynomials of degree* $n(n \geq 2)$*, with the same Gaussian curvature is not trivial.*

References

[1] M. Marden. The geometry of the zeros of a polynomial in complex variable. American Mathematical Society surveys 3, New York, 1949.

[2] J. Shomberg. A note on surfaces with radially symmetric nonpositive Gaussian curvature. Mathematica Bohemica. 130, P. 167 - 176, 2005.