A note on Ward-Horadam *H(x)*-binomials' recurrences

Andrzej Krzysztof Kwaśniewski

Member of the Institute of Combinatorics and its Applications, Winnipeg, Manitoba, Canada
PL-15-674 Białystok, Konwaliowa 11/11, Poland
e-mail: kwandr@gmail.com

Summary

As a matter of continuation of [45, 2010] - we deliver here *H(x)*-binomials' recurrence formula appointed by Ward-Horadam $H(x) = \langle H_n(x) \rangle_{n\geq 0}$ functions' sequence which comprises in $H \equiv H(x = 1)$ number sequences case the *V*-binomials' recurrence formula determined by the primordial Lucas sequence of the second kind $V = \langle V_n \rangle_{n\geq 0}$ as well as its well elaborated companion fundamental Lucas sequence of the first kind $U = \langle U_n \rangle_{n\geq 0}$ which gives rise in its turn to the *U*-binomials' recurrence as in [1, 1878] , [6, 1949], [8, 1964], [10, 1969], [14, 1989] or in [15, 1989] etc.

For the sake of combinatorial interpretations and in number theory $H(x = 1)$, $H_n(x = 1) \equiv H_n$ is usually considered to be natural or integer numbers valued sequence. Number sequences $H = H(x = 1) = \langle H_n \rangle_{n\geq 0}$ were recently called by several authors: Horadam sequences.

The list of references is mostly indicatory (see references therein) and is far from being complete.

AMS Classification Numbers: 05A10 , 05A30.

Keywords: extended Lucas polynomial sequences, generalized multinomial coefficients.

1 General Introduction

1.1. p, q **people are Lucas' followers people.** The are many authors who use in their investigation the fundamental Lucas sequence $U \equiv \langle n_{p,q} \rangle_{n\geq 0}$ - frequently with different notations - where $n_{p,q} = \sum_{j=0}^{n-1} p^{n-j-1} q^j = U_n$; see Definition 1 and then definitions that follow it. In regard to this a brief intimation is on the way.

To our knowledge it was François Édouard Anatole Lucas in [1, 1878] who was the first who had *not only* defined *fibonomial* coefficients as stated in [15, 1989] by Donald Ervin Knuth and Herbert Saul Wilf but who was the first who had defined $U_n \equiv n_{p,q}$-binomial coefficients $\binom{n}{k}_U \equiv \binom{n}{k}_{p,q}$ and had derived a recurrence for them: see page 27, formula (58) [1, 1878]. Then - referring to Lucas - the investigation relative to divisibility properties of relevant number Lucas sequences D, S as well as numbers' D - binomials and numbers' D - multinomials was continued in [82, 1913] by Robert Daniel Carmichel; see pp. 30, 35 and 40 in [82, 1913] for $U \equiv D = \langle D_n \rangle_{n \geq 0}$ and $\binom{n}{k_1, k_2, \ldots, k_s}_D$ - respectively. Note there also formulas (10) , (11) and (13) which might perhaps serve to derive explicit untangled form of recurrence for the *V- binomial coefficients* $\binom{n}{k}_V \equiv \binom{n}{k}_S$ denoted by primordial Lucas sequence $\langle S_n \rangle_{n \geq 0} = S \equiv V$. Number sequence $F(x = 1 = A)$ A - multinomial coefficients' *recurrences* are not present in that early works and up to our knowledge a special case of such appeared at first in [62, 1979] by Anthony G. Shannon. More on that - in what follows after Definition 3.

Significant peculiarity of Lucas originated sequences includes their importance for number theory (see middle-century paper [95] by John H. Halton and recent, this century papers [118, 2010] by Chris Smith and [119, 2010] by Kálmán Györy with Chris Smith and the reader may enjoy also the PhD Thesis [128, 1999] by Anne-Marie Decaillot-Laulagnet). This Lucas originated investigation amalgamates diverse areas of mathematics due to hyberbolic - trigonometric character of these Fonctions Numériques Simplement Priodiques i.e. fundamental and primordial Lucas sequences - as beheld in [1, 1878]. One may track then a piece of further constructions for example in [25, 1999]).

There in [25, 1999] tail formulas (3.12) and (3.14) illustrating proved and exploited by Éduard Lucas the complete analogy of the V_n and U_n symmetric functions of roots with the circular and hyperbolic functions of order 2 due to Lucas formulas (5) in [1] rewritten in terms of $\cosh$ and $\sinh$ functions as formulas (3.13) and (3.14) in [25] as resulting from de Moivre one parameter group introduced in [25] via (1.4) in order to pack compactly addition formulas (1.6), (1.7) in [25] equivalent to (49) and corresponding recurrence relations in [1] into abelian group "parcel" encompassing Tchebycheff polynomials of both kinds.

In this connection see the Section 2 in the recent Ward-Horadam people paper [79, 2009] by Tian-Xiao He, Peter Jau-Shyong Shiue. There in Proposition 2.7. illustrative Example 2.8. with Tchebycheff polynomials of the first kind the well known recurrence formula (2.28) is equivalent to abelian one-parameter de Moivre matrix group multiplication rule from which the corresponding recurrence (1.7) in [25, 1999] follows.

2.1. As has been foreshadowed in [45, 2010] we deliver here - continuing the note [45] - the *H(x)*-binomials' recurrence formula appointed by Ward-Horadam $H(x) = \langle H_n(x) \rangle_{n \geq 0}$ field of zero characteristic nonzero valued functions' sequence which comprises for $H \equiv H(x = 1)$ number sequences case - the *V*-binomials' recurrence formula determined by the primordial Lucas sequence of the second kind $V = \langle V_n \rangle_{n \geq 0}$ [45, 2010] as well as its well elaborated companion fundamental Lucas sequence of the first kind $U = \langle U_n \rangle_{n \geq 0}$ which gives rise in its turn to the *U*-binomials' recurrence as in [1, 1878] , [6, 1949], [8, 1964], [10, 1969], [14, 1989] or in [15, 1989] and so on.

We do it by following recent applicable work [2, 2009] by Nicolas A. Loehr and Carla D. Savage thought one may - for that purpose - envisage now easy extensions of particular p, q - cases considered earlier - as *for example* the following: the relevant recursions in [6, 1949], in [14, 1989], in [18, 1992] - (recursions (40) and (51)) , or [114, 2000] by John M. Holte (Lemmas 1,2 dealing with U-binomials provide a motivated example for observation Theorem 17 in [2]) One is invited also to track Lemma 1 in [115, 2001] by Hong Hu and Zhi-Wei Sun ; see also corresponding recurrences for p, q-binomials $\equiv U$-binomials in [1, 1878] or in [44, 2008] **v[1]** by Maciej Dziemiańczuk (compare there (1) and (2) formulas), or see Theorem 1 in [41, 2008] by Roberto Bagsarsa Corcino as well as track the proof of the Corollary 3. in [44, 2009] **v[2]** by Maciej Dziemiańczuk.

This looked for here *H(x)*-binomials' recurrence formula (recall: encompassing *V*-binomials for primordial Lucas sequence V) is not present neither in [1] nor in [2], nor in [3, 1915] , nor in [5, 1936], nor in [6, 1949]. Neither we find it in - quoted here contractually by a nickname as "Lucas (p, q)-people" - references [1-44]. Neither it is present in all other - quoted here contractually by a nickname as "Ward-Horadam -people" - references [49-79]. Ad "Lucas (p, q)-people" and "Ward-Horadam -people" references - (including these [n] with $n > 73$- the distinction which are which is quite contractual. The nicknames are nevertheless indicatively helpful. We shall be more precise soon - right with definitions are being started.

Meanwhile *H(x)*-binomials' recurrence formula for the Ward-Horadam sequence $H(x) = \langle H_n(x) \rangle_{n \geq 0}$ follows straightforwardly from the easily proved important observation - the Theorem 17 in [2, 2009] as already had it been remarked in [45, 2010] for the $H \equiv H(x = 1)$ case.

This paper formula may and should be confronted with Fontené obvious recurrence for complex valued A-binomials $\binom{n}{k}_A$, $A \equiv A(x = 1)$ in [3, 1915] i.e. with (6) or (7) identities in [10, 1969] by Henri W. Gould or with recurrence in [27, 1999] by Alexandru Ioan Lupas , which particularly also stem easily just from the definition of **any** $F(x)$-binomial coefficients arrays with $F(x) = \langle F_n(x) \rangle_{n \geq 0}$ staying for any field of characteristic zero nonzero valued functions'sequence ; $F_n(x) \neq 0, \ n \geq 0$. For $F = F(x = 1)$-multinomial coefficients automatic definition see [82, 1913] by Robert Daniel Carmichel or then [10, 1969] by Henri W. Gold and finally see [62, 1979] by Anthony G. Shannon, where recurrence is proved for $\binom{n}{k_1, k_2, \ldots, k_s}_U$ with U-Lucas fundamental being here complex valued number sequence. For $F(x)$ - multinomial coefficients see [46, 2004] and compare with $F(x)$-binomials from [27, 1999] or those from [47, 2001].

To this end we supply now two informations pertinent ad references and ad nomenclature.

3.1. Ad the number theory and divisibility properties references. For the sake of combinatorial interpretations of F- number sequences as well as their correspondent F-multinomial coefficients and also for the sake of the number theoretic studies of Charles Hermite [80] and with Thomas Jan Stieltjes in [81] or by Robert Daniel Carmichel [82, 1913] or [83, 1919] or that of Ward [88, 1936], [89, 1939], [90, 1937], [91, 1937], [49, 1954], [92, 1955], [93, 1959] and that of Lehmer [84, 1930], [85, 1933], [86, 1935] or this of Andrzej Bobola Maria Schinzel [97, 1974] and Others' studies *on divisibility properties* - these are the sub-cases $F_n \in \mathbb{N}$ or $F_n \in \mathbb{Z}$ which are being regularly considered at the purpose.

As for the "Others" - see *for example*: [94, 1959], [96, 1973], [98, 1974], [99, 1974], [100, 1974], [101, 1973], [61, 1977], [102, 1977], [63, 1979], [103, 1979], [104, 1980], [64, 1980], [15, 1989], [105, 1991], [109, 1992], [106, 1995], [107, 1999], [108], [110, 1995], [111, 1995], [112, 1995], [113, 1998], [115, 2001], [116, 2006], [117, 2009].

3.4. Ad Ward-Horadam naming. According to the authors of [79, 2009] it was Mansour [76] who called the sequence $H = \langle a_n \rangle\, n \geq 0$ defined by (1) a **Horadam's sequence**, as - accordingly to the author of [76] - the number sequence H was introduced in 1965 by Horadam [52] (for special case of Ward-Horadam number sequences see Section 2 in [59, 1974] and see also [77, 2009]), this however notwithstanding the ingress of complex numbers valued F-*binomials* and F-*multinomials* into Morgan Ward's systematic *Calculus of sequences* in [5, 1936] and then in 1954 Ward's introduction of "'*nomen omen*"' $W \equiv H$ in [49, 1954] integer valued sequences.

Perceive then the appraisal of adequate Morgan Wards' work in the domain by Henri W.Gould [10, 1959] and by Alwyn F. Horadam and Anthony G. Shannon in [60, 1976] or Derrick Henry Lehmer in [87, 1993]. On this occasion note also the **Ward-Horadam** number sequences **in** [50, 1965] and [53, 1965].

2 Preliminaries

Names: The Lucas sequence $V = \langle V_n \rangle_{n \geq 0}$ is called the Lucas sequence of the second kind - see: [61, 1977, Part I], or **primordial** - see [63, 1979].
The Lucas sequence $U = \langle U_n \rangle_{n \geq 0}$ is called the Lucas sequence of the first kind - see: [61, 1977, Part I], or **fundamental** - see p. 38 in [6, 1949] or see [62, 1979] and [63, 1979].

In the sequel we shall deliver the looked for recurrence for H-binomial coefficients $\binom{n}{k}_H$ determined by the Ward-Horadam sequence H - defined below.

In compliance with Edouard Lucas' [1, 1878] and twenty, twenty first century p,q-people's notation we shall at first review here in brief the general second order recurrence; (compare this review with the recent "Ward-Horadam" peoples' paper [79, 2009] by Tian-Xiao He and Peter Jau-Shyong Shiue or earlier p,q-papers [30, 2001] by Zhi-Wei Sun, Hong Hu, J.-X. Liu and [115, 2001] by Hong Hu and Zhi-Wei Sun). And with respect to natation: If in [1, 1878] François Édouard Anatole Lucas had been used $a = \mathbf{p}$ and $b = \mathbf{q}$ notation, he would be perhaps at first glance notified and recognized as a Great Grandfather of all the (p,q) - people. Let us start then introducing reconciling and matched denotations and nomenclature.

$$(1) \qquad H_{n+2} = P \cdot H_{n+1} - Q \cdot H_n, \ \ n \geq 0 \ and \ H_0 = a, \ H_1 = b.$$

which is sometimes being written in $\langle P, -Q\rangle \mapsto \langle s,t\rangle$ notation.

$$(2) \qquad H_{n+2} = s \cdot H_{n+1} + t \cdot H_n, \ \ n \geq 0 \ and \ H_0 = a, \ H_1 = b.$$

Simultaneously and collaterally we mnemonically pre adjust the starting point to discuss the $F(x)$ polynomials' case via - if entitled - antecedent "$\mapsto$ action": $H \mapsto H(x)$, $s \mapsto s(x)$, $t \mapsto t(x)$, etc.

$$(3) \quad H_{n+2}(x) = s(x) \cdot H_{n+1}(x) + t(x) \cdot H_n, \ n \geq 0, \ H_0 = a(x), H_1 = b(x).$$

enabling recovering explicit formulas also for sequences of polynomials correspondingly generated by the above linear recurrence of order 2 - with Tchebysheff polynomials and the generalized Gegenbauer-Humbert polynomials included. See for example Proposition 2.7 in the recent Ward-Horadam peoples' paper [79, 2009] by Tian-Xiao He and Peter Jau-Shyong Shiue.

The general solution of (1): $H(a,b;P,Q) = \langle H_n \rangle_{n\geq 0}$ is being called throught this paper - **Ward-Horadam number'sequence**.

The general solution of (3): $H(x) \equiv H(a, b(x); s(x), t(x)) = \langle H_n(x)\rangle_{n\geq 0}$ is being called throughout this paper - **Ward-Horadam functions' sequence**. It is then to be noted here that ideas germane to *special Ward-Horadam polynomials sequences* of the [71] paper were already explored in some details in [52]. For more on special Ward-Horadam *polynomials sequences* by Alwyn F. Horadam - consult then: [57], [65, 1985], [66] , [72] or see *for example* the following papers and references therein: recent papers [77, 2009] by Tugba Horzum and Emine Gökcen Kocer and [78, 2009] by Gi-Sang Cheon, Hana Kim and Louis W. Shapiro. For *Ward-Horadam functions sequences* [79, 2009] by Tiang-Xiao He and Peter J. -S. Shiue who however there then concetrate on on *special Ward-Horadam polynomials sequences* only.

In [127, 2010] Johann Cigler considers special Ward-Horadam *polynomials sequences* and among others he supplies the tiling combinatorial interpretation of these special Ward-Horadam *polynomials sequences* which are q-analogues of the Fibonacci and Lucas polynomials introduced in [125, 2002] and [126, 2003] by Johann Cigler.

In the paper [75, 2003] Johann Cigler introduces "abstract Fibonacci polynomials" - interpreted in terms of Morse coding sequences monoid with concatenation (monominos and dominos tiling then) Cigler's abstract Fibonacci polynomial sare monoid algebra over reals valued polynomials with straightforward Morse sequences i.e. tiling recurrence originated (1.6) "addition formula"

$$F_{m+n}(a,b) = F_{m+1}(a,b)\cdot F_m(a,b) + b\cdot F_{n-1}(a,b)\cdot F_n(a,b),$$

which is attractive and seductive to deal with within the context of this paper Theorem 1 below.

From the characteristic equation of (1)

$$x^2 = P\cdot x - Q, \tag{4}$$

written by some of $p.q$-people as

$$x^2 = s\cdot x + t \tag{5}$$

we readily find the Binet form solution of (1) (see (6) in [77, 2009]) which is given by (6) and (7).

$$(6)\quad H_n(a,b;P,Q) \equiv H_n(A,B;p,q) = Ap^n + Bq^n,\ n\geq 0, H_0 = A, H_1 = B.$$

where p,q are roots of (3) and we have assumed since now on that $p\neq q$. As for the case $p=q$ included see for example Proposition 2.1 in [79, 2009] and see references therein.

Naturally : $p+q = P\equiv s$, $p\cdot q = Q\equiv -t$ and

$$A = \frac{b-qa}{p-q}\ ,\ B = -\frac{b-pa}{p-q}. \tag{7}$$

hence we may and we shall use the following conventional identifications-abbreviations :

$$H\equiv H(a,b;P,Q)\equiv H(A,B;P,Q)\equiv H(A,B;p,q). \tag{8}$$

It is obvious that the exponential generating function for Ward-Horadam sequence H reads:

$$E_H(A,B;p,q)[x] = Aexp[p \cdot x] + Bexp[q \cdot x]. \tag{9}$$

The derivation of the formula for ordinary generating function for Ward-Horadam sequence is a standard task and so we have (compare with (5) in [77])

$$G_H(a,b;P,Q)[x] = \frac{a+(b-aP)x}{1-P\cdot x+Q\cdot x^2} = \frac{a+(b-a[p+q])x}{1-P\cdot x+p\cdot q\cdot x^2}. \tag{10}$$

where from we decide an identification-abbreviation

$$G_H[x] \equiv G_H(A,B;p,q)[x] \equiv G_H(a,b;P,Q)[x].$$

Naturally - in general $H(A,B;p,q) \neq H(A,B;q,p)$. If $H(A,B;p,q) = H(A,B;q,p)$ we then call the Ward-Horadam sequence symmetric and thus we arrive to Lucas *Théorie des Fonctions Numériques Simplement Priodiques* [1, 1878].

In [1, 1878] Edouard Lucas considers Lucas sequence of the second kind $V = \langle V_n \rangle_{n\geq 0}$ (second kind - see: [61, 1977, Part I]) as well as its till now well elaborated companion Lucas sequence of the first kind $U = \langle U_n \rangle_{n\geq 0}$ (first kind - see: [61, 1977, Part I]) which gives rise in its turn to the *U*-binomials' recurrence (58) in [1, 1878] (see then [6, 1949], [8, 1964], [10, 1969], [14, 1989] or in [15, 1989] etc.)

These sequences i.e ($A = B = 1$) the Lucas sequence of the second kind

$$H_n(2,P;p,q) = V_n = p^n + q^n. \tag{11}$$

and ($A = -B = 1$) the Lucas sequence of the first kind

$$H_n(0,1;p,q) = U_n = \frac{p^n - q^n}{p-q}, \tag{12}$$

where called by Lucas [1, 1878] the *simply periodic numerical functions* because of

[quote] *at the start, the complete analogy of these symmetric functions with the circular and hyperbolic functions.* [end of quote].

More ad Notation 1. The letters a,b $a \neq b$ in [1, 1878] denote the roots of the equation $x^2 = Px - Q$ then $(a,b) \mapsto (u,v)$ in [2, 2009] and u,v stay there for the roots of the equation $x^2 = \ell x - 1$.

We shall use here the identification $(a,b) \equiv (p,q)$ i.e. p,q denote the roots of $x^2 = Px - Q$ as is common in "'Lucas (p,q)-people"' publications.
For Lucas (p,q)-people then the following U-identifications are expediency natural:

Definition 1

$$(13) \quad n_{p,q} = \sum_{j=0}^{n-1} p^{n-j-1}q^j = U_n = \frac{p^n - q^n}{p-q}, \ 0_{p,q} = U_0 = 0, \ 1_{p,q} = U_1 = 1,$$

where p,q denote now the roots of the equation $x^2 = P \cdot x - Q \equiv x^2 = sx + t$ hence $p + q = s \equiv P$, $pq = Q \equiv -t$ and the empty sum convention was used for $0_{p,q} = 0$. Usually one assumes $p \neq q$. In general also $s \neq t$ - though according to the context [14, 1989] $s = t$ may happen to be the case of interest.

The Lucas U-binomial coefficients $\binom{n}{k}_U \equiv \binom{n}{k}_{p,q}$ are then defined as follows: ([1, 1878], [3, 1915], [5, 1936], [6, 1949], [8, 1964], [10, 1969] etc.)

Definition 2 *Let U be as in [1, 1878] i.e $U_n \equiv n_{p,q}$ then U-binomial coefficients for any $n,k \in \mathbb{N} \cup \{0\}$ are defined as follows*

$$(14) \qquad \binom{n}{k}_U \equiv \binom{n}{k}_{p,q} = \frac{n_{p,q}!}{k_{p,q}! \cdot (n-k)_{p,q}!} = \frac{n_{p,q}^{\underline{k}}}{k_{p,q}!}$$

where $n_{p,q}! = n_{p,q} \cdot (n-1)_{p,q} \cdot \ldots \cdot 1_{p,q}$ and $n_{p,q}^{\underline{k}} = n_{p,q} \cdot (n-1)_{p,q} \cdot \ldots \cdot (n-k+1)_{p,q}$.

Definition 3 *Let V be as in [1, 1878] i.e $V_n = p^n + q^n$, hence $V_0 = 2$ and $V_n = p + q = s$. Then V-binomial coefficients for any $n,k \in \mathbb{N} \cup \{0\}$ are defined as follows*

$$(15) \qquad \binom{n}{k}_V = \frac{V_n!}{V_k! \cdot V_(n-k)!} = \frac{V_n^{\underline{k}}}{V_k!}$$

where $V_n! = V_n \cdot V_{n-1} \cdot \ldots \cdot V_1$ and $V_n^{\underline{k}} = V_n \cdot V_{n-1} \cdot \ldots \cdot V_{n-k+1}$.

One automatically generalizes number F-binomial coefficients' array to functions $F(x)$-**multinomial** coefficients' array (see [46, 2004] and references to umbral calculus therein) while for *number sequences* F the $F = F(x=1)$-multinomial coefficients see p. 40 in [82, 1913] by Robert Daniel Carmichel , see [5, 1936] by Morgan Ward, [10, 1969] by Henri W. Gould or [62, 1979]) by Anthony G. Shannon where recursion for $U = F(x=1)$-multinomial coefficients is provided and where there specifically U denotes the fundamental Lucas sequence (i.e. the Lucas sequence of the first kind) and see also important paper [105, 1991] by Shiro Ando and Daihachiro Sato. The x-Fibonomial coefficients from [47, 2001] by Thomas M. Richardson are motivating example of functions $F(x)$-**binomial** coefficients' array from [46, 2004].

Definition 4 *Let $F(x)$ be any natural, or complex numbers' non zero valued functions' sequence i.e. $F_n(x) \in \mathbb{N}$ or and $F_n(x) \in \mathbb{C}$. The $F(x)$-***multinomial coefficient** *is then identified with the symbol*

$$\binom{n}{k_1, k_2, ..., k_s}_{F(x)} = \frac{F_n(x)!}{F_{k_1}(x)! \cdot ... \cdot F_{k_s}(x)!} \tag{16}$$

where $k_i \in \mathbb{N}$ and $\sum_{i=1}^{s} k_i = n$ for $i = 1, 2, ..., s$. Otherwise it is equal to zero,

and where $F_r(x)! = F_r(x) \cdot F_{r-1}(x) \cdot ... \cdot F_1(x)$.

Naturally for any natural n, k and $k_1 + ... + k_m = n - k$ the following holds

$$\binom{n}{k}_{F(x)} \cdot \binom{n-k}{k_1, k_2, ..., k_m}_{F(x)} = \binom{n}{k, k_1, k_2, ..., k_m}_{F(x)}, \tag{17}$$

$$\binom{n}{k_1, k_2, ..., k_m}_{F(x)} = \binom{n}{k_1}_{F(x)} \binom{n-k_1}{k_2}_{F(x)} \cdots \binom{n - k_1 - \cdots - k_{m-1}}{k_m}_{F(x)}.$$

More ad Notation 2.

We shall use further on the traditional , XIX-th century rooted notation under presentation in spite of being inclined to quite younger notation from [43, 2010] by Bruce E. Sagan and Carla D. Savage. This wise, economic notation is ready for straightforward record of combinatorial interpretations and combinatorial interpretations' substantiation in terms of popular text book tiling model since long ago used for example to visualize recurrence for Fibonacci-like sequences ; see for example [143, 1989] by Ronald Graham, Donald Ervin Knuth, and Oren Patashnik. The translation from François Édouard Anatole Lucas via Dov Jarden and Theodor Motzkin notation [6, 1949] and notation of Bruce E. Sagan and Carla D. Savage [43, 2010] is based on the succeeding identifications: the symbol used for U-**binomials** is $C\{...\}$ in place of $(...)_U$, the would be symbol for V-**binomials** i.e. $P\langle ... \rangle$ in place of $(...)_V$ is not considered at all in [43] while

$$\{n\} \equiv U_n \equiv n_{p,q}, \;\; \langle n \rangle \equiv V_n.$$

In Bruce E. Sagan and Carla D. Savage notation we would then write down the fundamental and primordial sequences' binomial coefficients as follows.

Definition 5 *Let $\{n\}$ be fundamental Lucas sequence as in [1, 1878] i.e $\{n\} \equiv U_n \equiv n_{p,q}$ then $\{n\}$-binomial coefficients for any $n, k \in \mathbb{N} \cup \{0\}$ are defined as follows*

$$F\{n, k\} = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{p,q} = \frac{\{n\}!}{\{k\}! \cdot \{n-k\}!} = \frac{\{n\}^{\underline{k}}}{\{k\}!} \tag{18}$$

where $\{n\}! = \{n\} \cdot \{n-1\} \cdot \ldots \cdot \{1\}$ *and* $\{n\}^{\underline{k}} = \{n\} \cdot \{n-1\} \cdot \ldots \cdot \{n-k+1\}$,

Definition 6 *Let* $\langle n \rangle$ *be primordial Lucas sequence as in [1, 1878] i.e* $\langle n \rangle \equiv V_n$ *then* $\langle n \rangle$*-binomial coefficients for any* $n, k \in \mathbb{N} \cup \{0\}$ *are defined as follows*

$$P\langle n, k \rangle = \left\langle {n \atop k} \right\rangle_{p,q} = \frac{\langle n \rangle !}{\langle k \rangle ! \cdot \langle n-k \rangle !} = \frac{\langle n \rangle^{\underline{k}}}{\langle k \rangle !}, \tag{19}$$

where $\langle n \rangle ! = \{n\} \cdot \{n-1\} \cdot \ldots \cdot \{1\}$ *and* $\{n\}^{\underline{k}} = \{n\} \cdot \{n-1\} \cdot \ldots \cdot \{n-k+1\}$.

The above consequent symbols $\left\{ {n \atop k} \right\}_{p,q}$ and $\left\langle {n \atop k} \right\rangle_{p,q}$ are - in not exceptional conflict - with second kind Stirling numbers notation and Euler numbers notation respectively in the spirit of [143, 1989] what extends on both p, q - extensions' notation.
Regarding the symbol $\left\{ {n \atop k} \right\}_{p,q}$ one draws the attention of a reader to [9, 1967] where Verner Emil Hoggatt, Jr. considers the C-binomial coefficients with indices in an arithmetic progression denoting them by symbols $\left\{ {n \atop k} \right\}_{u,k}$, where $\{u_n\}_{n \geq 0} = U$ with Ubeing the primordial Lucas sequence. For $\left\{ {n \atop k} \right\}$ corresponding notation see also: [10, 1969] by Henri W.Gould, [14, 1989] by Ira M. Gessel and Xavier Gérard Viennot , [36, 2005], [37, 2005], [38, 2006] by Jaroslav Seibert and Pavel Trojovský and [39, 2007] by Pavel Trojovský.

Whereas as in the *subset-subspace problem* (Example [Ex. q* ; 6] in subsection 4.3.) we rather need another natural notation. Namely for $q \neq 0$ introduce $q* = \frac{p}{q}$ and observe that

$$\binom{n}{k}_U \equiv \binom{n}{k}_{p,q} = q^{k(n-k)} \binom{n}{k}_{1,q*} \overset{q* \mapsto 1}{\rightarrow} \binom{n}{k}.$$

The V-binomial $P\langle n, k \rangle = \left\langle {n \atop k} \right\rangle_{p,q} \equiv \binom{n}{k}_V$ is not considered in [43, 2010].

3 $H(x)$-binomial coefficients' recurrence

3.1. Let us recall convention resulting from (3).

Recall. The general solution of (3): $H(x) \equiv H(a, b(x); s(x), t(x)) = \langle H_n(x) \rangle_{n \geq 0}$ is being called throughout this paper - **Ward-Horadam functions' sequence**.

From the characteristic equation of the recurrence (3)

$$z^2 - s(x) \cdot z - t(x) = 0 \tag{20}$$

we readily see that for $H_0 = a(x),\ H_1 = b(x),\ n \geq 0,$

$$H_n(x) \equiv H_n(a(x), b(x); p(x), q(x)) = A(x)p(x)^n + B(x)q(x)^n, \tag{21}$$

where $p(x), q(x)$ are roots of (20) and we have assumed that $p(x) \neq q(x)$ as well as that $p(x), q(x)$ are not roots of unity. Naturally:

$$A(x) = \frac{b(x) - q(x)a(x)}{p(x) - q(x)} \ , \ B = -\frac{b(x) - p(x)a(x)}{p(x) - q(x)}. \tag{22}$$

hence we may and we shall use the following conventional identifications-abbreviations

$$H(x) \equiv H(a(x), b(x); s(x), t(x)) \equiv H(A(x), B(x); s(x), t(x)). \tag{23}$$

As for the case $p(x) = q(x)$ included see for example Proposition 2.7 in [79, 2009].

Another explicit formula for Ward-Horadam functions sequences is the mnemonically extended formula (9) from [77, 2009] by Tugba Horzum and Emine Gökcen Kocer, where here down we use contractually the following abbreviations:

$H_n(x) \equiv H_n(a(x), b(x); s(x), t(x))$, $a(x) \equiv a$, $b(x) \equiv b$, $s(x) \equiv s$ and $t(x) \equiv t$

(24)

$$H_n(x) = a \sum_{0 \leq k \leq \lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} s^{n-2k} t^k + \left(\frac{b}{s} - a\right) \sum_{0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} s^{n-2k} t^k$$

Note and compare. The recurrence (1.1) and (1.2) in [71, 1996] defines a polynomials' subclass of Ward-Horadam functions sequences defined by (3). The standard Jacques Binet form (1.8) in [71, 1996] of the recurrence (1.1) and (1.2) solution for Ward-Horadam polynomials sequences in [71, 1996] is the standard Jacques Binet form (19), (20) of the recurrence (3) solution for Ward-Horadam functions sequences.

The recurrence (2.23) in [79, 2009] by Tian-Xiao He and Peter Jau-Shyong Shiue defines exactly the class of Ward-Horadam functions' second order sequences and the standard Jacques Binet form (19), (20) of the recurrence (3) solution for Ward-Horadam functions sequences $H(x)$ constitutes the content of their Proposition 2.7. - as has been mentioned earlier. No recurrences for $H(x)$-binomials neither for $H(x = 1)$-binomials are considered.

On Binet Formula - Historical Remark. We just quote Radoslav Rasko Jovanovic's information from

$$http://milan.milanovic.org/math/english/relations/relation1.html:$$

Quotation 1 *Binet's Fibonacci Number Formula was derived by Binet in 1843 although the result was known to Euler and to Daniel Bernoulli more than a century ago. ... It is interesting that A de Moivre (1667-1754) had written about Binet's Formula, in 1730, and had indeed found a method for finding formula for any general series of numbers formed in a similar way to the Fibonacci series.*

See also the book [145, 1989] by Steven Vajda.

3.2. The authors of [2] provide an easy proof of an observation named there Theorem 17 which extends automatically to the statement that the following recurrence holds for the general case of $\binom{r+s}{r,s}_{H(x)}$ $H(x)$-binomial arrays in multinomial notation.

Theorem 1 *Let us admit shortly the abbreviations:* $g_k(r,s)(x) = g_k(r,s)$, $k = 1,2$. *Let* $s, r > 0$. *Let* $F(x)$ *be any zero characteristic field nonzero valued functions' sequence (*$F_n(x) \neq 0$*). Then*

$$\binom{r+s}{r,\,s}_{F(x)} = g_1(r,s)\cdot\binom{r+s-1}{r-1,\,s}_{F(x)} + g_2(r,s)\cdot\binom{r+s-1}{r,\,s-1}_{F(x)} \tag{25}$$

where $\binom{r}{r,0}_{F(x)} = \binom{s}{0,s}_{F(x)} = 1$ *and*

$$F(x)_{r+s} = g_1(r,s)\cdot F(x)_r + g_2(r,s)\cdot F(x)_s. \tag{26}$$

are equivalent.

On the way historical note Donald Ervin Knuth and Herbert Saul Wilf in [15, 1989] stated that Fibonomial coefficients and the recurrent relations for them appeared already in 1878 Lucas work (see: formula (58) in [1, 1878] p. 27 ; for U-binomials which "Fibonomials" are special case of). More over on this very p. 27 Lucas formulated a conclusion from his (58) formula which may be stated in notation of this paper formula (2) as follows: *if* $s, t \in \mathbb{Z}$ and $H_0 = 0$, $H_1 = 1$ *then* $H \equiv U$ and $\binom{n}{k}_U \equiv \binom{n}{k}_{n_{p,q}} \in \mathbb{Z}$. Consult also in next century [144, 1910] by Paul Gustav Heinrich Bachmann or later on - [82, 1913] by Robert Daniel Carmichel [p. 40] or [6, 1949] by Dov Jarden and Theodor Motzkin where in all quoted positions it was also shown that $n_{p,q}$ - binomial coefficients are integers - for p and q representing distinct roots of (5) with their ratio being not a root of unity.

Let us take an advantage to note that Lucas Théorie des Fonctions Numériques Simplement Périodiques i.e. investigation exactly of fundamental U and primordial V sequences constitutes the far more non-accidental context for binomial-type coefficients exhibiting their relevance at the same time to number theory and to hyperbolic trigonometry (in addition to [1, 1878] see for example [25], [26] and [28]).

It seems to be the right place now to underline that the *addition formulas* for Lucas sequences below with respective hyperbolic trigonometry formulas and also consequently U-binomials'recurrence formulas - stem from commutative ring R identity: $(x - y)\cdot(x + y) \equiv x^2 - y^2, x, y \in R$.

Indeed. Taking here into account the **U**-*addition formula* i.e. the first of two trigonometric-like L-addition formulas (42) from [1, 1878] ($L[p,q] = L = U, V$ - see also [25, 1999] by A.K.Kwaśniewski and [26], [28]) i.e.

$$(27) \qquad 2U_{r+s} = U_r V_s + U_s V_r, \quad 2V_{r+s} = V_r V_s + U_s U_r$$

one readily recognizes that the U-binomial recurrence from the Corollary 18 in [2, 2009] is a case of the U-binomial recurrence (58) [1, 1878] which may be rewritten after François Édouard Anatole Lucas in multinomial notation and stated as follows: *according to the Theorem 1 case i.e. the Theorem 2 below the following is true*

$$2U_{r+s} = U_r V_s + U_s V_r$$

is equivalent to

$$(28) \qquad 2 \cdot \binom{r+s}{r,\, s}_{n_{p,q}} = V_s \cdot \binom{r+s-1}{r-1,\, s}_{n_{p,q}} + V_r \cdot \binom{r+s-1}{r,\, s-1}_{n_{p,q}}.$$

To this end see also Proposition 2.2. in [43, 2010] and compare it with both (28) and Example 3. below.

However there is no companion V-binomial recurrence i.e. for $\binom{r+s}{r,s}_V$ neither in [1, 1878] nor in [2, 2009] as well as all other quoted papers except for [45, 2010] - up to knowledge of this note author.

Consequently then there is no $H(x)$-binomial recurrence neither in [1, 1878] nor in [2] (2009) as well as all other quoted papers except for Final remark : p.5 in [45, 2010] up to this note author knowledge.

The End of *the on the way historical note*.

The looked for $H(x)$-binomial recurrence (29) accompanied by (30-33) might be then given right now in the form of (25) adapted to - Ward-Lucas functions'sequence case notation while keeping in mind that of course the expressions for $h_k(r,s)(x)$, $k = 1, 2$ below are designated by this $F(x) = H(x)$ choice and as a matter of fact are appointed by the recurrence (3).

For the sake of commodity let us admit shortly the abbreviations: $h_k(r,s)(x) = h_k(r,s) = h_k$, $k = 1, 2$. Then for $H(x)$ of the form (21) we evidently have what follows.

Theorem 2.

$$(29) \qquad \binom{r+s}{r,\, s}_{H(x)} = h_1(r,s)\binom{r+s-1}{r-1,\, s}_{H(x)} + h_2(r,s)\binom{r+s-1}{r,\, s-1}_{H(x)},$$

where $p(x) \neq q(x)$ and $\binom{r}{r,0}_{H(x)} = \binom{s}{0,s}_{H(x)} = 1$, *is equivalent to*

$$H(x)_{r+s} = h_1(r,s)H(x)_r + h_2(r,s)H(x)_s. \tag{30}$$

where $H_n(x)$ is explicitly given by (21) and (22). **The end** of the Theorem 2.

There might be various $h_1(r,s)(x) = h_1$ and $h_2(r,s)(x)_s = h_2$ solutions of (30) and (21). Compare (38) in Example 1 with (42) in Example 3 below. As the possible $h_1(r,s)(x) = h_1$ and $h_2(r,s)(x)_s = h_2$ formal solutions of (30) and (21) we may take

(31)

$$h_1(r,s)(x) = \frac{A(x) \cdot p(x)^{r+s}}{A(x) \cdot p^r + B(x) \cdot q(x)^r}, \qquad h_2(r,r)(x) = \frac{B(x) \cdot q(x)^{r+s}}{A(x) \cdot p^s + B(x) \cdot q(x)^s}.$$

As another possible $h_1(r,s)(x)$ and $h_2(r,s)(x)_s = h_2$ solutions of (30) and (21) we may take: **for** $r \neq s$

$$h_1(r,s) \cdot (p((x)^r q(x)^s - q(x)^r p(x)^s) = p(x)^{r+s} q(x)^s - q(x)^{r+s} p(x)^s, \tag{32}$$

$$h_2(r,s) \cdot (q(x)^r p(x)^s - p(x)^r q(x)^s) = p(x)^{r+s} q(x)^r - q(x)^{r+s} p(x)^r. \tag{33}$$

while for $r = s$ apply formula (31) with $r = s$.

Usually the specific features of particular cases of (21) and (22) allow one to infer the particular form of (30) hence the form of $h_1(r,s)(x) = h_1$ and $h_2(r,s)(x)_s = h_2$.

3.3. Three special cases examples.

Example 1. This is a particular case of the Theorem 2.

The recurrent relations (13) and (14) in Theorem 1 from [41, 2008] by Roberto Bagsarsa Corcino for $n_{p,q}$-binomial coefficients are special cases of this paper formula (29) as well as of Th. 17 in [2] with straightforward identifications of g_1, g_2 in (13) and in (14) in [41] or in this paper recurrence (30) for $H(x = 1) = U[p,q]_n = n_{p,q}$ sequence. Namely, recall here now in multinomial notation this Theorem 1 from [41, 2008] by Roberto Bagsarsa Corcino:

$$\binom{r+s}{r,s}_{p,q} = q^r \binom{r+s-1}{r-1,s}_{p,q} + p^s \binom{r+s-1}{r,s-1}_{p,q}, \tag{34}$$

$$\binom{r+s}{r,s}_{p,q} = p^r \binom{r+s-1}{r-1,s}_{p,q} + q^s \binom{r+s-1}{r,s-1}_{p,q}, \tag{35}$$

which is equivalent to

$$(s+r)_{p,q} = p^s r_{p,q} + q^r s_{p,q} = (r+s)_{q,p} = p^r s_{p,q} + q^s r_{p,q}, \tag{36}$$

what might be at once seen proved by noticing that

$$p^{r+s} - q^{r+s} \equiv p^s \cdot (p^r - q^r) + q^r \cdot (p^s - q^s).$$

Hence those mentioned straightforward identifications follow:

$$g_1 = q^r, \quad g_2 = p^s \ or \ \ g_1 = p^r, \quad g_2 = q^s. \tag{37}$$

The recurrence (36) in Lucas notation reads

$$U_{s+r} = p^s U_r + q^r U_s = U_{r+s} = p^r U_s + p^s U_r. \tag{38}$$

Compare it with equivalent recurrence (42) in order to notice that both h_1 and h_2 functions are different from case to case of recurrence (30) **equivalent realizations**.

Compare this example based on Theorem 1 in [41, 2008] by Roberto Corcino with with [44, 2008] **v[1]** by Maciej Dziemiańczuk (see there (1) and (2) formulas), and track as well - the simple combinatorial proof of the Corollary 3 in [44, 2009] **v[2]**) by Maciej Dziemiańczuk.

Example 2. This is a particular case of the Theorem 1.

Now let A be any natural numbers' or even complex numbers' valued sequence. One readily sees that also (1915) Fontené recurrence for Fontené-Ward generalized A-binomial coefficients i.e. equivalent identities (6) , (7) in [10] **are special cases of** this paper formula **(26)** as well as of Th. 17 in [2] with straightforward identifications of h_1, h_2 in this paper formula (25) while this paper recurrence (27) becomes trivial identity.
Namely, the identities (6) and (7) from [10, 1969] read correspondingly:

$$\binom{r+s}{r,s}_A = 1 \cdot \binom{r+s-1}{r-1,s}_A + \frac{A_{r+s} - A_r}{A_s} \binom{r+s-1}{r,s-1}_A, \tag{39}$$

$$\binom{r+s}{r,s}_A = \frac{A_{r+s} - A_s}{A_r} \cdot \binom{r+s-1}{r-1,s}_A + 1 \cdot \binom{r+s-1}{r,s-1}_A, \tag{40}$$

where $p \neq q$ and $\binom{r}{r,0}_L = \binom{s}{0,s}_L = 1$. And finally we have tautology identity

$$A_{s+r} \equiv \frac{A_{r+s} - A_s}{A_r} \cdot A_r + 1 \cdot A_s. \tag{41}$$

Example 2. becomes the general case of the Theorem 1. if we allow A to represent any zero characteristic field nonzero valued functions' sequence: $A = A(x) = \langle A_n(x)\rangle_{n\geq 0}\,,\ A_n(x) \neq 0$).

Example 3. This is a particular case of the Theorem 2.

The first example above is cognate to this third example in apparent way as might readily seen from François Édouard Anatole Lucas papers [1, 1878] or more recent article [115, 2001] by Hong Hu and Zhi-Wei Sun ; (see also $t = s$ case in [14, 1989] by Ira M. Gessel and Xavier Gérard Viennot on pp.23,24 .) In order to experience this let us start to consider now the number $H(x = 1) = U$ Lucas fundamental sequence *fulfilling* (**2**) with $U_0 = 0$ and $U_1 = 1$ as introduced in [1, 1878] and the - for example considered in [115, 2001]. There in [115, 2001] by Hong-Hu and Shi-Wei Sun - as a matter of fact - a kind of "pre-Theorem 17" from [2, 2009] is latent in the proof of Lemma 1 in [115]. We rewrite Lemma 1 by Hong-Hu and Shi-Wei Sun in multinomial notation and an arrangement convenient for our purpose here using sometimes abbreviation $U_n(p, q) \equiv U_n$.
(Note that the *addition formulas* for Lucas sequences hence consequently U-binomials'recurrence formulas [1, 1878] as well as $(p - q)\cdot(p^{j+k} - q^{j+k}) \equiv (p^{k+1} - q^{k+1})\cdot(p^j - q^j) - p\cdot q(p^{j-1} - q^{j-1}\cdot(p^k - q^k)$ - stem from commutative ring R identity: $(x - y)\cdot(x + y) \equiv x^2 - y^2, x, y \in R$.)
And so for $p \neq q$ and bearing in mind that $p\cdot q = -t$ - the following is true.

The identity (42) equivalent to

$$(p-q)\cdot(p^{j+k} - q^{j+k}) \equiv (p^{k+1} - q^{k+1})\cdot(p^j - q^j) - p\cdot q(p^{j-1} - q^{j-1}\cdot(p^k - q^k)$$

$$U_{j+k}(p,q) = U_{k+1}\cdot U_j(p,q) + tU_{j-1}\cdot U_k(p,q) \tag{42}$$

is equivalent to

$$\binom{j+k}{j,k}_U = U_{k+1}\cdot\binom{j+k-1}{j-1,k}_U + U_{j-1}\cdot\binom{j+k-1}{j,k-1}_U, \tag{43}$$

where p, q *are the roots of* (5) *and correspondingly the above Lucas fundamental sequence* $H_n = U_n(p, q)$ i.e. $U_0 = 0$ and $U_1 = 1$ *is given by its Binet form* (6),(7).

Compare (42) with equivalent recurrence (48) in order to notice that both h_1 and h_2 functions are different from case to case of recurrence (30) **equivalent realizations**.

Compare now: this paper recurrence formula (42) with recurrence formula (4) in [43, 2010], compare this paper recurrence formula (43) with Proposition 2.2. in [43, 2010] by Bruce E. Sagan and Carla D. Savage. Compare this paper recurrence (28) equivalent to (5) and proposition 2.2. in [43, 2010] and note that (5) in [43, 2010] is just the same - as (58) in [1, 1878] - the same except for notation. The translation from "younger" notation of Bruce E. Sagan and Carla D. Savage (from one - left hand - side) into more matured by tradition notation of François Édouard Anatole Lucas (from the other - right hand - side) is based on the identifications: the symbol used for U-binomials is {...} in place of $(...)_U$ and

$$\{n\} \equiv U_n \equiv n_{p,q},\ \langle n\rangle \equiv V_n.$$

For $s = t = 1$ we get Fibonacci $U_n = F_n$ sequence with recurrence (41) becoming the recurrence known from Donald Ervin Knuth and Herbert Saul Wilf masterpiece [15, 1989].

Example 3. becomes more general case of the Theorem 1. if we allow U to represent any zero characteristic field nonzero valued functions' sequence: $U(x) = \langle U_n(x)\rangle \_n \geq 0,\ U_n(x) = \frac{p(x)^n - q(x)^n}{p(x)-q(x)} \equiv n_{p(x),q(x)}, p(x) \neq q(x)$, where $p(x), q(x)$ denote the distinct roots of (20) and we have assumed as well that $p(x), q(x)$ are not roots of unity.

The End of three examples.

4 Snatchy information on F-binomials' and their relatives' combinatorial interpretations

4.1. In regard to **combinatorial interpretations** of L-binomial or F-multinomial coefficients or related arrays we leave that subject apart from this note. Nevertheless we direct the reader to some comprise papers and references therein; these are *for example* here the following:

Listing. **1.** [12, 1984] by Bernd Voigt: on common generalization of binomial coefficients, Stirling numbers and Gaussian coefficients .

Listing. **2.** [16, 1991] by Michelle L. Wachs and Dennis White and in [20, 1994] by Michelle L. Wachs: on p,q-Stirling numbers and set partitions.

Listing. **3.** [19, 1993] by Anne De Médicis and Pierre Leroux: on Generalized Stirling Numbers, Convolution Formulae and (p,q)-Analogues.

Listing. **4.** [120, 1998] John Konvalina: on generalized binomial coefficients and the Subset-Subspace Problem. Consult examples [Ex. q* ; 6] and [Ex. q* ; 7] in **4.3**. below. Then see also [121, 2000] by John Konvalina on an unified simultaneous interpretation of binomial coefficients of both kinds, Stirling numbers of both kinds and Gaussian binomial coefficients of both kinds.

Listing. **5.** Ira M. Gessel and Xavier Gérard Viennot in [14, 1989] deliver now well known their interpretation of the fibonomials in terms of non-intersecting lattice paths .

Listing. **6.** In [34, 2004] Jeffrey B. Remmel and Michelle L. Wachs derive a new rook theory interpretation of a certain class of generalized Stirling numbers and their (p, q)-analogues. In particular they prove that their (p, q)-analogues of the generalized Stirling numbers of the second kind may be interpreted in terms of colored set partitions and colored restricted growth functions.

Listing. **7.** [122, 2005] by Ottavio M. D'Antona and Emanuele Munarini deals with - in terms of weighted binary paths - combinatorial interpretation of the connection constants which is in particular unified, simultaneous combinatorial interpretation for Gaussian coefficients, Lagrange sum, Lah numbers, , q-Lah numbers, Stirling numbers of both kinds , q-Stirling numbers of both kinds. Notr the correspondence: weighted binary paths $\Leftrightarrow$ edge colored binary paths

Listing. **8.** Maciej Dziemiańczuk in [146, 2011] extends the results of John Konvalina from **4.** above. The Dziemiańczuk' ζ - analogues of the Stirling numbers arrays of both kinds cover ordinary binomial and Gaussian coefficients, p, q-Stirling numbers and other combinatorial numbers studied with the help of object selection, Ferrers diagrams and rook theory. The p, q-binomial arrays are special cases of ζ- numbers' arrays, too.

ζ -number of the first and the second kind is the number of ways to select k objects from k of n boxes without box repetition allowed and with box repetition allowed, respectively.

The weight vectors used for objects constructions and statements derivation are functions of parameter ζ.

Listing. **9.** As regards combinatorial interpretations via tilings in [123, 2003] and [124, 2010] - see **4.2.** below.

Listing. **10.** In [75, 2003] Johann Cigler introduces "abstract Fibonacci polynomials" - interpreted in terms of Morse coding sequences monoid with concatenation (monominos and dominos tiling then). Cigler's abstract Fibonacci polynomial sare monoid algebra over reals valued polynomials with straightforward Morse sequences i.e. tiling recurrence originated (1.6) "addition formula"

$$F_{m+n}(a,b) = F_{m+1}(a,b) \cdot F_m(a,b) + b \cdot F_{n-1}(a,b) \cdot F_n(a,b),$$

which is attractive and seductive to deal with within the context of this paper Theorem 1. The combinatorial tiling interpretation of the model is its construction framed in the Morse coding sequences monoid with concatenation (monominos and dominos tiling then).

Listing. **11.** In [127, 2010] Johann Cigler considers special Ward-Horadam *polynomials sequences* and reveals the tiling combinatorial interpretation of these special Ward-Horadam *polynomials sequences* in the spirit of Morse with monomino, domino alphabet monoid as here above in **10.**. Namely:

1. the q-Fibonacci polynomial $F_n(x,s,q) = \sum_{c\in\Phi_n} w(c) \equiv w(\Phi_n)$ is the weight function of the set Φ_n of all words (coverings) c of length $n-1$ in Morse (tiling) alphabet $\{a,b\}$ i.e. corresponding generation function for number of linear tilings as Φ_n clearly with the set of may be identified with the set of all linear tilings of $(n-1)\times 1$ rectangle or equivalently with Morse code sequences of length $n-1$.
Polynomials $F_n(x,s,q)$ satify this paper recursion (3) with $H_0(x) = 0$, $H_1(x) = 1$; $s(x) = x$ and $t(x) = s$.
The $F_n(x,s,q)$-binomial array $\left\{ {n \atop k} \right\}_{F_n(x,s,q)}$ is not considered in [127]. Similarly:

2. the q-Lucas polynomial $L_n(x,s,q) = \sum_{c\in\Lambda_n} w(c) \equiv w(\Lambda)$ is the weight function of the set Λ_n of all coverings c with arc monominos and dominos of the circle whose circumference has length n. Hence $L_n(x,s,q)$ is corresponding generation function for number of tilings of the circle whose circumference has length n. It may be then combinatorially seen that $w(\Lambda_n) = w(\Phi_{n+1}) + s\cdot w(\Phi_{n-1}))$ hence $L_n(x,s,q) = F_{n+1}(x,s,q) + s\cdot F_{n-1}(x,s,q)$.
Polynomials $L_n(x,s,q)$ satify this paper recursion (3) with $H_0(x) = 2$, $H_1(x) = x$; $s(x) = x$ and $t(x) = s$.
The $L_n(x,s,q)$-binomial array $\left\{ {n \atop k} \right\}_{L_n(x,s,q)}$ is not considered in [127].

Listing. **12.** In [43, 2010] by Bruce E. Sagan and Carla D. Savage the symbol $\{n\} \equiv U_n$ denotes the $n-th$ element of the fundamental Lucas sequence U satisfying this paper recurrence (2) with initial conditions $\{0\} = 0$, $\{1\} = 1$. Naturally $\{n\}$ is a polynomial in parameters s,t. So is also the U-binomial coefficient $\left\{ {n \atop k} \right\}_U \equiv \left\{ {n \atop k} \right\}_{p,q}$.
Similarly - the symbol $\langle n\rangle \equiv V_n$ denotes the $n-th$ element of the primordial Lucas sequence V satisfying this paper recurrence (2) with initial conditions $\langle 0\rangle = 2$, $\langle 1\rangle = s$. Naturally $\langle n\rangle$ is a polynomial in parameters s,t. So is also the V-binomial coefficient $\left\{ {n \atop k} \right\}_V \equiv \left\langle {n \atop k} \right\rangle_{p,q}$. V-binomials are not considered in [43, 2010]. Both fundamental and primordial sequences are interpreted via tilings similarly to the above in **11.** Johann Cigler attitude rooted in already text-books tradition.

An so: $\{n\}$ is generation function for number of linear tilings of $(n-1)\times 1$ rectangle or equivalently of number of Morse code sequences of length $n-1$.

$\langle n\rangle$ is generation function for number of circular tilings of the circle whose circumference has length n. Using naturally proved (just seen) relations Bruce E. Sagan and Carla D. Savage derive **two** combinatorial interpretations of the the same $\left\{ {m+n \atop m,n} \right\}_{p,q}$ via Theorem 3.1. from which we infer the following.

1. $\left\{ {m+n \atop m,n} \right\}_{p,q}$ is the weight of all linear tilings of all integer partitions λ inside the $m\cdot n$ rectangle

hence $\left\{ {m+n \atop m,n} \right\}_{p,q}$ is the generating function for numbers of such tilings of partitions.

2. $2^{m+n} \cdot \left\{ {m+n \atop m,n} \right\}_{p,q}$ is the weight of all circular tilings of all integer partitions λ inside the $m \cdot n$ rectangle

hence $\left\{ {m+n \atop m,n} \right\}_{p,q}$ is the generating function for numbers of such tilings of partitions.

Explanation. from [43, 2010]. *A linear tiling of a partition* λ is a covering of its Ferrers diagram with disjoint dominos and monominos obtained by linearly tiling each λ_i part. In circular tiling of a partition λ one performs circular tiling of each λ_i part

The above list is open and far from complete.

4.2.

Nevertheless, to this end let us discern in part- via indicative information - a part of Arthur T. Benjamin's recent contribution to the domain . Namely; in [123, 2003] by Arthur T. Benjamin and Jennifer J. Quinn track the tilings' Combinatorial Theorem 5, p.36. There for $H_n = U_n$ the number s from the recurrence (2) is interpreted as equal to the number of colors of squares and t from this very recurrence (2) equals to the number of colors of dominos while $H_n = U_{n+1}$ counts colored tilings of length n with squares and dominos. Similarly, also in [123, 2003] see the tilings' Combinatorial Theorem 6 , p.36. Here for $H_n = V_n$ the number s from the recurrence (2) should equal to the number of colors of a square and t from this very recurrence (2) equals to the number of colors of a domino while $H_n = V_n$ counts colored bracelets of length n tiled with squares and dominos. Bruce E. Sagan and Carla D. Savage in [43, 2010] refer to well known recurrences: Identity 73 on p. 38 in [123] - for (4) in [43] and Identity 94 p. 46 in [123] for (5) in [43]. Both (4) and (5) recurrences in [43, 2010] by Bruce E. Sagan and Carla D. Savage have been evoked in the illustrative Example 3. Section 3. above.

Partially based on [123, 2003] by Arthur T. Benjamin and Jennifer J. Quinn the paper [124, 2009] by Arthur T. Benjamin and Sean S. Plott should be notified and as being nominated by Arthur T. Benjamin and Sean S. Plott in errata [124, 2010] the present author feels entitled to remark on this errata.

4.2. According to errata [124, 2010] by Arthur T. Benjamin and Sean S. Plott [quote] " *The formula for* $\binom{n}{k}_F$ *should be multiplied by a factor of* F_{n-x_k}, *which accounts for the one remaining tiling that follows the* f_0 *tiling. Likewise, the formula for* $\binom{n}{k}_F$ *should be multiplied by* U_{n-x_k}." Our remark is that this errata is unsuccessful. If we follow this errata then $(x_{k-1} < x_k)$ we would have:

$$(44) \quad \binom{n}{k}_{errata} = \sum_{1 \leq x_1 < x_2 < \cdots < x_{k-1} \leq n-1} \prod_{i=1}^{k-1} F_{k-i}^{x_i - x_{i-1} - 1} F_{n-x_i-(k-i)+1} F_{n-x_k},$$

where $F_0 = 0$ and $x_0 = 0$. But the formula (44) implies for example

$$15 = \binom{5}{3}_F \neq \binom{5}{3}_{errata} = 11.$$

The task of finding the correct formula - due to the present author became a month ago an errand - exercise for Maciej Dziemiańczuk, a doctoral student from Gdańsk University in Poland. The result - to be quoted below as MD formula (46) - is his discovery, first announced in the form of a feedback private communication to the present author: (M. Dziemiańczuk on Mon, Oct 18, 2010 at 6:26 PM) however still not announced in public.

The source of an error in errata is that $\binom{n}{k}_F$ should be multiplied **not by** the factor of F_{n-x_k} *but by* the factor $F_{n-x_k+1} \equiv f_{n-x_k}$. Then we have

$$\binom{n}{k}_{now} = \sum_{1 \leq x_1 < x_2 < \cdots < x_{k-1} \leq n-1} \prod_{i=1}^{k-1} F_{k-i}^{x_i - x_{i-1} - 1} F_{n-x_i-(k-i)+1} F_{n-x_k+1},$$

Due to $x_{k-1} < x_k$ the above formula is equivalent to

$$\binom{n}{k}_{now} = \sum_{1 \leq x_1 < x_2 < \cdots < x_{k-1} < x_k \leq n} \prod_{i=1}^{k-1} F_{k-i}^{x_i - x_{i-1} - 1} F_{n-x_i-(k-i)+1} F_{n-x_k}, \tag{45}$$

and this in turn is evidently equivalent to the MD-formula (46) below i.e. (45) is equivalent to the corrected by Maciej Dziemiańczuk Benjamin and Plott formula from The Fibonacci Quarterly $46/47.1$ (2008/2009), 7-9.

Finally here now MD-formula follows:

$$\binom{n}{k}_F = \sum_{1 \leq x_1 < x_2 < \cdots < x_k \leq n} \prod_{i=1}^{k} F_{k-i}^{x_i - x_{i-1} - 1} F_{n-x_i-(k-i)+1}, \tag{46}$$

where $F_0 = 0$ and $x_0 = 0$.

Collaterally Maciej Dziemiańczuk supplies correspondingly correct formula for Lucas U - binomial coefficients $\binom{n}{k}_U$:

$$\binom{n}{k}_U = \sum_{\substack{1 \leq x_1 < x_2 < \cdots < x_{k-1} \leq n \\ x_k = x_{k-1}+1}} s^{x_k - k} \left(\prod_{i=1}^{k-1} U_{k-i}^{x_i - x_{i-1} - 1} U_{n-x_i-(k-i)+1} \right) U_{n-x_k+1} \tag{47}$$

$$= \sum_{1 \leq x_1 < x_2 < \cdots < x_k \leq n} s^{x_k - k} \prod_{i=1}^{k} U_{k-i}^{x_i - x_{i-1} - 1} U_{n-x_i-(k-i)+1}, \tag{48}$$

where $U_0^t = 0^t = \delta_{t,0}$.

4.3. p, q**-binomials versus** $q*$**-binomials combinatorial interpretation,** where $q* = \frac{p}{q}$ if $q \neq 0$.

In the first instance let us once for all switch off the uninspired $p \cdot q = 0$ case. Then obligatorily either $q \neq 0$ or $q \neq 0$. Let then $q* = \frac{p}{q}$. In this nontrivial case

$$\binom{n}{k}_{p,q} = q^{k(n-k)} \cdot \binom{n}{k}_{q*}. \tag{49}$$

Referring to the factor $q^{k(n-k)}$ as a kind of weight, one may transfer combinatorial interpretation statements on $q*$ binomials $\binom{n}{k}_{q*}$ onto combinatorial interpretation statements on p, q binomials $\binom{n}{k}_{p,q}$ through the agency of (49). Thence , apart from specific combinatorial interpretations uncovered for the class or subclasses of p, q-binomials there might be admitted and respected the "$q*$-overall" combinatorial interpretations transfered from $1, q*$-binomials i.e. from $q*$-binomials onto p, q-binomials.

By no means pretending to be the complete list here comes the skeletonized list of **[Ex. q* ; k]** examples, $k \geq 1$.

[Ex. q* ; 1] The $q*$-binomial coefficient $\binom{m+n}{m,n}_{q*}$ may be interpreted as a polynomial in $q*$ whose $q*^k$-th coefficient counts the number of distinct partitions of k elements which fit inside an $m \times n$ rectangle - see [129, 1976] by George Eyre Andrews.

On lattice path techniques - Historical Remark. It seams to be desirable now to quote here information from [133, 2010] by Katherine Humphreys based on [134, 1878] by William Allen Whitworth:

Quotation 2 *We find lattice path techniques as early as 1878 in Whitworth to help picture a combinatorial problem, but it is not until the early 1960's that we find lattice path enumeration presented as a mathematical topic on its own. The number of papers pertaining to lattice path enumeration has more than doubled each decade since 1960.*

[Ex. q* ; 2] The [Ex. q* ; 2] may be now compiled with [Ex. q* ; 1] above. For that to do recall that zigzag path is the shortest path that starts at $A = (0, 0)$ and ends in $B = (n, n-k)$ of the $n \times k$ rectangle; see: [130, 1962] by György Pólya [pp. 68-75], [130, 1969] by György Pólya and [132] by György Pólya and G. L. Alexanderson.

Let then $A_{n,k,\alpha}$ = the number of those $(0, 0) \longrightarrow (k, n-k)$ zigzag paths the area under which is α.

In [131, 1969] György Pólya using recursion for $q*$-binomial coefficients proved that

$$\binom{n}{k}_{q*} = \sum_{\alpha=0}^{k(n-k)} A_{n,k,\alpha} \cdot q *^{\alpha} .$$

from where György Pólya infers the following Lemma ([131, 1969], p.105) which is named Theorem (p. 104) in more detailed paper [132, 1971] by György Pólya and G. L. Alexanderson.

Quotation 3 *The number of those zigzag paths the area under which is* α *equals* $A_{n,k,\alpha}$.

[Ex. q* ; 3] The [Ex. q* ; 3] may be now compared with [Ex. q* ; 1]. The combinatorial interpretation of $\binom{r+s}{r,s}_{q*}$ from [Ex. q* ; 1] had been derived (pp. 106-107) in [132, 1971] by György Pólya and G. L. Alexanderson, from where - with advocacy from [135, 1971] by Donald Ervin Knuth - we quote the result.

(1971): $\binom{r+s}{r,s}_{q*}$ = *ordinary generating function in* α *powers of* $q*$ *for partitions of* α *into exactly* r *non-negative integers none of which exceeds* s ,

as derived in [132, 1971] by György Pólya and G. L. Alexanderson - see formula (6.9) in [132].

(1882): $\binom{n}{k}_{q*}$ = *ordinary generating function in* α *powers of* q *for partitions of* α *into at most* k *parts not exceeding* $(n-k)$,

as recalled in [135, 1971] by Donald Ervin Knuth and proved combinatorially in [136, 1882] by James Joseph Sylvester.

Let nonce : $r+s=n$, $r=k$ then **(1971)** $\equiv$ **(1882)** are equal due to

$$(50) \qquad \binom{n}{k}_{q*} = \sum_{\alpha=0}^{k(n-k)} A_{n,k,\alpha} \cdot q*^{\alpha} = \sum_{\alpha=0}^{r\cdot s} A_{r+s,r,\alpha} \cdot q*^{\alpha} = \binom{r+s}{r,\, s}_{q*} .$$

where for commodity of comparison formulas in two notations from two papers - we have been using contractually for a while: $r+s=n$, $r=k$ identifications.

[Ex. q* ; 4] The following was proved in [137, 1961] by Maurice George Kendall and Alan Stuart (see p.479 and p.964) and n [132, 1971] by György Pólya and G. L. Alexanderson (p.106).

The area under the zigzag path = The number of inversions in the very zigzag path coding sequence.

The possible extension of the above combinatorial interpretation onto three dimensional zigzag paths via "*three-nomials*" was briefly mentioned in [132] - see p.108.

[Ex. q* ; 5] The well known (in consequence - *finite geometries'*) interpretation of $\binom{n}{k}_{q*}$ coefficient due to Jay Goldman and Gian-Carlo Rota from [138, 1970] is now worthy of being recalled; see also [135, 1971] by Donald Ervin Knuth.

Let V_n be an n-dimensional vector space over a finite field of $q*$ elements. Then $\binom{n}{k}_{q*} =$ *the number of k-dimensional subspaces of V_n* .

[Ex. q* ; 6]
This example = the short substantial note [135, 1971] by Donald Ervin Knuth. Compile this example with the example [Ex. q* ; 5] above.
The essence of a coding of combinatorial interpretations via bijection between lattices is the construction of this *coding bijection* in [135]. Namely, let $GF(q*)$ be the Galois field of order $q*$ and let $V_n \equiv V = GF(q*)^n$ be the n-dimensional vector space over $GF(q*)$. Let $[n] = \{1, 2, ..., n\}$. Let $\ell(V)$ be the lattice of all subspaces of $V = GF(q*)^n$ while $\ell([n]) \equiv 2^{[n]}$ denotes the lattice of all subsets of $[n]$.
In [135] Donald Ervin Knuth constructs this *natural order and rank preserving* map Φ from the lattice $\ell(V)$ of subspaces onto the lattice $\ell([n]) \equiv 2^{[n]}$ of subsets of $[n]$.

$$\ell(V) \overset{\Phi}{\to} \ell([n]).$$

We bethink with some reason whether this Φ bijection coding might be an answer to the subset-subspace problem from subset-subspace problem from [120, 1998] by John Konvalina **?**

Quotation 4 *...the subset-subspace problem (see 6 , 9 , and 3) . The traditional approach to the subset-subspace problem has been to draw the following analogy: the binomial $\binom{n}{k}_F$ coefficient counts k-subsets of an n-set, while the analogous Gaussian $\binom{n}{k}_q$ coefficient counts the number of k-dimensional subspaces of an n-dimensional finite vector space over the field of q elements. The implication from this analogy is that the Gaussian coefficients and related identities tend to the analogous identities for the ordinary binomial coefficients as q approaches 1. The proofs are often algebraic or mimic subset proofs. But what is the combinatorial reason for the striking parallels between the Gaussian coefficients and the binomial coefficients?*

According to Joshef P. S. Kung [139, 1995] the Knuth's note is not the explanation:

Quotation 5 *... observation of Knuth yields an order preserving map from $L(V_n(q)$ to Boolean algebra of subsets, but it does not yield a solution to the* *still unresolved problem of finding a combinatorial interpretation of taking the limit* $q \longrightarrow 1$.

Well, perhaps this limit being performed by q-deformed Quantum Mechanics physicists might be of some help? There the so called q-quantum plain of q-commuting variables $x \cdot y - q \cdot y \cdot x = 0$ becomes a plane $\mathbb{F} \times \mathbb{F}$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}$,... p-adic fields included) of two commuting variables in the limit $q \longrightarrow 1$. For see [140, 1953] by Marcel-Paul Schützenberger. For quantum plains - see also [141, 1995] by Christian Kassel. It may deserve notifying that q - extension of of the "classical plane" of commuting variables ($q = 1$) seems in a sense ultimate as discussed in [142, 2001] by A.K. Kwaśniewski

[Ex. q* ; 7] Let us continue the above by further quotation from [120, 1998] on generalized binomial coefficients and the subset-subspace problem.

Quotation 6 *We will show that interpreting the Gaussian coefficients as generalized binomial coefficients of the second kind combinations with repetition reveals the combinatorial connections between not only the binomial coefficients and the Gaussian coefficients, but the Stirling numbers as well. Thus, the ordinary Gaussian coefficient tends to be an algebraic generalization of the binomial coefficient of the first kind, and a combinatorial generalization of the binomial coefficient of the second kind.*

Now in order to get more oriented go back to the begining of subsection **4.1.** and consult : Listing. **1.**, Listing. **2.**, Listing. **3.** which are earlier works and end up with [121, 2000] by John Konvalina on an unified simultaneous interpretation of binomial coefficients of both kinds, Stirling numbers of both kinds and Gaussian binomial coefficients of both kinds. Compare it then afterwards with Listing. **8.**.

References

[1] François Édouard Anatole Lucas, *Théorie des Fonctions Numériques Simplement Priodiques*, American Journal of Mathematics, Volume **1**, (**1878**): 184-240 (Translated from the French by Sidney Kravitz , Edited by Douglas Lind Fibonacci Association 1969.

[2] Nicholas A. Loehr, Carla D. Savage August 26, 2009 *Generalizing the combinatorics of binomial coefficients via ℓ-nomials* , Integers, Volume **9** (**2009**), A 45. with corrections noted (thanks to Bruce Sagan) $http : //www4.ncsu.edu/\ savage/PAPERS/605_lnom5_corrections.pdf$

[3] Fontené Georges *Généralisation d'une formule connue*, Nouvelles Annales de Mathématiques (4) 15 (1915): 112

[4] Derric Henry Lehmer , *Extended Theory of Lucas functions*, Ann. of Math. (2) 31 (**1930**): 418-448.

[5] Ward Morgan, *A calculus of sequences*, Amer.J. Math. **58** (**1936**): 255-266.

[6] Dov Jarden and Theodor Motzkin ,*The product of Sequences with the common linear recursion formula of Order 2*, Riveon Lematimatica **3** , (**1949**): 25-27.

[7] Leonard Carlitz, *Generating functions for powers of certain sequences of numbers*, Duke Math. J. **29** (1962): 521-538.

[8] Roseanna F. Torretto , J. Allen Fuchs *Generalized Binomial Coefficients*, The Fibonacci Quarterly,vol. **2**, (**1964**): 296-302 .

[9] Verner Emil Hoggatt, Jr. , *Fibonacci numbers and generalized binomial coefficients*, Fibonacci Quart. **5** (**1967**): 383-400.

[10] Henri W.Gould *The Bracket Function and Fontené-Ward Generalized binomial Coefficients with Applications to Fibonomial Coefficients*, The Fibonacci Quarterly vol.**7**, (**1969**): 23-40.

[11] Blagoj S. Popov, *Generating Functions for Powers of Certain Second-Order Recurrence Sequences*, The Fibonacci Quarterly, Vol. **15** No 3 (**1977**): 221-223

[12] Bernd Voigt *A common generalization of binomial coefficients, Stirling numbers and Gaussian coefficients* Publ. I.R.M.A. Strasbourg, **1984**, 229/S-08 Actes de Seminaire Lotharingien, p. 87-89. , In: Frolk, Zdenk (ed.): Proceedings of the 11th Winter School on Abstract Analysis. Circolo Matematico di Palermo, Palermo, **1984**, pp. 339-359.

[13] Luis Verde-Star, *Interpolation and combinatorial functions*. Studies in Applied Mathematics, **79**: (**1988**):65-92.

[14] Ira M. Gessel, Xavier Gérard Viennot, *Determinant Paths and Plane Partitions*, (**1989**):- Preprint; see(10.3) page 24

$$http://people.brandeis.edu/ gessel/homepage/papers/pp.pdf$$

[15] Donald Ervin Knuth, Herbert Saul Wilf, *The Power of a Prime that Divides a Generalized Binomial Coefficient*, J. Reine Angev. Math. 396 (**1989**) : 212-219.

[16] Michelle L. Wachs and Dennis White, *p,q-Stirling Numbers and Set Partition Statistics*, Journal of Combinatorial Theory, Series A **56**, (**1991**): 27–46.

[17] R. Chakrabarti and R. Jagannathan, *A (p,q)-Oscillator Realization of Two-parameter Quantum Algebras*, J. Phys. A: Math. Gen. **24**,(**1991**): L711.

[18] Jacob Katriel, Maurice Kibler, *Normal ordering for deformed boson operators and operator-valued deformed Stirling numbers*, J. Phys. A: Math. Gen. **25**,(**1992**):2683-2691.

[19] Anne de Medicis and Pierre Leroux, A unified combinatorial approach for q-(and p,q-)Stirling numbers, J. Statist. Plann. Inference **34** (**1993**): 89-105.

[20] Michelle L. Wachs: *sigma -Restricted Growth Functions and p,q-Stirling Numbers*. J. Comb. Theory, Ser. A **68**(2), (**1994**):470-480.

[21] SeungKyung Park*P-Partitions and q-Stirling Numbers* , Journal of Combinatorial Theory, Series A, **68** , (**1994**): 33-52.

[22] Diana L. Wells, *The Fibonacci and Lucas Triangles Modulo 2*, The Fibonacci Quarterly **32**.2 (**1994**): 111-123

[23] Anne De Médicis, Pierre Leroux, *Generalized Stirling Numbers. Convolution Formulae and p,q-Analogues*, Canad. J. Math. 47 , ((**1995**):474–499.

[24] Mirek Majewski and Andrzej Nowicki, *From generalized binomial symbols to Beta and Alpha sequences*, Papua New Guinea Journal of Mathematics, Computing and Education, 4, (**1998**): 73-78.

[25] Andrzej Krzysztof Kwaśniewski, *Higher Order Recurrences for Analytical Functions of Tchebysheff Type*, Advances in Applied Clifford Algebras; vol.**9** No1 (**1999**): 41-54. arXiv:math/0403011v1, [v1] Sun, 29 Feb 2004 04:09:32 GMT

[26] Wiesław, Wladysław Bajguz, Andrzej Krzysztof Kwaśniewski *On generalization of Lucas symmetric functions and Tchebycheff polynomials* ,Integral Transforms and Special Functions Vol.**8** Numbers 3-4, (**1999**): 165-174.

[27] Alexandru Ioan Lupas, *A guide of Fibonacci and Lucas polynomials*, Octogon, Math. Magazine, vol.**7** , No 1, (**1999**): 2-12.

[28] Wiesław, Wladysław Bajguz, *On generalized of Tchebycheff polynomials, Integral Transforms and Special Functions*, Vol. **9**, No. 2 (**2000**): pp. 91-98.

[29] Hong Hu and Zhi-Wei Sun, *An Extension of Lukas' Theorem*, Proc. Amer. Math. Soc. 129 (**2001**), no. 12, 3471-3478.

[30] Zhi-Wei Sun , H. Hu and J.-X. Liu, *Reciprocal sums of second-order recurrent sequences*, Fibonacci Quart., **39** (**2001**), no.3, 214-220.

[31] Eric R. Tou, *Residues of Generalized Binomial Coefficients Modulo a Product of Primes*, senior thesis, Spring (**2002**):, Department of Mathematics and Computer Science, Gustavus Adolphus College, St. Peter, MN, *http : //sites.google.com/site/erikrtou/home* , promotor John M. Holte.

[32] Karen Sue Briggs, *Q-analogues and p, q-analogues of rook numbers and hit numbers and their extensions*, Ph.D. thesis, University, of California, San Diego (**2003**).

[33] Karen Sue Briggs and J. B. Remmel, *A p,q-analogue of a Formula of Frobenius*, Electron. J. Comb. **10** (**2003**), No R9.

[34] J. B. Remmel and Michelle L. Wachs, *Rook Theory, Generalized Stirling Numbers and (p,q)-Analogues* , The Electronic Journal of Combinatorics **11** , (Nov 22, **2004**), No R 84.

[35] Karen Sue Briggs, *A Rook Theory Model for the Generalized p,q-Stirling Numbers of the First and Second Kind*, Formal Power Series and Algebraic Combinatorics, Series Formélles et Combinatoire Algébrique San Diego, California **2006**

[36] Jaroslav Seibert, Pavel Trojovský, *On some identities for the Fibonomial coefficients*, Mathematica Slovaca, vol. **55** Issue 1, (**2005**): 9-19.

[37] Jaroslav Seibert, Pavel Trojovský, *On certain identities for the Fibonomial coefficients*, Tatra Mt. Math. Publ. **32** (**2005**): 119-127.

[38] Jaroslav Seibert, Pavel Trojovský, *On sums of certain products of Lucas numbers*, The Fibonacci Quarterly, Vo. **42**, No 2, (**2006**): 172-180.

[39] Pavel Trojovský, **On some identities for the Fibonomial coefficients via generating Function**, Discrete Applied Mathematics, **155** (**2007**): 2017-2024.

[40] Jaroslav Seibert, Pavel Trojovský, *On Multiple Sums of Products of Lucas Numbers*, Journal of Integer Sequences, Vol. **10** (**2007**), Article 07.4.5

[41] Roberto Bagsarsa Corcino, *On p,q-Binomial Coefficients*, Integers: Electronic Journal of Combinatorial Number Theory **8** ,(**2008**): 1-16.

[42] M. Dziemiańczuk, *Generalization of Fibonomial Coefficients*, arXiv:0908.3248v1 [v1] Sat, 22 Aug (**2009**), 13:18:44 GMT

[43] Bruce E. Sagan, Carla D. Savage, *Combinatorial Interpretation of Binomial Coefficient Analogues Related to Lucas Sequences*, INTEGERS: Electronic Journal of Combinatorial Number Theory (**2010**), (**2010**), to appear

[44] M. Dziemiańczuk, *Report On Cobweb Posets' Tiling Problem*, arXiv:0802.3473v1 **[v1]** Sun, Sun, 24 Feb 2008 00:54:09 GMT, **[v2]** Thu, 2 Apr 2009 11:05:55 GMT

[45] Andrzej Krzysztof Kwaśniewski, *A note on V-binomials' recurrence for V-Lucas sequence companion to U-Lucas sequence*, arXiv:1011.3015v2 [math.CO] [v2] Sun, 21 Nov **2010**,10:02:13 GMT

[46] Andrzej Krzysztof Kwaśniewski, *On basic Bernoulli-Ward polynomials*, Bull. Soc. Sci. Lett. Lodz **54** Ser. Rech. Deform.45 (**2004**): 5-10. $arXiv : math/0405577v1$ [v1] Sun, 30 May 2004 00:32:47 GMT

[47] Thomas M. Richardson, *The Filbert Matrix* , The Fibonacci Quart. **39** no.3 (**2001**): 268-275. $arXiv : math/9905079v1$, [v1] Wed, 12 May 1999 20:04:41 GMT

[48] Emrah Kilic, Helmut Prodinger, *The generalized Filbert matrix*,The Fibonacci Quarterly **48**.1 (**2010**): 29-33.

[49] Morgan Ward, *Prime divisors of second order recurring sequences*, Duke Math. J. Volume 21, Number 4 (1954): 607-614.

[50] David Zeitlin, Generating functions for products of recursive sequences, Trans. Amer.Math. Soc. 116 (1965), 300-315.

[51] Alwyn F. Horadam, *A Generalized Fibonacci Sequence*, American. Mathematical. Monthly **68** (**1961**):455-59.

[52] Horadam, A.F. *Basic properties of a certain generalized sequence of numbers*, Fibonacci Quart., **3** (Oct. **1965**): 161-176.

[53] David Zeitlin, *Power Identities for Sequences Defined by* $W_{n+2} = dW_{n+1} - cW_n$, The Fibonacci Quarterly, Vol. 3, No. 4 (Dec. **1965**): 241-256.

[54] Alwyn F. Horadam, *Generating functions for powers of a certain generalized sequence of numbers*, Duke Math. J., **32** (**1965**): 437-446.

[55] M. Elmore, *Fibonacci functions*, Fibonacci Quart. **4** (**1967**): 371-382.

[56] Alwyn F. Horadam, *Special properties of the sequence* $W_n(a, b; p, q)$, Fibonacci Quarterly **5**, 424-434 (**1967**).

[57] Alwyn F. Horadam, *Tschebyscheff and other functions associated with the sequence* $\{W_n(a,b;p,q)\}$, Fibonacci Quart., **7** (**1969**): 14-22.

[58] R. Alter and Kenneth K. Kubota, *Multiplicities of second order linear recurrences* , Trans. Amer. Math. Soc. vol. **178** (**1973**): 271 - 284.

[59] J. E. Walton and A.F Horadam, *Some aspects of generalized Fibonacci numbers*, The Fibonacci Quarterly,**12**, (3): 241-250,**1974**.

[60] Horadam A. F., Shannon A.G., *Ward's Staudt-Clausen problem*, Math. Scand., **39** (**1976**), no. 2, 239-250 (1976)

[61] Kenneth K. Kubota, *On a conjecture of Morgan Ward, I, II, III*; Acta Arith. **33** (**1977**), 11 - 28, 29 - 48, 99 - 109.

[62] Anthony G. Shannon , A Recurrence Relation for Generalized Multinomial Coefficients, The Fibonacci Quartely, vol. 17, No 4, (1979): 344-346.

[63] A.G. Shannon and A. F. Horadam, *Special Recurrence Relations Associated with the Sequence* $\{w_n(a,b;p,q)\}$, The Fibonacci Quarterly 17.4 (**1979**): 294-299 .

[64] F. Beukers, *The multiplicity of binary recurrences*, Compositio. Math. **40** (**1980**): 251-267.

[65] A.F. Horadam and J.M. Mahon, *Pell and Pell-Lucas Polynomials*, The Fibonacci Quart., **23** (**1985**): 7-20.

[66] A. F. Horadam, *Associated Sequences of General Order*, The Fibonacci Quarterly **31**.2 (**1993**): 166-72.

[67] P. J. McCarthy , R. Sivaramakrishnan, *Generalized Fibonacci Sequences via Arithmetical Functions* , The Fibonacci Quarterly 2A (**1990**):363-70.

[68] S. P. Pethe , C. N. Phadte, *A Generalization of the Fibonacci Sequence*, Applications of Fibonacci Numbers Volume **5**, ed.G.E. Bergum, Kluwer Academic Publishers, (**1993**): 465-472.

[69] S.P. Pethe , R.M. Fernandes, *Two Generalized Trigonometric Fibonacci Sequences*, Ulam Quarterly, Volume **3**, Number1, (**1995**): 15-26.

[70] Gheorghe Udrea, *A Note on the Sequence* $(W_n)_{n\geq 0}$ of A.F. Horadam, Portugaliae Mathematica Vol. **53** Fasc. 2 , **1996**: 143-144.

[71] A. F. Horadam, *A Synthesis of Certain Polynomial Sequences*, In Applications of Fibonacci Numbers 6:215-29. Ed. G. E. Bergum, A. N. Philippou, and A. F. Horadam. Dordrecht: Kluwer, 1996.

[72] Horadam, A. F., *Extension of a Synthesis for a Class of Polynomial Sequences*, Fib. Quart. **34**, 68-74, **1996**.

[73] Clifford A. Reiter, *Exact Horadam Numbers with a Chebyshevish Accent*, Vector **16**, 122-131, **1999**.

[74] P. Haukkanen, *A note on Horadam's sequence*, The Fibonacci Quarterly **40**:4 (**2002**): 358-361.

[75] Johann Cigler, *q-Fibonacci polynomials*, Fibonacci Quart. v **41** No 1 (**2003**): 31-40.

[76] Toufik Mansour, *A formula for the generating functions of powers of Horadam's sequence*, The Australasian Journal of Combinatorics, vol. **30**, (**2004**): 207-212.

[77] Tugba Horzum , Emine Gökcen Kocer, *On Some Properties of Horadam Polynomials*, International Mathematical Forum, **4**, **2009** No 25, 1243-1252.

[78] Gi-Sang Cheon, Hana Kim and Louis W. Shapiro A generalization of Lucas polynomial sequence Discrete Applied Mathematics Vol. 157 (2009) 920-927 927

[79] Tian-Xiao He, Peter Jau-Shyong Shiue , *On Sequences of Numbers and Polynomials Defined by Linear Recurrence Relations of Order 2*, International Journal of Mathematics and Mathematical Sciences, Volume **2009**, Article ID 709386, 21 pages

[80] Charles Hermite, Problems 257-258. Jour, de math. Speciales (**1889**): 19-22; (1891):70" Solutions given by E. Catalan.

[81] Charles Hermite and Thomas Jan Stieltjes, Correspondance d'Hermite et de Stieltjes 1 (**1905**):415-16. Letter of 17 April **1889**.

[82] Robert Daniel **Carmichel**, *On the numerical factors of the arithmetic forms* $\alpha^n \pm \beta^n$, Ann. of Math. (2) **15** (**1913**): 30-70.

[83] L. E. Dickson, *History of the Theory of Numbers*, Vol. 1. Washington, D.C.: Carnegie Institution, **1919**; rpt. New York: Chelsea, 1952.

[84] Derric Henry **Lehmer** , *Extended Theory of Lucas functions*, Ann. of Math. (2) **31** (**1930**): 418-448

[85] Derric Henry Lehmer, Factorization of certain cyclotomic functions. Ann. of Math., 34:461-469, 1933.

[86] Derric Henry Lehmer, A certain class of polynomials , Ann. of Math. 36 (1935) p. 639

[87] Derric Henry Lehmer, The Mathematical Work of Morgan Ward, Mathematics of computation, vol. 61, number 203, July 1993, pages 307-311.

[88] Morgan Ward, *Note on divisibility sequences*, Bull. Amer. Math. Soc. Vol.**42**, No 12 (**1936**): 843-845.

[89] Ward Morgan, *A note on divisibility sequences*, Bull. Amer. Math. Soc. **45** (**1939**): 334-336.

[90] Morgan Ward, *Some Arithmetical Applications of Residuation* , American Journal of Mathematics, Vol. **59**, No. 4 (Oct., **1937**): 921-926.

[91] Morgan Ward, **Arithmetic functions on rings**, Annals of Mathematics, vol. **38**, No 3, (**1937**): 725-732.

[92] Morgan Ward, *The Intrinsic Divisors of Lehmer Numbers*, Ann. of Math. (2) 62 (1955): 230-236.

[93] Morgan Ward, *Tests for primality based on Sylvester's cyclotomic numbers*, Pacific J. Math. Vol. **9** , No. 4 (**1959**): 1269-1272.

[94] L. K. Durst, **Exceptional real Lehmer sequences**, Pacific J. Math.**9** (**1959**): 437-441.

[95] John H. Halton, *On the divisibility properties of Fibonacci numbers*, Fibonacci Quart. **15** (Oct. **1966**): 217-240.

[96] Ronald Alter, Kenneth K. Kubota , *Multiplicities of second order linear recurrences* , Trans. Amer. Math. Soc. vol. **178** (**1973**): 271 - 284.

[97] Andrzej Bobola Maria **Schinzel**, *Primitive divisors of the expression* $A^n - B^n$ *in algebraic number fields*, J. Reine Angew. Math. 268/269 (1974), 27-33.

[98] David Singmaster, *Notes on Binomial Coefficients I, A Generalization of Lucas' Congruence*, J. London Math. Soc. (**1974**) s2-8 (3): 545-548.

[99] David Singmaster, *Notes on Binomial Coefficients II, The Least N such that pe Divides an r-Nomial Coefficient of Rank n* , J. London Math. Soc. (**1974**) s2-8 (3): 549-554.

[100] David Singmaster, *Notes on Binomial Coefficients III, Any Integer Divides Almost All Binomial Coefficients*, J. London Math. Soc. (**1974**) s2-8 (3): 555-560.

[101] David Singmaster, *Notes on Binomial Coefficients IV, Proof of a Conjecture of Gould on the G.C.D.'s of Two Triples of Binomial Coefficients*, The Fibonacci Quarterly 11.3 (**1973**): 282-84.

[102] C. L. Stewart, *Primitive divisors of Lucas and Lehmer sequences*, in: Transcendence Theory:Advances and Applications (A. Baker and D.W. Masser, eds.), Academic press, New York,**1977**, pp. 79-92.

[103] A. F. Horadam, R.P. Loh, A. G. Shannon , *Divisibility Properties of Some Fibonacci-Type Sequences*, in A. F. Horadam and W.D. Wallis (eds.), Combinatorial Mathematics, VI:55-64. Berlin: Springer Verlag, **1979**.

[104] Laurence Somer, *The divisibility properties of primary Lucas recurrences with respect to primes*, The Fibonacci Quarterly **18** (4) (**1980**): 316-33.

[105] Shiro Ando and Daihachiro Sato, On the Proof of GCD and LCM Equalities Concerning the Generalized Binomial and Multinomial Coefficients, In *Applications of Fibonacci Numbers*, **4**: 9-16. Ed. G. E. Bergum et al. Dordrecht: Kluwer, **1991**.

[106] Henri W. Gould, Paula Schlesinger, Extensions of the Hermite g. c. d. theorems for binomial coefficients, Fibonacci Quarterly, 33(**1995**), 386-391.

[107] Discrete Mathematics - Special issue on selected papers in honor of Henry W. Gould Volume 204 Issue 1-3, June 6, 1**1999**, Editors: I. M. Gessel, L. W. Shapiro, D. G. Rogers

[108] Henri W. Gould, Articles in Professional Journals, An Abbreviated List *http : //www.math.wvu.edu/ gould/publications.html*

[109] Zhi-Wei Sun, *Reduction of unknowns in Diophantine representations*, Scientia Sinica (Ser. A) 35 (1992), no. 3, 257-269.

[110] Hilton Peter, Pedersen Jean, Vrancken Lue, *On Certain Arithmetic Properties of Fibonacci and Lucas Numbers*, The Fibonacci Quarterly vol. **33**.3 (**1995**): 211-217.

[111] W. A. Kimball and W. A. Webb, *Some congruences for generalized binomial coefficients*, Rocky Mountain J. Math. **25** (**1995**): 1079-1085.

[112] P. M. Voutier, *Primitive divisors of Lucas and Lehmer sequences*, Math. Comp. **64** (**1995**): 869-888.

[113] B. Wilson, *Fibonacci triangles modulo p*, Fibonacci Quart. **36** (**1998**), 194-203.

[114] John M. HOLTE, *Residues of generalized binomial coefficients modulo primes*, Fibonacci Quart. **38** (**2000**): 227 238.

[115] Hong Hu and Zhi-Wei Sun, *An Extension of Lucas' Theorem*, Proc. Amer. Math. Soc. **129** (**2001**), no. 12, 3471-3478.

[116] Mourad Abouzaid, *Les nombres de Lucas et Lehmer sans diviseur primitif*, J. Théor. Nombres Bordeaux **18** (**2006**): 299-313.

[117] Maciej Dziemiańczuk, Wiesław, Wladysław Bajguz, *On GCD-morphic sequences*, IeJNART: Volume (**3**), September (**2009**): 33-37. arXiv:0802.1303v1, [v1] Sun, 10 Feb 2008 05:03:40 GMT

[118] Chris Smith, *The terms in Lucas sequences divisible by their indices* J. Integer Seq. **13**, No. 2, Article ID 10.2.4, (**2010**) 18 pp.

[119] Kálmán Györy and Chris Smith , *The Divisibility of* $a^n - b^n$ *by Powers of n* , Integers **10**, 319-334 (**2010**).

[120] John Konvalina, *Generalized binomial coefficients and the subset-subspace problem* , Adv. in Appl. Math. Vol. **21** (**1998**): 228-240.

[121] John Konvalina, *A Unified Interpretation of the Binomial Coefficients, the Stirling Numbers and the Gaussian Coefficients*,The American Mathematical Monthly vol. **107**, No 10 , (**2000**):901-910.

[122] Ottavio M. D'Antona and Emanuele Munarini, *A combinatorial interpretation of the connection constants for persistent sequences of polynomials*, European Journal of Combinatorics, Volume **26**, Issue 7, October **2005**, Pages 1105-1118.

[123] Arthur T. Benjamin, Jennifer J. Quinn, *Proofs that really count: the art of combinatorial proof* , Washington DC, Mathematical Association of America , **2003**

[124] Arthur T. Benjamin, Sean S. Plott, *A Combinatorial Approach to Fibonomial Coefficients* , The Fibonacci Quarterly, Vol.**46/47** No.1 pp.7-9, February 2008/2009. **Errata**: Vol **48** No.3 pp.276, August **2010**.

[125] Johann Cigler, *Einige q-Analoga der Lucas - und Fibonacci-Polynome*, Sitzungsber. ÖAW 211 (**2002**): 3-20.

[126] Johann Cigler, *A new class of q - Fibonacci polynomials*, Electr. J. Comb. **10** (**2003**), R19

[127] Johann Cigler, *q-Lucas polynomials and associated Rogers-Ramanujan type identities*, arXiv:0907.0165v2 [v2] Tue, 16 Nov **2010** 12:21:13 GMT

[128] Anne-Marie Decaillot-Laulagnet, Universitè Paris Descartes Paris V phd **Thèse**; **1999**, *Edouard Lucas* (1842-1891): *le parcours original d'un scientifique française dans la deuxième moitié du XIXème siècle.*

[129] George Eyre Andrews, *The theory of partitions*, Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998. Reprint of the **1976** original.

[130] Pólya György, *Mathematical discovery* (Viley **1962**) v.1 .

[131] Pólya György *On the number of certain lattice polygons*, J. Combinatorial Theory v.**6** , **1969**, p. 102-105.

[132] Pólya György , G. L. Alexanderson, *Gaussian binomial coefficients*, Elem. Math. **26** (**1971**): 102-109.

[133] Katherine Humphreys, *A History and a Survey of Lattice Path Enumeration*, Journal of Statistical Planning and Inference, Volume **140**, Issue 8, August **2010**, Pages 2237-2254. Available online 21 January 2010.

[134] William Allen Whitworth, *Arrangements of m Things of One Sort and m Things of Another Sort under Certain Conditions of Priority*, Messenger of Math. **8**, (**1878**): 105-114.

[135] Donald Ervin Knuth, *Subspaces, Subsets and Partitions*, J. Comb. Theory **10** (**1971**):178-180.

[136] James Joseph Sylvester with insertions of F. Franklin , *A constructive theory of partitions, arranged in three acts, an interact and an exodion*, Amer. J. Math. **5**, (**1882**): 251-330.

[137] Maurice George Kendall, Alan Stuart, *The advanced theory of statistics*, London, **1961**, v.2

[138] Jay Goldman , Gian-Carlo Rota, *On the Foundations of Combinatorial Theory IV; finite-vector spaces and Eulerian generating functions*, Studies in Appl. Math. vol. **49**, (**1970**): 239-258.

[139] Joshef P. S. Kung, *The subset-subspace analogy*, in Gian-Carlo Rota on Combinatorics, J. Kung, Ed. , pp. 277-283, Birkhauser, Basel, **1995**.

[140] Marcel-Paul Schützenberger, *Une interprétation de certaines solutions de l'équation fonctionnelle* : $F(x+y) = F(x)F(y)$, C.R Acad.Sci.Paris 236 (**1953**): 352-353.

[141] Christian Kassel $Quantum\ groups$, Springer-Verlag, New York, (1995)

[142] Andrzej Krzysztof Kwaśniewski *On extended finite operator calculus of Rota and quantum groups* , Integral Transforms and Special Functions Vol. 2, No.4 (**2001**): 333-340.

[143] Ronald Graham, Donald, Ervin Knuth, and Oren Patashnik, *Concrete Mathematics: A Foundation for Computer Science* , (**1989**). Concrete Mathematics. Advanced Book Program (first ed.). Reading, MA: Addison-Wesley Publishing Company. pp. xiv+625., 2nd edition: January 1994 (1994). Concrete Mathematics (second ed.). Reading, MA: Addison-Wesley Publishing Company. pp. xiv+657.

[144] Paul Gustav Heinrich Bachmann, *Niedere Zahlentheorie* (Zweiter Teil), Leipzig, Teubner, **1910**; republished as Additive Zahlentheorie by Chelsea Publishing Co., New York, 1968.

[145] Steven Vajda, *Fibonacci and Lucas Number and the Golden Section - Theory and Applications*, John Wiley and Sons, **1989**.

[146] Maciej Dziemiańczuk, *First Remark on a* ζ *- Analogue of the Stirling Numbers*, INTEGERS: (**2011**) (to appear).