On Foundations of Newtonian Mechanics

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Abstract

Being based on V. Konoplev's axiomatic approach to continuum mechanics, the paper broadens its frontiers in order to bring together continuum mechanics with classical mechanics in a new theory of mechanical systems. There are derived motion equations of 'abstract' mechanical systems specified for mass–points, multibody systems and continua: Newton–Euler equations, Lagrange equations of II kind and Navier–Stokes ones.

Quasi–linear constitutive equations are introduced in conformity with V. Konoplev's definition of stress and strain (rate) matrices.

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I. INTRODUCTION

Classical mechanics is based on the axiom system introduced by I. Newton [1]. In result of generalizations made by L. Euler it is also used to studying the kinematical and dynamical behavior of physical objects modeled as a rigid body or their aggregates.

In the case of a parcel of air, water or rock consisting of a large number of particles, a corresponding discrete model, which can be constructed with the help of classical mechanics methods, would be hopelessly complicated. A different sort of models has been developed over the last three centuries to describe such physical systems. The model, called *continuous medium* or *continuum*, exploits the fact that in air, water and rock nearby particles behave similarly. The corresponding theory discounts the molecular structure of physical systems and regards matter as indefinitely divisible (here particles are characterized by their place volume and mass density). Thus the intent is to obtain a mathematical description of the macroscopic behavior of physical systems rather than to ascertain the ultimate physical basis of phenomena.

The analysis of the behavior of physical systems modeled as a continuum consists that we know as *continuum mechanics*.

A new architecture of mechanics is suggested in [2, 3] under the conditions that

: 1. there are no boxes or particles which can be rotated and deformed;

: 2. there are no mass–points (points with zero volume and non–zero mass).

The first condition makes it essentially various w.r.t. conventional continuum mechanics while the second condition deepens the conflict between classical mechanics and that of continua [4]:

> '. . . the dynamics of a continuous system must clearly include as a limiting case (corresponding to a medium of density everywhere zero except in one very small region) the mechanics of a single material particle. This at once shows that it is absolutely necessary that the postulates introduced for the mechanics of a continuous system should be brought into harmony with the modifications accepted above in the mechanics of the material particle'.

Following [2, 3, 5] we aim to remove the conflict by bringing together the continuum mechanics of Konoplev and the classical one into a theory of mechanical systems.

The new theory gives mathematical foundations to mechanics, which could be called *Newtonian* because it remains true to the principles of classical mechanics [1, 5] such as the absolute space and time, the concept of a mechanical system consisting of points in 3–dimensional space as well as those of the mass additivity, actions–at–a–distance and differential laws of motion, Galileo's principle of relativity, *etc.* (it sounds curiously, but I. Newton has defined mass, as well as force, as 'the measure of the same' – see, *e.g.*, definitions I, II and VI in [1] – as though he has foreseen application of the measure theory to mechanics in 20th century [2, 3, 5]).

It is built on relatively simple, transparent ideas, some conventional notions are used, but sometimes their sense is radically changed. We try to give all of them on *tabula rasa* without using any background in the field of mechanics. That is why no prior knowledge of continuum mechanics or the classical is required. It does not mean that we have done all our best in order to avoid any mechanical reminiscences. However giving no comments or motivations, we are about to point out all technical details of the introduced constructions (for, as Goethe has told, 'God is in the small things ...').

To demonstrate the new theory effectiveness we define the main classes of mechanical systems and deduce sufficiently many results known in the conventional mechanics: kinematics equations and Newton–Euler and Lagrange equations, stress–strain relations, *etc.*

We shall use the expression 'see also' in the case where a given statement differs in details from that of cited works and thus it is formally absent in them.

II. MAIN NOTIONS AND PRINCIPLES OF NEWTONIAN MECHANICS

The principle demand to a theory of mechanical systems is that 'the problem of mechanics comes to describing motions being in nature, namely, to their description in the most complete and simple form' [6]. Within the framework of this understanding the key concept of our theory is that a mechanical system is a set of points equiped with some fields: the mass, force, velocity ones, *etc.*

In what follows we shall use *Galilean spacetime* [7] introduced as a quadruple $\mathbf{G} = \{\mathbf{A}_4, \mathbf{V}_4, g, \tau\}$ where

: 1. $\mathbf{V}_4$ is a 4-dimensional vector space,

: 2. $\tau : \mathbf{V}_4 \to \mathbf{R}_1$ is a surjective linear map called the time map,

: 3. $g = \langle \cdot, \cdot \rangle$ is an inner product on $\ker\{\tau\}$ $(= \mathbf{R}_3)$, and

: 4. $\mathbf{A}_4$ is an affine normed space modeled on $\mathbf{V}_4$.

Introduce a point–wise spatial set $\mathbf{X}_3$ with the translation space $\mathbf{V}_3$ of 3–dimensional (free) vectors and a parameterization $t \in \mathbf{R}_1$ of the image of τ being in a point–wise time set $\mathbf{T}$ with the translation space $\mathbf{V}_+$ of 1–dimensional (free) vectors having one and the same sense.

A. Screw space

It is considered as conventional [8] that the screw calculus is not adapted for the description of continuous media, and '... being very attractive representation of a system of forces and rigid body motions with the help motors and screws, nevertheless it has no essential practical value ...' [9]. As a result in mechanics there is mainly absent the fundamental understanding (concept) that the interaction between mechanical systems is described with the help of screws.

The using of the screw concept is the key for the theory of mechanical systems (including, a continuum, a mass point and a rigid body) which is below constructed.

Let us introduce two fields λ and μ of vectors bounded with points of $\mathbf{X}_3$ such that for any x and $y \in \mathbf{X}_3$ with $\overrightarrow{\lambda}_x$ and $\overrightarrow{\lambda}_y \in \lambda$ and $\overrightarrow{\mu}_x$ and $\overrightarrow{\mu}_y \in \mu$ there are

$$\overrightarrow{\lambda}_x = \overrightarrow{\lambda}_y \stackrel{\text{def}}{=} \overrightarrow{\lambda}, \quad \overrightarrow{\mu}_x = \overrightarrow{\mu}_y + \overrightarrow{(x, y)} \times \overrightarrow{\lambda}$$

The set $l^\lambda = \{\lambda, \mu\}$ is called *screw* while $l^\lambda_x = \{\overrightarrow{\lambda}, \overrightarrow{\mu}_x\}$ is its representative at the point $x \in \mathbf{X}_3$. We shall use the notations $\mu(l^\lambda)$ and $\overrightarrow{\mu}_x$ for *moment component* μ of l^λ and its representative, respectively.

A screw is called *couple* if $\overrightarrow{\lambda} = 0$. An other special kind of screws is known as *sliding vector* (or *slider*) if in the case where $\lambda \neq 0$ the inner product $(\overrightarrow{\lambda}, \overrightarrow{\mu}_x) = 0$ for (all) x.

In the case where $\lambda \neq 0$ a screw l^λ is a slider iff there is at least a point at which the moment component $\mu(l^\lambda)$ of the screw is zero. A slider representative at a point x is called *axial* if $\mu_x(l^\lambda) = 0$.

Screws form a vector space $\mathbf{S}$ where any screw can be resolved in a sum of a slider and a couple [10].

B. Main measures of Newtonian mechanics (see also [2, 3])

Let us define the Lebesgue measure μ_t on σ–algebra of subsets in $\mathbf{T}$ while on σ–algebra of subsets in $\mathbf{X}_3$ there be so called *determinative* time–invariant measure

$$\mu_{LS}(A) = \mu_{ac}(A) + \mu_{pp}(A)$$

where $\mu_{ac}(A)$ is the absolutely continuous component w.r.t. Lebesgue measure μ_3 and $\mu_{pp}(A)$ is the pure point (discrete) component presented as $\mu_{pp}(A) = \sum_k \mu_{pp}(x_k)$ for points in an arbitrary subset $A \in \sigma_3$ such that $\mu_{pp}(x_k) \neq 0$. These points are called *pure*, the others being called *continuous* [11]. We assume μ_{ac} to be Lebesgue measure μ_3.

We shall further use the measure μ_{LS} for definition of points with mass, but without volume, and without masses bodies with volumes, but without masses and forces exerting on them.

Definition 1. *Let $A \in \sigma_3$, then the measure $m(A) : A \to \mathbf{R}_1$ is called* mass *(*measure of inertia*).*

Due to the Radon–Nikodym theorem [11] we may specify $m(A)$ as Lebesgue–Stieltjes integral

$$m(A) = \int \chi_A \rho_x \mu_{LS}(dx)$$

with a μ_{LS}–integrable (mass) density ρ_x w.r.t. the measure $\mu_{LS}(dx)$ (here χ_A is the characteristic function of A). The density can be time–varying.

The set $\mathbf{X}_3^c \subset \mathbf{X}_3$ is called *set of concentration* of the measure m on $\mathbf{X}_3$ if $m(B) = 0$ for everyone $\mu_{LS}-$measured set $B \subset \mathbf{X}_3 \setminus \mathbf{X}_3^c$.

We shall use the notion of *signed measure* [12] being a generalization of the concept of measure by allowing it to have negative values. Some authors call it *charge*, by analogy with electric charge, which is a familiar distribution that takes on positive and negative values.

Let $\eta(\cdot)$ be a function on σ_3 whose values are n–ples of signed measures. Then:

: 1. the function $\eta(\cdot)$ is called *vector signed measure* on σ_3;

: 2. a function $\zeta(\cdot,\cdot)$, defined on $\sigma_3 \times \sigma_3$ and being a vector signed measure by each of arguments, is called *vector signed bi–measure*;

: 3. the vector signed bi–measure $\zeta(\cdot,\cdot)$ is called skew if $\zeta(A,B) = -\zeta(B,A)$ for any A and $B \in \sigma_3$.

Definition 2. *Let A and $B \in \sigma_3$, then the skew vector signed bi–measure $\mathcal{F}(A, B)$: $(A, B) \to$ $\mathbf{S}$ is called* measure of action *of B on A (where $\mathbf{S}$) is the screw space.*

We specify $\mathcal{F}(A, B)$ as Lebesgue–Stieltjes integral

$$\mathcal{F}(A, B) = \int \chi_{_A} l^{\phi(x,B)} \mu_{LS}(dx) = \int \chi_{_B} l^{\psi(y,A)} \mu_{LS}(dy)$$

where representatives of μ_{LS}–integrable slider functions $l^{\phi(x,B)}$ and $l^{\psi(y,A)}$ are axial at all $x \in A$ and $y \in B$, respectively.

The set $A^e = \mathbf{X}_3 \setminus A$ is called *environment* of A. It is clear that

$$\mathcal{F}(A, A^e) = \mathcal{F}(A, A^e + A) \stackrel{\text{def}}{=} \int \chi_{_A} l^{\phi(x,A^e+A)} \mu_{LS}(dx) = \int \chi_{_A} l^{\phi(x,x^e)} \mu_{LS}(dx) \tag{1}$$

We shall assume that $l^{\phi(x,x^e)} \equiv 0$ on the set $\mathbf{X}_3 \setminus \mathbf{X}_3^c$.

Definition 3. *The slider function $l^{\phi(x,x^e)}$ is called* intensity *of the action of x^e upon $x \in \mathbf{X}_3^c$.*

Remark 1. *The intensity can also depend on the motion prehistory.*

Exemplify the introduced notion. Let the skew bi–measure $\mathcal{G}(A, B)$ be such that

$$\mathcal{G}(A, A^e) = \int \chi_{_A} l^{g(x,x^e)} \rho_x \mu_{LS}(dx), \quad g(x, x^e) = \gamma \int \chi_{x^e}(x - y) \frac{\rho_y \mu_{LS}(dy)}{\|x - y\|^3}$$

where γ is a positive (gravitational) constant, representatives of the μ_{LS}–integrable slider function $l^{g(x,x^e)}$ are axial at all $x \in \mathbf{X}_3^c$.

Definition 4 [3]. *The slider function $l^{\rho_x g(x,x^e)}$ is called* intensity of gravitating action *of x^e upon $x \in \mathbf{X}_3^c$.*

C. Fundamental principles of dynamics

Let σ_t be Borel σ–algebra of subsets in $\mathbf{T}$ while σ_3 be Borel σ–algebra of subsets in $\mathbf{X}_3$.

Let us fix some parameterization $t \in \mathbf{R}_1$ of $\mathbf{T}$, then the differentiable bijection: $\mathbf{X}_3 \to$ $\mathbf{X}_t \subset \mathbf{R}_3$ is called *motion*, t is a time instant.

For any point $x \in \mathbf{X}_3$ the motion defines the point $x(t) \in \mathbf{X}_t$. Introduce the radius–vector $\overrightarrow{x(t)} = \overrightarrow{(O_0, x(t))}$ called *position* of $x(t) \in \mathbf{X}_t$ w.r.t. O_0 and the vector $\overrightarrow{v}_x = \overrightarrow{x(t)}^{\bullet}$ called its *velocity* (to honor Newton, we use the superscript $^\bullet$ for derivatives by t). Thus we equip the set $\mathbf{X}_3$ with the fields of positions, velocities and the measuares of mechanics.

Let representatives of the slider function l^{v_x} be axial at all $x \in \mathbf{X}_3$.

Second Newton's law (see also [3, 13]). *There exist a Cartesian frame $\mathcal{E}_0$ with the origin O_0 and a parameterization $t \in \mathbf{R}_1$ of $\mathbf{T}$ such that motion of a point $x \in A^c$ is described by the following equations*

: *1. if the point x is continuous*

$$\rho_x(l_x^{v_x,0})^\bullet = l_x^{\phi(x,x^e),0} \tag{2}$$

: *2. if the point x is pure*

$$m_x(l_x^{v_x,0})^\bullet = \mu_{pp}(x) l_x^{\phi(x,x^e),0} \tag{3}$$

where $m_x = \rho_x \mu_{pp}(x)$ is mass *of the pure point (coordinate representations of vectors in $\mathcal{E}_0$ are marked with the superscript ^{0}).*

Henceforth we call the parameterization and the frame $\mathcal{E}_0$ *Galilean* (this formulation of second Newton's law is connected with first one and isolated systems nohow).

Remark 2. *In the case of time–varying densities of inertia (masses) relations (2)–(3) are invariant w.r.t.* Galilean group *[2, 3] while the traditional form of second Newton's law [1] does not. In such case relations (2)–(3) include in themselves slider functions of so called reactivity (see the well–known equation of Meschersky).*

Definition 5. *Elements of $\sigma-$algebra σ_3 answering the second Newton's law and the principles of causality, determinancy and relativity [7] are called* mechanical systems *[3], the set $\mathbf{X}_3, \mathbf{G}$, (σ_3, σ_t) and $(\mu_{LS}, m, \mathcal{F})$ being called* Universe of Newtonian mechanics.

In the given definition (see also [2, 3]), similarly to that of *probability space* [14], Universes of mechanics are separately defined for every mechanical problem.

In the motion equations the intensities are defined nohow, and any action intensity pictures some mechanical system [7] in depending on its 'constitution'. Sometimes some part of the intensities is implicitly given, while another one must be defined from the restriction or constraint imposed on a point, its velocity and, perhaps, derivative of the velocity, beforehand set, *i.e.*, not dependent on the law of point motion. In this case a point which motion is in agreement with constraints is called *constrained*.

Example. Let the vector $\phi(x, x^e)$ describe the action of x^e on x when constraints are absent and the constraints be given by the equation $\sigma(x^0, v^0, t) = 0$ where σ is a differentiable vector–function of the instant t, the position x^0 and the velocity v^0. After differentiating $\sigma(x^0, v^0, t) = 0$ we have

$$\frac{\partial\sigma}{\partial x^{0,T}}v^0 + \frac{\partial\sigma}{\partial v^{0,T}}v^{0\bullet} + \frac{\partial\sigma}{\partial t} = 0$$

and

$$v^{0\bullet} = -(\frac{\partial\sigma}{\partial v^{0,T}})^T[\frac{\partial\sigma}{\partial v^{0,T}}(\frac{\partial\sigma}{\partial v^{0,T}})^T]^{-1}(\frac{\partial\sigma}{\partial x^{0,T}}v^0 + \frac{\partial\sigma}{\partial t})$$

if the above inverse exists.

Hence there exists such slider function $l^{c(x,x^e)}$ (with representatives being axial at all $x \in A^c$) that

$$\rho_x(l_x^{v_x,0})^\bullet = l_x^{\phi(x,x^e),0} + l_x^{c(x,x^e),0} \tag{4}$$

if the point x is continuous or

$$m_x(l_x^{v_x,0})^\bullet = \mu_{pp}(x)[l_x^{\phi(x,x^e),0} + l_x^{c(x,x^e),0}] \tag{5}$$

if the point x is pure.

In this way we may introduce the following

Principle of constraint release. *Motion of any constrained point $x \in A^c$ is described by equations (4) and (5) (in the Galilean frame $\mathcal{E}_0$) with some μ_{LS}–integrable slider function $l^{c(x,x^e)}$ called* intensity of constraint action *upon $x \in A^c$.*

The principle of constraint release demarcates two categories of actions, namely, active and passive ones: it says that an active (motive) action creates motion while a passive one only puts obstacles in this motion. If we remove constraints then only active actions are kept.

Remark 3 *One must not suppose that the Principle of release from constraints and that of DAlembert eliminate the difference in the nature of active forces and passive ones (constraint actions and forces of inertia). It is only for the sake of convenience that we use these principles: only forces the resultant of which is f exert on the particle [15].*

III. A MASS–POINT

Consider a set $A \in \sigma_3$ consisting of a unique pure point of the measure μ_{LS} as a free *mass–point*. In this case from relation (3) follows the well–known *second Newton's law* (in the Galilean frame $\mathcal{E}_0$):

$$m_x v_x^{0\bullet} = f^0 \tag{6}$$

where $f = \mu_{pp}(x)\phi$ is called *force* exerting on the mass–point x.

Note that in the case where the mass is time–varying the force includes in itself that of reactivity (see the well–known equation of Meschersky).

Motion of a mass–point can be constrained. Let us give the description of constrained motion.

Variety of constraints contains so called ideal and non–ideal ones. Ideal constraints generate constraint actions having the direction and sense of the normal to the corresponding manifold. We shall assume that constraints are ideal (*Axiom of ideal constraints*), scleronomic and holonomic.

Ideal holonomic and scleronomic constraints force the point under consideration to move along with a certain manifold $\sigma(x^0) = 0$ having lower dimension than its configuration space. Let this manifold can be parameterized with some vector q. The vector q is called *generalized* one, its first derivative being called *generalized velocity* $q^\bullet$.

For any point x^0 of the manifold we have

$$x^0 = \eta(q)$$

If the columns of the following matrix [16]

$$\tau = \frac{\partial \eta}{\partial q^T}$$

are linearly independent, they form a basis of the linear space $\mathbf{T} = \mathbf{T}(q)$ being tangent to the manifold at a point q.

If the columns of the following matrix [16]

$$\nu = \frac{\partial \sigma}{\partial x^{0,T}}$$

are linearly independent, they form a basis of the linear space $\mathbf{N} = \mathbf{N}(q)$ being orthogonal to the manifold at a point q.

Thus $\mathbf{R}_3 = \mathbf{T} \times \mathbf{N}$ with the basis $[\tau, \nu]$. It is easy to see that

$$P_\tau \mathbf{R}_3 = \mathbf{T}, \quad P_\nu \mathbf{R}_3 = \mathbf{N}$$

where $P_\tau = (\tau^T \tau)^{-1} \tau^T$ and $P_\nu = \nu(\nu^T \nu)^{-1} \nu^T$ are projections.

It is obvious that $x^{0\bullet} = \tau q^{\bullet}$ and $x^{0\bullet\bullet} = \tau q^{\bullet\bullet} + (\frac{\partial\tau}{\partial q^T} q^{\bullet}) q^{\bullet}$ (here the matrix $\frac{\partial\tau}{\partial q^T} q^{\bullet}$ is square). As the constraint is supposed to be ideal and therefore $P_\tau r^0 = 0$, from (5) follows

$$m_x[q^{\bullet\bullet} + P_\tau(\frac{\partial\tau}{\partial q^T} q^{\bullet})q^{\bullet}] = P_\tau f^0 \tag{7}$$

Applying the projection P_ν to (5) we define the following equation

$$m_x P_\nu(\frac{\partial\tau}{\partial q^T} q^{\bullet})q^{\bullet} = P_\nu(f^0 + c^0) = P_\nu f^0 + c^0$$

From this relation follows that

$$c^0 = m_x P_\nu(\frac{\partial\tau}{\partial q^T} q^{\bullet})q^{\bullet} - P_\nu f^0$$

i.e., the constraint force is not a function of time, but it depends on the generalizing coordinates and velocities as well as on the active force.

It is clear that the theory above can be applied to mass–point systems.

IV. RIGID BODIES

Definition 6 (see also [17]). *A bounded closed set $A \in \sigma_3$ is called* rigid body *if*

: *1. constraints applied on its points keep distances between them not changing with time;*

: *2. the constraints are ideal.*

A rigid body may contain continuous and pure points.

Remark 4. *In elementary manuals of mechanics, transition from a mass point to a body as a point system is made somehow imperceptibly; constraint forces are not mentioned at all, and instead of a lawful exception there is an illegal, silent exclusion of these forces. They remain ordinarily without any attention and even without a mention, as if they did not exist at all.*

A. Motion of a rigid body (see also [2, 10])

Below we do not consider motion of separate points of a rigid body, as due to Euler the motion of a rigid body as a whole is characterized by so called generalized coordinates and velocities.

Quasi–velocities of a rigid body. Consider a rigid body $A_k \subset \sigma_3$ with an attached Cartesian frame $\mathcal{E}_k$ having an origin O_k.

For any vector $\lambda = \mathrm{col}\{\lambda_1, \lambda_2, \lambda_3\} \in \mathbf{R}_3$ there are defined the cross product matrix

$$\lambda^{\times} \stackrel{\text{def}}{=} \begin{bmatrix} 0 & -\lambda_3 & \lambda_2 \\ \lambda_3 & 0 & -\lambda_1 \\ -\lambda_2 & \lambda_1 & 0 \end{bmatrix} \tag{8}$$

and the relation $\lambda^0 = C_{0,k}\lambda^k$ where $C_{0,k} \in \mathcal{SO}(\mathbf{R}, 3)$.

Let x be an arbitrary point fixed in $\mathcal{E}_k$. Introduce the radius–vectors r_0 and $r_k \in \mathbf{R}_3$ of the point x w.r.t. the origins O_0 and O_k, respectively. Define $d_{0,k} = r_0 - r_k$. Then we may represent the relation $r_0 = d_{0,k} + r_k \in \mathbf{R}_3$ in $\mathcal{E}_0$ as $r_0^0 = d_{0,k}^0 + C_{0,k}r_k^k$. As r_k^k is time–constant, with differentiating the last relation we have $v_x^0 = v_{0,k}^0 + C_{0,k}^{\bullet} r_k^k$ where $v_x^0 = r_0^{0\bullet}$ and $v_{0,k}^0 = d_{0,k}^{0\bullet}$ are velocities of x and O_k w.r.t. O_0 in the frame $\mathcal{E}_0$, respectively. Hence

$$v_x^k = v_{0,k}^k + C_{k,0}C_{0,k}^{\bullet} r_k^k \tag{9}$$

Definition 7.

: *1. The vector* $\overrightarrow{v}_{0,k} = \overrightarrow{d}^{\bullet}_{0,k}$ *is called* velocity *of* $\mathcal{E}_k$–displacement *w.r.t. the frame* $\mathcal{E}_0$*, its coordinate column* $v_{0,k}^k$ *is called* quasi–velocity of $\mathcal{E}_k$–displacement *or* linear (translation) quasi–velocity*;*

: *2. the vector* $\overrightarrow{\omega}_{0,k}$ *with the coordinate column* $\omega_{0,k}^k$ *(in* $\mathcal{E}_k$*) is called* instantaneous velocity *of* $\mathcal{E}_k$–rotation *w.r.t. the frame* $\mathcal{E}_0$ *if (in according with (9))*

$$\omega_{0,k}^{k\times} = C_{k,0}C_{0,k}^{\bullet} \tag{10}$$

$\omega_{0,k}^k$ *is called* angular quasi–velocity*;*

: *3. the coordinate column*

$$V_{0,k} = \begin{pmatrix} v_{0,k}^k \\ \omega_{0,k}^k \end{pmatrix} \in \mathbf{R}_6$$

is called quasi–velocity of motion *of the frame* $\mathcal{E}_k$ *w.r.t. the frame* $\mathcal{E}_0$.

Algebraic theory of screws. Consider a vector $\lambda \in \mathbf{R}_3$ bounded with a point $x \in \mathbf{R}_3$ and define its moments $\widetilde{\mu}_0$ and $\widetilde{\mu}_k \in \mathbf{R}_3$ w.r.t. the points O_0 and O_k, respectively. As $\widetilde{\mu}_k = \widetilde{\mu}_0 + \lambda \times d_{0,k}$, the set $\{\lambda, \widetilde{\mu}\}$ is a slider (with representatives $\{\lambda, \widetilde{\mu}_0\}$ and $\{\lambda, \widetilde{\mu}_k\}$).

Let us define:

: 1. the coordinate columns λ^0, λ^k, $\widetilde{\mu}^0_0 = r^{0\times}_0 \lambda^0$, $\widetilde{\mu}^k_k = r^{k\times}_k \lambda^k$ in the frames $\mathcal{E}_0$ and $\mathcal{E}_k$;

: 2. (*Plücker*) coordinate columns (of 1–st type) $\pi^{\lambda,0}_0 = \mathrm{col}\{\lambda^0, \widetilde{\mu}^0_0\}$ and $\pi^{\lambda,k}_k = \mathrm{col}\{\lambda^k, \widetilde{\mu}^k_k\}$;

: 3. the following matrices

$$T^0_{0,k} = \begin{bmatrix} I & O \\ d^{0\times}_{0,k} & I \end{bmatrix}, \quad C^{\otimes}_{0,k} = \begin{bmatrix} C_{0,k} & O \\ O & C_{0,k} \end{bmatrix} \tag{11}$$

where I is the unit matrix, O is the null matrix.

Theorem 1 [3]. *Matrices* $L^{(1)}_{0,k} = T^0_{0,k} C^{\otimes}_{0,k} = C^{\otimes}_{0,k} T^k_{0,k}$ *form the multiplicative group* $\mathcal{L}^{(1)}(\mathbf{R}, 6)$ *such that* $\pi^{\lambda,0}_0 = L^{(1)}_{0,k} \pi^{\lambda,k}_k$ *and*

$$L^{(1)\bullet}_{0,k} = L^{(1)}_{0,k} \varPhi_{0,k} \tag{12}$$

where $\varPhi_{0,k} = \begin{bmatrix} \omega^{k\times}_{0,k} & O \\ v^{k\times}_{0,k} & \omega^{k\times}_{0,k} \end{bmatrix}$.

Proof. The columns $\pi^{\lambda,0}_0$ and $\pi^{\lambda,k}_k$ can be represented as follows

$$\pi^{\lambda,0}_0 = \begin{bmatrix} I \\ r^{0\times}_0 \end{bmatrix} \lambda^0, \quad \pi^{\lambda,k}_k = \begin{bmatrix} I \\ r^{k\times}_k \end{bmatrix} \lambda^k$$

From $r^0_0 = d^0_{0,k} + r^0_k$ follows that

$$L^{(1)}_{0,k} \pi^{\lambda,k}_k = T^0_{0,k} C^{\otimes}_{0,k} \begin{bmatrix} I \\ r^{k\times}_k \end{bmatrix} \lambda^k = T^0_{0,k} \begin{bmatrix} I \\ r^{0\times}_k \end{bmatrix} \lambda^0 = \pi^{\lambda,0}_0$$

Relation (12) is true as from (11) follows that $L^{(1)\bullet}_{0,k} = T^{0\bullet}_{0,k} C^{\otimes}_{0,k} + T^0_{0,k} C^{\otimes\bullet}_{0,k} = T^0_{0,k} C^{\otimes}_{0,k} (C^{\otimes}_{k,0} C^{\otimes\bullet}_{0,k} + C^{\otimes}_{k,0} T^{0\bullet}_{0,k} C^{\otimes}_{0,k}) = L^{(1)}_{0,k} \varPhi_{0,k}$.

These matrices form a group as there are $L^{(1)}_{0,p} L^{(1)}_{p,k} = T^0_{0,p} C^{\otimes}_{0,p} T^p_{p,k} C^{\otimes}_{p,k} = T^0_{0,p} T^0_{p,k} C^{\otimes}_{0,p} C^{\otimes}_{p,k} = T^0_{0,k} C^{\otimes}_{0,k} = L^{(1)}_{0,k}$ for a subindex p and $L^{(1),-1}_{0,k} = (T^0_{0,k} C^{\otimes}_{0,k})^{-1} = C^{\otimes,T}_{0,k} (T^0_{0,k})^{-1} = C^{\otimes}_{k,0} T^0_{k,0} C^{\otimes,T}_{0,k} C^{\otimes}_{k,0}$ $= T^k_{k,0} C^{\otimes}_{k,0} = L^{(1)}_{k,0}$.

Sum of $\pi^{\lambda,0}_0$–kind columns (for different triples λ^0 and points x) is an element of $6-$dimensional linear space $\mathbf{R}_6$.

Using different Cartesian frames $\mathcal{E}_k$ we are brought to say that the rotation group $\mathcal{SO}(\mathbf{R}, 3)$ restores $\overrightarrow{\lambda} \in \mathbf{V}_3$ from coordinate triples λ^k. In the similar way we may use the group $\mathcal{L}^{(1)}(\mathbf{R}, 6)$ in order to define the vector space $\mathbf{S}_1$ from sums of $\pi^{\lambda,k}_k$ with different triples λ^k and points x. Its elements are called *wrenches*.

The columns $\pi_k^{\lambda,k}$ generate the element $\pi^\lambda \in \mathbf{S}_1$ called *slider* (of 1–st type).

Introduce columns of the kind $\mathrm{col}\{\widetilde{\mu}_x^k, \lambda^k\} = \widehat{I}\pi_k^{\lambda,k}$ where $\widehat{I} = \begin{bmatrix} O & I \\ I & O \end{bmatrix}$. Then the group $\mathcal{L}^{(2)}(\mathbf{R}, 6)$ of matrices $L_{0,k}^{(2)} = \widehat{I}L_{0,k}^{(1)}\widehat{I}$ generates the element called *slider* (of 2–nd type). Their sums generate the vector space $\mathbf{S}_2$ with elements called *twists*. It is clear that the wrench and twist spaces are isomorphic to the space $\mathbf{S}$.

Remark 5. *The quasi–velocity of motion of the frame $\mathcal{E}_k$ w.r.t. the frame $\mathcal{E}_0$ is the twist representative because for any points x and $y \in \mathbf{R}_3$*

$$\begin{pmatrix} v_x^k \\ \omega_{0,k}^k \end{pmatrix} = \begin{bmatrix} I & \overrightarrow{(x,y)}^{k\times} \\ O & I \end{bmatrix} \begin{pmatrix} v_y^k \\ \omega_{0,k}^k \end{pmatrix}$$

Newton–Euler equation. From relations (4) and (5) follows that

: 1. at each continuous point $x \in A_k^c$

$$\rho_x(\pi_0^{v_x,0})^\bullet = \pi_0^{\phi(x,x^e),0} + \pi_0^{c(x,x^e),0}$$

: 2. at each pure point $x \in A_k^c$

$$m_x(\pi_0^{v_x,0})^\bullet = \mu_{pp}(x)[\pi_0^{\phi(x,x^e),0} + \pi_0^{c(x,x^e),0}]$$

Hence we have the screw form of the second Newton's law

$$\int \chi_{A_k}(\pi_0^{v_x,0})^\bullet \rho_x \mu_{LS}(dx) = \mathcal{F}_k^0 \stackrel{\text{def}}{=} \int \chi_{A_k} \pi_0^{\phi(x,x^e),0} \mu_{LS}(dx) \tag{13}$$

as the constraints are considered as ideal and thus $\int \chi_{A_k} \pi_0^{c(x,x^e),0} \mu_{LS}(dx) = 0$ [17].

Due to (1) there is $\mathcal{F}_k^0 = \int \chi_{A_k} \pi_0^{\phi(x,A_k^e),0} \mu_{LS}(dx)$, *i.e.*, we may neglect interactions between the points of A_k.

Remark 6. *These interactions are not constraint actions.*

Lemma 1 [3]. *There is the following relation*

$$\pi_k^{v_x,k} = \Theta_k^x V_{0,k}, \quad \Theta_k^x = \begin{bmatrix} I & -r_k^{k\times} \\ r_k^{k\times} & -(r_k^{k\times})^2 \end{bmatrix}$$

Proof. The statement is true as from relations (9) and (10) follows

$$\pi_k^{v_x,k} = \begin{bmatrix} I \\ r_k^{k\times} \end{bmatrix} v_x^k = \begin{bmatrix} I \\ r_k^{k\times} \end{bmatrix} \left(v_{0,k}^k + \omega_{0,k}^{k\times} r_k^k\right) = \begin{bmatrix} I & -r_k^{k\times} \\ r_k^{k\times} & -(r_k^{k\times})^2 \end{bmatrix} \begin{pmatrix} v_{0,k}^k \\ \omega_{0,k}^k \end{pmatrix}$$

where the relation $\omega_{0,k}^{k\times} r_k^k = -r_k^{k\times}\omega_{0,k}^k$ is used.

From the lemma we have $\left(\pi_0^{v_x,0}\right)^\bullet = \left(L_{0,k}^{(1)}\pi_k^{v_x,k}\right)^\bullet = L_{0,k}^{(1)}(\Theta_k^x V_{0,k}^\bullet + \Phi_{0,k}\Theta_k^x V_{0,k})$. That is why from (13) we arrive at the following

Theorem 2 [3]. *The motion of A_k (w.r.t. $\mathcal{E}_0$ in the frame $\mathcal{E}_k$) is described by the (*Newton–Euler*) equation*

$$\Theta_k V_{0,k}^\bullet + \Phi_{0,k}\Theta_k V_{0,k} = \mathcal{F}_k^k \tag{14}$$

where $\Theta_k = \int \chi_{A_k}\Theta_k^x \rho_x \mu(dx)$, $\mathcal{F}_k^k = L_{k,0}^{(1)}\mathcal{F}_k^0$.

It is easy to see that the matrices of relation (14) depend on rotation matrices (and linear and angular quasi–velocities, too) that is why equation (14) must be considered along with the (Euler kinematical) relation

$$C_{0,k}^\bullet = C_{0,k}\omega_{0,k}^{k\times} \tag{15}$$

Euler angles. We may avoid to use equation (15) as the rotation matrix $C_{0,k}$ can be parameterized with the help of some vector $\vartheta_{0,k} \in \mathbf{R}_3$ and (see [2])

$$\omega_{0,k}^k = D_{0,k}\vartheta_{0,k}^\bullet \tag{16}$$

where the matrix $D_{0,k} = D_{0,k}(\vartheta_{0,k})$ is known.

Indeed, for Cartesian frames $\mathcal{E}_0$ and $\mathcal{E}_k$ of the space $\mathbf{R}_3$ with the origin O_0 and O_k and the bases $[\mathbf{e}^0]$ and $[\mathbf{e}^k]$ let the rotation matrix $C_{0,k}$ be such that for any vector $\overrightarrow{\lambda}$ there is $\lambda^0 = C_{0,k}\lambda^k$.

First define, *e.g.*, the auxiliary vector $e = e_3^{k\times}e_1^0$. Second with the rotation around the axis e_1^0 at the angle φ between the vectors e_2^0 and e we obtain $E^1 = [e_1^1, e_2^1, e_3^1] = E^0 C_1$ where $E^0 = [e_1^0, e_2^0, e_3^0]$, $e_1^1 = e_1^0$ and $e_2^1 = e$. Then with the rotation around the axis $e_2^1 = e$ at the angle ϑ between the vectors e_3^1 and e_3^k we obtain $E^2 = [e_1^2, e_2^2, e_3^3] = E^1 C_2$ where $e_2^2 = e_2^1 = e$ and $e_3^3 = e_3^k$. At last the rotation around the axis e_3^k at the angle ψ between the vectors e_2^2 and e_2^k gives $E^k = \left[e_1^k, e_2^k, e_3^k\right] = E^2 C_3$. Thus the rotation matrix $C_{0,k}$ is defined by the relation $C_{0,k} = C_1 C_2 C_3$ where

$$C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}, C_2 = \begin{bmatrix} \cos\vartheta & 0 & \sin\vartheta \\ 0 & 1 & 0 \\ -\sin\vartheta & 0 & \cos\varphi \end{bmatrix}, C_3 = \begin{bmatrix} \cos\psi & -\sin\varphi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{17}$$

are so called the simplest rotation matrices with Euler angles φ, ϑ, and ψ [18].

Note that at the second step we could choose the angle between e_1^0 and e. Moreover we could start with any auxiliary vector $e = e_k^{0\times} e_l^k$ where k, l are any naturals 1, 2, 3 (the case where $k = l$ is admissible, too). It means that there are 12 ways to represent the matrix $C_{0,k}$ as a product of three matrices of type (17) with the help of different Euler angles [18].

Our choice of representing $C_{0,k}$ given above has the advantage that

$$C_{0,k} \approx I + \vartheta^{\times}$$

for small $\vartheta = \mathrm{col}\{\varphi, \vartheta, \psi\}$ (among 12 representations of $C_{0,k}$ there are 3! with the same property).

Lemma 2 [3] *For relation (16) there is the matrix*

$$D_{0,k}(\vartheta_{0,k}) = \left[C_3^T C_2^T e_1^0, C_3^T e_2^0, e_3^0\right], \quad \vartheta_{0,k} = \begin{pmatrix} \vartheta_1 \\ \vartheta_2 \\ \vartheta_3 \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} \varphi \\ \vartheta \\ \psi \end{pmatrix} \tag{18}$$

Proof. From (10) follows that $\omega_{0,k}^{k\times} = C_{0,k}^T C_{0,k}^{\bullet} = C_3^T C_2^T C_1^T (C_1^{\bullet} C_2 C_3 + C_1 C_2^{\bullet} C_3 + C_1 C_2 C_3^{\bullet}) = C_3^T C_2^T (C_1^T C_1^{\bullet}) C_2 C_3 + C_3^T (C_2^T C_2^{\bullet}) C_3 + (C_3^T C_3^{\bullet})$. Due to (17) the relations in the parentheses above are $C_i^T C_i^{\bullet} = e_i^{0\times} \vartheta_i^{\bullet}$ and $\omega_{0,k}^{k\times} = C_3^T C_2^T e_1^{0\times} C_2 C_3 \vartheta_1^{\bullet} + C_3^T e_2^{0\times} C_3 \vartheta_2^{\bullet} + e_3^{0\times} \vartheta_3^{\bullet}$ or $\omega_{0,k}^k = C_3^T C_2^T e_1^0 \vartheta_1^{\bullet} + C_3^T e_2^0 \vartheta_2^{\bullet} + e_3^0 \vartheta_3^{\bullet}$ or $\overrightarrow{\omega}_{0,k} = \sum_{i=1}^3 \overrightarrow{\theta}_i \vartheta_i^{\bullet}$ where $\theta_1 = C_3^T C_2^T e_1^0$, $\theta_2 = C_3^T e_2^0$ and $\theta_3 = e_3^0$. Thus the matrices of relation (30) prove to be expressed through Euler angles.

Fedorov vector–parameter. Let us consider an other parameterization. The number triple $f \in \mathbf{R}_3$ is called *Fedorov vector–parameter* of a rotation matrix C [19] if it is answered with the following skew matrix (see (8))

$$f^{\times} = (C - I)(C + I)^{-1} \tag{19}$$

Some authors prefer the involutory form $(C - I)^{-1}(C + I)$.

The inverse (Cayley) transformation restores the rotation matrix

$$C=(I+f^{\times})(I-f^{\times})^{-1} \tag{20}$$

It can be proven that $Cf^{\times}C^T = f^{\times}$ or $Cf = f$, *i.e.*, f is an eigenvector of C.

With using (20) as a function of f, it is easy to be verified (for example, by means of Maple©) that there are true the following relations

$$f^{\times} = \frac{C - C^T}{1 + \operatorname{tr} C} \tag{21}$$

$$C = \frac{(1 - \|f\|^2)I + 2ff^T + 2f^{\times}}{1 + \|f\|^2} \tag{22}$$

There is no frame w.r.t. which Fedorov vector–parameter is defined. It can be put in correspondence with many other objects of different nature, *e.g.*, with Euler angles.

Gibbs vectors. In a Cartesian frame $\mathcal{E}_0$ of the space $\mathbf{R}_3$ there is a point that can be defined by f as coordinates of the point or by its (bounded) radius–vector $\overrightarrow{g}$ (in the space $\mathbf{V}_3$ of geometrical - free - vectors) for which this triple serves as decomposition coefficients on this basis, *i.e.*, there is the coordinate triple $g = f$. Thus with the help of the canonical basis of $\mathbf{R}_3$ we define the vector $\overrightarrow{g} \in \mathbf{V}_3$ known as that of *Gibbs* [18].

Rodrigues vector. Consider a Cartesian frame $\mathcal{E}_k$ with an origin O_k and define the vectors $\overrightarrow{r}_{0,p}$ and $\overrightarrow{r}_{p,k} \in \mathbf{V}_3$ such that $r^p_{0,p} = f_{0,p}$ in $\mathcal{E}_p$ and $r^k_{p,k} = f_{p,k}$ in $\mathcal{E}_k$. We will call them *Rodrigues vectors* [20] ($2\overrightarrow{r}_{p,k}$ – *vector of finite rotation of* $\mathcal{E}_k$ *w.r.t.* $\mathcal{E}_p$ [21]).

Thus noone may identify Fedorov vector–parameter and Gibbs and Rodrigues vectors.

Composition rules for Fedorov vector–parameters and Gibbs and Rodrigues vectors. Henceforth we shall consider the structure of indices used in rigid body kinematics where any rotation matrices $C_{0,p}$ and $C_{p,k}$ depicture rotations of Cartesian frames $\mathcal{E}_p$ and $\mathcal{E}_k$ w.r.t. $\mathcal{E}_0$ and $\mathcal{E}_p$, respectively. According to (21), for the matrices $C_{0,p}$ and $C_{p,k}$ it is possible to construct vector–parameters $f_{0,p}$ and $f_{p,k}$.

Let us consider the product of two rotation matrices $C_{0,p}$ and $C_{p,k}$ with the vector–parameters $f_{0,k}$ and $f_{p,k}$. Then [19]

$$C_{0,k}=C_{0,p}C_{p,k} \Leftrightarrow f_{0,k} = \langle f_{0,p}, f_{p,k}\rangle = \frac{1}{1 - f^T_{0,p}f_{p,k}}(f_{0,p} + f_{p,k} + f^{\times}_{0,p}f_{p,k}) \tag{23}$$

Indeed, introduce the following notation

$$f_{0,k} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad f_{0,p} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \qquad f_{p,k} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

where elements of the columns are known functions of time.

Let us rewrite relation (23) in the following form

$$(1 - ap - bq - cr)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} p \\ q \\ r \end{pmatrix} + \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

It is easy to be verified, for example, by means of Maple© that

$$C_{0,k} = \begin{bmatrix} 1 & -z & y \\ z & 1 & -x \\ -y & x & 1 \end{bmatrix}\begin{bmatrix} 1 & z & -y \\ -z & 1 & x \\ y & -x & 1 \end{bmatrix}^{-1} =$$

$$\begin{bmatrix} 1 - ap - bq - cr & -(c + r - bp + aq) & (b + q + cp - ar) \\ (c + r - bp + aq) & 1 - ap - bq - cr & -(a + p + br - cq) \\ -(b + q + cp - ar) & (a + p + br - cq) & 1 - ap - bq - cr \end{bmatrix} \times$$

$$\begin{bmatrix} 1 - ap - bq - cr & (c + r - bp + aq) & -(b + q + cp - ar) \\ -(c + r - bp + aq) & 1 - ap - bq - cr & (a + p + br - cq) \\ (b + q + cp - ar) & -(a + p + br - cq) & 1 - ap - bq - cr \end{bmatrix}^{-1}$$

and $C_{0,k} - C_{0,p}C_{p,k} = 0$ where we use

$$C_{0,p} = \begin{bmatrix} 1 & -c & b \\ c & 1 & -a \\ -b & a & 1 \end{bmatrix}\begin{bmatrix} 1 & c & -b \\ -c & 1 & a \\ b & -a & 1 \end{bmatrix}^{-1}, \quad C_{p,k} = \begin{bmatrix} 1 & -r & q \\ r & 1 & -p \\ -q & p & 1 \end{bmatrix}\begin{bmatrix} 1 & r & -q \\ -r & 1 & p \\ q & -p & 1 \end{bmatrix}^{-1}$$

As the vector product of $\overrightarrow{g}_{0,p}$ and $\overrightarrow{g}_{p,k}$ is answered with a vector having the coordinate representation $g^{\times}_{0,p}g_{p,k} = f^{\times}_{0,p}f_{p,k}$ in $\mathcal{E}_0$, from (23) follows that the composition rule (23) takes the following form [19]

$$\overrightarrow{g}_{0,k} = \langle \overrightarrow{g}_{0,p}, \overrightarrow{g}_{p,k} \rangle = \frac{1}{1 - (\overrightarrow{g}_{0,p}, \overrightarrow{g}_{p,k})}\left(\overrightarrow{g}_{0,p} + \overrightarrow{g}_{p,k} + \overrightarrow{g}_{0,p} \times \overrightarrow{g}_{p,k}\right) \tag{24}$$

For vector–parameters in $\mathbf{R}_3$, the operation of vector product is not defined, as well as the matrix multiplication is not defined for vectors in $\mathbf{V}_3$. That is why the composition rules (23) and (24) are formally different, but speaking more precisely rule (23) proves to be the coordinate form of (24).

For Rodrigues vectors we may write down the following composition rule [20, 21]

$$\overrightarrow{r}_{0,k} = \langle \overrightarrow{r}_{0,p}, \overrightarrow{r}_{p,k} \rangle = \frac{1}{1 - (\overrightarrow{r}_{0,p}, \overrightarrow{r}_{p,k})} (\overrightarrow{r}_{0,p} + \overrightarrow{r}_{p,k} - \overrightarrow{r}_{0,p} \times \overrightarrow{r}_{p,k}) \tag{25}$$

Indeed, in the frame $\mathcal{E}_k$ from (25) follows

$$r^k_{0,k} = \frac{1}{1 - r^{k,T}_{0,p} r^k_{p,k}} \left(r^k_{0,p} + r^k_{p,k} + r^{k\times}_{p,k} r^k_{0,p} \right) \tag{26}$$

As $r^k_{0,p} = C_{k,p} r^p_{0,p}$, $r^k_{0,k} = f_{0,k}$, $r^p_{0,p} = f_{0,p}$, $r^k_{p,k} = f_{p,k}$, $f_{p,k} = C_{k,p} f_{p,k}$ and $(I + f^{\times}_{p,k}) C_{k,p} = I - f^{\times}_{p,k}$, relations (25) and (26) coincide one with another.

Gibbs vector $\overrightarrow{g}_{0,k}$ coincides with Rodrigues vector $\overrightarrow{r}_{0,k}$ while in general it is not the same for $\overrightarrow{g}_{p,k}$ (introduced with the help of the canonical basis of $\mathbf{R}_3$) and $\overrightarrow{r}_{p,k}$ (having the same coordinates in $\mathcal{E}_p$ and $\mathcal{E}_k$, but not in $\mathcal{E}_0$), *i.e.*, Gibbs and Rodrigues vectors are essentially various.

Relation (24) for Gibbs vectors and relation (25) for Rodrigues vectors are formally different, but they are only geometrical and kinematical interpretations (modifications or forms) of the composition rule (23). In the calculating relation their using gives nothing new as their coordinate representations lead to the same rule (23). But no one may identify Fedorov vector–parameter or Gibbs vector with that of Rodrigues (see Fig. 1.1).

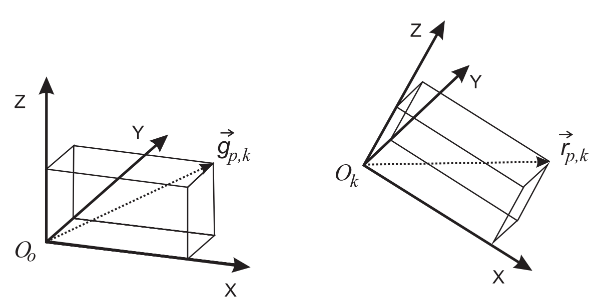

FIG. 1. Gibbs and Rodrigues vectors

Relation between angular quasi–velocity and vector–parameter. For $C_{0,k} = C_1$

there are $\omega_{0,k}^k = \varphi^\bullet \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $f_1 = \tan\frac{\varphi}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, *i.e.*, the vector of angular velocity and that of Rodrigues are collinear.

As $f_1^{\times\bullet} = \frac{1}{2}\left(1 + \tan^2\frac{\varphi}{2}\right)\varphi^\bullet \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{1}{2}(1 + \|f_1\|^2)\varphi^\bullet \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, we have $f_1^\bullet = \frac{1}{2}\left(1 + \|f_1\|^2\right)\varphi^\bullet \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\omega_{0,k}^k = \frac{2}{1+\|f_1\|^2} f_1^\bullet$. This result depends on the special kind of $C_{0,k} = C_1$ as for the product $C_{0,k} = C_1 C_2 C_3$ there is the following relation [21]

$$\overrightarrow{\omega}_{0,k} = \frac{2}{1 + \|\overrightarrow{r}_{0,k}\|^2}(\overrightarrow{r}^\bullet_{0,k} + \overrightarrow{r}_{0,k} \times \overrightarrow{r}^\bullet_{0,k}) \tag{27}$$

or

$$\omega_{0,k}^k = \frac{2}{1 + \|f_{0,k}\|^2}(f^\bullet_{0,k} + f^\times_{0,k} f^\bullet_{0,k}) \tag{28}$$

Relations (16) or (28) permits us to pass from angular quasi–velocities to Euler angles (see also [2]) or Fedorov vector–parameters in Newton–Euler equations and to obtain Lagrange equations of the second kind.

Canonical generalized coordinates and velocities [2].

Definition 8: *1. The vectors* $q_{0,k} = \mathrm{col}\{d_{0,k}^k, \vartheta_{0,k}\}$, $\quad q^\bullet_{0,k} = \mathrm{col}\{d_{0,k}^{k\bullet}, \vartheta^\bullet_{0,k}\}$ *are called* canonical generalized coordinates *and* velocities *of the frame $\mathcal{E}_k$ in the motion w.r.t. the frame $\mathcal{E}_0$;*

: *2. the relation*

$$V_{0,k} = M_{0,k} q^\bullet_{0,k}, \quad M_{0,k} = \mathrm{diag}\{I, D_{0,k}\} \tag{29}$$

is called equation of kinematics *of $\mathcal{E}_k$–frame w.r.t. $\mathcal{E}_0$.*

Then from relation (14) follows the (*Lagrange*) equation

$$\mathcal{A}(q_{0,k}) q^{\bullet\bullet}_{0,k} + \mathcal{B}(q_{0,k}, q^\bullet_{0,k}) q^\bullet_{0,k} = M_{0,k}^T \mathcal{F}_k^k \tag{30}$$

where $\mathcal{A}(q_{0,k}) = M_{0,k}^T \varTheta_k M_{0,k}$, $\mathcal{B}(q_{0,k}, q^\bullet_{0,k}) = M_{0,k}^T(\varTheta_k M^\bullet_{0,k} + \varPhi_{0,k}\varTheta_k M_{0,k})$.

Note that as a rule the matrix $M^\bullet_{0,k}$ can be analytically calculated (see, *e.g.*, [3]).

Systems of consecutively connected bodies [22]. Let us have a system of $n+1$ consecutively connected bodies A_k with attached Cartesian frames $\mathcal{E}_k$ having the origins O_k

$(k=\overline{0,n})$. The first body is immobile and is indexed by 0, and the bodies are associated with the rotation matrices $C_{k-1,k}$.

We may consider Newton–Euler equations where 'absolute' quasi–velocities $V_{0,k}$ of the bodies w.r.t. the main frame $\mathcal{E}_0$ are used. But in practice there are only the 'relative' quasi–velocities $V_{k-1,k}$ of the frame $\mathcal{E}_k$ w.r.t. $\mathcal{E}_{k-1}$.

Thus we must connect the 'absolute' quasi–velocities with 'relative' ones.

Twist composition rule. *There is the following composition rule* [3]

$$\{\overrightarrow{v}_{0,k},\overrightarrow{\omega}_{0,k}\}=\{\overrightarrow{v}_{0,k-1},\overrightarrow{\omega}_{0,k-1}\}+\{\overrightarrow{v}_{k-1,k},\overrightarrow{\omega}_{k-1,k}\}$$

Proof. As the bodies are connected consecutively there is the relation $C_{0,k}=C_{0,k-1}C_{k-1,k}$ is true. With differentiating it we have $\omega^k_{0,k}=\omega^k_{0,k-1}+\omega^k_{k-1,k}=C_{k,k-1}\omega^{k-1}_{0,k-1}+\omega^k_{k-1,k}$. Besides define the vectors $\overrightarrow{d}_{0,k-1}=\overrightarrow{(O_0,O_{k-1})}$ and $\overrightarrow{d}_{k-1,k}=\overrightarrow{(O_{k-1},O_k)}$, then $d^0_{0,k}=d^0_{0,k-1}+d^0_{k-1,k}$, $v^0_{0,k}=v^0_{0,k-1}+d^{0\bullet}_{k-1,k}$, $d^0_{k-1,k}=C_{0,k-1}d^{k-1}_{k-1,k}$, $d^{0\bullet}_{k-1,k}=v^0_{k-1,k}+C^\bullet_{0,k-1}d^{k-1}_{k-1,k}=C_{0,k-1}\omega^{k-1\times}_{0,k-1}d^{k-1}_{k-1,k}+v^0_{k-1,k}=v^0_{k-1,k}+C_{0,k-1}\omega^{k-1\times}_{0,k-1}C_{0,k-1}d^{k-1}_{k-1,k}=v^0_{k-1,k}-d^{0\times}_{k-1,k}\omega^0_{0,k-1}=v^0_{k-1,k}+d^{0\times}_{k,k-1}\omega^0_{0,k-1}$. Hence $v^0_{0,k}=v^0_{0,k-1}+v^0_{k-1,k}+d^{0\times}_{k,k-1}\omega^0_{k-1,k}$, $v^k_{0,k}=C_{k,k-1}v^{k-1}_{0,k-1}+C_{k,k-1}d^{k-1\times}_{k,k-1}\omega^{k-1}_{0,k-1}+v^k_{k-1,k}$, $V_{0,k}=\begin{bmatrix}C_{k,k-1}&O\\O&C_{k,k-1}\end{bmatrix}\begin{bmatrix}I&d^{k-1\times}_{k,k-1}\\O&I\end{bmatrix}V_{0,k-1}+V_{k-1,k}=L^{(2)}_{k,k-1}V_{0,k-1}+V_{k-1,k}$.

From the rule we have

$$V_{0,k}=\sum_{p=1}^{p=k}L^{(2)}_{k,p}V_{p-1,p},\quad V_{p-1,p}=\begin{pmatrix}v^p_{p-1,p}\\ \omega^p_{p-1,p}\end{pmatrix}\tag{31}$$

where $L^{(2)}_{k,p}=\begin{bmatrix}C_{k,p}&O\\O&C_{k,p}\end{bmatrix}\begin{bmatrix}I&d^{p\times}_{k,p}\\O&I\end{bmatrix}$, $L^{(2)}_{k,k}=I$.

From (31) follows the *equation of kinematics*

$$\mathcal{V}_a=\mathcal{L}^{(2)}\mathcal{V}_r\tag{32}$$

where $\mathcal{V}_a$=col$\{V_{0,1},\ldots,V_{0,k},\ldots,V_{0,n}\}$, $\mathcal{V}_r$=col$\{V_{0,1},\ldots,V_{k-1,k},\ldots,V_{n-1,n}\}$, $\mathcal{L}^{(2)}$ is the triangular matrix with blocks $L^{(2)}_{k,p}$ being functions of 'relative' frame rotations and translations.

Let rotation matrices $C_{k-1,k}$ be parameterized with the help of some vectors $\vartheta_{k-1,k}\in\mathbf{R}_3$ and (see (16))

$$\omega^k_{k-1,k}=D_{k-1,k}\vartheta^\bullet_{k-1,k}$$

where the matrices $D_{k-1,k} = D_{k-1,k}(\vartheta_{k-1,k})$ are known.

Then we may introduce *canonical generalized coordinates* and *velocities* of the frame $\mathcal{E}_k$ in the motion w.r.t. the frame $\mathcal{E}_{k-1}$:

$$q_{k-1,k} = \mathrm{col}\{d^k_{k-1,k}, \vartheta_{k-1,k}\}, \quad q^\bullet_{k-1,k} = \mathrm{col}\{d^{k\bullet}_{k-1,k}, \vartheta^\bullet_{k-1,k}\}$$

and the *equation of kinematics*

$$V_{k-1,k} = M_{k-1,k} q^\bullet_{k-1,k}, \quad M_{k-1,k} = \mathrm{diag}\{I, D_{k-1,k}\}$$

Introduce the vector q with the entries $q_{k-1,k}$. Then from relation (32) follows

$$\mathcal{V}_a = \mathcal{L}^{(2)} \mathcal{M} q \tag{33}$$

where $\mathcal{M}$ is the diagonal matrix with block entries $M_{k-1,k}$.

Thus with the help of relation (14) we have the following Lagrange equation

$$\mathcal{A}(q) q^{\bullet\bullet} + \mathcal{B}(q, q^\bullet) q^\bullet = \mathcal{M}^T \mathcal{L}^{(2),T} \mathcal{F}$$

where $\mathcal{A}(q)=\mathcal{M}^T\mathcal{L}^{(2),T}\Theta\mathcal{L}^{(2)}\mathcal{M}$, $\mathcal{B}(q,q^\bullet)=\mathcal{M}^T\mathcal{L}^{(2),T}[\Theta(\mathcal{L}^{(2)}\mathcal{M})^\bullet + \Phi\Theta\mathcal{L}^{(2)}\mathcal{M}]$, Θ, Φ and $\mathcal{F}$ are diagonal matrices and columns with block entries Θ_k, $\Phi_{k-1,k}$ and $\mathcal{F}^k_k$.

Note that as a rule the matrix $(\mathcal{L}^{(2)}\mathcal{M})^\bullet$ can be analytically calculated (see, *e.g.*, [3]).

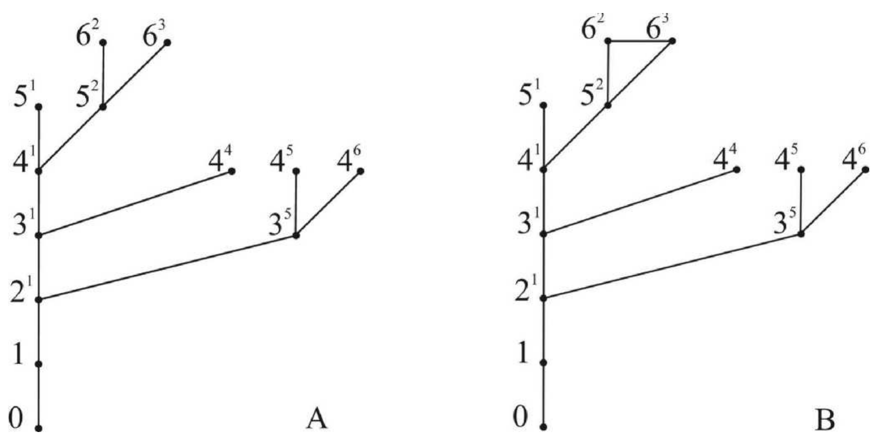

FIG. 2. Multibody system graphs

Multibody systems with tree–like structure. Consider a multibody system with tree–like structure given by the graph in Fig. 1.2A. Let vertices j^i represent the system bodies or the origins of the attached Cartesian frames $\mathcal{E}^i_j$ where the index i numbers the tree–tops, the index j numbers the bodies from the base to the corresponding tree–tops. Introduce $V_{k,j}^{m,i}$ as quasi–velocities characterizing rotation and translation of the frames $\mathcal{E}_j^i$ w.r.t. $\mathcal{E}_k^m$. Then we have the sets $\{V_{0,1}^{0,1}, V_{1,2}^{1,1}, V_{2,3}^{1,1}, V_{3,4}^{1,1}, V_{4,5}^{1,1}\}, \{V_{0,1}^{0,1}, V_{1,2}^{1,1}, V_{2,3}^{1,1}, V_{3,4}^{1,1}, V_{4,5}^{1,2}, V_{5,6}^{2,2}\}$, $\{V_{0,1}^{0,1}, V_{1,2}^{1,1}, V_{2,3}^{1,1}, V_{3,4}^{1,1}, V_{4,5}^{1,2}, V_{5,6}^{2,3}\}$, $\{V_{0,1}^{0,1} V_{1,2}^{1,1}, V_{2,3}^{1,1}, V_{3,4}^{1,4}\}$, $\{V_{0,1}^{0,1} V_{1,2}^{1,1} V_{2,3}^{1,5} V_{3,4}^{5,5}\}$, $\{V_{0,1}^{0,1}, V_{1,2}^{1,1}, V_{2,3}^{1,5}, V_{3,4}^{5,6}\}$ and $\{V_{0,1}^{0,1}, V_{0,2}^{1,1}, V_{0,3}^{1,1}, V_{0,4}^{1,1}, V_{0,5}^{1,1}\}, \{V_{0,1}^{0,1}, V_{0,2}^{1,1}, V_{0,3}^{1,1}, V_{0,4}^{1,1}, V_{0,5}^{1,2}, V_{0,6}^{2,2}\}, \{V_{0,1}^{0,1}, V_{0,2}^{1,1}, V_{0,3}^{1,1}, V_{0,4}^{1,1}, V_{0,5}^{1,2}, V_{0,6}^{2,3}\}, \{V_{0,1}^{0,1}, V_{0,2}^{1,1}, V_{0,3}^{1,1}, V_{0,4}^{1,4}\}, \{V_{0,1}^{0,1}, V_{0,2}^{1,1}, V_{0,3}^{1,5}, V_{0,4}^{5,5}\}, \{V_{0,1}^{0,1}, V_{0,2}^{1,1}, V_{0,3}^{1,5}, V_{0,4}^{5,6}\}$ with the same subscripts as in the case of consecutively connected bodies for the relative and absolute quasi–velocities.

This case is considered above that is why we arrive at relations (32) and (33) with the known matrix $\mathcal{L}^{(2)}$, $\mathcal{M}$, $\mathcal{P}$ and

$$\begin{aligned}
\mathcal{V}_a &= \mathrm{col}\{V_{0,1}^{0,1}, V_{0,2}^{1,1}, V_{0,3}^{1,1}, V_{0,4}^{1,1}, V_{0,5}^{1,1}, V_{0,5}^{1,2}, V_{0,6}^{2,2}, V_{0,6}^{2,3}, V_{0,4}^{1,4}, V_{0,3}^{1,5}, V_{0,4}^{5,5}, V_{0,4}^{5,6}\}\\
\mathcal{V}_r &= \mathrm{col}\{V_{0,1}^{0,1}, V_{1,2}^{1,1}, V_{2,3}^{1,1}, V_{3,4}^{1,1}, V_{4,5}^{1,1}, V_{4,5}^{1,2}, V_{5,6}^{2,2}, V_{5,6}^{2,3}, V_{3,4}^{1,4}, V_{2,3}^{1,5}, V_{3,4}^{5,5}, V_{3,4}^{5,6}\}\\
q &= \mathrm{col}\{q_{0,1}^{0,1}, q_{1,2}^{1,1}, q_{2,3}^{1,1}, q_{3,4}^{1,1}, q_{4,5}^{1,1}, q_{4,5}^{1,2}, q_{5,6}^{2,2}, q_{5,6}^{2,3}, q_{3,4}^{1,4}, q_{2,3}^{1,5}, q_{3,4}^{5,5}, q_{3,4}^{5,6}\}\\
\widetilde{q} &= \mathrm{col}\{\widetilde{q}_{0,1}^{0,1}, \widetilde{q}_{1,2}^{1,1}, \widetilde{q}_{2,3}^{1,1}, \widetilde{q}_{3,4}^{1,1}, \widetilde{q}_{4,5}^{1,1}, \widetilde{q}_{4,5}^{1,2}, \widetilde{q}_{5,6}^{2,2}, \widetilde{q}_{5,6}^{2,3}, \widetilde{q}_{3,4}^{1,4}, \widetilde{q}_{2,3}^{1,5}, \widetilde{q}_{3,4}^{5,5}, \widetilde{q}_{3,4}^{5,6}\}
\end{aligned}$$

where $q_{k,j}^{m,i}$ are generalized coordinates characterizing motion of $\mathcal{E}_j^i$ w.r.t. $\mathcal{E}_k^m$.

Remark 7 *The results obtained can be immediately applied to systems with loops,* e.g., *if in the system under consideration (see Fig. 1.2B) the vertex* 6^2 *is connected with* 6^3 *by the edge* $(6^2, 6^3)$*. In this case relation (37) is the same, but in the case where constraints are considered there are the following additional constraints* $\overrightarrow{(5^2, 6^2)} + \overrightarrow{(6^2, 6^3)} + \overrightarrow{(6^2, 5^2)} = 0$ *and* $C_{5,6}^{2,2} C_{6,6}^{2,3} C_{6,5}^{2,2} = I$.

Computing efficiency of the vector–parameter method for matrix product. Due to rule (23) for the product $C_{p,k} = C_1 C_2 C_3$ we have

$$f_{p,k} = \frac{1}{1 - (\tan\frac{\varphi}{2}\tan\frac{\psi}{2})\tan\frac{\vartheta}{2}} \begin{pmatrix} -\tan\frac{\varphi}{2} + (\tan\frac{\psi}{2}\tan\frac{\vartheta}{2}) \\ -\tan\frac{\vartheta}{2} - (\tan\frac{\varphi}{2}\tan\frac{\psi}{2}) \\ -\tan\frac{\psi}{2} + (\tan\frac{\varphi}{2}\tan\frac{\vartheta}{2}) \end{pmatrix} \tag{34}$$

Using the notation: $c_\zeta = \cos\zeta$, $s_\zeta = \sin\zeta$, $\zeta = \varphi$, ϑ, and ψ – we may write

$$C_{p,k} = C_1 C_2 C_3 = \begin{bmatrix} (c_\vartheta c_\psi) & c_\varphi s_\psi + (s_\varphi c_\psi) s_\vartheta & (s_\varphi s_\psi) - c_\varphi c_\psi s_\vartheta \\ -(c_\vartheta s_\psi) & (c_\varphi c_\psi) - (s_\varphi s_\psi) s_\vartheta & (s_\varphi c_\psi) + (c_\varphi s_\psi) s_\vartheta \\ s_\vartheta & -(s_\varphi c_\vartheta) & (c_\varphi c_\vartheta) \end{bmatrix} \tag{35}$$

Define the rotation matrices $C_{k-1,k}$ for $k = \overline{0,n}$ with the help of relations (35) and construct their consecutive products $C_{0,k} = C_{0,k-1}C_{k-1,k}$. For $k = \overline{1,n}$ with the help of vector–parameters $f_{k-1,k}$ (see relation (34)) define $f_{0,k}$ (see relation (23)) and the rotation matrices $C_{0,k}$ (see relation (22)). The corresponding numbers of multiplications (divisions) $N_\times$, additions N_+ and transcendental functions N_{tr} are given in Table 1.1.

The most important point of kinematics problems is what variables are measured. In the case where such variables are Euler angles (in the set $3-2-1$) the vector–parameter method proves to be more numerically effective in calculating matrix products than the direct product one (see also [23]).

If we may measure vector–parameters then with using them in the capacity of generalized coordinates it leads to the highest effectiveness of describing multibody dynamics in the computational respect.

Table 1.1. Computing efficiency of direct product (I) and vector–parameter method (II) in the case where Euler angles are given in the set $3-2-1$.

Relation	(21)	(22)	(23)	(34)	(35)	I	II
$N_\times$	1	10	10	8	12	$39n-27$	$28n-10$
N_+	13	23	10	4	4	$22n-18$	$37n-10$
N_{tr}	–	–	–	3	6	$6n$	$3n$

Motion equations for systems with tree–like structure and simple constraints. Consider a system with ideal holonomic constraints such that there are given m_k time–constant entries of $q_{k-1,k}$ – this case is used in mechanical systems such as, *e.g.*, manipulators. For any time–varying entry i of $q_{k-1,k}$ let us define the 6−dimensional column p_i with 1 at the place i and 0 otherwise. This column is called *basis vector of mobility* [2]. Define $(6-m_k)\times 6-$dimensional matrix $P_{k-1,k}$ with such columns. It means that we may define $(6-m_k)-$dimensional time–varying vector $\widetilde{q}_{k-1,k}$ such that $q^{\bullet}_{k-1,k} = P_{k-1,k}\widetilde{q}^{\bullet}_{k-1,k}$.

Thus we arrive at the following equation of kinematics

$$V_{k-1,k} = M_{k-1,k}P_{k-1,k}\widetilde{q}^{\bullet}_{k-1,k}$$

w.r.t. the time–varying generalized coordinates $\widetilde{q}_{k-1,k}$ and velocities $\widetilde{q}^{\bullet}_{k-1,k}$.

We may introduce the block–column $\widetilde{q} = \mathrm{col}\{\widetilde{q}_{0,1},\ldots,\widetilde{q}_{k-1,k},\ldots,\widetilde{q}_{n-1,n}\}$ and the diagonal matrix $\mathcal{P}$ with block entries $P_{k-1,k}$ (see (29)). Then there is the following relation

$$\mathcal{V}_r = \mathcal{M}\mathcal{P}\widetilde{q}^{\bullet}$$

and *Lagrange equation* system of II kind

$$A(\widetilde{q})\widetilde{q}^{\bullet\bullet} + B(\widetilde{q},\widetilde{q}^{\bullet})\widetilde{q}^{\bullet} = \mathcal{P}^T\mathcal{M}^T\mathcal{L}^{(2),T}\mathcal{F} \tag{36}$$

where $A(\widetilde{q}) = \mathcal{P}^T\mathcal{M}^T\mathcal{L}^{(2),T}\Theta\mathcal{L}^{(2)}\mathcal{M}\mathcal{P}$, $B(\widetilde{q},\widetilde{q}^{\bullet}) = \mathcal{P}^T\mathcal{M}^T\mathcal{L}^{(2),T}(I\frac{d}{dt}+\Phi)(\Theta\mathcal{L}^{(2)}\mathcal{M}\mathcal{P})$.

V. A CONTINUUM

Suppose the set $A\in\mathbf{X}_3$ has no pure point of the measure μ_{LS}, $\mu_{LS}(dx) = \mu_3(dx)$ and $m(dx) = \rho_x\mu_3(dx)$ in A^c.

Strain matrix and its rate. Given $x(t)$ and $y(t)\in A^c$ in the instant $t\in\mathbf{T}$, define the vector $h(t) = y^0(t) - x^0(t)$ (in $\mathcal{E}_0$). If $h(t)$ is small we have

$$v^0_y(t) \cong v^0_x(t) + dv^0_x/dx^0h(t)$$

Define the matrix $Z_x(t)$ as the solution of the following equation

$$Z^{\bullet}_x(t) = dv^0_x/dx^0$$

with initial data $Z_x = I$ for $t = t_0$.

Definition 9 [3]. *The matrices Z_x and $Z^{\bullet}_x$ are called* strain *one and its* rate *at the point $x(t)\in A^c$ in the instant t, respectively.*

There is no reason to consider the strain matrix and its rate as important (kinematical) characteristics of continuum motion.

Stress matrix. Let us define (see also [2, 3]):

: 1. a section S between the set A and an arbitrary plane P;

: 2. the vector bi–measure

$$\mathcal{D}(A) = \int \chi_{_A} l^{\Delta(x,x^e)} \mu_3(dx)$$

where representatives of the slider function l^{Δ} are axial at all $x \in A$;

: 3. on the set S the slider function $l^{\delta(x)}$ of the measure $\mathcal{D}$ w.r.t. Lebesgue 2−dimensional measure $\mu_2(dx)$ on Borel σ–algebra σ_2 of open subsets of S such that

$$\mathcal{D}(S) = \int \chi_{_S} l^{\delta(x,x^e)} \mu_2(dx)$$

: 4. 3×3–matrix–function T_x of x and t which can be differentiable by x the necessary number of times and such that the vector $\delta(x, x^e)$ has the coordinate representation

$$\delta^0(x, x^e) = T_x n_x^0$$

where n_x is the normal to the plane P at the point x;

: 5. the entries of $\Delta^0(x, x^e)$ being connected with the rows T_x^j $(j = \overline{1,3})$ of the matrix T_x by the following relation (in the frame $\mathcal{E}_0$)

$$\Delta^0(x, x^e) = \operatorname{Div} T_x \stackrel{\text{def}}{=} \operatorname{col}\{\operatorname{div} T_x^1, \operatorname{div} T_x^2, \operatorname{div} T_x^3\}$$

Remark 8. *One may see that the measure $\mathcal{D}(A)$ is introduced under the influence of Gauss–Ostrogradsky divergence theorem [5], but here it is said nothing about the properties of T_x,* e.g., *about its symmetry.*

Definition 10 [3].

: *1. The slider function $l^{\Delta(x,x^e)}$ is called* intensity of stress action *upon $x \in A^c$;*

: *2. T_x is called* stress matrix.

Notion of continuum.

Definition 11 *A matrix–function of entries of some matrices is called* isotropic *if it is invariant w.r.t. $\mathcal{S}O(\mathbf{R}, 3)$.*

Let us note the set of all isotropic maps from the strain matrix Z_x or its rate $Z_x^\bullet$ to the stress matrix T_x as $\alpha^2(Z_x, Z_x^\bullet)$ and use below the notation $T_x \in \alpha^2(Z_x, Z_x^\bullet)$.

Definition 12. *Suppose that*

$$l^{\phi(x,x^e)} = \rho_x l^{g(x,x^e)}, \quad l^{c(x,x^e)} = l^{\Delta(x,x^e)} \tag{37}$$

the stress matrix T_x belongs to $\alpha^2(Z_x, Z_x^\bullet)$ and the measure of inertia is time–constant on A^c, i.e., $\frac{d}{dt}m(dx) = 0$. *Then the set A is called* continuous medium *or* continuum.

Motion of continuum. Due to relations (4) and (37) the equation of continuum motion at a point $x \in A^c$ is of the form (see also [3, 5]) (in the Galilean frame $\mathcal{E}_0$)

$$\rho_x v_x^{0\bullet} = \rho_x g^0 + \operatorname{Div} T_x, \quad \rho_x^\bullet + \operatorname{div}\rho_x v_x^0 = 0, \quad T_x \in \alpha^2(Z_x, Z_x^\bullet)$$

where $\operatorname{Div} T_x$ is the constraint action [24].

Some stress–strain or constitutive relations.

Introduce the following matrices $E_1 = (\operatorname{tr} U_x)I$, $E_2 = \operatorname{sym} U_x = 0.5(U_x + U_x^T)$ and $E_3 = \operatorname{ant} U_x = 0.5(U_x - U_x^T)$ where $U_x = Z_x$ or $U_x = Z_x^\bullet$. These 3 matrices are linearly independent.

It is easy to see that for 3×3–matrices P and Q the aggregate PU_xQ is an isotropic quasi–linear function of U_x if P and Q are proportional to I with scalar coefficients being isotropic (*i.e.*, invariant w.r.t. rotations).

Theorem 3 [25]. *All isotropic quasi–linear 3×3–matrix functions of entries of U_x are given by the following map ($U_x \to T$)*

$$T = r_1 E_1 + r_2 E_2 + r_3 E_3 \tag{38}$$

where r_i are invariant w.r.t. rotations (they can be functions of the time, invariants of U_x and so on).

Thus we may define the following relation

$$T_x = -r_0 E_0 + r_1 E_1 + r_2 E_2 + r_3 E_3, \quad E_0 = I \tag{39}$$

as *constitutive*. It is convectional the invariant w.r.t. rotations r_i to be called *rheological coefficients* w.r.t. the set of E_i.

We may construct another set of linearly independent matrices, *e.g.*,

$$\widehat{E}_1 = (\operatorname{tr} U_x) I, \quad \widehat{E}_2 = U_x, \quad \widehat{E}_3 = U_x^T$$

Then from relation (38) follows that we may take the constitutive relation

$$T_x = -\widehat{r}_0 \widehat{E}_0 + \widehat{r}_1 \widehat{E}_1 + \widehat{r}_2 \widehat{E}_2 + \widehat{r}_3 \widehat{E}_3, \quad \widehat{E}_0 = I$$

with the invariants $\widehat{r}_0$, $\widehat{r}_1$, $\widehat{r}_2$ and $\widehat{r}_3$ (w.r.t. rotations) called *rheological coefficients w.r.t.* the set of $\widehat{E}_i$. Thus the 'structure' of constitutive relations is not unique.

If $U_x = Z_x$ and $r_0 = 0$ the continuum is called *elastic material*, if $U_x = Z_x^\bullet$ and $r_0 > 0$ (called Pascal pressure) the continuum is called *viscous fluid* [26].

Remark 9. *Continua defined by relation (39) coincide with the continua used in continuum mechanics in the following cases* [5, 26]

: *1. the Pascal pressure r_0 is positive and $r_1 = r_2 = r_3 = 0$ (ideal fluid);*

: *2. r_0 is non–negative and $r_3 = 0$ (continua of Navier–Stokes–Lame type);*

: *3. r_0 is non–negative and $r_1 \operatorname{tr} I + r_2 = 0 \to \operatorname{tr} T_x = -r_0 \operatorname{tr} I$ (continua used in some theories).*

In the case of 2×2–matrices it is easy to see that for matrices P and Q the aggregate $P U_x Q$ is an isotropic quasi–linear map of U_x if P and Q are of the kind $aI + \widetilde{a}\widetilde{I}$ where $\widetilde{I} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, the scalar coefficients a and $\widetilde{a}$ are isotropic (*i.e.*, invariant w.r.t. rotations).

Let us construct the following isotropic quasi–linear map

$$T = r_1 (\operatorname{tr} U_x) I + \widetilde{r}_1 (\operatorname{pf} U_x) \widetilde{I} + r_2 U_x + \widetilde{r}_2 U_x^T + r_3 \widetilde{I} U_x + \widetilde{r}_3 U_x^T \widetilde{I} + r_4 \widetilde{I} U_x \widetilde{I} + \widetilde{r}_4 \widetilde{I} U_x^T \widetilde{I} \qquad (40)$$

where r_i and $\widetilde{r}_i$ are invariants (w.r.t. rotations), $\operatorname{pf} U_x = \operatorname{tr}\{\widetilde{I}\, U_x\}$.

It is easy to see that there are only 6 terms being linearly independent (all isotropic affine functions of entries of the matrix U_x are given by relation (40)). Therefore we may reduce function (40), *e.g.*, to the following one

$$T = r_1 (\operatorname{tr} U_x) I + \widetilde{r}_1 (\operatorname{pf} U_x) \widetilde{I} + r_2 U_x + r_3 \widetilde{I} U_x + \widetilde{r}_3 U_x^T \widetilde{I} + r_4 \widetilde{I} U_x \widetilde{I}$$

where all terms are linearly independent. That is why we may define the following constitutive relation

$$T_x = -r_0 I - \widetilde{r}_0 \widetilde{I} + r_1 (\operatorname{tr} U_x) I + \widetilde{r}_1 (\operatorname{pf} U_x) \widetilde{I} + r_2 U_x + r_3 \widetilde{I} U_x + \widetilde{r}_3 U_x^T \widetilde{I} + r_4 \widetilde{I} U_x \widetilde{I} \qquad (41)$$

with invariants (w.r.t. rotations) r_i and $\widetilde{r}_i$.

Let us stop at 2− and 3−dimensional constitutive relations with the same 'structure'. With the help of routine calculations we see the following statement to be true.

Theorem 4. *In 2– and 3–dimensional cases let constitutive relations be of the (39)–form and $(r_1 \mathrm{tr} I + r_2) r_2 r_3 \neq 0$ where I is used as 2– and 3–dimensional identity matrices, respectively. Then there exists the inverse map*

$$U_x = n_0 I + n_1 (\mathrm{tr}\, T_x) I + n_2 \,\mathrm{sym}\, T_x + n_3 \,\mathrm{ant}\, T_x$$

where

$$n_0 = \frac{r_0}{r_1 \mathrm{tr} I + r_2}, \ n_1 = \frac{-r_1}{r_2 (r_1 \mathrm{tr} I + r_2)}, \ n_2 = \frac{1}{r_2}, \ n_3 = \frac{1}{r_3}$$

Remark 10. *The three coefficients*

$$\varepsilon = \frac{1}{n_2 - n_1} = \frac{r_2 (r_1 \mathrm{tr} I + r_2)}{r_1 (\mathrm{tr} I - 1) + r_2}, \ \mu = \frac{1}{2n_2} = \frac{r_2}{2}, \ \nu = \frac{n_1}{n_1 - n_2} = \frac{r_1}{r_1 (\mathrm{tr} I - 1) + r_2}$$

can be called Young modulus, *ε,* shear or rigidity one, *μ, and* Poisson ratio, *ν, respectively. Note that there is the known relation $\varepsilon = 2\mu(1 + \nu)$.*

Divergence of the stress matrix.

Introduce the next notations

$$u_x = \mathrm{col}\{u_{x1}, u_{x2}, u_{x3}\}, \quad u_k^j = (\frac{\partial}{\partial x_k^0} u_{xj}), \quad U_x = \begin{bmatrix} u_1^1 & u_2^1 & u_3^1 \\ u_1^2 & u_2^2 & u_3^2 \\ u_1^3 & u_2^3 & u_3^3 \end{bmatrix}$$

Then with the help of routine calculations in 3–dimensional case we have $\mathrm{Div} U_x = \nabla^2 u_x$ and

$$\mathrm{Div} U_x^T = \begin{pmatrix} u_{11}^1 + u_{12}^2 + u_{13}^3 \\ u_{21}^1 + u_{22}^2 + u_{23}^3 \\ u_{31}^1 + u_{32}^2 + u_{33}^3 \end{pmatrix} = \nabla^2 u_x + \mathrm{cirl}\, \mathrm{cirl}\, u_x$$

as

$$\mathrm{cirl}\, u_x = \nabla \times u_x = \nabla \times \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_2^3 - u_3^2 \\ u_3^1 - u_1^3 \\ u_1^2 - u_2^1 \end{pmatrix}$$

$$\mathrm{cirl}\,\mathrm{cirl}\,u_x = \nabla \times (\nabla \times u_x) = \begin{pmatrix} u^1_{11} + u^2_{12} + u^3_{13} \\ u^1_{21} + u^2_{22} + u^3_{23} \\ u^1_{31} + u^2_{32} + u^3_{33} \end{pmatrix} - \nabla^2 u_x$$

where $\nabla^2 u_x = \mathrm{col}\{\nabla^2 u_{x1}, \nabla^2 u_{x2}, \nabla^2 u_{x3}\}$. That is why $\mathrm{Div}\,\mathrm{sym}\,U_x = \nabla^2 u_x + \frac{1}{2}\mathrm{cirl}\,\mathrm{cirl}\,u_x$, $\mathrm{Div}\,\mathrm{ant}\,U_x = -\frac{1}{2}\mathrm{cirl}\,\mathrm{cirl}\,u_x$ and in the case of (39) we have

$$\begin{aligned} \mathrm{Div}\,T_x = &\,\mathrm{grad}\,r_0 + \mathrm{div}\,u_x\,\mathrm{grad}\,r_1 + 2\,\mathrm{sym}\,U_x\,\mathrm{grad}\,r_2 + 2\,\mathrm{ant}\,U_x\,\mathrm{grad}\,r_3 \\ &+ r_1 \mathrm{grad}\,\mathrm{div}\,u_x + 2r_2\nabla^2 u_x + (r_2 - r_3)\mathrm{cirl}\,\mathrm{cirl}\,u_x \end{aligned} \tag{42}$$

where $\mathrm{div}\,u_x = \mathrm{tr}\,U_x$.

In 2–dimensional case there are $\mathrm{Div}\,U_x = \nabla^2 u_x$ and

$$\mathrm{Div}\,U_x^T = \begin{pmatrix} u^1_{11} + u^2_{12} \\ u^1_{21} + u^2_{22} \end{pmatrix} = \nabla^2 u_x + \begin{pmatrix} u^2_{12} - u^1_{12} \\ u^1_{21} - u^2_{21} \end{pmatrix}$$

That is why $\mathrm{Div}\,\mathrm{sym}\,U_x = \nabla^2 u_x + \frac{1}{2}\begin{pmatrix} u^2_{12} - u^1_{12} \\ u^1_{21} - u^2_{21} \end{pmatrix}$, $\mathrm{Div}\,U_x = -\frac{1}{2}\begin{pmatrix} u^2_{12} - u^1_{12} \\ u^1_{21} - u^2_{21} \end{pmatrix}$ and in the case of (39) we have relation (42) where the term $(r_2 - r_3)\mathrm{cirl}\,\mathrm{cirl}\,u_x$ must be replaced with $(r_2 - r_3)\begin{pmatrix} u^2_{12} - u^1_{12} \\ u^1_{21} - u^2_{21} \end{pmatrix}$.

A constitutive relation is called *correct* if map (38) or (41) has inverse (see also [3]). Navier–Stokes–Lame continua are incorrect as $r_2 r_3 (r_1 \mathrm{tr} I + r_2) = 0$.

VI. BRIEF COMMENTS

Continuum mechanics is closely connected with Riemann integral theory. In continuum mechanics as well as in Riemann theory there is realized the idea of approximating an area by summing rectangular strips (segments, squares or boxes), then using some kind of limit process to obtain the exact area required. It is safe to say that we may name the well known mechanics of continua as that of Cauchy (due to the man who created it).

The Riemann integral, natural though it is, has been superseded by the Lebesgue or Lebesgue–Stieltjes integral and other more recent theories of integration. In this way, V. Konoplev suggested a new architecture of continuum mechanics based on Lebesgue integral and his algebraic theory of sliders. As result in Konoplev mechanics there do not arise boxes or particles which can be rotated by the laws of Newtonian mechanics as well as there are no imaged surfaces with stresses over them and other concepts of Cauchy mechanics. But with introducing the measures as Lebesgue integrals he was forced to exclude mass–points from consideration.

Unlike V. Konoplev, we use Lebesgue–Stieltjes integral in order to introduce main mechanics measures and classes of mechanical systems such that mass–points, rigid bodies and continua (under the special assumption about interaction in mechanical systems and their 'constitution'). In this way we become closer to mechanics of C. Truesdell.

CONCLUSION

It is a first attempt to represent elements of Konoplev's axiomatics and its (possibly debatable) modification in the form of a journal paper. One must realize the difficulties and gaps issued from this goal.

It is impossible to separate the theory given above from that of Konoplev. That is why the paper author prefers to yield the palm to Prof. V. Konoplev but carries full responsibility for all lacks of his paper. This is the place to express his sincere thanks to Prof. V. Konoplev for the collaboration of many years.

The author would be highly grateful with whoever would bring any element likely to be able to make progress the development, and thus the comprehension, of the paper. Any comments, reviews, critiques, or objections are invited to be sent to the author by e–mail.

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