ON SMALL DEVIATIONS OF GAUSSIAN PROCESSES USING MAJORIZING MEASURES

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Abstract. We give two examples of periodic Gaussian processes, having entropy numbers of exactly same order but radically different small deviations. Our construction is based on classical Knopp's result yielding of existence of continuous nowhere differentiable functions, and more precisely on Loud's functions. We also obtain a general lower bound for small deviations using the majorizing measure method. We show on examples that our bound is sharp. We also apply it to Gaussian independent sequences and to the generic class of ultrametric Gaussian processes.

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1. Introduction–Preliminaries

The relatively recent small deviations theory of Gaussian processes and of more general processes is a very active and interactive domain of research, having connections with statistics and operator theory. This is also completing the one of large deviations, which has been earlier extensively investigated. The large deviations theory is essentially based on the Borel-Sudakov-Tsirelson isoperimetric inequality, regularity methods (metric entropy method, majorizing measure method) as well as Slepian comparison lemma. Although some of these tools are relevant in the study of small deviations, this one also relies upon intrinsic devices: Laplace transform, Tauberian theorems, subadditive lemma, and most importantly, Kathri-Sidák's inequality implying for any Gaussian vector $(X_1, \ldots, X_J)$ that

$$\prod_{j=1}^{J} \mathbb{P}\{|X_j| \le z\} \le \mathbb{P}\big\{\sup_{j=1}^{J} |X_j| \le z\big\}. \tag{1.1}$$

Talagrand's well-known lower bound [8] is based on this device. Let $\{X(t), t \in T\}$ be a Gaussian process and let as customary $d(s,t) = \|X(s) - X(t)\|_2$, $s,t \in T$. Recall that the entropy number $N(T,d,\varepsilon)$ is the minimal number (possibly infinite) of d-balls of radius $\varepsilon > 0$ enough to cover T. Assume there exists a nonnegative function ϕ on $\mathbb{R}^+$ such that $N(T,d,\varepsilon) \le \phi(\varepsilon)$, and moreover $c_1\phi(\varepsilon) \le \phi(\frac{\varepsilon}{2}) \le c_2\phi(\varepsilon)$ for some constants $1 < c_1 \le c_2 < \infty$. Then, for some $K > 0$ and every

$\varepsilon > 0$,

$$\mathbb{P}\big\{ \sup_{s,t\in T} |X(s) - X(t)| \le \varepsilon \big\} \ge e^{-K\phi(\varepsilon)}. \tag{1.2}$$

This estimate has been recently improved in [1] where a much larger set of size's functions ϕ is permitted. The basic idea is the use of inequality (1.1) to control the Laplace transform of some standard approximating chaining sum, and next to apply de Bruijn's Tauberian result. As moreover, some general links between the Kolmogorov's entropy function $H(\varepsilon) = \log N(T, d, \varepsilon)$ (relatively to the unit ball of the associated reproducing Hilbert space) and $-\log \mathbb{P}\big\{ \sup_{t\in T} |X(t)| \le \varepsilon \big\}$ have been earlier established by Kuelbs and Li (see [4] or [5]), there seems to be a kind of dual behavior between small deviations and entropy numbers of a Gaussian process.

However, this is not exactly so. The convenient estimate (1.2) is indeed known to not always provide sharp lower estimates, whereas is some cases it is quite sharp. See [4] §3.4, [5] §2-3. A typical instance is $X(t) = g|t|^\alpha$, $t \in [0, 1]$ where $0 < \alpha \le 1$. We have $d(s, t) \le |t - s|^\alpha$, so that $N(T, d, \varepsilon) \le C\varepsilon^{1/\alpha}$. However $\mathbb{P}\big\{ \sup_{0\le s,t\le 1} |X(s) - X(t)| \le \varepsilon \big\} \approx \varepsilon$. In fact, much more can be said. In section 2, we show that there exist two sample continuous periodic Gaussian processes, with entropy numbers of *exactly* same order, but having radically different small deviations. There also exist aperiodic sample continuous Gaussian processes for which this duality breakes even more dramatically. In Section 3, we establish a new general lower bound for small deviations by using the majorizing measure method. We show on examples that our bound is sharp. We also apply it to Gaussian independent sequences and to the generic class of ultrametric Gaussian processes.

Notation and convention. All Gaussian processes we consider are supposed to be *centered*. The letter g is used to denote throughout a standard Gaussian random variable. Further $g_1, g_2, \ldots$ will always denote a sequence of i.i.d. standard Gaussian random variables. The notation $f(t) \preceq h(t)$ (resp. $f(t) \succeq h(t)$) near $t_0 \in \overline{\mathbb{R}}$ means that for t in a neighborhood of t_0, $|f(t)| \le c|h(t)|$, (resp. $|f(t)| \ge c|h(t)|$) for some constant $0 < c < \infty$. We write $f(t) \approx h(t)$ when $f(t) \preceq h(t)$ and $f(t) \succeq h(t)$.

2. Examples Breaking the Duality with Entropy Numbers

By considering two kind of processes, one of type $X(t) = gf(t)$ and the second as in example given in [5] (see (2.5),(2.6) and also [4] section 3.4), we will prove the following striking result.

Theorem 2.1. *Let $0 < \alpha < 1$. There exists two cyclic continuous Gaussian processes $X_1(t), X_2(t)$, $t \ge 0$ such that, as $\varepsilon \to 0$,*

$$N([0, 1], d_{X_i}, \varepsilon) \approx \varepsilon^{1/\alpha}, \qquad i = 1, 2.$$

However,

$$\mathbb{P}\{ \sup_{0\le t\le 1} |X_1(t)| \le \varepsilon\} \approx \varepsilon, \qquad \log \mathbb{P}\{ \sup_{0\le t\le 1} |X_2(t)| \le \varepsilon\} \approx -(\log \frac{1}{\varepsilon})^2.$$

Therefore the sole knewledge of entropy numbers of the process is, in general unsufficient for estimating its small deviations. The proof is essentially based on two lemmas.

To begin, recall a classical result from real analysis, namely the existence of continuous nowhere differentiable functions, see Knopp's construction in [3]. In [6], Loud has given an example of a function $f(t)$ which satisfies, for every real t, a Lipschitz condition of order precisely α $(0<\alpha<1)$. The proof is based on the method used in [3], as well as van der Waerden's construction [10]. More precisely, if $0<\alpha<1$, there exists a continuous periodic function f and a pair of positive constants K_1, K_2 such that

a) *for any t and any h, $|f(t+h)-f(t)| \le K_1|h|^\alpha$,*
b) *for any t and infinitely many, arbitrary small h,*

$$|f(t+h)-f(t)| \ge K_2|h|^\alpha. \tag{2.1}$$

Let $\varphi(t,h)$ be the saw-tooth function equal to 0 for even multiples of h, to 1 for odd multiples of h and linear otherwise. Loud's function is defined as follows: let A be an integer such that $2^{2A(1-\alpha)}>2$ and put

$$f(t) = \sum_{n=1}^{\infty} 2^{-2\alpha An}\varphi(t, 2^{-2An}). \tag{2.2}$$

Then f satisfies (2.1). Notice that f is 2^{-2A}–periodic. The leading idea in Loud's proof is that for every pair of values t and h, at most one or two terms of the series (2.2) makes a significant contribution to the difference $f(t+h)-f(t)$. Further, it is of interest to notice that property *b)* is established for the values $h=2^{-2An}$, $n>1$. From this and by considering $X(\omega,t)=g(\omega)f(t)$, it follows easily that

Lemma 2.2. *For any $0<\alpha\le 1$, there exists a cyclic Gaussian process $X(t), t\ge 0$ with sample paths verifying a Lipschitz condition of order precisely α. Moreover, as $\varepsilon\to 0$*

$$N([0,1], d_X, \varepsilon) \asymp \varepsilon^{-\frac{1}{\alpha}} \quad \textit{whereas} \quad \mathbb{P}\{\sup_{0\le t\le 1}|X(t)|\le\varepsilon\} \asymp \varepsilon.$$

Now consider the following example. Let $0<\alpha<1$, $p\ge 2$ be some integer and let A integer be such that $p^{2(1-\alpha)A}>2$. For each integer k, let $\varphi_k(t) = p^{-2\alpha Ak}\varphi(t, p^{-2Ak})$. Put

$$f(t) = \sum_{k=1}^{\infty}\varphi_k(t), \qquad X(t) = \sum_{k=1}^{\infty} g_k\varphi_k(t). \tag{2.3}$$

For proving Theorem 2.1 as well as Proposition 2.6, the lemma below providing estimates of both the increments of f and of its random counterpart X is useful.

Lemma 2.3. *a) For all $0\le s,t\le 1$,*

$$c_1|s-t|^\alpha \le \|X(s)-X(t)\|_2 \le c_2|s-t|^\alpha,$$

where $c_1=p^{-2A}$, $c_2=\big(\frac{p^{4A\alpha}}{1-p^{-4(1-\alpha)A}}+\frac{1}{1-p^{-4\alpha A}}\big)^{1/2}$.

b) For all $0\le s,t\le 1$ such that $|s-t|=p^{-2A(m+1)}$, $m\ge 1$ integer,

$$|f(s)-f(t)| \ge \kappa_p|s-t|^\alpha.$$

And for all $0 \le s,t \le 1$,

$$|f(s)-f(t)| \le \mathcal{K}_p |s-t|^{\alpha}.$$

Further $\kappa_p = p^{-2(1-\alpha)A}\frac{1-2p^{-2(1-\alpha)A}}{1-p^{-2(1-\alpha)A}}$ *and* $\mathcal{K}_p = \big(\frac{p^{4A\alpha}}{1-p^{-4(1-\alpha)A}} + \frac{1}{1-p^{-4\alpha A}}\big)$.

Proof. This is just reproducing Loud's proof for $p \neq 2$, which we do because the way the constants depend on p and α matters in what follows. Given any function f, we denote for any t and Δt, $\Delta f = f(t+\Delta t) - f(t)$. Let m be the integer such that $p^{-2A(m+1)} < \Delta t \le p^{-2Am}$. The slope of $\varphi_k(t)$ is $\pm\, p^{2(1-\alpha)Ak}$, so that

$$|\Delta\varphi_k| \le p^{2(1-\alpha)Ak}|\Delta(t)| \le p^{2(1-\alpha)Ak-2Am} = p^{-2(1-\alpha)A(m-k)-2A\alpha m}.$$

Moreover φ_k has maximal oscillation $p^{-2\alpha An}$. Therefore

$$\begin{aligned}
|\Delta f(t)| &\le \sum_{k=1}^{\infty}|\Delta\varphi_k(t)| \le \sum_{k=1}^{m} p^{-2(1-\alpha)A(m-k)-2A\alpha m} + \sum_{k=m+1}^{\infty} p^{-2\alpha Ak} \\
&\le \frac{p^{-2A\alpha m}}{1-p^{-2(1-\alpha)A}} + \frac{p^{-2\alpha A(m+1)}}{1-p^{-2\alpha A}} \\
&\le |\Delta t|^{\alpha}\Big(\frac{p^{2A\alpha}}{1-p^{-2(1-\alpha)A}} + \frac{1}{1-p^{-2\alpha A}}\Big).
\end{aligned} \tag{2.4}$$

And

$$\begin{aligned}
\|\Delta X(t)\|_2^2 &= \sum_{k=1}^{\infty}[\Delta\varphi_k(t)]^2 \le \sum_{k=1}^{m} p^{-4(1-\alpha)A(m-k)-4A\alpha m} + \sum_{k=m+1}^{\infty} p^{-4\alpha Ak} \\
&\le \frac{p^{-4A\alpha m}}{1-p^{-4(1-\alpha)A}} + \frac{p^{-4\alpha A(m+1)}}{1-p^{-4\alpha A}} \\
&\le |\Delta t|^{2\alpha}\Big(\frac{p^{4A\alpha}}{1-p^{-4(1-\alpha)A}} + \frac{1}{1-p^{-4\alpha A}}\Big).
\end{aligned}$$

In the other direction, fix t and let $\Delta t = p^{-2A(m+1)}$. By periodicity, $\Delta\varphi_k = 0$ is $k > m$. And $\Delta\varphi_k = \pm\, p^{2(1-\alpha)Ak-2Am}$, if $k \le m$. Thus

$$\begin{aligned}
|\Delta f(t)| &= p^{-2A(m+1)}\big[\pm\, p^{2(1-\alpha)Am} \pm p^{2(1-\alpha)A(m-1)} \pm \ldots \pm p^{2(1-\alpha)A}\big] \\
&= p^{-2A-2\alpha Am}\big[\pm 1 \pm p^{-2(1-\alpha)A} \pm \ldots \pm p^{-2(1-\alpha)A(m-1)}\big].
\end{aligned}$$

As $r := p^{-2(1-\alpha)A} < 1/2$, it follows that

$$\big|\pm 1 \pm p^{-2(1-\alpha)A} \pm \ldots \pm p^{-2(1-\alpha)A(m-1)}\big| \ge 1 - \frac{r}{1-r} = \frac{1-2r}{1-r}.$$

As $|\Delta t|^{\alpha} = p^{-2\alpha-2\alpha Am}$, we therefore get

$$|\Delta f(t)| \ge p^{-2A-2\alpha Am}\frac{1-2p^{-2(1-\alpha)A}}{1-p^{-2(1-\alpha)A}} = |\Delta t|^{\alpha}p^{-2(1-\alpha)A}\frac{1-2p^{-2(1-\alpha)A}}{1-p^{-2(1-\alpha)A}}. \tag{2.5}$$

The corresponding estimate for ΔX is very easy. Let m be such that $p^{-2A(m+1)} \le |\Delta t| < p^{-2Am}$. We have $\Delta\varphi_m(t) = \pm p^{-2A\alpha m}p^{-2Am}\Delta t$. Thus

$$\begin{aligned}
\|\Delta X(t)\|_2^2 &\ge [\Delta\varphi_m(t)]^2 = p^{-4A\alpha m}p^{-4Am}p^{-4A(m+1)} \\
&= p^{-4A}p^{-4A\alpha m} \ge p^{-4A}|\Delta t|^{2\alpha}.
\end{aligned}$$

Hence the lower part with $c_1 = p^{-2A}$. □

The following known estimate will be used. We give a proof because it is elementary and may be easily adapted (up to some extend) to other non geometric coeffcients.

Lemma 2.4. *Given any* $0<\rho<1$,

$$\log \mathbb{P}\{\sum_{n=1}^{\infty}|g_n|\rho^n \le \varepsilon\} \approx (\log\frac{1}{\varepsilon})^2, \qquad as\ \varepsilon\to 0.$$

Proof. We begin with the lower bound. Let $H=\frac{\sqrt{\rho}}{1-\sqrt{\rho}}$. Plainly,

$$\mathbb{P}\{\sum_{n=1}^{\infty}|g_n|\rho^n \le \varepsilon_0\} \ge \prod_{n=1}^{\infty}\mathbb{P}\{|g|<\frac{\varepsilon_0}{H}\rho^{-n/2}\}.$$

Thus it suffices to estimate the product $\prod_{n=1}^{\infty}\mathbb{P}\{|g|<\varepsilon\delta^n\}$ with $\varepsilon=\frac{\varepsilon_0}{H}, \delta=\rho^{-1/2}$, $\delta>1$. Let a be such that $\mathbb{P}\{|g|\ge a\}\le\frac{1}{2}$, and put $N=\sup\{n:\delta^n\le\frac{a}{\varepsilon}\}$. Then

$$\prod_{n=1}^{N}\mathbb{P}\{|g|<\varepsilon\delta^n\}\ge\mathbb{P}\{|g|<\varepsilon\}^N\ge\exp\big\{-C_\delta(\log\frac{1}{\varepsilon})^2\big\}.$$

Now,

$$\begin{aligned}\sum_{n=N+1}^{\infty}\mathbb{P}\{|g|\ge\varepsilon\delta^n\} &= \int_a^{\infty}\{\sum_{a\le\varepsilon\delta^n\le u}1\}e^{-u^2/2}\mathrm{d}u\\ &\le C_\delta\int_a^{\infty}\big\{1\vee\log u\big\}e^{-u^2/2}\mathrm{d}u<\infty.\end{aligned}$$

As $\log(1-x)\ge -2x$ if $0\le x\le 1/2$ and $\mathbb{P}\{|g|>\varepsilon\delta^n\}\le 1/2$ if $n>N$, we get

$$\prod_{n=N+1}^{\infty}\mathbb{P}\{|g|<\varepsilon\delta^n\}\ge\exp\big\{-\sum_{n=N+1}^{\infty}\mathbb{P}\{|g|\ge\varepsilon\delta^n\}\big\}\ge c_\delta>0.$$

Thus $\prod_{n=1}^{\infty}\mathbb{P}\{|g|<\varepsilon\delta^n\}\ge c_\delta(\log\frac{1}{\varepsilon})^2$. To get the upper bound is faster. Let $N'=\sup\{n:\delta^n\le\frac{1}{\sqrt{\varepsilon}}\}$. Then

$$\begin{aligned}&\prod_{n=1}^{\infty}\mathbb{P}\{|g|<\varepsilon\delta^n\}\le\prod_{n=1}^{N'}\mathbb{P}\{|g|<\varepsilon\delta^n\}\le\mathbb{P}\{|g|<\sqrt{\varepsilon}\}^{N'}\\ =\ &\exp\Big\{-N'\log\frac{1}{\mathbb{P}\{|g|<\sqrt{\varepsilon}\}}\Big\}\le\exp\big\{-C_\delta(\log\frac{1}{\varepsilon})^2\big\}.\end{aligned}\tag{2.6}$$

□

We can now prove Theorem 2.1. Take X_1 as in Lemma 2.2. Let $p=2$ in (2.3) and choose $X_2=X$. The entropy numbers clearly verify $N([0,1],d_{X_i},\varepsilon)\approx\varepsilon^{1/\alpha}$, $i=1,2$. First, by using Lemma 2.4,

$$\mathbb{P}\Big\{\sup_{0\le t\le 1}|X_2(t)|\le\varepsilon\Big\}\ge\mathbb{P}\Big\{\sum_{k=1}^{\infty}2^{-2\alpha Ak}|g_k|\le\varepsilon\Big\}\ge e^{-C_\alpha\log^2\frac{1}{\varepsilon}}.$$

Next we notice

$$\varphi_j(2^{-2Ak}) = \begin{cases} 2^{-2A\alpha j}\, 2^{-2A(k-j)} & \text{if } j \le k, \\ 0 & \text{if } j > k. \end{cases}$$

Thus $X_2(2^{-2Ak}) = 2^{-2Ak}\sum_{j=1}^{k} g_j 2^{2A(1-\alpha)j}$. And as

$$2^{2Ak}X_2(2^{-2Ak}) - 2^{2A(k-1)}X_2(2^{-2A(k-1)}) = g_k 2^{2A(1-\alpha)k},$$

it follows from (2.6) that

$$\mathbb{P}\Big\{\sup_{0\le t\le 1}|X_2(t)| \le \varepsilon\Big\} \le \prod_{k=1}^{\infty}\mathbb{P}\big\{|g_k| \le 2^{2A\alpha k}(1+2^{-2A})\varepsilon\big\} \le \exp\big\{-C_\alpha(\log\frac{1}{\varepsilon})^2\big\}.$$

This achieves the proof.

Remark 2.5. Let $\psi(t) = 1 - |2\{t\} - 1|$ where $\{t\}$ denote the fractional part of t. Lifshits has considered the following example

$$X(t) = g_0 t + \sum_{n=1}^{\infty} g_n 2^{-\frac{\alpha}{2}n}\psi(\{2^n t\}) \qquad t \in [0,1]. \tag{2.7}$$

It is observed in [5] that $\|X(s) - X(t)\|_2 \ge c|t-s|^{\frac{\alpha}{2}}$ whereas

$$\log\mathbb{P}\big\{\sup_{t\in T}|X(t)| \le \varepsilon\big\} \approx -\log^2\frac{1}{\varepsilon}.$$

As we said at the beginning, our second process is of same type since $\psi(t) = \varphi(t, \frac{1}{2})$.

A class of examples. If τ is a piecewise C^2 expanding map on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, by the Lasota-Yorke theorem there exists a τ-invariant probability measure μ which is absolutely continuous with respect to Lebesgue measure. So is the case for ψ. This leads us to introduce the following family of processes: let $\{a_n, n \ge 1\} \in \ell_1$, $f \in L^1(\mathbb{T}, \mu)$ and put

$$X(t) = \sum_{n=1}^{\infty} a_n g_n f(\psi^n(t)).$$

We have just considered the case $f(t) = t$. It would be certainly very informative to describe the small deviations of this class of Gausian processes. By Birkhoff's theorem,

$$\frac{1}{n}\sum_{k=0}^{n-1} f(\psi^n(t)) \to \int_{\mathbb{T}} f\,\mathrm{d}\mu \qquad \text{almost everywhere.}$$

The rate of this convergence, which for specific f only maybe explicited, certainly plays a role since by using Abel summation we formally have

$$X(t) = \sum_{n=1}^{\infty} n(a_n - a_{n+1})g_n\big[\frac{1}{n}\sum_{k=0}^{n-1} f(\psi^n(t))\big] \sim \big(\int_{\mathbb{T}} f\,\mathrm{d}\mu\big)\sum_{n=1}^{\infty} n(a_n - a_{n+1})g_n.$$

Shao and Li [4] argued from example (2.7) that stationarity (in turn periodicity) should play a big role in upper estimates for small deviations.

We show that Loud's functions can be used to build aperiodic examples breaking the duality even more dramatically. The intuitive idea behind the construction is that adding infinitely many functions with periods q_n^{-1}, where q_n are mutually coprime integers, produces aperiodic functions.

Proposition 2.6. *There exists an aperiodic sample continuous Gaussian process* $\{X(t), 0 \le t \le 1\}$ *such that*

$$\lim_{\varepsilon\to 0} \frac{\log N([0,1], d_X, \varepsilon)}{\log \frac{1}{\varepsilon}} = \infty \quad \textit{whereas} \quad \liminf_{\varepsilon\to 0} \frac{\log \mathbb{P}\{\sup_{0\le t\le 1} |X(t)| \le \varepsilon\}}{(\log \frac{1}{\varepsilon})^2} > -\infty.$$

Proof. Let $\mathcal{P}$ be an infinite set of mutually coprime integers larger than 2. Let $0 < \alpha_p < 1/2$, $\alpha_p \downarrow 0$. Take $A = 1$, then condition $p^{2(1-\alpha_p)A} > 2$ holds. We further assume that

$$\lim_{p\to\infty} \alpha_p \log p = 0, \qquad 2^{hp}\alpha_p \log p \uparrow \infty \quad (\forall h > 0). \tag{2.8}$$

Let $\varphi_{p,k}(t) = p^{-2\alpha_p k}\varphi(t, p^{-2k})$, $k = 1, \dots$ and put $f_p = \sum_{k=1}^{\infty} \varphi_{p,k}$. Then f_p is p^{-2}-periodic. Now let $\{a_p, p \in \mathcal{P}\}$ be a sequence of reals such that $\sum_p a_p^2 < \infty$, and consider the Gaussian process

$$X(t) = \sum_p g_p a_p f_p(t) \tag{2.9}$$

Since $\mathcal{P}$ is a set of mutually coprime integers, periodicity is destroyed and so by considering its covariance, X is no longer periodic.

By Lemma 2.3, $|f_p(s)-f_p(t)| \ge \kappa_p |s-t|^{\alpha_p}$, whenever $|s-t| = p^{-2(m+1)}$, m integer. By assumption (2.8), $p^{\alpha_p} \sim 1$ as $p \to \infty$, so that $\kappa_p = p^{-2(1-\alpha_p)} \frac{1-2p^{-2(1-\alpha_p)}}{1-p^{-2(1-\alpha_p)}} \sim p^{-2}$. Moreover,

$$\|f_p\|_\infty \le \sum_{k=1}^{\infty} p^{-2\alpha_p k} = \frac{p^{-2\alpha_p}}{1-p^{-2\alpha_p}} \le \frac{1}{1-e^{-2\alpha_p \log p}} \le \frac{C}{\alpha_p \log p}.$$

Let $0 \le s, t \le 1$ be such that $|s-t| = p^{-2(m+1)}$. Then

$$\|X(s)-X(t)\|_2^2 = \sum_q a_q^2 |f_q(s)-f_q(t)|^2 \ge a_p^2 |f_p(s)-f_p(t)|^2 \ge a_p^2 \kappa_p^2 |s-t|^{2\alpha_p}.$$

Thereby $\|X(s)-X(t)\|_2 \ge C a_p p^{-2} |s-t|^{\alpha_p}$. Now let $\alpha > 0$. We choose m integer so that

$$m+1 \sim \frac{p\alpha \log 2}{2\alpha_p \log p}.$$

Then $|s-t|^{\alpha_p} = p^{-2\alpha_p(m+1)} \sim 2^{-p\alpha}$. Let β, γ be such that $0 < \beta < \alpha < \alpha+\beta < \gamma$, and choose $a_p = 2^{-\beta p}$. Then, for all p large enough

$$\|X(s)-X(t)\|_2 \ge C 2^{-(\alpha+\beta)p} p^{-2} \ge 2^{-\gamma p}.$$

Put $\varepsilon = 2^{-\gamma p}$. Then

$$N([0,1], d_X, \varepsilon) \ge p^{2(m+1)} = e^{2(m+1)\log p} \ge 2^{c\, p \frac{\alpha}{\alpha_p}} \gg 2^{\frac{p}{\alpha}} = \varepsilon^{-\frac{1}{\alpha}}$$

Let $0 < \beta' < \beta$. Now as $2^{hp}\alpha_p \log p \uparrow \infty$ for any $h > 0$, it follows that

$$|X(t)| \le \sum_p |g_p| 2^{-\beta p} \|f_p\|_\infty \le C \sum_p |g_p| \frac{2^{-\beta p}}{\alpha_p \log p} \le C \sum_p |g_p| 2^{-\beta' p}.$$

Therefore, by using Lemma 2.4

$$\mathbb{P}\big\{\sup_{0\le t\le 1}|X(t)|\le\varepsilon\big\}\ge\mathbb{P}\big\{\sum_p |g_p|2^{-\beta' p}\le\varepsilon/C\big\}\ge e^{-C(\log\frac{1}{\varepsilon})^2}.$$

□

3. A General Lower Bound Using Majorizing Measures

The results from the previous section suggest the search of lower bounds for small deviations by using the majorizing measure method. It is known from the general theory of Gaussian processes that this is the paramount method for studying the regularity of Gaussian processes. And also that in general, entropy numbers are not a sufficiently precise tool. A classical example is provided by independent Gaussian sequences. See [7],[9],[11]. Generally speaking, once having Kathri-Sidák's inequality in hands, the argument leading to lower bounds is relatively direct. A well appreciation of the used chaining technic is however necessary. In [12], we obtained a general lower estimate of small deviations by using majorizing measure method. Since the result is relevant there and in the next section, we present a slightly updated formulation of it and provide a proof.

Let $X=\big\{X(t),t\in T\big\}$ be a centered Gaussian process, with basic probability space $(\Omega,\mathcal{A},\mathbb{P})$, and let $d(s,t)=\|X(s)-X(t)\|_2$, $D=\mathrm{diam}(T,d)$. We assume that $\sigma=\sup_{t\in T}\|X(t)\|_2<\infty$ and that X is d-separable. Let $\Pi_0\preceq\Pi_1\preceq\ldots$ be a sequence of finite measurable ordered partitions of T (Π_{n+1} is a refinement of Π_n) such that

$$\max_{\pi\in\Pi_n}\max_{u,v\in\pi} d(u,v)\le 2^{-n}D,\qquad n=0,1,\ldots \tag{3.1}$$

Let $N_n=\#\{\Pi_n\}$. For any $\pi\in\Pi_n$, let $\bar\pi$ be such as $\pi\subset\bar\pi\in\Pi_{n-1}$. If $t\in T$, we also define $\pi_n(t)$ by the relations $t\in\pi_n(t)\in\Pi_n$. Introduce now a majorizing measure condition.

There exists a probability measure μ on T such that:

$$\lim_{n\to\infty}\sup_{t\in T}\sum_{m>n}2^{-m}\Big(\log\frac{1}{\mu(\pi_m(t))}\Big)^{1/2}=0. \tag{3.2}$$

The following technical ingredient will be useful in the proof. Let $v(m)>0$ be such that $\sum_{m=0}^{\infty}v(m)^{-1}<\infty$, and put

$$H(n)=\sup_{t\in T}\sum_{m>n}(2^{-m}D)\big(\log\frac{v(m)}{\mu(\pi_m(t))}\big)^{1/2}.$$

Then $H(n)$ is finite and $H(n)\to 0$ as $n\to\infty$. The additional term v is often of little unconvenience since, at least on standard examples, one may take $v(m)\gg\sup_t\mu(\pi_m(t))$, (see next section).

Theorem 3.1. *For $0<\varepsilon\sigma<H(0)$, let $n(\varepsilon)$ be such that $H(n(\varepsilon))\le\varepsilon\sigma$. Then,*

$$\mathbb{P}\Big\{\sup_{t\in T}|X(t)|\le 2\varepsilon\sigma\Big\}\ge Ce^{-N_{n(\varepsilon)}(\log\frac{1}{\varepsilon})}.$$

Proof. Since X is d-separable, it suffices to produce a proof for a countable d-dense subset of T, which we will call again T. Put

$$X_\pi = \int_\pi X(u)\frac{\mu(du)}{\mu(\pi)}, \qquad X_n(t) = \int_{\pi_n(t)} X(u)\frac{\mu(du)}{\mu(\pi_n(t))}.$$

These Gaussian random variables are the bricks of the majorizing measure method. By (3.1), $\|X_n(t) - X_{n-1}(t)\|_2 \le 2^{-n}$. Elementary considerations then yield that $X(t) \overset{a.s.}{=} \lim_{n\to\infty} X_n(t)$. Thus $X(t) - X_n(t) \overset{a.s.}{=} \sum_{m=n+1}^\infty \big(X_m(t) - X_{m-1}(t)\big)$ and we have the bound

$$|X(t)| \quad \le \quad \sup_{\pi\in\Pi_n} |X_\pi| + \sum_{m=n+1}^{\infty} \big|X_m(t) - X_{m-1}(t)\big|. \tag{3.3}$$

Put

$$A_m = \Big\{\forall t \in T, \frac{|X_m(t) - X_{m-1}(t)|}{\|X_m(t) - X_{m-1}(t)\|_2} \le \big(\log\frac{v(m)}{\mu(\pi_m(t))}\big)^{1/2}\Big\}.$$

Then by using (3.3) and the fact that $\|X_\pi\|_2 \le \sigma$ for all $\pi \in \Pi_n$ and n,

$$\begin{aligned}
& \mathbb{P}\Big\{\Big\{\sup_{\pi\in\Pi_n}\frac{|X_\pi|}{\|X_\pi\|_2} \le \varepsilon\Big\} \cap \bigcap_{m>n} A_m\Big\} \\
\le\ & \mathbb{P}\Big\{\forall t \in T,\ |X(t)| \le \varepsilon\sigma + 2\sum_{m>n} 2^{-m} D\big(\log\frac{v(m)}{\mu(\pi_m(t))}\big)^{1/2}\Big\} \\
\le\ & \mathbb{P}\Big\{\sup_{t\in T}|X(t)| \le \varepsilon\sigma + 2\sup_{t\in T}\sum_{m>n} 2^{-m} D\big(\log\frac{v(m)}{\mu(\pi_m(t))}\big)^{1/2}\Big\} \\
=\ & \mathbb{P}\Big\{\sup_{t\in T}|X(t)| \le \varepsilon\sigma + 2H(n)\Big\}.
\end{aligned}$$

Now by noticing that

$$A_m = \Big\{\forall \pi \in \Pi_m, \frac{|X_\pi - X_{\bar\pi}|}{\|X_\pi - X_{\bar\pi}\|_2} \le \big(\log\frac{v(m)}{\mu(\pi)}\big)^{1/2}\Big\}$$

and using Kathri-Sidák's inequality (1.1), we get

$$\begin{aligned}
& \mathbb{P}\Big\{\{\sup_{\pi\in\Pi_n}\frac{|X_\pi|}{\|X_\pi\|_2} \le \varepsilon\} \cap \bigcap_{m>n} A_m\Big\} \\
\ge\ & \mathbb{P}\{|g| \le \varepsilon\}^{N_n} \prod_{\substack{m>n \\ \pi\in\Pi_m}} \mathbb{P}\big\{|g| \le \big(2\log\frac{v(m)}{\mu(\pi)}\big)^{1/2}\big\} \\
\ge\ & \exp\Big\{-N_n \log\frac{1}{\mathbb{P}\{|g|\le\varepsilon\}} - c\sum_{\substack{m>n \\ \pi\in\Pi_m}} \mathbb{P}\big\{|g| > \big(2\log\frac{v(m)}{\mu(\pi)}\big)^{1/2}\Big\} \\
\ge\ & \exp\Big\{-N_n \log\frac{1}{\mathbb{P}\{|g|\le\varepsilon\}} - c\sum_{m>n}\sum_{\pi\in\Pi_m}\frac{\mu(\pi)}{v(m)}\Big\} \\
=\ & \exp\Big\{-N_n \log\frac{1}{\mathbb{P}\{|g|\le\varepsilon\}} - c\sum_{m>n}\frac{1}{v(m)}\Big\} \\
\ge\ & C\exp\Big\{-N_n\log\frac{1}{\varepsilon}\Big\}.
\end{aligned}$$

Consequently,

$$\mathbb{P}\Big\{\sup_{t\in T}|X(t))|\le \varepsilon\sigma+2H(n)\Big\}\ge C\exp\Big\{-N_n\log\frac{1}{\varepsilon}\Big\}. \qquad (3.4)$$

Choose $n=n(\varepsilon)$. We obtained

$$\mathbb{P}\Big\{\sup_{t\in T}|X(t)|\le 2\varepsilon\sigma\Big\}\ge C\exp\Big\{-N_{n(\varepsilon)}(\log\frac{1}{\varepsilon})\Big\}.$$

□

Let $\delta:[0,1]\to\mathbb{R}^+$ be increasing, $\delta(0)=0$, and verifying the integral condition

$$\int_0^D\big(\log\frac{1}{\delta(u)}\big)^{1/2}\mathrm{d}u<\infty.$$

Choose $v(m)=\frac{1}{\omega(2^{-m}D)}$ where $\omega(t)>0$ is increasing and verifies $\int_0^1\frac{\omega(t)}{t}\mathrm{d}t<\infty$.

Corollary 3.2. *Assume there exists a family $\{\Pi_n,n\ge 0\}$ of finite measurable ordered partitions of T satisfying (3.1) and a probability measure μ on T such that:*

$$\min\{\mu(\pi),\pi\in\Pi_m\}\ge\delta(2^{-mD})/2 \qquad (\forall m\ge 0).$$

Let $n(\varepsilon)=\sup\Big\{n:2\int_0^{\varepsilon_n}\big(\log\frac{2}{\delta^{-1}(u)\omega(u)}\big)^{1/2}\mathrm{d}u\le\varepsilon\sigma\Big\}$. *Then*

$$\mathbb{P}\Big\{\sup_{t\in T}|X(t)|\le 2\varepsilon\sigma\Big\}\ge C\exp\big\{-N_{n(\varepsilon)}(\log\frac{1}{\varepsilon})\big\}.$$

Proof. We have

$$\begin{aligned}\sum_{m>n}(2^{-m}D)\big(\log\frac{v(m)}{\mu(\pi_m(t))}\big)^{1/2} &\le \sum_{m>n}(2^{-m}D)\big(\log\frac{2}{\delta^{-1}(2^{-m}D)\omega(2^{-m}D)}\big)^{1/2}\\ &\le 2\int_0^{\varepsilon_n}\big(\log\frac{2}{\delta^{-1}(u)\omega(u)}\big)^{1/2}\mathrm{d}u.\end{aligned}$$

Therefore

$$\mathbb{P}\Big\{\sup_{t\in T}|X(t)|\le 2\varepsilon\sigma\Big\}\ge C\exp\Big\{-N_{n(\varepsilon)}(\log\frac{1}{\varepsilon})\Big\}.$$

□

Example. Consider Gaussian processes $X(t),t\in[0,1]$, which satisfy the increment condition:

$$\|X(s)-X(t)\|_2\le\delta(|s-t|), \qquad (\forall s,t\in[0,1]).$$

For $m=0,1\dots$, let Π_m be a partition of $[0,1]$ by consecutive intervals of length less or equal to $\varepsilon_m=\delta^{-1}(2^{-m}D)$, $D=d(1)$. One can arrange it so that each interval has length greater than $\varepsilon_m/2$. Let μ be the Lebesgue measure. Then $\mu(\pi)\ge\delta^{-1}(2^{-m}D)/2$ if $\pi\in\Pi_m$. Thus Corollary 3.2 applies. In the particular case $d(u)=|\log u|^{\beta}$ with $\beta>1/2$, this gives

$$\log\Big|\log\mathbb{P}\Big\{\sup_{t\in T}|X(t)|\le 2\varepsilon\sigma\Big\}\Big|=\mathcal{O}(\varepsilon^{-\frac{2}{2\beta-1}}). \qquad (3.5)$$

That estimate can also be deduced from the very recent work [1] (Theorem 3 with $\gamma=\beta^{-1}$), where a growth condition on entropy numbers (namely on the induced Gaussian metric) is given.

4. Gaussian Independent Sequences

Let $\varphi(n) \uparrow \infty$ with n and consider the Gaussian sequence $G(\varphi) = \{G_n, n \in \overline{\mathbb{N}}\}$ defined by

$$G_n = \frac{g_n}{\varphi(n)}, \qquad G_\infty = 0.$$

It is known ([7] p.102) that already on these elementary examples, the metric entropy approach fails to describe their regularity. As

$$\limsup_{n\to\infty} \frac{|g_n|}{\sqrt{2\log n}} \stackrel{a.s.}{=} 1, \tag{4.1}$$

$G(\varphi)$ is sample bounded if $\varphi(n) = \sqrt{\log n}$, and sample continuous on $\overline{\mathbb{N}}$ if and only if

$$\sqrt{\log n} = o(\varphi(n)). \tag{4.2}$$

We begin with a general remark. From Talagrand's representation of bounded or continuous Gaussian processes ([7], theorems 2-3), we know that a Gaussian process $\{X(t), t \in T\}$ is sample bounded if and only if there exists a (not necessarily independent) Gaussian sequence $\{\xi_n, n \ge 1\}$ with $\|\xi_n\|_2 \le Ka(\log n + a^2/b^2)^{-1/2}$, and that for each $t \in T$ one can write

$$X(t) = \sum_{n=1}^\infty \alpha_n(t)\xi_n$$

where $\alpha_n(t) \ge 0$, $\sum_{n=1}^\infty \alpha_n(t) \le 1$ and the series converges a.s. and in L^2. And if T is a compact metric space, $\{X(t), t \in T\}$ is sample continuous if and only if its covariance function is continuous, and the same representation holds with $\|\xi_n\|_2 = o(\sqrt{\log n})$. Thus by Kathri-Sidák's inequality,

$$\mathbb{P}\big\{\sup_{t\in T}|X(t)| \le \varepsilon\big\} \ge \mathbb{P}\Big\{\sup_{n=1}^{\infty}|\xi_n| \le \varepsilon\Big\} \ge \prod_{n=1}^\infty \mathbb{P}\big\{|\xi_n| \le \varepsilon\big\}.$$

This consequently makes the study of small deviations of sequences $G(\varphi)$ of particular interest in this general context. We shall show that Theorem 3.1 allows to get sharp lower bounds. The sequence of ordered partitions associated to φ is based on an intrinsic sieve of $\mathbb{N}$, and as to the majorizing measure we will construct, it turns up to be very simple.

We notice that $\|G_n - G_m\|_2 = (\frac{1}{\varphi(n)^2} + \frac{1}{\varphi(m)^2})^{1/2}$ and

$$D = \sup_{n,m\ge1}\|G_n - G_m\|_2 = (\frac{1}{\varphi(1)^2} + \frac{1}{\varphi(2)^2})^{1/2}, \qquad \sigma = \sup_{n\ge1} = \|G_n\|_2 = \varphi_1^{-1}.$$

Theorem 4.1. *Assume that (4.2) holds,* $\log\varphi(m) = \mathcal{O}(\log m)$ *and*

$$\int_0^D \big(\log\varphi^{-1}(\frac{1}{u})\big)^{1/2}\mathrm{d}u < \infty. \tag{4.3}$$

Let $\varepsilon_n = 2^{-n}D$ *and put* $H(n) = \int_0^{\varepsilon_n}\big(\log\varphi^{-1}(\frac{1}{u})\big)^{1/2}\mathrm{d}u$, $n \ge 0$. *For* $0 < \frac{\varepsilon}{\varphi(1)} < H(1)$*, let* $n(\varepsilon)$ *be such that* $H(n(\varepsilon)) \le \frac{\varepsilon}{\varphi(1)}$*. There exists an absolute constant* C

such that,

$$\mathbb{P}\Big\{\sup_{n\ge 1}|G_n| \le \frac{2}{\varphi(1)}\varepsilon\Big\} \ge Ce^{-\varphi^{-1}(\frac{1}{\varepsilon_{n(\varepsilon)}})(\log\frac{1}{\varepsilon})}.$$

Condition $\log\varphi(m) = \mathcal{O}(\log m)$ is technical. Notice that it only excludes cases that are too regular, typically when $\varphi(m)$ increases exponentially.

Proof. Let $F_n = \varphi^{-1}(\frac{1}{\varepsilon_n})$, $n \ge 0$. We notice that $F_1 = \varphi^{-1}(\varphi(1)) = 1$. For $u \ge 1$, let $\nu(u)$ denote the unique integer such that $F_{\nu(u)} \le u < F_{\nu(u)+1}$.

Lemma 4.2. *Let $B(u,\varepsilon) = \{v \ge 1 : \|G_u - G_v\|_2 \le \varepsilon\}$. Then,*

$$\begin{aligned} &B(u,\varepsilon_n) = \{u\} && (\forall n > \nu(u)),\\ &B(u,\varepsilon_n) \supseteq [F_{n+1},\infty) && (\forall n < \nu(u)). \end{aligned}$$

Proof. Plainly $\varepsilon_{\nu(u)+1} < \frac{1}{\varphi(u)} \le \varepsilon_{\nu(u)}$. If $n > \nu(u)$, then for any v, $\|G_u - G_v\|_2 > \frac{1}{\varphi(u)} > \varepsilon_{\nu(u)+1} \ge \varepsilon_n$. Hence $B(u,\varepsilon_n) = \{u\}$. Now notice that if $m \le \nu(u)$, then $v \ge F_m = \varphi^{-1}(\frac{1}{\varepsilon_m})$ implies that $\frac{1}{\varphi(v)} \le \varepsilon_m$, and so

$$\|G_u - G_v\|_2 \le (\varepsilon_{\nu(u)}^2 + \varepsilon_m^2)^{1/2} \le \sqrt{2}\varepsilon_m < \varepsilon_{m-1}.$$

Hence with $n = m-1$ the second assertion. □

Let $\Pi_0 = \mathbb{N}$. For $\nu \ge 1$, let Π_ν be the finite partition of $\mathbb{N}$ defined by:

$$\pi \in \Pi_\nu \iff \pi = \{u\},\ u < F_\nu \text{ or } \pi = [F_\nu,\infty).$$

Then $\#\{\Pi_\nu\} = F_\nu$ and $\Pi_{\nu+1}$ is a refinement of Π_ν. Further, assumption (3.1) is satisfied since by Lemma 4.2

$$\max_{\pi\in\Pi_\nu}\max_{u,v\in\pi} d(u,v) \le \varepsilon_\nu.$$

Let μ be the probability measure on $\mathbb{N}$ defined by $\mu\{t\} = ct^{-2}$, $c = (\sum_{t=1}^\infty t^{-2})^{-1}$. Let $t \ge 1$, we set $\pi_m(t) = \{t\}$ if $t < F_m$ and $\pi_m(t) = [F_m,\infty)$ otherwise. It follows that

$$\mu(\pi_m(t)) \ge \begin{cases} Ct^{-2} & \text{if} \quad m > \nu(t)\\ CF_m^{-1} & \text{if} \quad m \le \nu(t). \end{cases} \tag{4.4}$$

Fix some integer n and let $t \ge 1$. If $n > \nu(t)$, then $t < F_n = \varphi^{-1}(\frac{1}{\varepsilon_n})$ and

$$\sum_{m=n}^\infty \varepsilon_m\big(\log\frac{1}{\mu(\pi_m(t))}\big)^{1/2} \le C\big(\sum_{m=n}^\infty \varepsilon_m\big)\big(\log t\big)^{1/2} \le C\varepsilon_n\big(\log\varphi^{-1}(\frac{1}{\varepsilon_n})\big)^{1/2}.$$

Now let $n \le \nu(t)$. If $\nu(t) \ge m \ge n$, then $\mu(\pi_m(t)) \ge CF_m^{-1} \ge CF_{\nu(t)}^{-1}$ and as $t < F_{\nu(t)+1}$, we may write

$$\begin{aligned} \sum_{m=n}^\infty \varepsilon_m\big(\log\frac{1}{\mu(\pi_m(t))}\big)^{1/2} \quad &\le \quad C\sum_{m=n}^{\nu(t)} \varepsilon_m\big(\log\varphi^{-1}(\frac{1}{\varepsilon_m})\big)^{1/2}\\ &\quad + \big(\sum_{m>\nu(t)} \varepsilon_m\big)(\log t)^{1/2}\\ &\le \quad C\sum_{m=n}^{\nu(t)} \varepsilon_m\big(\log\varphi^{-1}(\frac{1}{\varepsilon_m})\big)^{1/2} + C\varepsilon_{\nu(t)}(\log t)^{1/2} \end{aligned}$$

$$\begin{aligned} &\leq C\sum_{m=n}^{\nu(t)+1}\varepsilon_m\big(\log\varphi^{-1}(\frac{1}{\varepsilon_m})\big)^{1/2}\\ &\leq C\int_{\varepsilon_{\nu(t)+2}}^{\varepsilon_n}\big(\log\varphi^{-1}(\frac{1}{u})\big)^{1/2}\mathrm{d}u. \end{aligned}$$

Thereby

$$\sup_{t\geq 1}\sum_{m=n}^{\infty}\varepsilon_m\big(\log\frac{1}{\mu(\pi_m(t))}\big)^{1/2}\leq C\int_0^{\varepsilon_n}\big(\log\varphi^{-1}(\frac{1}{u})\big)^{1/2}\mathrm{d}u\to 0,$$

as $n\to\infty$, by assumption. Condition (3.2) is thus realized. It remains to choose v. We first observe that if $\log v(m)=\mathcal{O}\big(\log\varphi^{-1}(m)\big)$, then

$$\begin{aligned} \sum_{m>n}2^{-m}D\big(\log\frac{v(m)}{\mu(\pi_m(t))}\big)^{1/2} &\leq \sum_{m>n}2^{-m}D\big(\log\frac{1}{\mu(\pi_m(t))}\big)^{1/2}\\ &\quad+\sum_{m>n}2^{-m}D\big(\log v(m)\big)^{1/2}\\ &\leq C\int_0^{\varepsilon_n}\big(\log\varphi^{-1}(\frac{1}{u})\big)^{1/2}\mathrm{d}u. \end{aligned}$$

Next, clearly $\sum_{m=0}^{\infty}v(m)^{-1}<\infty$ if $v(m)=m^2$. This imposes that $\log m=\mathcal{O}\big(\log\varphi^{-1}(m)\big)$ or $\log\varphi(m)=\mathcal{O}(\log m)$, which is precisely assumed. Consequently,

$$H(n)=\sup_{t\in T}\sum_{m>n}(2^{-m}D)\big(\log\frac{v(m)}{\mu(\pi_m(t))}\big)^{1/2}=\int_0^{\varepsilon_n}\big(\log\varphi^{-1}(\frac{1}{u})\big)^{1/2}\mathrm{d}u.$$

Let $n(\varepsilon)$ be such that $H(n(\varepsilon))\leq\frac{\varepsilon}{\varphi(1)}$. By applying Theorem 3.1, we deduce that

$$\mathbb{P}\Big\{\sup_{t\geq 1}|G_t|\leq 2\varepsilon\sigma\Big\}\geq Ce^{-N_{n(\varepsilon)}(\log\frac{1}{\varepsilon})}.$$

□

The following corollary easily follows.

Corollary 4.3. *a) Let $\varphi(t)=(\log t)^{\beta}$, $\beta>1/2$. Then,*

$$\log\Big|\log\mathbb{P}\Big\{\sup_{n\geq 1}\frac{|g_n|}{\varphi(n)}\leq\varepsilon\Big\}\Big|\preceq\varepsilon^{-\frac{2}{2\beta-1}}.$$

b) Let $\varphi(t)=(\log t)^{\frac{1}{2}}(\log\log t)^{1+h}$, $h>0$. Then,

$$\log\log\Big|\log\mathbb{P}\Big\{\sup_{n\geq 1}\frac{|g_n|}{\varphi(n)}\leq\varepsilon\Big\}\Big|\preceq\varepsilon^{-\frac{1}{h}}.$$

5. Ultrametric Gaussian Processes

For ultrametric Gaussian processes, a general upper bound of small deviations can be established. And by using Theorem 3.1, this is completed with a sharp lower bound. A metric space (T,d) is called ultrametric when d satisfies the strong triangle inequality:

$$d(s,t)\leq\max\big(d(s,u),d(u,t)\big),\qquad(\forall s,t,u\in T).$$

Thus two balls of same radius are either disjoint or identical. Let $B(t,u) = \{s \in T : d(s,t) \le u$, and let $v \le u$. It also follows that $s \in B(t,u) \Rightarrow B(s,v) \subset B(t,u)$. When (T,d) is separable, it is easy to show that (T,d) embeds continuously into a projective limit of sets, itself endowed with an ultrametric structure. Since we need the construction, we briefly recall it. Let $D = \mathrm{diam}(T,d)$. Let S_n be the set of centers of balls forming a minimal covering of (T,d) with closed balls of radius $\varepsilon_n = 2^{-n}D$, $n = 0,1,\ldots$. Notice that each ball $B(t,\varepsilon_n)$ contains at least one element of S_{n+1}, thereby a ball $B(s,\varepsilon_{n+1})$ for some $s \in S_{n+1}$. Otherwise, there is one ball $B(t_0,\varepsilon_n)$, say, such that $\min\{d(t_0,s), s \in S_{n+1}\} > \varepsilon_n > \varepsilon_{n+1}$, which contradicts the fact that S_{n+1} realizes a covering of T of order ε_{n+1}. Consider for $n = 0,1,\ldots$ the mappings $\theta_n : T \to S_n$, $\Pi_{n,n-1} : S_n \to S_{n-1}$ respectively defined by $d(s,\theta_n(s)) \le \varepsilon_n$ and $d(t,\Pi_{n,n-1}(t)) \le \varepsilon_{n-1}$. Next we define $\Pi_{n,k} : S_n \to S_k$ for $n \ge k$ as follows: $\Pi_{n,n} = \mathrm{Id}(S_n)$ and

$$\Pi_{n,k} = \Pi_{n,n-1} \circ \ldots \circ \Pi_{k+1,k}.$$

The following elementary lemma arises from the construction itself, so we omit the proof.

Lemma 5.1. *The pair $\big((S_n),(\Pi_{n,k})\big)$ defines a projective system of sets and we have the relations*

$$\theta_k = \Pi_{n,k} \circ \theta_n, \qquad (\forall n \ge k \ge 0).$$

Let $L = \varprojlim \big((S_n),(\Pi_{n,k})\big)$ denote its projective limit, $G = \prod_{k=0}^{\infty} S_k$. Let Π_k be the restriction to L of the projection of G onto S_k, $k = 0,1,\ldots$. Put for any two elements s,t of L

$$\delta(s,t) = \varepsilon_{n(s,t)},$$

where $n(s,t) = \sup\{k \ge 0 : \Pi_k(s) = \Pi_k(t)\}$. Then (L,δ) is a compact ultrametric space. Moreover, the mapping $\ell : (T,d) \to (L,\delta)$ defined by $\ell(t) = \{\theta_k(t), k \ge 0\}$ a continuous embedding from (T,d) to (L,δ), and we have the relations

$$\frac{1}{2}\delta(\ell(s),\ell(t)) \le d(s,t) \le \delta(\ell(s),\ell(t)), \qquad (\forall s,t \in T).$$

The projective limit L and thereby T, is easily visualized as a tree with branches in G, anytwo of them separating at offshoots of high "$n(s,t)$". One can attach to any such tree an ultrametric Gaussian process. These classes of processes have been much investigated by Fernique, see [2]. Let $\{g_n, n \in \Sigma S_k\}$ be a sequence of independent Gaussian standard random variables. We put

$$Z(t) = \sum_{n=0}^{\infty} \varepsilon_n g_{\Pi_n(t)}, \qquad (\forall t \in T).$$

Theorem 5.2. *a) For some absolute constant $\gamma > 0$, we have for $\varepsilon \le D$,*

$$\mathbb{P}\Big\{\sup_{s,t\in L} |Z(s) - Z(t)| \le \varepsilon\Big\} \quad \le \quad e^{-\gamma N(T,2\varepsilon)}.$$

b) Assume that condition (3.2) is fulfilled. Then, with the notation of Theorem 3.1, letting $\sigma = 2D/\sqrt{3}$,

$$\mathbb{P}\Big\{\sup_{t\in T} |Z(t)| \le 2\varepsilon\sigma\Big\} \ge Ce^{-N_{n(\varepsilon)}(\log \frac{1}{\varepsilon})}.$$

Proof. a) The assumption made implies that from each offshoot of S_n grows at least one new branch. A plain calculation yields that $d_Z(s,t) := \|Z(s)-Z(t)\|_2 = \varepsilon_{n(s,t)}(3/2)^{1/2}$, $s,t\in T$. Further, we notice that

$$Z(t)-Z(s)=\sum_{n>n(s,t)}^{\infty}\varepsilon_n g_{\Pi_n(t)}.$$

Write $S_n = \{s_{n,j}, 1\le j\le N_n\}$ where we set $N_n = N(T,\varepsilon_n)$. Let $L_n\subset L$, $L_n=\{t_{n,j}, 1\le j\le N_n\}$ be such that $\Pi_n(t_{n,j}) = s_{n,j}$ for each j. Then $\mathbb{E}\,(Z(t_{n,i})-Z(t_{n,i-1}))^2 = (3/2)\varepsilon_n^2$, and since the random variables g_n are independent, we observe that

$$\mathbb{E}\,(Z(t_{n,2i})-Z(t_{n,2i-1}))(Z(t_{n,2j})-Z(t_{n,2j-1}))=0, \qquad (\forall 1\le j<i\le N_n/2). \tag{5.1}$$

So that the covariance matrix of $\{Z(t_{n,2i})-Z(t_{n,2i-1}), 1\le i\le N_n/2\}$ is diagonal with all diagonal entries equal to $(3/2)\varepsilon_n^2$. Consequently

$$\begin{aligned}\mathbb{P}\Big\{\sup_{s,t\in L}|Z(s)-Z(t)|\le\varepsilon_n\Big\} &\le \mathbb{P}\Big\{\sup_{1\le i\le N_n/2}|Z(t_{n,2i})-Z(t_{n,2i-1})|\le\varepsilon_n\Big\}\\ &= \mathbb{P}\Big\{\sup_{1\le i\le N_n/2}\frac{|Z(t_{n,2i})-Z(t_{n,2i-1})|}{\|Z(t_{n,2i})-Z(t_{n,2i-1})\|_2}\le c\Big\}\\ &\le e^{-\gamma N(T,\varepsilon_n)},\end{aligned}$$

c, γ being absolute constants. Let $0<\varepsilon\le \operatorname{diam}(T,d)$, and let n be such that $\varepsilon_{n+1}<\varepsilon\le\varepsilon_n$. Then

$$\begin{aligned}\mathbb{P}\Big\{\sup_{s,t\in L}|Z(s)-Z(t)|\le\varepsilon\Big\} &\le \mathbb{P}\Big\{\sup_{s,t\in L}|Z(s)-Z(t)|\le\varepsilon_n\Big\}\le e^{-\gamma N(T,\varepsilon_n)}\\ &\le e^{-\gamma N(T,2\varepsilon)}.\end{aligned}$$

b) This is a direct consequence of Theorem 3.1. □

References

[1] Aurzada F., Lifshits M. (2008): *Small deviation probability via chaining*, Stoch. Proc. & Appl. **118**, 2344–2368. 1–26.

[2] Fernique X. (1976): *Caractérisation de processus à trajectoires majorées ou continues*, Séminaire de Probabilités **XII** (Univ. Strasbourg 1976/77), 651–706.

[3] Knopp K. (1918): *Ein einfaches Verfahren zur Bildung stetiger nirgends differenzierbar Funktionen*, Math. Zeit. **2**, 1–26.

[4] Li W.V., Shao Q-M. (2005): *Gaussian Processes: Inequalities, Small Balls Probabilities and Applications*, in D.N. Shanbhag et al. (Eds.), Stochastic Processes: Theory and Methods, in: Handb. Statist. **19**, Elsevier, Amsterdam 2001, 533–597.

[5] Lifshits M. (1999): *Asymptotic behavior of small balls probabilities,* in: Prob. theory and Math. Statist. Proc. VII, International Vilnius Conference, VSP/TEV, 453–468.

[6] Loud W.S. (1951): *Functions with prescribed Lipschitz conditions*, Proc Amer. Math. Soc.. **2**, 358–360.

[7] Talagrand M. (1987): *Regularity of Gaussian processes*, Acta Math. **159** no. 1-2, 99–149.

[8] Talagrand M. (1993): *New Gaussian estimates for enlarged balls*, Geom. and Funct. Anal. **3**, 502–526.

[9] Talagrand M. (2005): *The generic chaining*, Springer Monogr. Math., Springer-Verlag, Berlin.

[10] Van der Waerden B.L. (1930): *Ein einfaches Beispiel einer nicht-differenzierbaren stetigen Funktion*, Math. Zeit. **32** 474–475.

[11] Weber M. (2009): *Dynamical Systems and Processes*, European Mathematical Society Publishing House, IRMA Lectures in Mathematics and Theoretical Physics **14** xiii+761p.
[12] Weber M. (2005): *Small deviations of Gaussian processes. Unpublished Notes*.

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